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MODELLING IN FRACTIONAL-ORDER SYSTEMS WITH APPLICATIONS IN ENGINEERING

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To our Teachers

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Preface

Recently fractional calculus has gained much attention of the scientists due to its application in the fields of science and engineering like fluid dynamics, bio engineering, heat transform Fuzzy analysis. Modelling of dynamical problems with fractional order differential equations are the base of different existing systems, solving these fraction order differential systems are challenging. It is worth mentioning that various aspects of fractional order (singular/non-singular kernels) modelling that may include deterministic or uncertain (viz. fuzzy or interval or stochastic) scenarios are also important to understand the behaviour of the physical systems. As such, the aim of this book will be to include computation and modelling for obtaining exact and/or numerical solutions for fractional order systems.

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Chapter 1

A Mathematical Analysis of a Caputo Fractional-order Cholera Model and its Sensitivity Analysis

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Abstract

This chapter presents a formulation of a Caputo fractional-order model to study the dynamics of cholera. Our primary objective is to investigate the behavior of the disease through a comprehensive analysis. Firstly, we establish the existence and uniqueness of the proposed model using the fixed-point theorem. Additionally, we conduct an analysis to ascertain the positivity and boundedness of the model, which confirms its mathematical and epidemiological well-posedness. Furthermore, we perform a sensitivity analysis to identify key parameters that significantly influence the basic reproduction number. This analysis allows us to determine the important features and their impact on the disease dynamics. Finally, employing effective numerical techniques, we generate various graphical results for the model using appropriate parameter values. By employing the Caputo fractional-order model, conducting rigorous analyses, and employing advanced numerical tools, this chapter provides valuable insights into the dynamics of cholera. The findings contribute to our understanding of the disease and aid in the development of effective control and prevention strategies.

Keywords

Cholera, mathematical model, sensitivity analysis, numerical simulations, fixed point theorems

1.1. Introduction

Cholera disease is a highly infectious and deadly disease that belongs to the family bacterium Vibrio cholerae. The issue at hand continues to be a matter of great significance in the realm of global health, especially in areas where sanitation is insufficient and the availability of potable water is limited [28]. Understanding the epidemiology and transmission dynamics of cholera requires essential prerequisite strategies in the form of effective prevention and control measures. Implementing effective measures for preventing and controlling cholera outbreaks is of utmost importance in managing their impact on public health. Employing key strategies such as vaccination, the provision of clean water and sanitation facilities, and conducting health education campaigns are crucial in this regard.

Extensive research conducted over the years, in accordance with data available from the World Health Organization (WHO) [27], has revealed the profound impact of the cholera outbreak on both children and mothers. These two groups are identified as the most vulnerable to the disease. The magnitude of the problem has been brought to light through these studies, and it is estimated that globally there are approximately 3-5 million cases of cholera reported annually. Out of these cases, it is alarming to note that 11.4 percent involve children below the age of 5, who are particularly susceptible to the severe effects of the disease. It is imperative to acknowledge that elderly individuals also contribute to this statistic. The estimated annual fatality rate for cholera is a tragic occurrence, ranging from 100,000 to 120,000 individuals. This sobering reality emphasizes the pressing need to address and efficiently manage this debilitating disease. The significance of targeting interventions and public health initiatives that prioritize the well-being and protection of vulnerable populations, particularly children and mothers, cannot be overstated.

Understanding the fundamental mechanisms of communicable disease transmission is crucial for implementing reliable precautionary and mitigation strategies against cholera outbreaks. Mathematical models enable researchers and policymakers to assess the effectiveness of interventions such as vaccination campaigns, sanitation improvements, and behavior change initiatives. By studying the model outputs, decision-makers can make informed choices regarding resource allocation and prioritize interventions based on their potential impact on controlling cholera outbreaks. In this regard, mathematical modeling offers a unique approach to comprehending the dynamics of communicable diseases at a fundamental level. Mathematical models are widely utilized to gain insights into environmental and epidemic issues. [14, 15, 35, 29, 18]. They serve as valuable scientific tools for evaluating and comparing mitigation and prevention strategies, as well as for assessing the impacts of various biological, sociocultural, and ecological factors on disease spread. By employing mathematical models, researchers can simulate and analyze the complex interactions involved in the transmission of cholera and other diseases [1, 3, 12, 6]. These models take into account variables such as population dynamics, disease characteristics, environmental factors, and human behavior. They allow for the exploration of different scenarios, providing valuable predictions and insights into the potential outcomes of various intervention strategies, see for example, [7, 8, 16, 22, 17, 34, 26].

Fractional calculus is a field of mathematics that focuses on the derivatives and integrals of non-integer orders. Its origins can be traced back to the works of Newton and Leibniz, and it has grown into a powerful tool used to comprehend and model complex systems. Since its inception, fractional calculus has discovered a wide range of practical uses across various disciplines, such as physics, engineering, finance, and biology, see, [8, 13, 20, 21, 24, 30, 31, 32]. The notion of fractional derivatives allows for a more comprehensive understanding of systems and phenomena that exhibit fractal, anomalous, or memory-like behavior. It provides a mathematical framework to describe processes with long-range dependence, non-locality, and fractional dynamics. Fractional calculus has proven particularly useful in modeling complex systems and phenomena where traditional calculus fails to capture the underlying dynamics accurately [2, 9, 33, 4, 5, 10].

Our objective is to investigate the transmission mechanics of cholera by considering the Caputo fractional-order setting, which has not been extensively studied in this context. To achieve this, we employed fixed-point theorems to explore the existence and uniqueness of solutions in the proposed model. In addition, we conducted a thorough sensitivity analysis of the corresponding basic reproduction number to examine the key sensitive parameters. Furthermore, to gain insights into the effectiveness of control strategies, we conducted numerical simulations. These simulations allowed us to evaluate various optimal control strategies, considering multiple controls. By systematically analyzing the outcomes of these simulations, we aimed to identify the most effective interventions for mitigating the spread of cholera.

1.2. Preliminaries Concepts

Some of the foremost theoretical aspects of fractional-order derivatives, which are key in proving the existence and uniqueness of the proposed model, were reviewed in this part.

Definition 1.1: [30] Suppose that $g \in L^1[0,a]$ for $0 < t < a$. If ρ $(0 < \rho < 1)$, then the operator defined by

$$\mathcal{I}_0^\rho g(t) = \frac{1}{\Gamma(\rho)} \int_0^t g(t)(t-x)^{\rho-1} dx, \tag{1.1}$$

is called the Riemann-Liouville fractional integral of the function g of order ρ such that $\Gamma(\cdot)$ denotes the gamma function defined by

$$\Gamma(z) = \int_0^\infty v^{z-1} e^{-v} dv, z \in \mathbb{C}/\{0,1,2,\ldots\}.$$

Definition 1.2: [30] Suppose that the function $f \in C^n[0,T]$, $n \in \mathbb{N}$ and $(0 < r < 1)$. The operator

$${}^C\mathcal{D}_0^\rho g(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{1}{(t-x)^\rho} \frac{d}{dt} f(x) dx, \quad t > 0, \tag{1.2}$$

is referred to the Caputo fractional derivative of order ρ of the function g. Note that if $r \to 1$ then ${}^C\mathcal{D}_0^\rho g(t) = \frac{d}{dt} g(t)$.

Lemma 1.1: *[30] If $g : [0, a] \to \mathbb{R}$ is continuous for any $z \in C^1[0, a]$. Thus, $z(t)$ is a solution of the following Caputo fractional-order differential equation:*

$$\begin{cases} {}^C\mathcal{D}_0^r z(t) = g(t), \ t \in [0, a], \ 0 < \rho \leq 1, \\ z(0) = z_0, \end{cases}$$

if and only if $z(t)$ obeys the following integral equation:

$$z(t) = z_0 - \frac{1}{\Gamma(\rho)} \int_0^t g(t)(t-x)^{\alpha-1} dx.$$

1.3. Formulation of the Caputo Fractional-order Cholera Model

We study the dynamic behavior of the cholera disease model as seen in [25]. The classical cholera disease model is governed by the following system of ordinary differential equations:

$$\begin{aligned} \frac{d\mathcal{S}(t)}{dt} &= \Pi - (\alpha_1 \mathcal{I} + \alpha_2 \mathcal{V} + d_1 + r_1)\mathcal{S} + \beta_2 \mathcal{R} \\ \frac{d\mathcal{R}(t)}{dt} &= (\alpha_1 \mathcal{I} + \alpha_2 \mathcal{V})\mathcal{S} - (d_1 + d_2 + \beta_1 + r_2)\mathcal{I} \\ \frac{d\mathcal{R}(t)}{dt} &= r_2 \mathcal{I} - r_1 \mathcal{S} - (\beta_2 + d_1)\mathcal{R} \\ \frac{d\mathcal{V}(t)}{dt} &= \beta_1 \mathcal{I} - r_3 \delta \mathcal{V}, \end{aligned} \tag{1.3}$$

with the initial conditions

$$\mathcal{S}(0) = \mathcal{S}_0 \geq 0, \ \mathcal{I}(0) = \mathcal{I}_0 \geq 0, \ \mathcal{R}(0) = \mathcal{R}_0 \geq 0, \ \text{and} \ \mathcal{V}(0) = \mathcal{V}_0 \geq 0, \tag{1.4}$$

The meaning of each state variables as well as the parameters associated with the model is given respectively, in Table 1.1.

Table 1.1: States variables and Parameters.

CompartmentDescription

$\mathcal{S}$	Population of human who are susceptible with cholera
$\mathcal{I}$	Population of people infected with cholera
$\mathcal{R}$	Population of people recovered
$\mathcal{V}$	Volume of vibrio bacteria in the ecosystem

Parameters	Biological Meanings
Π	Population recruitment rate of susceptible human individuals
α_1	Contact rate of between susceptible and infectious individuals
α_2	Rate at which susceptible human associated with contaminated water
β_1	Rate at which infectious humans contaminate the ecosystem
β_2	Loss of immunity by recovered individuals
d_1	Natural death rate
d_2	Disease-induced death rate
r_1	vaccination rate
r_2	Recovery rate
r_3	Treatment rate of water bodies in the ecosystem
δ	Decay rate of Vibrios
a_1	Adjustment rate of infection
a_2	Adjustment rate of Vibrio ingestion in the ecosystem
C_h	Density rate of Vibrio in the ecosystem
C_v	Density rate of infection among individuals
k_h	Rate of exposure by humans
k_e	Rate of exposure to contaminated water
C_w	Rate of compliance with hygienic water
C_s	Rate of compliance with environmental sanitation

Therefore, the proposed model in the setting of Caputo fractional derivative takes the form:

$$
\begin{aligned}
{}^{C}\mathcal{D}_0^{\rho}\mathcal{S} &= \Pi - (\alpha_1\mathcal{I} + \alpha_2\mathcal{V} + d_1 + r_1)\mathcal{S} + \beta_2\mathcal{R} \\
{}^{C}\mathcal{D}_0^{\rho}\mathcal{I} &= (\alpha_1\mathcal{I} + \alpha_2\mathcal{V})\mathcal{S} - (d_1 + d_2 + \beta_1 + r_2)\mathcal{I} \\
{}^{C}\mathcal{D}_0^{\rho}\mathcal{R} &= r_2\mathcal{I} - r_1\mathcal{S} - (\beta_2 + d_1)\mathcal{R} \\
{}^{C}\mathcal{D}_0^{\rho}\mathcal{V} &= \beta_1\mathcal{I} - r_3\delta\mathcal{V},
\end{aligned} \tag{1.5}
$$

where $\alpha_1 = \frac{C_h}{(1+a\mathcal{I})}$, $\alpha_2 = \frac{C_v}{V+b}$, and $C_h = k_h(1 - C_w)$, $C_v = k_e(1 - C_s)$ respectively.

1.3.1. Theoretical Analysis of the Caputo fractional-order Cholera Disease Model

This subsection is devoted to investigate the existence and uniqueness of solutions to the Caputo fractional-order model (2.1) by utilizing the concepts of fixed point theorems. To this purpose, we denote $\mathbb{B}(J)$ the Banach space of all continuous real-valued function defined on $J = [0, a]$ with sub norm and $\mathcal{Q} = \mathcal{B}(J) \times \mathcal{B}(J) \times \mathcal{B}(J) \times \mathcal{B}(J)$ with the norm $\|(\mathcal{S}, \mathcal{I}, \mathcal{R}, \mathcal{V})\| = \|\mathcal{S}\| + \|\mathcal{I}\| + \|\mathcal{R}\| + \|\mathcal{V}\|$, and $\|\mathcal{S}\| = \sup_{t\in J} |\mathcal{S}(t)|$, $\|\mathcal{I}\| = \sup_{t\in J} |\mathcal{I}(t)|$, $\|\mathcal{R}\| = \sup_{t\in J} |\mathcal{R}(t)|$, $\|\mathcal{V}\| = \sup_{t\in J} |\mathcal{V}(t)|$. Firstly, utilizing Caputo operator on the model (2.1) yields

$$
\begin{aligned}
\mathcal{S}(t) &= \mathcal{S}(0) + {}^{C}\mathcal{D}^{\rho}\left[\Pi - (\alpha_1\mathcal{I} + \alpha_2\mathcal{V} + d_1 + r_1)\mathcal{S} + \beta_2\mathcal{R}\right], \\
\mathcal{I}(t) &= \mathcal{I}(0) + {}^{C}\mathcal{D}^{\rho}\left[\alpha_1\mathcal{I} + \alpha_2\mathcal{V})\mathcal{S} - (d_1 + d_2 + \beta_1 + r_2)\mathcal{I}\right], \\
\mathcal{R}(t) &= \mathcal{R}(0) + {}^{C}\mathcal{D}^{\rho}\left[r_2\mathcal{I} - r_1\mathcal{S} - (\beta_2 + d_1)\mathcal{R}\right], \\
\mathcal{V}(t) &= \mathcal{V}(0) + {}^{C}\mathcal{D}^{\rho}\left[\beta_1\mathcal{I} - r_3\delta\mathcal{V}\right].
\end{aligned} \tag{1.6}
$$

denoting

$$
\begin{aligned}
g_1 &= \Pi - (\alpha_1\mathcal{I} + \alpha_2\mathcal{V} + d_1 + r_1)\mathcal{S} + \beta_2\mathcal{R}, \\
g_2 &= (\alpha_1\mathcal{I} + \alpha_2\mathcal{V})\mathcal{S} - (d_1 + d_2 + \beta_1 + r_2)\mathcal{I}, \\
g_3 &= r_2\mathcal{I} - r_1\mathcal{S} - (\beta_2 + d_1)\mathcal{R}, \\
g_4 &= \beta_1\mathcal{I} - r_3\delta\mathcal{V}.
\end{aligned} \tag{1.7}
$$

Thus, using the Caputo fraction operator, systems (1.6) takes the form

$$\begin{aligned}
\mathcal{S}(t) &= \mathcal{S}(0) + \mathcal{M}(\rho)\int_0^t (t-x)^{-\rho} g_1(\rho, x, \mathcal{S}(t))dx,\\
\mathcal{I}(t) &= \mathcal{I}(0) + \mathcal{M}(\rho)\int_0^t (t-x)^{-\rho} g_2(\rho, x, \mathcal{I}(t))dx,\\
\mathcal{R}(t) &= \mathcal{R}(0) + \mathcal{M}(\rho)\int_0^t (t-x)^{-\rho} g_3(\rho, x, \mathcal{R}(t))dx,\\
\mathcal{V}(t) &= \mathcal{V}(0) + \mathcal{M}(\rho)\int_0^t (t-x)^{-\rho} g_4(\rho, x, \mathcal{V}(t))dx.
\end{aligned} \tag{1.8}$$

It clear that $g_1(\mathcal{S}, x)$, $g_2(\mathcal{I}, x)$, $g_3(\mathcal{R}, x)$, and $g_4(\mathcal{V}, x)$ satisfy the Lipschitz condition if and only if $\mathcal{S}(t)$, $\mathcal{I}(t)$, $\mathcal{R}(t)$ and $\mathcal{V}(t)$ have an upper bound. Let $\mathcal{S}(t)$ and $\mathcal{S}^*(t)$ be two functions, then

$$\|g_1(\rho, x, \mathcal{S}(t)) - g_1(\rho, x, \mathcal{S}^*(t))\| = \| - (\alpha_1\mathcal{I} + \alpha_2\mathcal{V} + d_1 + r_1)(\mathcal{S}(t) - \mathcal{S}^*(t))\|. \tag{1.9}$$

Putting $\kappa_1 = \| - (\alpha_1\mathcal{I} + \alpha_2\mathcal{V} + d_1 + r_1)\|$, gives

$$\|g_1(\rho, x, \mathcal{S}(t)) - g_1(\rho, x, \mathcal{S}^*(t))\| \leq \kappa_1\|\mathcal{S}(t) - \mathcal{S}^*(t)\|. \tag{1.10}$$

Similarly putting $\kappa_2 = \|(\alpha_1-\alpha_2)-(d_1+d_2+\beta_1)+r_2\|$, $\kappa_3 = \|-(\beta_2 d_1)\|$ and $\kappa_4 = \|-r_2\delta\|$, we obtain

$$\begin{aligned}
\|g_2(\rho, x, \mathcal{I}(t)) - g_2(\rho, x, \mathcal{I}^*(t))\| &\leq \kappa_2\|\mathcal{I}(t) - \mathcal{I}^*(t)\|,\\
\|g_3(\rho, x, \mathcal{R}(t)) - g_3(\rho, x, \mathcal{R}^*(t))\| &\leq \kappa_3\|\mathcal{R}(t) - \mathcal{R}^*(t)\|,\\
\|g_4(\rho, x, \mathcal{V}(t)) - g_4(\rho, x, \mathcal{V}^*(t))\| &\leq \kappa_4\|\mathcal{V}(t) - \mathcal{V}^*(t)\|.
\end{aligned} \tag{1.11}$$

Thus, the Lipschitz condition is satisfied. Recursively, (1.8) can be expressed as

$$\begin{aligned}
\mathcal{S}_n(t) &= \mathcal{M}(\rho)\int_0^t \frac{g_1(\rho, x, \mathcal{S}_n(t))}{(t-x)^\rho}dx,\\
\mathcal{I}_n(t) &= \mathcal{M}(\rho)\int_0^t \frac{g_2(\rho, x, \mathcal{I}_n(t))}{(t-x)^\rho}dx,\\
\mathcal{R}_n(t) &= \mathcal{M}(\rho)\int_0^t \frac{g_3(\rho, x, \mathcal{R}_n(t))}{(t-x)^\rho}dx,\\
\mathcal{V}_n(t) &= \mathcal{M}(\rho)\int_0^t \frac{g_4(\rho, x, \mathcal{V}_n(t))}{(t-x)^\rho}dx,
\end{aligned} \tag{1.12}$$

associated with the initial conditions $\mathcal{S}_0(t) = \mathcal{S}(0)$, $\mathcal{I}_0(t) = \mathcal{I}(0)$, $\mathcal{R}_0(t) = \mathcal{R}(0)$, and

$\mathcal{V}_0(t) = \mathcal{V}(0)$. Now, the difference between the successive terns gives

$$\begin{aligned}
\Phi_{\mathcal{S},n}(t) &= \mathcal{S}_n(t) - \mathcal{S}_{n-1}(t) = \mathcal{M}(\rho)\int_0^t \frac{g_1(\rho,x,\mathcal{S}_{n-1}(x)) - g_1(\rho,x,\mathcal{S}_{n-2}(x))}{(t-x)^\rho}dx,\\
\Phi_{\mathcal{I},n}(t) &= \mathcal{I}_n(t) - \mathcal{I}_{n-1}(t) = \mathcal{M}(\rho)\int_0^t \frac{g_2(\rho,x,\mathcal{I}_{n-1}(x)) - g_2(\rho,x,\mathcal{I}_{n-2}(x))}{(t-x)^\rho}dx,\\
\Phi_{\mathcal{R},n}(t) &= \mathcal{R}_n(t) - \mathcal{R}_{n-1}(t) = \mathcal{M}(\mathcal{R})\int_0^t \frac{g_3(\rho,x,\mathcal{R}_{n-1}(x)) - g_3(\rho,x,\mathcal{R}_{n-2}(x))}{(t-x)^\rho}dx,\\
\Phi_{\mathcal{D},n}(t) &= \mathcal{V}_n(t) - \mathcal{V}_{n-1}(t) = \mathcal{M}(\rho)\int_0^t \frac{g_4(\rho,x,\mathcal{V}_{n-1}(x)) - g_4(\rho,x,V_{n-2}(x))}{(t-x)^\rho}dx.
\end{aligned} \tag{1.13}$$

Consider

$$\begin{aligned}
\mathcal{S}_n(t) &= \sum_{k=0}^{n}\Phi_{\mathcal{S}_n,k}(t),\\
\mathcal{I}_n(t) &= \sum_{k=0}^{n}\Phi_{\mathcal{I}_n,k}(t),\\
\mathcal{R}_n(t) &= \sum_{k=0}^{n}\Phi_{\mathcal{R}_n,k}(t),\\
\mathcal{V}_n(t) &= \sum_{k=0}^{n}\Phi_{\mathcal{V}_n,k}(t),
\end{aligned} \tag{1.14}$$

and in view of equations (1.10), (1.11) and the relation $\Phi_{\mathcal{S},n-1}(t) = \mathcal{S}_{n-1}(t) - \mathcal{S}_{n-2}(t)$, $\Phi_{\mathcal{I},n-1}(t) = \mathcal{I}_{n-1}(t) - \mathcal{I}_{n-2}(t)$, $\Phi_{\mathcal{R},n-1}(t) = \mathcal{R}_{n-1}(t) - \mathcal{R}_{n-2}(t)$, $\Phi_{\mathcal{V},n-1}(t) = \mathcal{V}_{n-1}(t) - \mathcal{V}_{n-2}(t)$, we get

$$\begin{aligned}
\|\Phi_{\mathcal{S},n}(t)\| &= \mathcal{M}(\rho)\int_0^t \frac{\|\Phi_{\mathcal{S},n-1}(x)\|}{(t-x)^\rho}dx,\\
\|\Phi_{\mathcal{I},n}(t)\| &= \mathcal{M}(\rho)\int_0^t \frac{\|\Phi_{\mathcal{I},n-1}(x)\|}{(t-x)^\rho}dx,\\
\|\Phi_{\mathcal{R},n}(t)\| &= \mathcal{M}(\rho)\int_0^t \frac{\|\Phi_{\mathcal{R},n-1}(x)\|}{(t-x)^\rho}dx,\\
\|\Phi_{\mathcal{V},n}(t)\| &= \mathcal{M}(\rho)\int_0^t \frac{\|\Phi_{\mathcal{V},n-1}(x)\|}{(t-x)^\rho}dx.
\end{aligned} \tag{1.15}$$

Thus from the above results, we can state and prove the following theorem:

Theorem 1.1: *The Caputo fractional-order cholera disease model* (2.1) *has a unique solution such that*

$$\frac{\mathcal{M}(\rho)}{\rho}a^\rho\kappa_j < 1,\ j = 1,\dots,4, \tag{1.16}$$

is true when $t \in [0, a]$.

Proof. From the above analysis, the functions $\mathcal{S}(t),\ \mathcal{I}(t),\ \mathcal{R}(t)$ and $\mathcal{V}(t)$ are bounded and $g_1,\ g_2,\ g_3, g_4$ obeys the Lipschitz condition. Therefore, in view of equations (1.15), and with the help of the recursive principle, we write

$$\begin{aligned}
\|\Phi_{\mathcal{S},n}(t)\| &\leq \|\mathcal{S}_0(t)\| \left(\frac{\mathcal{M}(\rho)}{\rho} a^\rho \kappa_1\right)^n, \\
\|\Phi_{\mathcal{I},n}(t)\| &\leq \|\mathcal{I}_0(t)\| \left(\frac{\mathcal{M}(\rho)}{\rho} a^\rho \kappa_2\right)^n, \\
\|\Phi_{\mathcal{R},n}(t)\| &\leq \|\mathcal{R}_0(t)\| \left(\frac{\mathcal{M}(\rho)}{\rho} a^\rho \kappa_3\right)^n, \\
\|\Phi_{\mathcal{V},n}(t)\| &\leq \|\mathcal{V}_0(t)\| \left(\frac{\mathcal{M}(\rho)}{\rho} a^\rho \kappa_4\right)^n.
\end{aligned} \tag{1.17}$$

Which implies that $\|\Phi_{\mathcal{S},n}(t)\| \to 0,\ \|\Phi_{\mathcal{I},n}(t)\| \to 0,\ \|\Phi_{\mathcal{R},n}(t)\| \to 0$, and $\|\Phi_{\mathcal{V},n}(t)\| \to 0$ for $n \to \infty$. Moreover, make use of the triangle inequality and the system (1.16) for any p, yields

$$\begin{aligned}
\|\mathcal{S}_{n+p}(t) - \mathcal{S}_n\| &\leq \sum_{k=n+1}^{n+p} j_1^k = \frac{j_1^{n+1} - j_1^{n+p+1}}{1 - j_1}, \\
\|\mathcal{I}_{n+p}(t) - \mathcal{I}_n\| &\leq \sum_{k=n+1}^{n+p} j_2^k = \frac{j_2^{n+1} - j_2^{n+p+1}}{1 - j_2}, \\
\|\mathcal{R}_{n+p}(t) - \mathcal{R}_n\| &\leq \sum_{k=n+1}^{n+p} j_3^k = \frac{j_3^{n+1} - j_3^{n+p+1}}{1 - j_3}, \\
\|\mathcal{V}_{n+p}(t) - \mathcal{V}_n\| &\leq \sum_{k=n+1}^{n+p} j_4^k = \frac{j_4^{n+1} - j_4^{n+p+1}}{1 - j_4},
\end{aligned} \tag{1.18}$$

such that $\frac{\mathcal{M}(\rho)}{\rho} a^\rho L_j \leq 1$. Thus, $\mathcal{S}_n,\ \mathcal{I}_n,\ \mathcal{R}_n,\ \mathcal{V}_n$ are Cauchy sequences in $\mathcal{B}(J)$. Though, it can be concluded that they are uniformly convergent. Through the limit theorem, the limit of the sequences (1.12) is the unique solution of the fractional-order cholera disease model (2.1).

1.3.2. Positivity and boundedness of solution.

Positivity and boundedness of solutions are important features of epidemiological models. We show that for all $t > 0$, all state variables are nonnegative, implying that any trajectory that begins with a positive initial condition will remain positive for all $t > 0$. From systems (2.1), we have

$$\begin{aligned} {}^{C}\mathcal{D}^{\rho}\mathcal{S}(t)|_{\mathcal{S}=0} &= \Pi + \beta_2\mathcal{R} \geq 0, \\ {}^{C}\mathcal{D}^{\rho}\mathcal{I}(t)|_{\mathcal{I}=0} &= \alpha_2\mathcal{V}\mathcal{S} \geq 0, \\ {}^{C}\mathcal{D}^{\rho}\mathcal{R}(t)|_{\mathcal{R}=0} &= r_2\mathcal{I} - r_1\mathcal{S} \geq 0, \\ {}^{C}\mathcal{D}^{\rho}\mathcal{V}(t)|_{\mathcal{V}=0} &= \beta_1\mathcal{I} \geq 0, \end{aligned} \tag{1.19}$$

on each hyperplane bounding the non-negative orthant. Moreover, Let $\mathcal{N}(t) = \mathcal{S}(t) + \mathcal{I}(t) + \mathcal{R}(t)$ be the total number of human population. Then, adding the first three equations of (2.1) gives

$${}^{C}\mathcal{D}^{\rho}\mathcal{N}(t) = \Pi - d_1\mathcal{N}, \tag{1.20}$$

then one has

$$\mathcal{N}(t) \leq \left(\mathcal{N}(0) - \frac{\Pi}{d_1}\right)\mathcal{E}_{\rho}(-d_1t^{\rho}) + \frac{\Pi}{d_1}.$$

Hence, the biological feasible region for the Caputo fractional-order model (2.1) is

$$\Omega = \left\{(\mathcal{S}(t), \mathcal{I}(t), \mathcal{R}(t)) \in \mathbb{R}^3_+ : 0 \leq \mathcal{N}(t) \leq \frac{\Pi}{d_1}\right\} \tag{1.21}$$

Therefore, the region Ω is positively invariant so that no solution path moves beyond the boundary of Ω. Thus proposed fractional-order model (2.1) is both mathematically and epidemiologically well-posed.

1.4. Sensitivity analysis in relation to R_0

In the present section, we utilize the forward sensitivity index to examine the sensitivity of the biological parameters in the proposed model (2.1) with respect to the basic reproduction number R_0. The basic reproduction number is crucial in determining the transmission of cholera disease, and it is essential to reduce it to a value below one to effectively control the infection.

The objective of the present investigation is to ascertain the sensitivity status of every parameter and enhance the output of the model. This will enable us to pinpoint the most crucial parameters and develop effective measures for controlling the spread of the cholera disease. We denote by

$$\Gamma_{\theta}^{R_0} = \frac{\theta}{R_0} \times \frac{\partial R_0}{\partial \theta}, \tag{1.22}$$

the normalized local sensitivity index of the output R_0 with respect to a parameter $\theta \in \{\Pi, \alpha_1, d_1, d_2, r_1, r_2,\}$ where

$$R_0 = \frac{\alpha_1 \Pi - (d_1 + r_1)(d_1 + d_2 + \Pi + r_2)}{(d_1 + r_1)}. \tag{1.23}$$

The results obtained from the sensitivity analysis (see, respectively, Table 1.2 and Figure 2.11) indicate that the recruitment rate of susceptible individuals Π and the effective contact rate between susceptible and infectious individuals α_1 are the most influential parameters contributing to the increase of the basic reproduction number R_0. Any increase or decrease in these parameters will correspondingly affect the value of R_0.

Furthermore, the natural death rate d_1, the death rate due to disease d_2, the rate of vaccination r_1, and the rate of recovery r_2 have an indirect negative impact on the basic reproduction number R_0. This implies that increasing these parameters will result in a reduction in the basic reproduction number R_0. Therefore, identifying optimal strategies for adjusting these parameters will play a crucial role in controlling the spread of the disease in the future.

Table 1.2: Forward normalized sensitivity indices.

Parameters	Index value
Π	1.0009
α_1	4.5482
d_1	-0.00048355
d_2	-0.00046992

r_1	-4.5477
r_2	-0.00041368

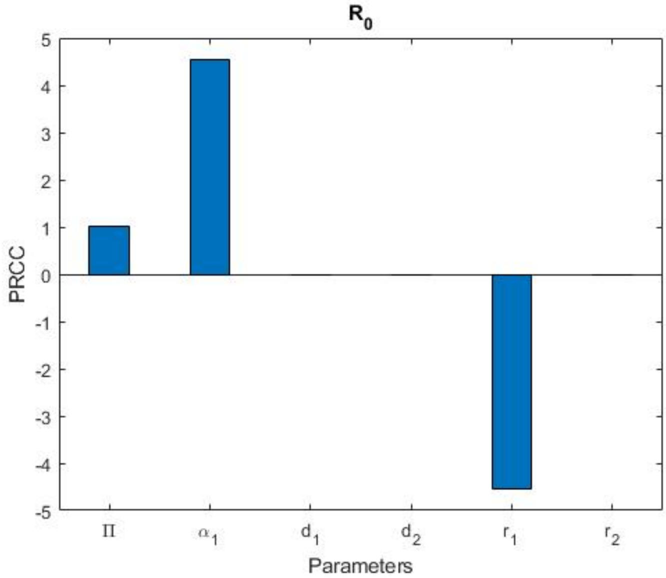

Figure 1.1: Sensitivity analysis of the parameter values.

1.5. Numerical simulations and discussions

The utilization of numerical schemes is crucial in both classical and fractional-order models, as they provide valuable insights into solution trajectories and additional information about solution paths. In this study, we have adopted the recent and effective numerical scheme proposed by Li et al. (2015) to accomplish this objective. The incorporation of such features ensures the secure and reliable application of this method in our simulations. For a more comprehensive analysis of the accuracy, stability, and convergence of this method, see [11, 23, 19].

Table 1.3, provides the values of the running parameters used during the numerical simulations of the proposed model (2.1).

Table 1.3: Numerical values for parameters of model (2.1).

Parameters	Parameters value
Π	10
a	0.2125
b	0.878455198
β_1	0.001
β_2	0.000001
d_1	0.000048
d_2	0.00132473
r_1	0.46791193
r_2	0.00116620
r_3	0.0001
δ	0.27707643
a_1	0.0346
a_2	0.0678
ϑ_1	0.0645
ϑ_2	0.0234
C_w	1
C_s	0.09
k_h	0.56341910
k_e	0.091371323

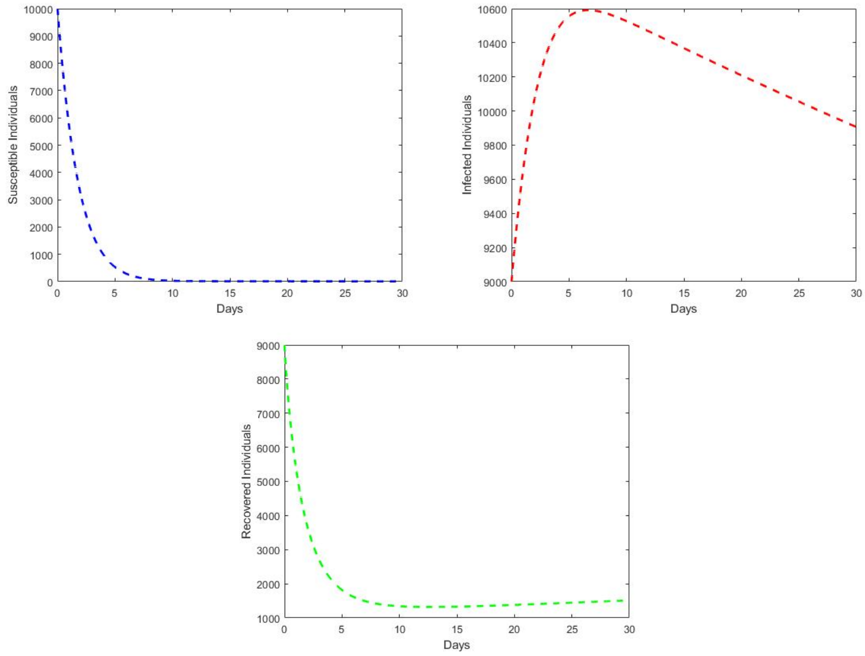

Figure 1.2: Profiles for behavior of each state variable for the classical version of the model.

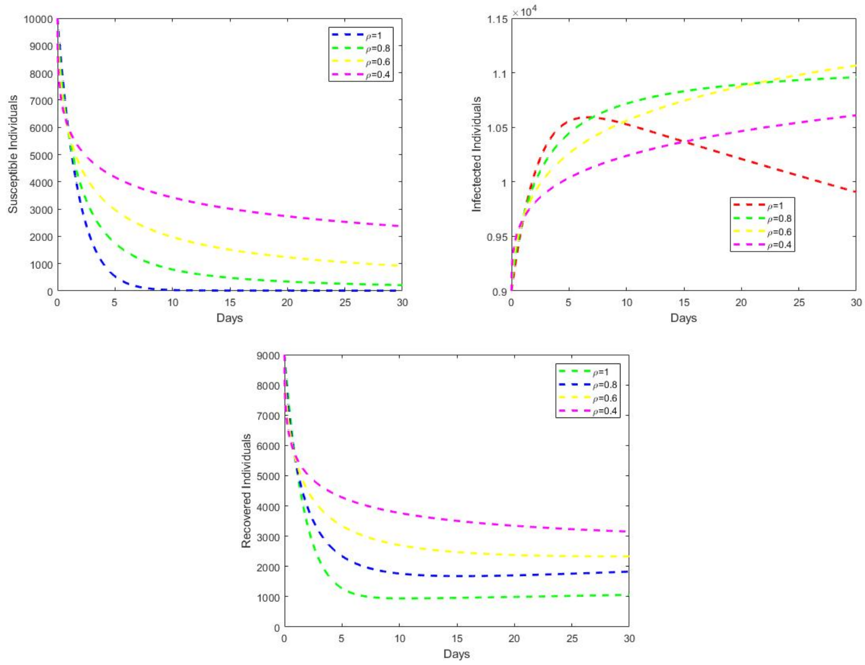

Figure 1.3: Profiles for behavior of each state variable for the proposed model (2.1), when $\rho = 1, 0.8, 0.6, 0.4$.

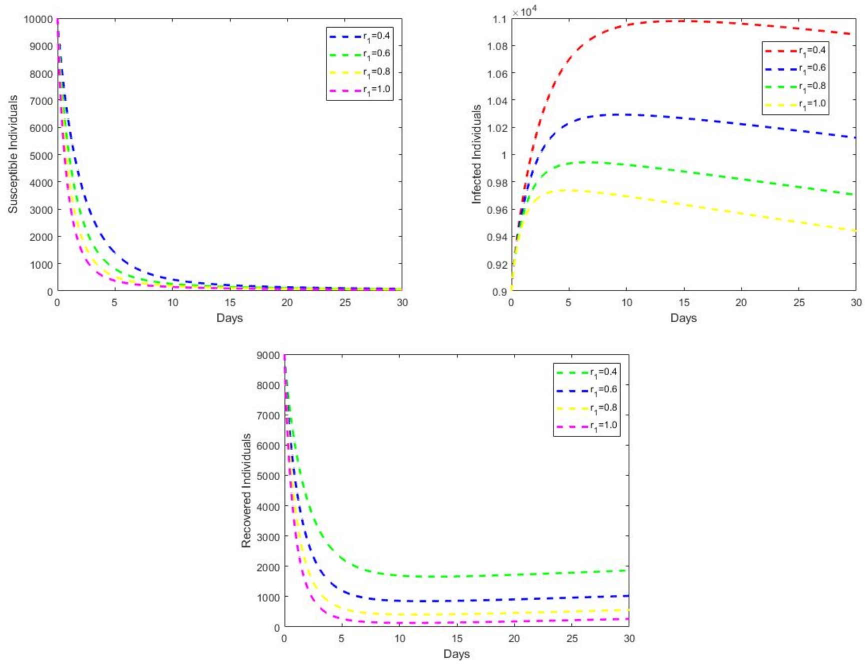

Figure 1.4: Effect of vaccination rate r_1 on the susceptible, infected and recovered compartments when $\rho = 0.9$.

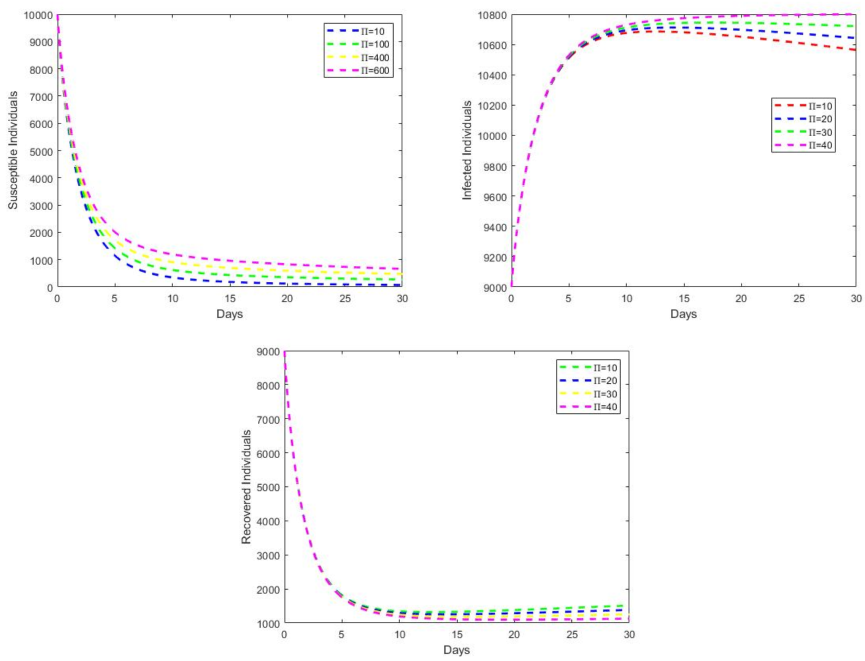

Figure 1.5: Effect of recruitment rate Π on the susceptible, infected and recovered compartments when $\rho = 0.9$.

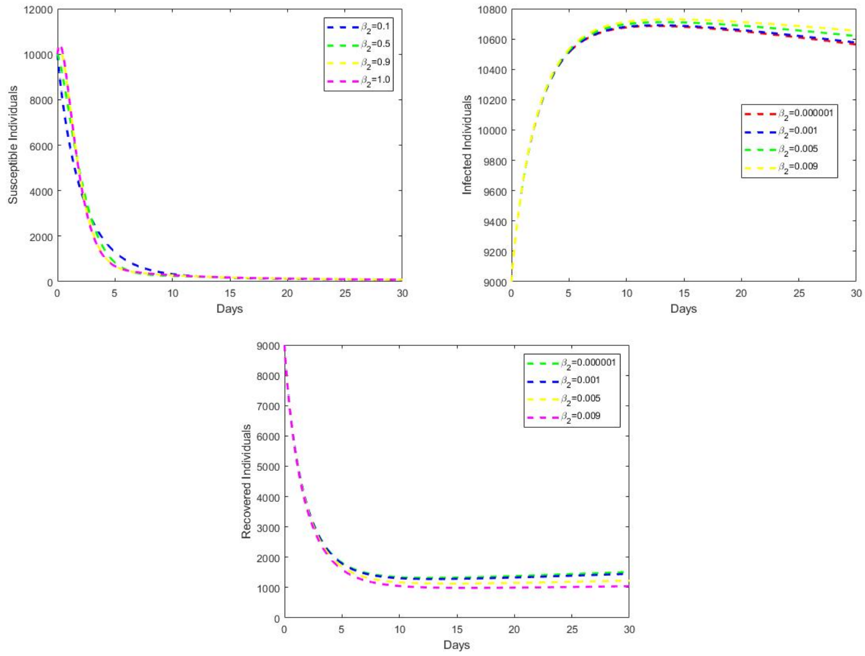

Figure 1.6: Effect of loss of immunity β_2 on the susceptible, infected and recovered compartments when $\rho = 0.9$.

The physical representation of the classical and Caputo fractional-order cholera models of the individual state variables of the model is illustrated in Fig. 1.2 and Fig. 1.3, respectively. In Fig. 1.3, we vary the order of the derivative $\rho = 1, 0.8, 0.6, \ and\ 0.4$ to visualize the complex dynamical behavior of each compartment. The results demonstrate an increase in the number of susceptible individuals $\mathcal{S}(t)$ and recovered individuals $\mathcal{R}(t)$ while the number of individuals infected with cholera $\mathcal{I}(t)$ decreases. Therefore, the proposed fractional-order cholera disease model (2.1), provides valuable insight into the complex dynamics of disease transmission and allows us to visualize the memory effect when varying the order of the derivative. This information can be useful in developing more effective strategies for controlling and preventing the spread of the virus in the future.

In Fig. 1.4, it can be observed that the fractional order ρ is fixed at 0.9, while the rate of vaccination r_1 is varied. As the value of r_1 increases, the number of susceptible individuals $\mathcal{S}(t)$ and infected individuals $\mathcal{I}(t)$ decreases, while the number of recovered individuals $\mathcal{R}(t)$ increases. This indicates that when the government or policymakers increase the rate of vaccination, the number of individuals at risk of contracting the disease, as well as the number of individuals infected with the disease, will decrease significantly.

In Fig. 1.5, the different dynamical phenomena are depicted when varying the recruitment rate Π. It can be observed that as more individuals are recruited into the susceptible population, the number of susceptible and infected individuals increases, while the number of recovered individuals decreases. This analysis emphasizes the importance of controlling the recruitment rate, especially during an outbreak, as it directly affects the dynamics of the disease spread.

The results depicted in Fig. 1.6 demonstrate the impact of individuals losing their immunity after recovering from the disease. The findings indicate that an increase in β_2 results in a decrease in the number of susceptible individuals, subsequently leading to an increase in the number of infected individuals. Additionally, as the number of infected individuals rises, there is a significant increase in the number of individuals who have recovered. These outcomes emphasize the significance of prioritizing intervention mechanisms to effectively prevent and control infection within the community.

1.6. Conclusions

In this chapter, we formulated and analyzed a mathematical model using a system of Caputo fractional-order differential equations to explore the impact of sensitive parameters as a strategy for disease control. We utilized fixed-point results to establish the existence and uniqueness of the proposed model. Furthermore, we demonstrated that the model is both mathematically and epidemiologically well-posed, as its solutions remained positive and bounded. The sensitivity analysis conducted in this study revealed that reducing the rate of recruitment of susceptible individuals Π and the effective contact rate λ between susceptible and infectious individuals (as shown in Table 1.2 and Fig. 2.11, respectively) are crucial factors in decreasing the basic reproduction number and mitigating the spread of the disease. Additionally, the numerical results demonstrated the advantages of employing a fractional-order model with memory effects over a classical-order model (as illustrated in Fig. 1.2 and Fig. 1.3). Furthermore, by varying certain parameters of the model, we were able to visualize the effects of these key parameters on controlling the spread of cholera disease, as shown in Fig. 1.4–Fig. 1.6. Based on these findings, we recommend that policymakers and health practitioners prioritize the use of effective media coverage to conduct widespread awareness campaigns on preventive measures, regardless of whether there is an ongoing epidemic or not. **Author contributions**

The authors contributed equally in writing this chapter. All authors read and approved the final chapter.

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Conflicts of interest

The author declare no conflict of interest.

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Chapter 2

Univalence Criteria for Integral Operators Defined by Rabotnov Fractional Exponential Functions

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Abstract

In this paper, we define some integral operators by using Rabotnov fractional exponential function. We use Kudriasov conditions for functions to be univalent to derive the univalence criteria for these integral operators.

2.1. Introduction and preliminaries

Geometric function theory heavily relies on special functions. The classic Bieberbach conjecture solution by L. de Branges may be the most well-known use. According to L.V. Ahlfors [2] in the proceedings of the 1986 meeting to commemorate the proof of L. de Branges' theorem, the surprising use of generalised hypergeometric functions by L. de Branges has generated a great deal of interest, and the geometric properties of the generalised, Gauss, and Kummer hypergeometric functions have been studied by many authors in the last few decades. Although the geometric characteristics of these functions are intriguing in and of themselves, they have shown to be helpful in numerous other geometric function theory issues.

Let $\mathcal{A}$ be the class of functions of the form

$$\mathfrak{g}(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2.1}$$

analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ and $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. In 1949, a Russian researcher Yuriy Nicholaevich Rabotnov, who carried his work in solid mechanics consisted of a broad range of topics, including as plasticity, creep theory, heredity mechanics, failure mechanics, nonelastic stability, composites, and shell theory introduced a function by utilizing two parameter Mittag-Leffler function $\mathbb{E}_{s,u}$. Today, this function is recognised on his name as the Rabotnov fractional exponential function or simply Rabotnov function It is defined as follows

$$R_{s,u}(z) = z^s \sum_{n=0}^{\infty} \frac{u^n}{\Gamma\left(\left(n+1\right)\left(1+s\right)\right)} z^{n(1+s)}, \quad s, u, z \in \mathbb{C}. \tag{2.2}$$

It is clear that this series will converge at any argument values. Noting that it becomes the typical exponential $exp(uz)$ for $s = 0$. The Mittag-Leffler function, which is well-known and frequently used in fractional calculus has a special case called the Rabotnov function. Following is a possible way to express the relationship between the Rabotnov function and the Mittag-Leffler function.

$$R_{s,u}(z) = z^s E_{1+s,1+s}(uz^{1+s}),$$

where E is Mittag-Leffler function and $s,\ u,\ z \in \mathbb{C}$. The two parameter Mittag-Leffler function $\mathbb{E}_{s,u}$ which can be considered as a simple generalization of classical Mittag-Leffler function $\mathbb{E}_{s,u}$ is given as

$$E_{s,u}\left(z\right) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(sn+u\right)}, \ s,\ u \in \mathbb{C},\ z \in \mathbb{C}. \tag{2.3}$$

The Mittag-Leffler functions described in(2.3) originally appeared in Wiman [21] work. These functions were later investigated by Agarwal [1]. The geometric characteristics and uses of the the function $E_{s,u}$ and some related functions have recently piqued scholars' curiosity. Yasar [22] investigated the characteristics of generalised Mittag-Leffler functions. The specific geometrical characteristics of this function were discussed by

Bansal [4]. Partial sums of normalised Mittag-Leffler functions were the focus of Raducanu's [17] work. The fact that the one- and two-parametric Mittag-Leffler functions are fractional extensions of the fundamental functions is another significant and intriguing aspect of Mittag-Leffler functions. That is $E_1(\pm z) = E_{1,1}(\pm z) = e^{\pm z}$, $E_{2,2}(z) = sinh(\sqrt{z})/\sqrt{z}$, $E_{1,2}(z) = (e^z - 1)/z$, $E_{2,1}(z) = cosh(\sqrt{z})$.

The function $R_{s,u}$ is not in class $\mathcal{A}$, therefore we take the transformation

$$\mathbb{R}_{s,u}(z) = z^{1/(1+s)}\Gamma\left(1+s\right)R_{s,u}(z^{1/(1+s)}) = z + \sum_{n=2}^{\infty}\frac{u^{n-1}\Gamma\left(1+s\right)z^n}{\Gamma\left(k\left(s+1\right)\right)},\ z \in \mathbb{U}. \tag{2.4}$$

The geometric properties of $\mathbb{R}_{s,u}$ have recently been discussed by Eker and Ece [9] and Eker et al [20]. Partial sums of generalized function of $\mathbb{R}_{s,u}$ have been studied by Frasin [11].

The univalence of integral operators involving special functions were first introduced by Baricz and Frasin in 2010 [5]. These integral operators were described using the normalized Bessel functions. The convexity and strongly convexity properties of these integral operator were explored by Frasin, Arif, and Raza [3, 10]. Authors have recently examined the families of one parameter integral operators employing a variety of special functions, including the Mittag-Leffler functions [19], Lommel functions [14], Struve functions [13], Dini functions [8] and generalized Bessel function [6, 7]. The Kudriasov type univalence conditions for the integral operators defined by generalised Bessel functions were studied by Raza et al [18], see also[15, 16]. We study the following integral operators defined by $\mathbb{R}_{s,u}$ and given by

$$\mathbb{F}_{s_i,u,\varsigma_i,\kappa}\left(z\right) = \left\{\kappa\int_0^z t^{\kappa-1}\prod_{i=1}^{n}\left(\frac{\mathbb{R}_{s_i,u}\left(t\right)}{\mathfrak{g}_i\left(t\right)}\right)^{\varsigma_i}dt\right\}^{1/\kappa},\quad \kappa,\ \varsigma_i \in \mathbb{C}, \tag{2.5}$$

$$\mathbb{H}_{s_i,u,\varsigma_i,\kappa}\left(z\right) = \left\{\kappa\int_0^z t^{\kappa-1}\prod_{i=1}^{n}\left(\frac{\mathbb{R}'_{s_i,u}\left(t\right)}{\mathfrak{g}'_i\left(t\right)}\right)^{\varsigma i}dt\right\}^{1/\kappa},\qquad \kappa,\ \varsigma_i \in \mathbb{C}, \tag{2.6}$$

and

$$\mathbb{I}_{s_i,u,\varsigma_i,\delta_i,\kappa}\left(z\right) = \left\{\kappa\int_0^z t^{\kappa-1}\prod_{i=1}^{n}\left(\frac{\mathbb{R}_{s_i,u}\left(t\right)}{t}\right)^{\varsigma_i}\left(\mathfrak{g}'_i\left(t\right)\right)^{\delta_i}dt\right\}^{1/\kappa}\qquad \kappa,\ \varsigma_i,\ \delta_i \in \mathbb{C}. \tag{2.7}$$

In order to derive our main results, we need the following lemmas.

Lemma 2.1: *[16] If $\mathfrak{g} \in \mathcal{A}$ such that*

$$\left(\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}\,(\zeta)}\right)\left|\frac{z\mathfrak{g}''(z)}{\mathfrak{g}'(z)}\right| \leq 1, \mathfrak{Re}(\zeta) > 0,$$

then for $\kappa \in \mathbb{C}$, $\mathfrak{Re}(\kappa) \geq \mathfrak{Re}(\zeta)$ the function

$$F_\kappa(z) = \left(\kappa \int_0^z t^{\kappa-1}\mathfrak{g}'(t)\,dt\right)^{\frac{1}{\kappa}} \in \mathcal{S}.$$

Lemma 2.2: *[12] If $\mathfrak{g}(z) = z + a_2 z^2 + \ldots.$ is analytic in $\mathbb{U}$ and if*

$$\left|\frac{\mathfrak{g}''(z)}{\mathfrak{g}'(z)}\right| \leq \mathbb{L} \simeq 3.05, \quad z \in \mathbb{U}$$

then $\mathfrak{g}$ is univalent in $\mathbb{U}$.

Remark 2.1: Here $\mathbb{L}$ is the approximate solution of $8\sqrt{y(y-2)^3} - 3(4-y)^2 - 12 = 0$. Its value is 3.03902118847875. Kudriasov used the value of $\mathbb{L}$ as 3.05.

Lemma 2.3: *[6] If $s > -1$ and $u \in \mathbb{C}$, then the function $\mathbb{R}_{s,u} : \mathbb{U} \to \mathbb{C}$ given by* (2.4) *holds the following*

(i)

$$\left|\frac{z\mathbb{R}'_{s,u}(z)}{\mathbb{R}_{s,u}(z)} - 1\right| \leq \frac{\frac{|u|}{1+s}e^{\frac{|u|}{1+s}}}{2 - e^{\frac{|u|}{1+s}}}, \; z \in \mathbb{U},$$

(ii)

$$\left|\frac{z\mathbb{R}''_{s,u}(z)}{\mathbb{R}'_{s,u}(z)}\right| \leq \frac{\frac{|u|(2s+|u|+2)e^{\frac{|u|}{1+s}}}{(1+\zeta)^2}}{2 - \frac{(s+|u|+1)e^{\frac{|u|}{1+s}}}{1+s}}, \qquad |u| < \ln\left(\frac{2(1+s)}{1+s+|u|}\right)^{1+s}, \; z \in \mathbb{U}.$$

2.2. Main Results

Theorem 2.1: *Let $i=1,\cdots,n,\ s_i>-1,\ u\in\mathbb{C}$ with $|u|<\ln\left(\frac{2(1+s)}{1+s+|u|}\right)^{1+s}$ and $\varsigma_i\in\mathbb{C}$ and suppose $s=\min\{s_1,s_2,\dots s_n\}$ and if $\mathfrak{g}_i\in\mathcal{A}$ with*

$$\left|\frac{\mathfrak{g}_i''(z)}{\mathfrak{g}_i'(z)}\right|\le\mathbb{L}\simeq 3.05,\quad z\in\mathbb{U},$$

such that the relation satisfy

$$\frac{1}{\mathfrak{Re}\zeta}\left(5+\frac{\frac{|u|}{1+s}e^{\frac{|u|}{1+s}}}{2-e^{\frac{|u|}{1+s}}}\right)\sum_{i=1}^{n}|\varsigma_i|<1, \tag{2.8}$$

for $0<\mathfrak{Re}(\zeta)<1$ and for $\mathfrak{Re}(\zeta)\ge 1$

$$\left[\frac{1}{\mathfrak{Re}(\zeta)}\left(1+\frac{\frac{|u|}{1+s}e^{\frac{|u|}{1+s}}}{2-e^{\frac{|u|}{1+s}}}\right)+4\right]\sum_{i=1}^{n}|\varsigma_i|<1. \tag{2.9}$$

Then for $\kappa\in\mathbb{C}$ such that $\mathfrak{Re}(\kappa)\ge\mathfrak{Re}(\zeta)>0$, the function $\mathbb{F}_{s_i,u,\varsigma_i,\kappa}(z)$ defined by (2.5) is univalent.

Proof. Consider the function

$$\mathbb{F}_{s_i,u,\varsigma_i}(z)=\int_0^z\prod_{i=1}^{n}\left(\frac{\mathbb{R}_{s_i,u}(t)}{\mathfrak{g}_i(t)}\right)^{\varsigma_i}dt. \tag{2.10}$$

It is clear that $\mathbb{F}_{s_i,u,\varsigma_i}(0)=\mathbb{F}'_{s_i,u,\varsigma_i}-1=0$. It follows easily that

$$\frac{\mathbb{F}''_{s_i,u,\varsigma_i}(z)}{\mathbb{F}'_{s_i,u,\varsigma_i}(z)}=\sum_{i=1}^{n}\varsigma_i\left\{\frac{\mathbb{R}'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}-\frac{\mathfrak{g}_i'(z)}{\mathfrak{g}_i(z)}\right\}.$$

Therefore, we obtain

$$\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)}\left|\frac{z\mathbb{F}''_{s_i,u,\varsigma_i}(z)}{\mathbb{F}'_{s_i,u,\varsigma_i}(z)}\right|\le\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}|\varsigma_i|\left\{\left|\frac{z\mathbb{R}'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}\right|+\left|\frac{z\mathfrak{g}_i'(z)}{\mathfrak{g}_i(z)}\right|\right\}. \tag{2.11}$$

Now using Lemma 2.2, it follows that $\mathfrak{g}_i\in\mathcal{S}$, $i=1\dots n$, and

$$\left|\frac{z\mathfrak{g}_i'(z)}{\mathfrak{g}_i(z)}\right|\le\frac{1+|z|}{1-|z|}. \tag{2.12}$$

By virtue of (2.11) and (2.12), we find that

$$\begin{aligned}\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)}\left|\frac{z\mathbb{F}''_{s_i,u,\varsigma_i}(z)}{\mathbb{F}'_{s_i,u,\varsigma_i}(z)}\right| &\leq \frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}|\varsigma_i|\left\{\left|\frac{z\mathbb{R}'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}\right|+\frac{1+|z|}{1-|z|}\right\}\\ &\leq \frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}|\varsigma_i|\left|\frac{z\mathbb{R}'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}\right|\\ &\quad+\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)}\frac{2}{1-|z|}\sum_{i=1}^{n}|\varsigma_i|.\end{aligned}$$

Firstly, we take the case

$$\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}|\varsigma_i|\left|\frac{z\mathbb{R}'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}\right|.$$

Now

$$\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}|\varsigma_i|\left|\frac{zR'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}\right| \leq \frac{1}{\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}|\varsigma_i|\left|\frac{z\mathbb{R}'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}\right|.$$

Using Lemma 2.3 (i), we have

$$\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}|\varsigma_i|\left|\frac{z\mathbb{R}'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}\right| \leq \frac{1}{\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}|\varsigma_i|\left\{1+\frac{\frac{|u|}{1+s_i}e^{\frac{|u|}{1+s_i}}}{2-e^{\frac{|u|}{1+s_i}}}\right\}.$$

Consider $h:(-1,\infty)\to\mathbb{R},\ h(y)=\frac{\frac{|u|}{1+y}e^{\frac{|u|}{1+y}}}{2-e^{\frac{|u|}{1+y}}}$. It is decreasing function, therefore

$$\frac{\frac{|u|}{1+s_i}e^{\frac{|u|}{1+s_i}}}{2-e^{\frac{|u|}{1+s_i}}} \leq \frac{\frac{|u|}{1+s}e^{\frac{|u|}{1+s}}}{2-e^{\frac{|u|}{1+s}}}.$$

Hence

$$\begin{aligned}&\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}|\varsigma_i|\left|\frac{z\mathbb{R}'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}\right| \leq\\ &\frac{1}{\mathfrak{Re}(\zeta)}\left\{1+\frac{\frac{|u|}{1+s}e^{\frac{|u|}{1+s}}}{2-e^{\frac{|u|}{1+s}}}\right\}\sum_{i=1}^{n}|\varsigma_i|.\end{aligned} \tag{2.13}$$

Now, we take

$$\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)}\frac{2}{1-|z|}\sum_{i=1}^{n}|\varsigma_i|.$$

For this part, the following cases arise:

(1) When $0 < \mathfrak{Re}(\zeta) < 1$, cosider the function $v : (0,1) \to \mathbb{R}$ defined by

$$v(y) = 1 - \mathfrak{b}^{2y},$$

where $y = \mathfrak{Re}(\zeta)$ and $|z| = \mathfrak{b}$. Then it is increasing and

$$1 - |z|^{2\mathfrak{Re}\zeta} \leq 1 - |z|^2,$$

therefore

$$\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)} \frac{2}{1-|z|} \sum_{i=1}^{n} |\varsigma_i| \leq \frac{4}{\mathfrak{Re}(\zeta)} \sum_{i=1}^{n} |\varsigma_i|. \tag{2.14}$$

From (2.13) and (2.14), for $0 < \mathfrak{Re}(\zeta) < 1$ we have

$$\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)} \left| \frac{z\mathbb{F}''_{s_i,u,\varsigma_i}(z)}{\mathbb{F}'_{s_i,u,\varsigma_i}(z)} \right| \leq \frac{1}{\mathfrak{Re}\zeta} \left(5 + \frac{\frac{|u|}{1+s} e^{\frac{|u|}{1+s}}}{2 - e^{\frac{|u|}{1+s}}} \right) \sum_{i=1}^{n} |\varsigma_i|. \tag{2.15}$$

(2) For the case $\mathfrak{Re}(\zeta) \geq 1$, thake the function $w : [1, \infty) \to \mathbb{R}$ defined by

$$w(y) = \frac{1-\mathfrak{b}^{2y}}{y},$$

where $y = \mathfrak{Re}(\zeta)$ and $|z| = \mathfrak{b}$. Then w is an decreasing function and

$$\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)} \leq 1 - |z|^2,$$

therefore

$$\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)} \frac{2}{1-|z|} \sum_{i=1}^{n} |\varsigma_i| \leq 4 \sum_{i=1}^{n} |\varsigma_i|. \tag{2.16}$$

Combining (2.15) and (2.16) for $\mathfrak{Re}(\zeta) \geq 1$, we get

$$\left(\frac{1-|z|^{2\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\zeta)} \right) \left| \frac{z\mathbb{F}''_{s_i,u,\varsigma_i}(z)}{\mathbb{F}'_{s_i,u,\varsigma_i}(z)} \right| \leq \frac{1}{\mathfrak{Re}(\zeta)} \left(1 + 4\mathfrak{Re}(\zeta) + \frac{\frac{|u|}{1+s} e^{\frac{|u|}{1+s}}}{2 - e^{\frac{|u|}{1+s}}} \right) \sum_{i=1}^{n} |\varsigma_i|. \tag{2.17}$$

From (2.8) , (2.15) and (2.9) and (2.17) , we obtain

$$\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}\left|\frac{z\mathbb{F}''_{s_i,u,\varsigma_i}(z)}{\mathbb{F}'_{s_i,u,\varsigma_i}(z)}\right|<1.$$

Now from (2.10) , we have $\mathbb{F}'_{s_i,u,\varsigma_i}(z)=\prod_{i=1}^{n}\left(\frac{\mathbb{R}_{s_i,u}(t)}{\mathfrak{g}_i(t)}\right)^{\varsigma_i}$. Therefore using Lemma 2.1 we get the required result.

Theorem 2.2: *Let* $i=1,\cdots,n,\ s_i>-1,\ u\in\mathbb{C}$ *with* $|u|<\ln\left(\frac{2(1+s)}{1+s+|u|}\right)^{1+s}$ *and* $\varsigma_i\in\mathbb{C}$ *and suppose* $s=\min\{s_1,s_2,\ldots s_n\}$ *and if* $\mathfrak{g}_i\in\mathcal{A}$*. Suppose* $s=\min\{s_1,s_2,\ldots s_n\}$ *and if* $\mathfrak{g}_i\in\mathcal{A}$ *with*

$$\left|\frac{\mathfrak{g}''_i(z)}{\mathfrak{g}'_i(z)}\right|\leq\mathbb{L}\simeq 3.05,\ \ z\in\mathbb{U}$$

and these numbers satisfy the relation

$$\left\{\frac{1}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\frac{\frac{|u|(2s+|u|+2)e^{\frac{|u|}{1+s}}}{(1+s)^2}}{2-\frac{(s+|u|+1)e^{\frac{|u|}{1+s}}}{1+s}}+\frac{2\mathbb{L}}{(2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)+1)^{(2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)+1)/2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}\right\}\sum_{i=1}^{n}|\varsigma_i|<1. \tag{2.18}$$

Then for $\kappa\in\mathbb{C}$ *such that* $,\mathfrak{Re}(\varsigma)\mathfrak{Re}(\kappa)\geq\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)>0,$ *the function* $\mathbb{H}_{s_i,u,\varsigma_i,\kappa}(z)$ *defined by* (2.6) *is univalent in* $\mathbb{U}$.

Proof. Take

$$\mathbb{H}_{s_i,u,\varsigma_i}(z)=\int_0^z\prod_{i=1}^{n}\left(\frac{\mathbb{R}'_{si_i,u}(t)}{\mathfrak{g}'_i(t)}\right)^{\varsigma_i}dt.$$

Clearly $\mathbb{H}_{s_i,u,\varsigma_i}\in\mathcal{A}$, that is $\mathbb{H}_{s_i,u,\varsigma_i}(0)=\mathbb{H}'_{s_i,u,\varsigma_i}-1=0$. Now

$$\frac{\mathbb{H}''_{s_i,u,\varsigma_i}(z)}{\mathbb{H}'_{s_i,u,\varsigma_i}(z)}=\sum_{i=1}^{n}\varsigma_i\left\{\frac{\mathbb{R}''_{si_i,u}(z)}{\mathbb{R}'_{si_i,u}(z)}-\frac{\mathfrak{g}''_i(z)}{\mathfrak{g}'_i(z)}\right\}.$$

Therefore, we obtain

$$\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}\left|\frac{z\mathbb{H}''_{s_i,u,\varsigma_i}(z)}{\mathbb{H}'_{s_i,u,\varsigma_i}(z)}\right|\leq\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}\sum_{i=1}^{n}|\varsigma_i|\left\{\left|\frac{z\mathbb{R}''_{si_i,u}(z)}{\mathbb{R}'_{si_i,u}(z)}\right|+|z|\left|\frac{\mathfrak{g}''_i(z)}{\mathfrak{g}'_i(z)}\right|\right\}. \tag{2.19}$$

This implies that

$$\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}\left|\frac{z\mathbb{H}''_{s_i,u,\varsigma_i}(z)}{\mathbb{H}'_{s_i,u,\varsigma_i}(z)}\right| \leq \frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}\sum_{i=1}^{n}|\varsigma_i|\left|\frac{z\mathbb{R}''_{s i_i,u}(z)}{\mathbb{R}'_{s i_i,u}(z)}\right| + \frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}|z|\sum_{i=1}^{n}|\varsigma_i|\left|\frac{\mathfrak{g}''_i(z)}{\mathfrak{g}'_i(z)}\right|.$$

Using the Lemma 2.2 and Lemma 2.3 (ii), we get

$$\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}\left|\frac{z\mathbb{H}''_{s_i,u,\varsigma_i}(z)}{\mathbb{H}'_{s_i,u,\varsigma_i}(z)}\right| \leq \frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}\sum_{i=1}^{n}|\varsigma_i|\frac{\frac{|u|(2s_i+|u|+2)e^{\frac{|u|}{1+s_i}}}{(1+s_i)^2}}{2-\frac{(s_i+|u|+1)e^{\frac{|u|}{1+s_i}}}{1+s_i}} + \frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}|z|\,\mathbb{L}\sum_{i=1}^{n}|\varsigma_i|.$$

Consider the function $h:[0,1]\to\mathbb{R}$, given by

$$h(y)=y\left(1-y^{2\mathfrak{b}}\right)/\mathfrak{b},$$

where $y=|z|$, $\mathfrak{b}=\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)$. Then

$$\max_{y\in[0,1]}h(y)=\frac{2}{(2\mathfrak{b}+1)^{(2\mathfrak{b}+1)/2\mathfrak{b}}}.$$

Also consider the function $l:(-1,\infty)\to\mathbb{R}$, $l(y)=\frac{\frac{|u|(2y+|u|+2)e^{\frac{|u|}{1+y}}}{(1+y)^2}}{2-\frac{(y+|u|+1)e^{\frac{|u|}{1+y}}}{1+y}}$. It is decreasing function, therefore

$$\frac{\frac{|u|(2s_i+|u|+2)e^{\frac{|u|}{1+s_i}}}{(1+s_i)^2}}{2-\frac{(s_i+|u|+1)e^{\frac{|u|}{1+s_i}}}{1+s_i}} \leq \frac{\frac{|u|(2s+|u|+2)e^{\frac{|u|}{1+s}}}{(1+s)^2}}{2-\frac{(s+|u|+1)e^{\frac{|u|}{1+s}}}{1+s}}.$$

This implies that

$$\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\left|\frac{z\mathbb{H}''_{s_i,u,\varsigma_i}(z)}{\mathbb{H}'_{s_i,u,\varsigma_i}(z)}\right| \leq \frac{1}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\frac{\frac{|u|(2s+|u|+2)e^{\frac{|u|}{1+s}}}{(1+s)^2}}{2-\frac{(s+|u|+1)e^{\frac{|u|}{1+s}}}{1+s}}\sum_{i=1}^{n}|\varsigma_i| + \frac{2\mathbb{L}}{(2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)+1)^{(2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)+1)/2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}\sum_{i=1}^{n}|\varsigma_i|.$$

Using (2.18) and Lemma 2.1, we have the required result.

Theorem 2.3: *Let $i=1,\cdots,n$, $s_i>-1$, $u\in\mathbb{C}$ with $|u|<\ln\left(\frac{2(1+s)}{1+s+|u|}\right)^{1+s}$ and $\varsigma_i,\delta_i\in\mathbb{C}$ and suppose $s=\min\{s_1,s_2,\dots s_n\}$ and if $\mathfrak{g}_i\in\mathcal{A}$ with*

$$\left|\frac{\mathfrak{g}_i''(z)}{\mathfrak{g}_i'(z)}\right|\le\mathbb{L}\simeq 3.05,\quad z\in\mathbb{U}$$

and these numbers satisfy the relation

$$\frac{1}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\frac{\frac{|u|}{1+s}e^{\frac{|u|}{1+s}}}{2-e^{\frac{|u|}{1+s}}}\sum_{i=1}^{n}|\varsigma_i|+\frac{2\mathbb{L}}{(2\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta+1)^{(2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)+1)/2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}\sum_{i=1}^{n}|\delta_i|<1. \tag{2.20}$$

Then for $\kappa\in\mathbb{C}$ such that $\mathfrak{Re}(\varsigma)\mathfrak{Re}(\kappa)\ge\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)>0$, the function $\mathbb{I}_{s_i,u,\varsigma_i,\delta_i,\kappa}(z)$ defined by (2.7) is univalent.

Proof. Consider the function

$$\mathbb{I}_{s_i,u,\varsigma_i,\delta_i}(z)=\int_0^z\prod_{i=1}^{n}\left(\frac{\mathbb{R}_{s_i,u}(t)}{t}\right)^{\varsigma_i}\left(\mathfrak{g}_i'(t)\right)^{\delta_i}dt. \tag{2.21}$$

Clearly, $\mathbb{I}_{s_i,u,\varsigma_i,\delta_i}\in\mathcal{A}$, that is $\mathbb{I}_{s_i,u,\varsigma_i,\delta_i}(0)=\mathbb{I}'_{s_i,u,\varsigma_i,\delta_i}-1=0$. Now

$$\frac{z\mathbb{I}''_{s_i,u,\varsigma_i,\delta_i}(z)}{\mathbb{I}'_{s_i,u,\varsigma_i,\delta_i}(z)}=\sum_{i=1}^{n}\varsigma_i\left(\frac{\mathbb{R}'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}-1\right)+\sum_{i=1}^{n}\delta_i\left\{\frac{z\mathfrak{g}_i''(z)}{\mathfrak{g}_i'(z)}\right\}.$$

This implies that

$$\begin{aligned}&\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\left|\frac{z\mathbb{I}''_{s_i,u,\varsigma_i,\delta_i}(z)}{\mathbb{I}'_{s_i,u,\varsigma_i,\delta_i}(z)}\right|\\ &\le\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}\left\{|\varsigma_i|\left|\frac{\mathbb{R}'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}-1\right|+|z|\,|\delta_i|\left|\frac{\mathfrak{g}_i''(z)}{\mathfrak{g}_i'(z)}\right|\right\}.\end{aligned} \tag{2.22}$$

Hence

$$\begin{aligned}\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\left|\frac{z\mathbb{I}''_{s_i,u,\varsigma_i,\delta_i}(z)}{\mathbb{I}'_{s_i,u,\varsigma_i,\delta_i}(z)}\right| &\le \frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}|\varsigma_i|\left|\frac{\mathbb{R}'_{s_i,u}(z)}{\mathbb{R}_{s_i,u}(z)}-1\right|\\ &\quad+\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}|z|\sum_{i=1}^{n}|\delta_i|\left|\frac{\mathfrak{g}_i''(z)}{\mathfrak{g}_i'(z)}\right|.\end{aligned}$$

By applying the Lemma 2.2 and Lemma 2.3 (i), we get

$$\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}\zeta}\left|\frac{z\mathbb{I}''_{s_i,u,\varsigma_i,\delta_i}(z)}{\mathbb{I}'_{s_i,u,\varsigma_i,\delta_i}(z)}\right| \leq$$

$$\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\sum_{i=1}^{n}|\varsigma_i|\frac{\frac{|u|}{1+s_i}e^{\frac{|u|}{1+s_i}}}{2-e^{\frac{|u|}{1+s_i}}}+\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}|z|\,\mathbb{L}\sum_{i=1}^{n}|\delta_i|.$$

Also, we have

$$\max_{y\in[0,1]} h(y)=\frac{2}{(2\mathfrak{b}+1)^{(2\mathfrak{b}+1)/2\mathfrak{b}}},$$

and

$$\frac{\frac{|u|}{1+s_i}e^{\frac{|u|}{1+s_i}}}{2-e^{\frac{|u|}{1+s_i}}}\leq\frac{\frac{|u|}{1+s}e^{\frac{|u|}{1+s}}}{2-e^{\frac{|u|}{1+s}}}.$$

Therefore

$$\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\left|\frac{z\mathbb{I}''_{s_i,u,\varsigma_i,\delta_i}(z)}{\mathbb{I}'_{s_i,u,\varsigma_i,\delta_i}(z)}\right| \leq \frac{1}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\frac{\frac{|u|}{1+s}e^{\frac{|u|}{1+s}}}{2-e^{\frac{|u|}{1+s}}}\sum_{i=1}^{n}|\varsigma_i| + \frac{2\mathbb{L}}{(2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)+1)^{(2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)+1)/2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}\sum_{i=1}^{n}|\delta_i| \tag{2.23}$$

Using (2.20) and (2.23), we get

$$\frac{1-|z|^{2\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}}{\mathfrak{Re}(\varsigma)\mathfrak{Re}(\zeta)}\left|\frac{z\mathbb{I}''_{s_i,u,\varsigma_i,\delta_i}(z)}{\mathbb{I}'_{s_i,u,\varsigma_i,\delta_i}(z)}\right|<1.$$

Now from (2.21), it is clear that $\mathbb{I}'_{s_i,u,\varsigma_i,\delta_i}(z)=\prod_{i=1}^{n}\left(\frac{\mathbb{R}_{s_i,u}(t)}{t}\right)^{\varsigma_i}(\mathfrak{g}'_i(t))^{\delta_i}$. By applying Lemma 2.1, we achieve the desired outcome.

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Chapter 3

Newly Discovered Inclusions through Generalized Mittag-Leffler Functions in Double Fractional Integrals: Novel Discoveries

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Abstract In this chapter, we explore new fractional versions of various classical inequalities, including those of Hermite-Hadamard, Pachpatte, and Hermite-Hadamard-Fejer-like forms. We introduce a novel set of generalized AB-fractional operators that involve double integrals with a generalized Mittag-Leffler mapping as a kernel. These operators include various established fractional operators and lead to the creation of new fractional operators under suitable conditions. Using these operators, we formulate new bounds for Hermite-Hadamard-type inequalities, achieved through interval-valued coordinated pre-invex functions. This chapter concludes with some numerical and visual analysis of the main results that depict the validity and significance of the primary outcomes of the chapter.

Keywords:

Hermite–Hadamard inequality; interval-valued pre-invex function; interval-valued Mittag–Leffler fractional double AB-integrable functions; interval-valued co-ordinated pre-invex function.

3.1. Introduction and Preliminaries

Let us begin by reviewing some initial concepts and outcomes.

Definition 3.1: If a set $\mathcal{C} \subseteq \mathbb{R}$ satisfies the following condition, it is referred to as a convex set:

$$(1-\varrho)\mathfrak{v}_1 + \varrho\mathfrak{v}_2 \in \mathcal{C}, \quad \forall \mathfrak{v}_1, \mathfrak{v}_2 \in \mathcal{C}, \varrho \in [0,1].$$

Definition 3.2: A function $\Pi : \mathcal{C} \to \mathbb{R}$ is said to be convex, if

$$\Pi\big((1-\varrho)\mathfrak{v}_1 + \varrho\mathfrak{v}_2\big) \leq (1-\varrho)\Pi(\mathfrak{v}_1) + \varrho\Pi(\mathfrak{v}_2), \quad \forall \mathfrak{v}_1, \mathfrak{v}_2 \in \mathcal{C},\ \varrho \in [0,1].$$

Since the time of Archimedes, traditional concepts of convexity have been the subject of intense research due to their practical applications across a variety of scientific domains. This area of study has been explored through multiple mathematical approaches including functional analysis, topological vector spaces, fixed point theory, operator theory, advanced mathematical analysis, error analysis, and optimality theory. Of particular interest in current investigations is the study of convex functions, which has led to numerous generalizations, extensions, and modifications of convex mappings. These mappings have proven to be extremely useful in nonlinear and applied analysis, relative entropy in quantum mechanics, electrical networking, optimization, information theory, and particularly in the theory of inequality. The theory of convex mappings has had a significant impact on the development of inequalities, as the concept of convexity allows for the direct formulation of many fundamental inequalities.

One notable outcome of the theory of inequalities is the widely recognized Jensen's inequality, which is closely related to convex functions.

Theorem 3.1: *If* $\Pi : I = [\mathfrak{v}_1, \mathfrak{v}_2] \subseteq \mathbb{R} \to \mathbb{R}$ *is a convex function, then*

$$\Pi\left(\sum_{i=1}^{n} \widetilde{\mu}_i \mathfrak{x}_i\right) \leq \sum_{i=1}^{n} \widetilde{\mu}_i \Pi(\mathfrak{x}_i),$$

where $\mathfrak{x}_i \in [\mathfrak{v}_1, \mathfrak{v}_2]$ *and* $\widetilde{\mu}_i \in [0,1]$, $(i = \overline{1,n})$ *with* $\sum_{i=1}^{\infty} \widetilde{\mu}_i = 1$.

For more detail, see [13].

We will now bring to mind another interesting outcome that gives us both the essential and adequate condition for a function to be convex, as well as bounds for the mean integral. This outcome is commonly referred to as Hermite-Hadamard's inequality.

Theorem 3.2: *If* $\Pi : [\mathfrak{v}_1, \mathfrak{v}_2] \subseteq \mathbb{R} \to \mathbb{R}$ *is a convex function, then*

$$\Pi\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right) \leq \frac{1}{\mathfrak{v}_2 - \mathfrak{v}_1} \int_{\mathfrak{v}_1}^{b} f(\mathfrak{x}) \mathrm{d}\mathfrak{x} \leq \frac{\Pi(\mathfrak{v}_1) + \Pi(\mathfrak{v}_2)}{2}.$$

Let's revisit the notion of invex sets and pre-invex functions.

Definition 3.3*([40])*: If a bifunction $\widetilde{\eta} : K \times K \to \mathbb{R}$ exists such that a set K satisfies the following condition, then K is referred to as invex:

$$\mathfrak{x} + \varrho\widetilde{\eta}(\mathfrak{y}, \mathfrak{x}) \in K, \quad \forall\, \mathfrak{x}, \mathfrak{y} \in K,\ \forall\, \varrho \in [0,1].$$

Definition 3.4*([40])*: If bifunction $\widetilde{\eta}(\cdot,\cdot)$ exists, a function Π defined on an invex set K is considered to be pre-invex if it satisfies the following condition:

$$\Pi(\mathfrak{x} + \varrho\widetilde{\eta}(\mathfrak{y}, \mathfrak{x})) \leq (1 - \varrho)\Pi(\mathfrak{x}) + \varrho\Pi(\mathfrak{y}), \quad \forall\, \mathfrak{x}, \mathfrak{y} \in K,\ \forall\, \varrho \in [0,1].$$

Condition C, introduced by Mohan and Neogy [30], is an extremely useful condition that will significantly contribute to proving some of our primary results. To ensure a comprehensive understanding, we will revisit the specifics of Condition C.

For an invex set $K \subset \mathbb{R}$ with respect to a bifunction $\widetilde{\eta}(.,.)$, the following condition holds for any $\varrho \in [0,1]$ and $\mathfrak{x}, \mathfrak{y} \in A$:

$$\widetilde{\eta}(\mathfrak{y}, \mathfrak{y} + \varrho\widetilde{\eta}(\mathfrak{x}, \mathfrak{y})) = -\varrho\widetilde{\eta}(\mathfrak{x}, \mathfrak{y})$$

$$\widetilde{\eta}(\mathfrak{x},\mathfrak{y}+\varrho\widetilde{\eta}(\mathfrak{x},\mathfrak{y}))=(1-\varrho)\widetilde{\eta}(\mathfrak{x},\mathfrak{y}).$$

Also

$$\widetilde{\eta}(\mathfrak{y}+\varrho_2\widetilde{\eta}(\mathfrak{x},\mathfrak{y}),\mathfrak{y}+\varrho_1\widetilde{\eta}(\mathfrak{x},\mathfrak{y}))=(\varrho_2-\varrho_1)\widetilde{\eta}(\mathfrak{x},\mathfrak{y}),$$

For any $\varrho_1,\varrho_2\in[0,1]$.

Matloka [28] expanded the concept of pre-invexity to include coordinated pre-invexity.

Definition 3.5: If $(\mathfrak{x},\mathfrak{y})\in A\times B$, then $A\times B$ is considered to be an invex set with respect to bifunctions $\widetilde{\eta}1$ and $\widetilde{\eta}2$ if the following condition is satisfied for every $(\widehat{w},z)\in A\times B$ and $\varrho,s\in[0,1]$:

$$(\mathfrak{x}+\varrho\widetilde{\eta}_1(\widehat{w},\mathfrak{x}),\mathfrak{y}+s\widetilde{\eta}_2(\mathfrak{y},z))\in A\times B.$$

With this in mind, Matloka et al. [28] proposed the coordinated pre-invex function class as follows:

Definition 3.6: If a function $\Pi:\Delta=[\mathfrak{v}_1,\mathfrak{v}_2]\times[\mathfrak{v}_3,\mathfrak{v}_4]\subset\mathbb{R}^2\to\mathbb{R}$ is a coordinated pre-invex function with respect to bifunctions $\widetilde{\eta}1$ and $\widetilde{\eta}2$, then:

$$\begin{aligned}&\Pi((\mathfrak{v}_1+\varrho\widetilde{\eta}_1(\mathfrak{v}_1,\mathfrak{v}_2),\mathfrak{v}_3+\widetilde{\eta}_2s(\mathfrak{v}_3,\mathfrak{v}_4)))\\&\leq(1-\varrho)(1-s)\Pi(\mathfrak{v}_1,\mathfrak{v}_3)+(1-\varrho)sf(\mathfrak{v}_1,\mathfrak{v}_4)+\varrho(1-s)\Pi(\mathfrak{v}_2,\mathfrak{v}_3)+tsf(\mathfrak{v}_2,\mathfrak{v}_4).\end{aligned}$$

In recent times, fractional calculus (F.C) has emerged as an effective tool for investigating various mathematical and physical models. The combination of F.C and integral inequality theory has become a highly active field of research. For instance, Sarikaya et al. [35] skillfully employed fractional calculus concepts to establish fractional analogues of Hermite-Hadamard's inequality. This paper opened up new avenues in this direction, leading to extensive research. Du et al. [14] utilized the ideas of (s,m)-preinvex functions to derive variants of Hermite-Hadamard's inequality. Iqbal et al. [19] employed the concepts of conformable fractional calculus to obtain new refinements of Hermite-Hadamard's inequality. Khurshid et al. [23] derived conformable fractional

Hermite-Hadamard's inequality using the class of pre-invex functions. Lei et al. [26] established some new bounds related to Fej'er-Hermite-Hadamard-type inequalities and identified their corresponding applications. Liao et al. [27] investigated Sugeno integral concerning α-pre-invex functions. Set et al. [36] obtained several Fej'er-Hermite-Hadamard-type inequalities for conformable fractional integrals. Zhang et al. [42] established some new k-fractional integral inequalities containing multiple parameters through generalized (s, m)-preinvexity. Mohammed et al. [29] established generalized Hermite-Hadamard inequalities via tempered fractional integrals. Srivastava et al. [38] developed new Chebyshev-type inequalities via a general family of fractional integral operators with a modified Mittag-Leffler kernel. Fernandez et al. [17] investigated series representations for fractional-calculus operators involving generalized Mittag-Leffler functions.

In 1924, Burkil [7] examined the characteristics of interval-valued mappings, which were later extended to multi-valued mappings by Kolmogorov [24]. However, due to a lack of applications in science, this idea remained relatively unknown for many years. In 1966, Moore [31] published an exceptional monograph on interval analysis, making it applicable to error analysis and serving as a launching point for further research in this field. Nikodem et al. [32] introduced the concept of interval-valued convexity, and in [5], the authors developed some inequalities involving interval-valued mappings. Zhang et al. [41] investigated set-valued Jensen-like inequalities. Zhao's work on inequalities associated with interval-valued convexities of various kinds is considered ground breaking in this field. In [43], Zhao et al. derived some inequalities involving h-interval-valued mappings. In [46], the authors formulated new generalized variants of inequalities in the setting of time-scale calculus. Zhao et al. [47, 45] analyzed Chebyshev-type inclusions regarding interval-valued convex mappings and used the concept of harmonically interval-valued convexity to draw new refinements of existing results. In [44], Zhao et al. considered the concept of coordinated interval-valued convexity to evaluate new inequalities in the rectangular form in $\mathbb{R}^2$. Bin-Mohsin et al. used the concept of harmonically interval-valued and coordinated harmonically interval-valued convexity to establish some Hermite-Hadamard-type inclusions. For readers interested in exploring this topic further, references [9, 10, 12, 11, 34, 18, 22, 21, 39, 20, 6] provide additional resources.

In order to understand the main results of this paper, it is necessary to review some foundational concepts from special functions and fractional calculus.

Definition 3.7*([16])*: The Mittag–Leffler function denoted by $E_\alpha(\cdot)$ is defined as:

$$E_\alpha(\varrho)=\sum_{n=0}^{\infty}\frac{\varrho^n}{\Gamma(\alpha n+1)},$$

where $\varrho,\alpha\in\mathbb{C},\Re(\alpha)>0$ and $\Gamma(\cdot)$ is the gamma function.

Definition 3.8*([16])*: Let $\widetilde{\mu},\alpha,l,\widetilde{\gamma},\mathfrak{v}_3\in\mathbb{C},\Re(\widetilde{\mu}),\Re(\alpha),\Re(l)>0,\Re(\mathfrak{v}_3)>\widetilde{\gamma}>0$ with $p\geq 0,\widetilde{\delta}>0$ and $0<k\leq\widetilde{\delta}+\Re(\widetilde{\mu})$. Then the extended generalized Mittag–Leffler function $E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l}(\varrho;p)$ is defined by:

$$E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l}(\varrho;p)=\sum_{n=0}^{\infty}\frac{\beta_p(\widetilde{\gamma}+nk,\mathfrak{v}_3-\widetilde{\gamma})}{\beta(\widetilde{\gamma},\mathfrak{v}_3-\widetilde{\gamma})}\frac{(\mathfrak{v}_3)_{nk}}{\Gamma(\widetilde{\mu}n+\alpha)}\frac{\varrho^n}{(l)_{n\widetilde{\delta}}},$$

where $\beta_p(\cdot,\cdot)$ is defined by

$$\beta_p(\mathfrak{x},\mathfrak{y})=\int_0^1\varrho^{\mathfrak{x}-1}(1-\varrho)^{\mathfrak{y}-1}e^{-\frac{p}{\varrho(1-\varrho)}}\mathrm{d}\varrho,$$

and $(\mathfrak{v}_3)_{nk}:=\frac{\Gamma(\mathfrak{v}_3+nk)}{\Gamma(\mathfrak{v}_3)}$.

Definition 3.9*([16])*: Let $\widehat{w},\widetilde{\mu},\alpha,l,\widetilde{\gamma},\mathfrak{v}_3\in\mathbb{C},\Re(\widetilde{\mu}),\Re(\alpha),\Re(l)>0,\Re(\mathfrak{v}_3)>\widetilde{\gamma}>0$ with $p\geq 0,\widetilde{\delta}>0$ and $0<k\leq\widetilde{\delta}+\Re(\widetilde{\mu})$. Let $\aleph\in L_1[\mathfrak{v}_1,\mathfrak{v}_2]$ and $\mathfrak{x}\in[\mathfrak{v}_1,\mathfrak{v}_2]$. Then the generalized left-hand side fractional integral operator containing Mittag–Leffler function $E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l}(\varrho;p)$ is defined by:

$$\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+}\right)\{\aleph(\mathfrak{x};p)\}=\int_{\mathfrak{v}_1}^{\mathfrak{x}}(\mathfrak{x}-\varrho)^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,\widehat{w},l}(\widehat{w}(\mathfrak{x}-\varrho)^{\widetilde{\mu}};p)\aleph(\varrho)\mathrm{d}\varrho.$$

The generalized right-hand side fractional integral operator is given as follows:

$$\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2-}\right)\{\aleph(\mathfrak{x};p)\}=\int_{\mathfrak{x}}^{\mathfrak{v}_2}(\varrho-\mathfrak{x})^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,\widehat{w},l}(\widehat{w}(\varrho-\mathfrak{x})^{\widetilde{\mu}};p)\aleph(\varrho)\mathrm{d}\varrho.$$

In [1] Atangana-Baleanu presented the following new integrals which are known as Atangana-Baleanu fractional integrals.

Definition 3.10: The fractional integral related to the new nonlocal kernel of a mapping $\Pi \in L^1(\mathfrak{v}_1, \mathfrak{v}_2)$ is defined as follows:

$$
{}^{AB}_{\mathfrak{v}_1}I^{\alpha}_{\varrho}\Pi(\varrho) = \frac{1-\alpha}{B(\alpha)}\Pi(\varrho) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\mathfrak{v}_1}^{\varrho}\Pi(\mathfrak{x})(\varrho-\mathfrak{x})^{\alpha-1}\mathrm{d}\mathfrak{x},
$$

$$
where \ \mathfrak{v}_2 > \mathfrak{v}_1, \alpha \in [0,1].
$$

The right hand side of integral operator as follows:

$$
{}^{AB}I^{\alpha}_{\mathfrak{v}_2}\Pi(\varrho) = \frac{1-\alpha}{B(\alpha)}\Pi(\varrho) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\varrho}^{\mathfrak{v}_2}\Pi(\mathfrak{x})(\mathfrak{x}-\varrho)^{\alpha-1}\mathrm{d}\mathfrak{x}.
$$

Here, $\Gamma(\alpha)$ is the gamma function. $B(\alpha) > 0$ is called the normalization function.

Definition 3.11 *([15])***:** Assume that $\widehat{F}(\mathfrak{x}) = [\underline{\Pi}(\mathfrak{x}), \overline{\Pi}(\mathfrak{x})]$, $\mathfrak{x} \in \Lambda^0$ is the interval-valued function, where Λ^0 is the interior of $\Lambda \subset \mathbb{R}$. We call $\widehat{F}(\mathfrak{x})$ Lebesgue integrable if the functions $\underline{\Pi}(\mathfrak{x})$ as well as $\overline{\Pi}(\mathfrak{x})$ are both measurable along with Lebesgue integrable defined over Λ^0. Furthermore, we can write $\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}\widehat{F}(\mathfrak{x})\mathrm{d}\mathfrak{x}$ as follows:

$$
\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}\widehat{F}(\mathfrak{x})\mathrm{d}\mathfrak{x} = \left[\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}\underline{\Pi}(\mathfrak{x})\mathrm{d}\mathfrak{x}, \int_{\mathfrak{v}_1}^{\mathfrak{v}_2}\overline{\Pi}(\mathfrak{x})\mathrm{d}\mathfrak{x}\right].
$$

Now we suppose that the rectangle $\overline{\Omega} = [\mathfrak{v}_1, \mathfrak{v}_2] \times [\mathfrak{v}_3, \mathfrak{v}_4]$ where $\mathfrak{v}_1 < \mathfrak{v}_2$ and $\mathfrak{v}_3 < \mathfrak{v}_4$. A group of numbers $\{\widehat{w}_{i-1}, \nu_i, \widehat{w}_i\}_{i=1}^m$ is termed as tagged partition π_1 with respect to interval $[\mathfrak{v}_1, \mathfrak{v}_2]$ if: $\pi_1 : \mathfrak{v}_1 = \widehat{w}_0 < \widehat{w}_1 < \widehat{w}_2 < ... < \widehat{w}_m = \mathfrak{v}_2$ and if $\widehat{w}_{i-1} \leq \nu_i \leq \widehat{w}_i, i = i, 2, 3, ...m$. If $\overline{\Delta}\widehat{w}_i < \widetilde{\delta}$ for every i, then the partition π_1 is known as $\widetilde{\delta}$- fine. If $P(\widetilde{\delta}, [\mathfrak{v}_1, \mathfrak{v}_2])$ is collection of all $\widetilde{\delta}-$partitions of interval $[\mathfrak{v}_1, \mathfrak{v}_2]$. Suppose that $\{\widehat{w}_{i-1}, \nu_i, \mathfrak{x}_i\}_{i=1}^m$ belongs to $\widetilde{\delta}$-fine π_1 regarding $[\mathfrak{v}_1, \mathfrak{v}_2]$ and $\{z_{j-1}, \theta_j, \theta_j\}_{j=1}^n$ is another $\widetilde{\delta}$-fine π_2 regarding $[\mathfrak{v}_3, \mathfrak{v}_4]$, then the series rectangles
$\overline{\Delta} = [\widehat{w}_{i-1}, \widehat{w}_i] \times [\theta_{j-1}, \theta_j]$ are the partitions of $\overline{\Omega}$, and the points $(\nu_i, \lambda_j) \in \overline{\Delta}^{\circ}$. Furthermore letting $\pi(\widetilde{\delta}, \overline{\Omega})$ be the set of all $\widetilde{\delta}-$fine partitions of rectangle Ω along with $\pi_1 \times \pi_2$, in which $\pi_1 \in \pi(\widetilde{\delta}, [\mathfrak{v}_1, \mathfrak{v}_2])$ along with $\pi_2 \in P(\widetilde{\delta}, [\mathfrak{v}_3, \mathfrak{v}_4])$. We consider using the notation $\Delta A_{i,j}$ denotes the area of the rectangle $\widetilde{\delta}_{i,j}$. Within every rectangle $\overline{\Delta}_{i,j}$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, Let us take taking into account arbitrary ν_i, λ_j and we have $S(\Pi, \pi, \widetilde{\delta}, \overline{\Omega}) = \sum_{i=1}^m \sum_{j=1}^n \Pi(\nu_i, \lambda_j)\overline{\Delta}A_{i,j}$, which is named as $S(\Pi, P, \widetilde{\delta}, \overline{\Omega})$ is the integral sum of Π in connection $P(\widetilde{\delta}, \overline{\Omega})$.

Theorem 3.3*([15])*: *Assume that the interval valued function* $\Pi : \overline{\Omega} = [\mathfrak{v}_1, \mathfrak{v}_2] \times [\mathfrak{v}_3, \mathfrak{v}_4] \to \mathbb{R}_I, p$ *then* Π *is called double integrable defined on rectangle* $\overline{\Omega}$ *with interval valued double integral* $U = \int\int_{\overline{\Omega}} \Pi(\mathfrak{x}, \mathfrak{y}) \mathrm{d}A$*, if for every* $\epsilon > 0$ *there exist* $\widetilde{\delta} > 0$ *satisfying that*

$\mathfrak{v}_4(S(\Pi, P, \widetilde{\delta}, \overline{\Omega})) < \epsilon$ *for each* $P \in P(\widetilde{\delta}, \overline{\Omega})$.

Theorem 3.4*([15])*: *Assume that* $\widehat{F} : \overline{\Omega} \to \mathbb{R}_I$ *is interval valued double integrable defined on* $\overline{\Omega}$*, then*

$$\int\int_{\overline{\omega}} \widehat{F}(\mathfrak{x}, \mathfrak{y}) \mathrm{d}A = \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \int_{\mathfrak{v}_3}^{d} \widehat{F}(\mathfrak{x}, \mathfrak{y}) \mathrm{d}\mathfrak{y} \mathrm{d}\mathfrak{x}.$$

Now, we recall the AB like generalized integral operator involving five parameter Mittag-Leffler function and some results which will play important role in obtaining the our main outcomes.

Definition 3.12*([4])*: Assume that $\widehat{F} : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathbb{R}_I$ is an interval-valued function satisfying that $\widehat{F}(\mathfrak{x}) = [\underline{\Pi}(\mathfrak{x}), \overline{\Pi}(\mathfrak{x})]$, in which the functions $\underline{\Pi}(\mathfrak{x})$ and $\overline{\Pi}(\mathfrak{x})$ are both Riemann integrable defined on the interval $[\mathfrak{v}_1, \mathfrak{v}_2]$. The interval-valued left-sided as well as right-sided generalized AB type fractional operator involving generalized Mittag–Leffler functions are defined by

$$\begin{aligned}&\left(\epsilon^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}_{\widetilde{\mu}, \alpha, l, \widehat{w}, \mathfrak{v}_1 +}\right) \{\widehat{F}(\mathfrak{v}_2; p)\} \\ &= \frac{1-\alpha}{B(\alpha)} \widehat{F}(\mathfrak{v}_2) - \frac{\alpha}{B(\alpha)} \int_{\mathfrak{v}_1}^{\mathfrak{x}} (\mathfrak{x} - \varrho)^{\alpha - 1} E^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}_{\widetilde{\mu}, \alpha, \widehat{w}, l} (\widehat{w}(\mathfrak{x} - \varrho)^{\widetilde{\mu}}; p) \widehat{F}(\varrho) \mathrm{d}\varrho, \quad \mathfrak{x} > \mathfrak{v}_1\end{aligned}$$

and

$$\begin{aligned}&\left(\epsilon^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}_{\widetilde{\mu}, \alpha, l, \widehat{w}, \mathfrak{v}_2 -}\right) \{\widehat{F}(\mathfrak{v}_1; p)\} \\ &= \frac{1-\alpha}{B(\alpha)} \widehat{F}(\mathfrak{v}_1) - \frac{\alpha}{B(\alpha)} \int_{\mathfrak{x}}^{\mathfrak{v}_2} (\varrho - \mathfrak{x})^{\alpha - 1} E^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}_{\widetilde{\mu}, \alpha, \widehat{w}, l} (\widehat{w}(\varrho - \mathfrak{x})^{\widetilde{\mu}}; p) \widehat{F}(\varrho) \mathrm{d}\varrho, \quad \mathfrak{x} < \mathfrak{v}_2,\end{aligned}$$

with $\alpha \in [0, 1]$ and $B(\alpha) > 0$ is the normalization function, where $B(0) = B(1) = 1$. Obviously, we observe that

$$\left(\epsilon^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}_{\widetilde{\mu}, \alpha, l, \widehat{w}, \mathfrak{v}_1 +}\right) \{\widehat{F}(\mathfrak{v}_2; p)\} = \left[\left(\epsilon^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}_{\widetilde{\mu}, \alpha, l, \widehat{w}, \mathfrak{v}_1 +}\right) \{\underline{\Pi}(\mathfrak{v}_2; p)\}, \left(\epsilon^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}_{\widetilde{\mu}, \alpha, l, \widehat{w}, \mathfrak{v}_1 +}\right) \{\overline{\Pi}(\mathfrak{v}_2; p)\}\right],$$

and

$$\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2^-}\right)\{\widehat{F}(\mathfrak{v}_1;p)\} = \left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2^-}\right)\{\underline{\Pi}(\mathfrak{v}_1;p)\}, \left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2^-}\right)\{\overline{\Pi}(\mathfrak{v}_1;p)\}\right].$$

Now we recall Hermite-Hadamard's inequality involving generalized convexity, which is given as:

Theorem 3.5 *([4])*: *Assume that the function $\widehat{F} : [\mathfrak{v}_1, \mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)] \to \mathbb{R}^+_I$ is interval-valued pre-invex and $\widehat{F}(\mathfrak{x}) = [\underline{\Pi}(\mathfrak{x}), \overline{\Pi}(\mathfrak{x})]$. Then, the successive inclusion relations hold true:*

$$\begin{aligned}
&\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2}\right) E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha+1,\widehat{w},l}(\widehat{w}(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}};p) \\
&\supseteq \frac{B(\alpha)}{2\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\right)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1);p)\} + \left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\right)\{\widehat{F}(\mathfrak{v}_1;p)\}\right. \\
&\left. -\frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)))\right] \\
&\supseteq \frac{(\widehat{F}(\mathfrak{v}_1)+\widehat{F}(\mathfrak{v}_2)}{2} E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha+1,\widehat{w},l}(\widehat{w}(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}};p). \qquad (3.1)
\end{aligned}$$

Next, we rewrite the Pachpatte type inclusions.

Theorem 3.6 *([4])*: *Suppose that the two functions $\widehat{F}, G : [\mathfrak{v}_1, \mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)] \to \mathbb{R}^+_I$ are both interval-valued pre-invex and $\widehat{F}(\mathfrak{x}) = [\underline{\Pi}(\mathfrak{x}), \overline{\Pi}(\mathfrak{x})]$, and $G(\mathfrak{x}) = [\underline{g}(\mathfrak{x}), \overline{g}(\mathfrak{x})]$. Then, the successive inclusion relation holds true:*

$$\begin{aligned}
&\frac{B(\alpha)}{\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\right)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1);p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1);p)\}\right. \\
&\quad + \left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\right)\{\widehat{F}(\mathfrak{v}_1;p)G(\mathfrak{v}_1;p)\} \qquad (3.2) \\
&- \frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1)G(\mathfrak{v}_1)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))) \\
&\supseteq 2E_0[-\mathbb{P}(\mathfrak{v}_1,\mathfrak{v}_2)+\mathbb{Q}(\mathfrak{v}_1,\mathfrak{v}_2)] + \mathbb{P}(\mathfrak{v}_1,\mathfrak{v}_2)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha+1,\widehat{w},l}(\widehat{w}(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}};p), \qquad (3.3)
\end{aligned}$$

where

$$\mathbb{P}(\mathfrak{v}_1,\mathfrak{v}_2) := \widehat{F}(\mathfrak{v}_1)G(\mathfrak{v}_1)+\widehat{F}(\mathfrak{v}_2)G(\mathfrak{v}_2)$$

$$\mathbb{Q}(\mathfrak{v}_1,\mathfrak{v}_2) := \widehat{F}(\mathfrak{v}_1)G(\mathfrak{v}_2)+\widehat{F}(\mathfrak{v}_2)G(\mathfrak{v}_1)$$

and

$$E_0 := \sum_{n=0}^{\infty} \frac{\beta_p(\widetilde{\gamma}+nk, \mathfrak{v}_3-\widetilde{\gamma})}{\beta(\widetilde{\gamma}, \mathfrak{v}_3-\widetilde{\gamma})} \frac{(\mathfrak{v}_3)_{nk}}{(\widetilde{\mu}n+\alpha+1)(\widetilde{\mu}n+\alpha+2)\Gamma(\widetilde{\mu}n+\alpha)} \frac{\varrho^n}{(l)_{n\widetilde{\delta}}}.$$

Theorem 3.7 *([4])*: *Under the assumptions of Theorem 3.6, the following successive inclusion relation holds true:*

$$\begin{aligned}
&\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2}\right) G\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2}\right) E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha+1,\widehat{w},l}(\widehat{w}(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}};p)\\
&\supseteq \frac{B(\alpha)}{4\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}} \left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\right)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1);p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1);p)\}\right.\\
&+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\right)\{\widehat{F}(\mathfrak{v}_1;p)G(\mathfrak{v}_1;p)\} - \frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1)G(\mathfrak{v}_1)\\
&\left.+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)))\right] + \frac{1}{2}E_0[\mathbb{P}(\mathfrak{v}_1,\mathfrak{v}_2)-\mathbb{Q}(\mathfrak{v}_1,\mathfrak{v}_2)]\\
&+\frac{1}{4}\mathbb{Q}(\mathfrak{v}_1,\mathfrak{v}_2)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha+1,\widehat{w},l}(\widehat{w}(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}};p). \qquad (3.4)
\end{aligned}$$

Now, we are in position to introduce the following definitions that will be used in the sequel.

Definition 3.13: Let $\widehat{F} : [\mathfrak{v}_1,\mathfrak{v}_2]\times[\mathfrak{v}_3,\mathfrak{v}_4] \subset \mathbb{R}^2 \to \mathbb{R}_I$ be a given interval-valued function in two variables forms satisfying that $\widehat{F}(\varrho,s) = [\underline{\Pi}(\varrho,s),\overline{\Pi}(\varrho,s)]$, and let $\widehat{F}$ be interval Riemann integrable on the rectangle $[\mathfrak{v}_1,\mathfrak{v}_2]\times[\mathfrak{v}_3,\mathfrak{v}_4]$. The interval-valued generalized AB type double integral operator involving generalized Mittag-Leffler function are defined by

$$\begin{aligned}
&\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha_1,l,\widehat{w},\mathfrak{v}_1^+,\mathfrak{v}_3^+}\right)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4;p)\} = \frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4) + \frac{\alpha\beta}{B(\alpha)B(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}\int_{\mathfrak{v}_3}^{\mathfrak{v}_4}(\mathfrak{v}_2-\varrho)^{\alpha-1}(\mathfrak{v}_4-s)^{\beta-1}\\
&\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_2-\varrho)^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_4-s)^{\widetilde{\mu}_2};p)\widehat{F}(\varrho,s)\mathrm{d}s\mathrm{d}\varrho, \quad \textit{where} \quad \mathfrak{v}_2>\mathfrak{v}_1,\ \mathfrak{v}_4>\mathfrak{v}_3.\\
&\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha_1,l,\widehat{w},\mathfrak{v}_1^+,\mathfrak{v}_4^-}\right)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3;p)\} = \frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3) + \frac{\alpha\beta}{B(\alpha)B(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}\int_{\mathfrak{v}_3}^{\mathfrak{v}_4}(\mathfrak{v}_2-\varrho)^{\alpha-1}(s-\mathfrak{v}_3)^{\beta-1}\\
&E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_2-\varrho)^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(s-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}(\varrho,s)\mathrm{d}s\mathrm{d}\varrho, \quad \textit{where} \quad \mathfrak{v}_2>\mathfrak{v}_1,\ \mathfrak{v}_4>\mathfrak{v}_3.\\
&\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha_1,l,\widehat{w},\mathfrak{v}_2^-,\mathfrak{v}_3^+}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)\} = \frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4) + \frac{\alpha\beta}{B(\alpha)B(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}\int_{\mathfrak{v}_3}^{\mathfrak{v}_4}(\varrho-\mathfrak{v}_1)^{\alpha-1}(\mathfrak{v}_4-s)^{\beta-1}\\
&E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\varrho-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}(\mathfrak{v}_4-s)^{\widetilde{\mu}_2};p)\widehat{F}(\varrho,s)\mathrm{d}s\mathrm{d}\varrho, \quad \textit{where} \quad \mathfrak{v}_2>\mathfrak{v}_1,\ \mathfrak{v}_4>\mathfrak{v}_3.\\
&\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha_1,l,\widehat{w},\mathfrak{v}_2^-,\mathfrak{v}_4^-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)\} = \frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3) + \frac{\alpha\beta}{B(\alpha)B(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}\int_{\mathfrak{v}_2}^{\mathfrak{v}_4}(\varrho-\mathfrak{v}_1)^{\alpha-1}(s-\mathfrak{v}_3)^{\beta-1}\\
&E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\varrho-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(s-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}(\varrho,s)\mathrm{d}s\mathrm{d}\varrho, \quad \textit{where} \quad \mathfrak{v}_2>\mathfrak{v}_1,\ \mathfrak{v}_4>\mathfrak{v}_3.
\end{aligned}$$

Here, $\widehat{w}=(\widehat{w}_1,\widehat{w}_2), \alpha_1=(\alpha,\beta), \widetilde{\mu}=(\widetilde{\mu}_1,\widetilde{\mu}_2),\ \varrho, s\in[0,1]$, and $B(\alpha), B(\beta)$ are normalizing functions with $B(\alpha), B(\beta) > 0$, and $B(0) = B(1) = 1$.

Definition 3.14: Given an interval-valued function $\widehat{F} : K \to \mathbb{R}_I$, where $K \subset \mathbb{R}$ is an invex set, we call that $\widehat{F}$ is an interval-valued pre-invex function if and only if

$$\widehat{F}(\mathfrak{x} + (1-\varrho)\widetilde{\eta}(\mathfrak{y}, \mathfrak{x})) \supseteq \varrho \widehat{F}(\mathfrak{x}) + (1-\varrho)\widehat{F}(\mathfrak{y}),$$

holds true for each $\mathfrak{x}, \mathfrak{y} \in K$ and for all $\varrho \in [0, 1]$.

Definition 3.15*([25])*: Suppose that $\widehat{F} : \Omega \to \mathbb{R}_I^+$ belongs to an interval-valued function regarding two variable forms, where $\Omega := [\mathfrak{v}_1, \mathfrak{v}_1 + \widetilde{\eta}(\mathfrak{v}_2, \mathfrak{v}_1)] \times [\mathfrak{v}_3, \mathfrak{v}_3 + \widetilde{\eta}(\mathfrak{v}_4, \mathfrak{v}_3)]$ is a given rectangle. We say that $\widehat{F}$ is an interval-valued co-ordinated pre-invex function if and only if

$$\begin{aligned}&\widehat{F}(\mathfrak{v}_1 + (1-\varrho)\widetilde{\eta}(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + (1-s)\widetilde{\eta}(\mathfrak{v}_4, \mathfrak{v}_3))\\ &\supseteq ts\widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3) + \varrho(1-s)\widehat{F}(\mathfrak{v}_1, \mathfrak{v}_4) + (1-\varrho)s\widehat{F}(\mathfrak{v}_2, \mathfrak{v}_3) + (1-\varrho)(1-s)\widehat{F}(\mathfrak{v}_2, \mathfrak{v}_4),\end{aligned}$$

holds for every $[\mathfrak{v}_1, \mathfrak{v}_1 + \widetilde{\eta}(\mathfrak{v}_2, \mathfrak{v}_1)], [\mathfrak{v}_3, \mathfrak{v}_3 + \widetilde{\eta}(\mathfrak{v}_4, \mathfrak{v}_3)] \in \Omega$ and $\varrho, s \in [0, 1]$.

The primary objective of this research paper is to introduce novel Hermite-Hadamard type inequalities utilizing the concept of generalized interval-valued Mittag-Leffler fractional double AB-integrals in combination with the class of interval-valued coordinated pre-invex functions. Our aim is to offer fresh ideas and techniques that may serve as inspiration to researchers interested in this area of study.

3.2. Main Results

This section will present new discoveries regarding Hermite-Hadamard, Pachpatte, and Fejer inequalities using generalized interval-valued convexity over a rectangular form in $\mathbb{R}^2$ in the context of fractional calculus.

3.2.1. Generalized fractional Hermite-Hadamar's type inclusions

We now present a novel inequality of Hermite-Hadamard type for functions that are coordinated interval-valued pre-invex.

Theorem 3.8: *Suppose that* $\widehat{F}: K \times K \subset \mathbb{R}^2 \to \mathbb{R}_I^+$ *is a given interval-valued co-ordinated pre-invex function defined on* $[\mathfrak{v}_1, \mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1)] \times [\mathfrak{v}_3, \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)]$ *along with* $0 \leq \mathfrak{v}_1 < \mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1),\ 0 \leq \mathfrak{v}_3 < \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)$*, and* $\widehat{F}(\mathfrak{x}, \mathfrak{y}) = [\underline{\Pi}(\mathfrak{x}, \mathfrak{y}), \overline{\Pi}(\mathfrak{x}, \mathfrak{y})]$*. Then, the successive inclusion relations hold true:*

$$
\begin{aligned}
&\widehat{F}\left(\frac{2\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1)}{2}, \frac{2\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)}{2}\right) \\
&\quad \times E_{\widetilde{\mu}_1, \alpha+1, \widehat{w}_1, l}^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^{\widetilde{\mu}_1}; p) E_{\widetilde{\mu}_2, \beta+1, \widehat{w}_2, l}^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^{\widetilde{\mu}_2}; p) \\
&\supseteq \frac{B(\alpha)B(\beta)}{4\alpha\beta(\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^{\beta}} \Bigg[\left(\epsilon_{\widetilde{\mu}, \alpha, l, \widehat{w}, \mathfrak{v}_1^+, \mathfrak{v}_3^+}^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}\right)\{\widehat{F}(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3); p)\} \\
&\quad + \left(\epsilon_{\widetilde{\mu}, \alpha, l, \widehat{w}, \mathfrak{v}_1^+, (\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^-}^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}\right)\{\widehat{F}(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3; p)\} \\
&\quad + \left(\epsilon_{\widetilde{\mu}, \alpha, l, \widehat{w}, (\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^-, \mathfrak{v}_3^+}^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}\right)\{\widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3); p)\} \\
&\quad + \left(\epsilon_{\widetilde{\mu}, \alpha, l, \widehat{w}, (\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^-, (\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^-}^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}\right)\{\widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3; p)\} \\
&\quad - \frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\Big(\widehat{F}(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)) + \widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)) \\
&\quad + \widehat{F}(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3) + \widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3)\Big)\Bigg] \\
&\supseteq \left[\frac{\widehat{F}(\mathfrak{v}_2, \mathfrak{v}_4) + \widehat{F}(\mathfrak{v}_1, \mathfrak{v}_4) + \widehat{F}(\mathfrak{v}_2, \mathfrak{v}_3) + \widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3)}{4}\right] \\
&\quad \times E_{\widetilde{\mu}_1, \alpha+1, \widehat{w}_1, l}^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^{\widetilde{\mu}_1}; p) E_{\widetilde{\mu}_2, \beta+1, \widehat{w}_2, l}^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^{\widetilde{\mu}_2}; p).
\end{aligned}
$$

Proof. By the definition of interval-valued co-ordinated preinvexity of $\widehat{F}$, we have

$$
\begin{aligned}
&\widehat{F}(\mathfrak{x} + \varrho\widetilde{\eta}_1(\mathfrak{y}, \mathfrak{x}), u + s\widetilde{\eta}_2(v, u)) \\
&\supseteq (1-\varrho)(1-s)\widehat{F}(\mathfrak{x}, u) + (1-\varrho)s\widehat{F}(\mathfrak{x}, v) + (1-s)\varrho\widehat{F}(\mathfrak{y}, u) + ts\widehat{F}(\mathfrak{y}, v).
\end{aligned}
$$

If we consider to take $\mathfrak{x} = \mathfrak{v}_1 + \varrho\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{y} = \mathfrak{v}_1 + (1-\varrho)\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), u = \mathfrak{v}_3 + s\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)$

and $v = \mathfrak{v}_3 + (1-s)\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)$ along with $\varrho = s = \frac{1}{2}$, then we derive

$$\widehat{F}\left(\frac{2\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1)}{2}, \frac{2\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)}{2}\right)$$
$$\supseteq \frac{1}{4}[\widehat{F}(\mathfrak{v}_1 + \varrho\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + s\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)) + \widehat{F}(\mathfrak{v}_1 + \varrho\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + (1-s)\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))$$
$$+ \widehat{F}(\mathfrak{v}_1 + (1-\varrho)\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + s\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)) + \widehat{F}(\mathfrak{v}_1 + (1-\varrho)\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + (1-s)\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))]. \tag{3.5}$$

Multiplying both sides of (3.5) by $\varrho^{\alpha-1}s^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^{\widetilde{\mu}_1}\varrho^{\widetilde{\mu}_1}; p)$ $E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^{\widetilde{\mu}_2}s^{\widetilde{\mu}_2}; p)$, then by integrating the resulting inclusion with regard to (ϱ, s) on $[0,1]\times[0,1]$, it yields that

$$\widehat{F}\left(\frac{2\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1)}{2}, \frac{2\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)}{2}\right)\int_0^1\int_0^1 \varrho^{\alpha-1}s^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^{\widetilde{\mu}_1}\varrho^{\widetilde{\mu}_1}; p)$$
$$\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^{\widetilde{\mu}_2}s^{\widetilde{\mu}_2}; p)\mathrm{d}s\mathrm{d}\varrho$$
$$\supseteq \frac{1}{4}\left[\int_0^1\int_0^1 \varrho^{\alpha-1}s^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^{\widetilde{\mu}_1}\varrho^{\widetilde{\mu}_1}; p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^{\widetilde{\mu}_2}s^{\widetilde{\mu}_2}; p)\right.$$
$$\times[\widehat{F}(\mathfrak{v}_1 + \varrho\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + s\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)) + \widehat{F}(\mathfrak{v}_1 + \varrho\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + (1-s)\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))]\mathrm{d}s\mathrm{d}\varrho$$
$$+\int_0^1\int_0^1 \varrho^{\alpha-1}s^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^{\widetilde{\mu}_1}\varrho^{\widetilde{\mu}_1}; p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}((\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^{\widetilde{\mu}_2}s^{\widetilde{\mu}_2}; p)$$
$$\times\left[\widehat{F}(\mathfrak{v}_1 + (1-\varrho)\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + s\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))\right.$$
$$\left.\left.+\widehat{F}(\mathfrak{v}_1 + (1-\varrho)\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + (1-s)\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))\right]\mathrm{d}s\mathrm{d}\varrho\right].$$

This implies

$$\widehat{F}\left(\frac{2\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1)}{2}, \frac{2\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)}{2}\right)$$
$$\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^{\widetilde{\mu}_1}; p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^{\widetilde{\mu}_2}; p)$$
$$\supseteq \frac{1}{4(\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^{\beta}}\left[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}\right.$$
$$\times(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1) - \mathfrak{x})^{\alpha-1}(\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3) - \mathfrak{y})^{\beta-1}$$
$$\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1) - \mathfrak{x})^{\widetilde{\mu}_1}; p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3) - \mathfrak{y})^{\widetilde{\mu}_2}; p)\widehat{F}(\mathfrak{x}, \mathfrak{y})\mathrm{d}\mathfrak{y}\mathrm{d}\mathfrak{x}$$
$$+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1) - \mathfrak{x})^{\alpha-1}(\mathfrak{y} - \mathfrak{v}_3)^{\beta-1}$$
$$\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1) - \mathfrak{x})^{\widetilde{\mu}_1}; p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{y} - \mathfrak{v}_3)^{\widetilde{\mu}_2}; p)\widehat{F}(\mathfrak{x}, \mathfrak{y})\mathrm{d}\mathfrak{y}\mathrm{d}\mathfrak{x}$$

$$+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)$$
$$\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{x},\mathfrak{y})d\mathfrak{y}d\mathfrak{x}$$
$$+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}(\mathfrak{y}-\mathfrak{v}_3)^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)$$
$$\left.\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{x},\mathfrak{y})d\mathfrak{y}d\mathfrak{x}\right].$$

After simplifying, we acquire the first inclusion relation.

For the proof of second inequality, taking into account the interval-valued co-ordinated preinvexity of the function $\widehat{F}$, we have

$$\begin{aligned}&\widehat{F}(\mathfrak{v}_1+\varrho\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+s\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\widehat{F}(\mathfrak{v}_1+\varrho\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+(1-s)\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\&\quad+\widehat{F}(\mathfrak{v}_1+(1-\varrho)\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+s\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\&\quad+\widehat{F}(\mathfrak{v}_1+(1-\varrho)\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+(1-s)\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)) \quad (3.6)\\&\supseteq\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3). \quad (3.7)\end{aligned}$$

Multiplying both sides of (3.6) by $\varrho^{\alpha-1}s^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1}\varrho^{\widetilde{\mu}_1};p)$ $E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}((\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2}s^{\widetilde{\mu}_2};p)$, then by integrating the resulting inclusion with regard to (ϱ,s) on $[0,1]\times[0,1]$, after simple computations, it yields the required second relation. Thus the proof is accomplished.

Remark 3.1: If we take $\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)=\mathfrak{v}_2-\mathfrak{v}_1$ and $\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)=\mathfrak{v}_4-\mathfrak{v}_3$ in (3.8), we get the result for interval-valued convex function.

$$\begin{aligned}&\widehat{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2},\frac{\mathfrak{v}_3+\mathfrak{v}_4}{2}\right)\\&\quad E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_2-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_4-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\\&\supseteq\frac{B(\alpha)B(\beta)}{4\alpha\beta(\mathfrak{v}_2-\mathfrak{v}_1)^{\alpha}(\mathfrak{v}_4-\mathfrak{v}_3)^{\beta}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,\mathfrak{v}_3^+}\right)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4;p)\}\right.\\&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,\mathfrak{v}_4^-}\right)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3;p)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2^-,\mathfrak{v}_3^+}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)\}\\&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2^-,\mathfrak{v}_4^-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)\}-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\\&\quad\left.\times\left(\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\right)\right]\\&\supseteq\left[\frac{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)}{4}\right]\end{aligned}$$

$$\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_2-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_4-\mathfrak{v}_3)^{\widetilde{\mu}_2};p).$$

Remark 3.2: By choosing the $\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)=\mathfrak{v}_2-\mathfrak{v}_1$, $\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)=\mathfrak{v}_4-\mathfrak{v}_3$, $p=0,\alpha_1=(0,0)$ and $\widehat{w}=(0,0)$ in (3.8), we get the partial result obtained in [46].

We now present a new midpoint Hermite-Hadamard's inequality for coordinated interval-valued pre-invex functions.

Theorem 3.9: *Under the assumptions of Theorem 3.8, the following inclusion relations hold true:*

$$\begin{aligned}
&\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\\
&\quad\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)\\
&\supseteq \frac{2^{\alpha+\beta+n_1\widetilde{\mu}_1+n_2\widetilde{\mu}_2}B(\alpha)B(\beta)}{4\alpha_1\beta(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2}\right)^+,\left(\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)^+}\right)\right.\\
&\left\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\right\}\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2}\right)^+,\left(\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)^-}\right)\left\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\right\}\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2}\right)^-,\left(\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)^+}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2}\right)^-,\left(\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)^-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&\quad-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\Big(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\
&\quad\left.+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\Big)\right]\\
&\supseteq\frac{1}{4}(\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3))\\
&\quad\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p).
\end{aligned}$$

Proof. In view of the interval-valued co-ordinated preinvexity of the function $\widehat{F}$, if we consider $\mathfrak{x}=\mathfrak{v}_1+\frac{2-\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{y}=\mathfrak{v}_1+\frac{\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),u=\mathfrak{v}_3+\frac{2-s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3),v=\mathfrak{v}_3+\frac{s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)$, as well as $\varrho=s=\frac{1}{2}$, then we have

$$\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\supseteq\frac{1}{4}\left[\widehat{F}\left(\mathfrak{v}_1+\frac{2-\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\frac{2-s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)\right.$$

$$+\widehat{F}\left(\mathfrak{v}_1+\frac{\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\frac{2-s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)+\widehat{F}\left(\mathfrak{v}_1+\frac{2-\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\frac{s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)$$
$$+\widehat{F}\left(\mathfrak{v}_1+\frac{\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\frac{s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)\Bigg] \tag{3.8}$$

Multiplying both sides of (3.8) by $(\frac{\varrho}{2})^{\alpha-1}(\frac{s}{2})^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1}(\frac{\varrho}{2})^{\widetilde{\mu}_1};p)$ $E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}((\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2}(\frac{s}{2})^{\widetilde{\mu}_2};p)$, then by integrating the resulting inclusion with regard to (ϱ,s) on

$[0,1]\times[0,1]$, it yields that

$$\begin{aligned}
&\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\int_0^1\int_0^1(\frac{\varrho}{2})^{\alpha-1}(\frac{s}{2})^{\beta-1}\\
&\quad\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1}\left(\frac{\varrho}{2}\right)^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}((\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2}\left(\frac{s}{2}\right)^{\widetilde{\mu}_2};p)\mathrm{d}s\mathrm{d}\varrho\\
&\supseteq\frac{1}{4}\left[\int_0^1\int_0^1(\frac{\varrho}{2})^{\alpha-1}(\frac{s}{2})^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1}\left(\frac{\varrho}{2}\right)^{\widetilde{\mu}_1};p)\right.\\
&\quad\left.\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}((\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2}\left(\frac{s}{2}\right)^{\widetilde{\mu}_2};p))\right]\\
&\quad\times\left[\widehat{F}\left(\mathfrak{v}_1+\frac{2-\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3\right.\right.\\
&\quad\left.\left.+\frac{2-s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)+\widehat{F}\left(\mathfrak{v}_1+\frac{\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\frac{2-s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)\right]\mathrm{d}s\mathrm{d}\varrho\\
&\quad+\int_0^1\int_0^1(\frac{\varrho}{2})^{\alpha-1}(\frac{s}{2})^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1}\left(\frac{\varrho}{2}\right)^{\widetilde{\mu}_1};p)\\
&\quad E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}((\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2}\left(\frac{s}{2}\right)^{\widetilde{\mu}_2};p)\\
&\quad\times\left[\widehat{F}\left(\mathfrak{v}_1+\frac{2-\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\frac{s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)\right.\\
&\quad\left.\left.+\widehat{F}\left(\mathfrak{v}_1+\frac{\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\frac{s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)\right]\mathrm{d}s\mathrm{d}\varrho\right].
\end{aligned}$$

After comparing the above relation with newly defined fractional integrals, we acquire the first inclusion relation.

For the proof of second inequality, taking into account the interval-valued co-ordinated preinvexity of the function $\widehat{F}$, we have

$$\begin{aligned}
&\widehat{F}\left(\mathfrak{v}_1+\frac{2-\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\frac{2-s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)+\widehat{F}\left(\mathfrak{v}_1+\frac{2-\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\frac{s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)\\
&\quad+\widehat{F}\left(\mathfrak{v}_1+\frac{\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\frac{2-s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)+\widehat{F}\left(\mathfrak{v}_1+\frac{\varrho}{2}\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\frac{s}{2}\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)\\
&\supseteq[\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)].
\end{aligned}\tag{3.9}$$

Multiplying both sides of (3.9) by $(\frac{\varrho}{2})^{\alpha-1}(\frac{s}{2})^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1}(\frac{\varrho}{2})^{\widetilde{\mu}_1};p)$ $E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}((\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2}(\frac{s}{2})^{\widetilde{\mu}_2};p)$, then by integrating the resulting inclusion with regard to (ϱ,s) on

$[0,1]\times[0,1]$, we obtain the second inclusion.

Thus the proof is accomplished.

Remark 3.3: If we take $\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)=\mathfrak{v}_2-\mathfrak{v}_1$ and $\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)=\mathfrak{v}_4-\mathfrak{v}_3$ in Theorem 3.9, we get the result for interval-valued convex function.

$$\begin{aligned}
&\widehat{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2},\frac{\mathfrak{v}_3+\mathfrak{v}_4}{2}\right)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_2-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_4-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\\
&\supseteq\frac{2^{\alpha+\beta+n_1\widetilde{\mu}_1+n_2\widetilde{\mu}_2}B(\alpha)B(\beta)}{4\alpha_1\beta(\mathfrak{v}_2-\mathfrak{v}_1)^{\alpha}(\mathfrak{v}_4-\mathfrak{v}_3)^{\beta}}\Bigg[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}\right)^+,\left(\frac{\mathfrak{v}_3+\mathfrak{v}_4}{2}\right)^+}\right)\left\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4;p)\right\}\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}\right)^+,\left(\frac{\mathfrak{v}_3+\mathfrak{v}_4}{2}\right)^-}\right)\left\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3;p)\right\}\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}\right)^-,\left(\frac{\mathfrak{v}_3+\mathfrak{v}_4}{2}\right)^+}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)\}\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2}\right)^-,\left(\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)^-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&\quad-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\left(\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\right)\Bigg]\\
&\supseteq\frac{1}{4}(\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3))\\
&\quad\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_2-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_4-\mathfrak{v}_3)^{\widetilde{\mu}_2};p).
\end{aligned}$$

We now present another fractional variants of Hermite-Hadamard's result.

Theorem 3.10: *Suppose that* $\widehat{F}:[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]\times[\mathfrak{v}_3,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)]\subset\mathbb{R}^2\to\mathbb{R}$ *is an interval-valued co-ordinated pre-invex function defined over the rectangle* $[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]\times[\mathfrak{v}_3,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)]$ *together with* $0\le\mathfrak{v}_1<\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)$, $0\le\mathfrak{v}_3<\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)$, *and* $\widehat{F}(\mathfrak{x},\mathfrak{y})=[\underline{\Pi}(\mathfrak{x},\mathfrak{y}),\overline{\Pi}(\mathfrak{x},\mathfrak{y})]$. *Then, the following inclusion relations hold true:*

$$\begin{aligned}
&\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\\
&\supseteq\frac{B(\beta)}{4L_2\beta(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_3+}\right)\left\{\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)\right\}\right.\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\right)\left\{\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{v}_3\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{(1-\beta)}{B(\beta)}\left(\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right),\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{v}_3\right)\right)\Bigg]\\
&+\frac{B(\alpha)}{4L_1\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\right)\left\{\widehat{F}\left(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\right\}\right.\\
&+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2^-}\right)\left\{\widehat{F}\left(\mathfrak{v}_1,\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\right\}\\
&\left.-\frac{(1-\alpha)}{B(\alpha)}\left(\widehat{F}\left(\mathfrak{v}_1,\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right),\widehat{F}\left(\mathfrak{v}_2,\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\right)\right]\\
\supseteq\;&\frac{B(\alpha)B(\beta)}{4L_1L_2\alpha\beta(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,\mathfrak{v}_3^+}\right)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\right.\\
&+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\right)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}\\
&+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,\mathfrak{v}_3^+}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\Big(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\
&\left.+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\Big)\right]\\
\supseteq\;&\frac{B(\alpha)}{8L_1\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)\}\right.\\
&+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\right)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\right)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)\}\\
&\left.-\frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4))\right]\\
&+\frac{B(\beta)}{8L_2\beta(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\right)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)\}\right.\\
&+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_3^+}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_3^+}\right)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\}\\
&\left.-\frac{(1-\beta)}{B(\beta)}(\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)))\right]\\
\supseteq\;&\frac{[\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)]}{4},
\end{aligned}
\tag{3.10}
$$

where

$$
L_1 := E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p),
$$

and

$$
L_2 := E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p).
$$

Proof. From the interval-valued co-ordinated convexity of the function $\widehat{F}$, it yields that the function $\widehat{F}_{\mathfrak{x}}:[\mathfrak{v}_3,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)]\to\mathbb{R}$, $\widehat{F}_{\mathfrak{x}}(\mathfrak{y})=\widehat{F}(\mathfrak{x},\mathfrak{y})$ is convex defined over $[\mathfrak{v}_3,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)]$ for each $\mathfrak{x}\in[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]$. Utilizing the inclusion relations in

(3.1), we find that

$$\begin{aligned}
&\widehat{F}_{\mathfrak{x}}\left(\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)\\
&\supseteq \frac{B(\beta)}{2\beta(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_3+}\right)\{\widehat{F}_{\mathfrak{x}}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\right)\right.\\
&\quad\left.\times\{\widehat{F}_{\mathfrak{x}}(\mathfrak{v}_3)\}-\frac{(1-\beta)}{B(\beta)}(\widehat{F}_{\mathfrak{x}}(\mathfrak{v}_3)+\widehat{F}_{\mathfrak{x}}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)))\right]\\
&\supseteq \frac{(\widehat{F}_{\mathfrak{x}}(\mathfrak{v}_3)+\widehat{F}_{\mathfrak{x}}(\mathfrak{v}_4))}{2}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p).
\end{aligned}$$

It means that

$$\begin{aligned}
&\widehat{F}\left(\mathfrak{x},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)\\
&\supseteq \frac{1}{2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\left[\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{x})^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{x})^{\widetilde{\mu}_2};p)\right.\\
&\quad\left.\times\widehat{F}(\mathfrak{x},\mathfrak{y})d\mathfrak{y}+\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{x}-\mathfrak{v}_3)^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{x}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{x},\mathfrak{y})d\mathfrak{y}\right]\\
&\supseteq \frac{(\widehat{F}(\mathfrak{x},\mathfrak{v}_3)+\widehat{F}(\mathfrak{x},\mathfrak{v}_4))}{2}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p).
\end{aligned}\tag{3.11}$$

Multiplying both sides of (3.11) by $\frac{1}{2(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)$ and $\frac{1}{2(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\beta}}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)$, respectively, and integrating the resulting inclusion relation regarding $\mathfrak{x}$ over $[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]$, we have

$$\begin{aligned}
&\frac{E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)}{2(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}\\
&\quad\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\widehat{F}\left(\mathfrak{x},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)d\mathfrak{x}\\
&\supseteq \frac{1}{4(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\left[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{x})^{\beta-1}\right.\\
&\quad\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{x})^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{y})d\mathfrak{y}d\mathfrak{x}\\
&\quad+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}(\mathfrak{x}-\mathfrak{v}_3)^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_3)^{\widetilde{\mu}_1};p)\\
&\quad\left.\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{y})d\mathfrak{y}d\mathfrak{x}\right]\\
&\supseteq \frac{E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)}{4(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\left[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\right.
\end{aligned}$$

$$\times\widehat{F}(\mathfrak{x},\mathfrak{v}_3)\mathrm{d}\mathfrak{x}+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}E_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{v}_4)\mathrm{d}\mathfrak{x}\Bigg]. \tag{3.12}$$

And

$$\begin{aligned}
&\frac{E_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)}{2(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}\\
&\quad\times E_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)\widehat{F}\left(\mathfrak{x},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\mathrm{d}\mathfrak{x}\\
&\supseteq\frac{1}{4(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Bigg[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}\\
&\quad\times(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{x})^{\beta-1}\\
&\quad E_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}((\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{x})^{\widetilde{\mu}_2};p)E_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{y})\mathrm{d}\mathfrak{y}\mathrm{d}\mathfrak{x}\\
&\quad+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}(\mathfrak{x}-\mathfrak{v}_3)^{\beta-1}\\
&\quad\times E_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_2(\mathfrak{x}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)E_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{y})\mathrm{d}\mathfrak{y}\mathrm{d}\mathfrak{x}\Bigg]\\
&\supseteq\frac{E_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)}{4(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\\
&\quad\times\Bigg[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}E_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{v}_3)\mathrm{d}\mathfrak{x}\\
&\quad+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}E_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{v}_4)\mathrm{d}\mathfrak{x}\Bigg].
\end{aligned}\tag{3.13}$$

By similar argument applying on the function $\widehat{F}_{\mathfrak{y}}:[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]\to\mathbb{R}$, $\widehat{F}_{\mathfrak{y}}(\mathfrak{x})=\widehat{F}(\mathfrak{x},\mathfrak{y})$, it yields that

$$\begin{aligned}
&\frac{E_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)}{2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\alpha}}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{y}-\mathfrak{v}_3)^{\beta-1}E_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\\
&\quad\times\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{y}\right)\mathrm{d}\mathfrak{y}\\
&\supseteq\frac{1}{4(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Bigg[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}(\mathfrak{x}-\mathfrak{v}_3)^{\beta-1}\\
&\quad\times E_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)E_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{y})\mathrm{d}\mathfrak{y}\mathrm{d}\mathfrak{x}\\
&\quad+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}(\mathfrak{y}-\mathfrak{v}_3)^{\beta-1}E_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\\
&\quad\times E_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{y})\mathrm{d}\mathfrak{y}\mathrm{d}\mathfrak{x}\Bigg]\\
&\supseteq\frac{E_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)}{4(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Bigg[\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{y}-\mathfrak{v}_3)^{\beta-1}E_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{v}_1,\mathfrak{y})\mathrm{d}\mathfrak{y}
\end{aligned}$$

$$+\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{y}-\mathfrak{v}_3)^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{v}_2,\mathfrak{y})d\mathfrak{y}\Bigg]. \tag{3.14}$$

And

$$\begin{aligned}
&\frac{E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)}{2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\alpha}}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}\\
&\quad\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)\widehat{F}\left(\frac{2\mathfrak{v}_1+\varrho\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{y}\right)d\mathfrak{y}\\
&\supseteq\frac{1}{4(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Bigg[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}\\
&\quad\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{y})d\mathfrak{y}d\mathfrak{x}\\
&\quad+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}\\
&\quad\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{y})d\mathfrak{y}d\mathfrak{x}\Bigg]\\
&\supseteq\frac{E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)}{4(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\\
&\quad\times\Bigg[\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{v}_1,\mathfrak{y})d\mathfrak{y}\\
&\quad+\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{v}_2,\mathfrak{y})d\mathfrak{y}\Bigg].
\end{aligned}\tag{3.15}$$

Adding the inclusion relations (3.12)-(3.15), we obtain second and third inclusion relations in (3.10).

Now, by using the first relation in (3.1), we find that

$$\begin{aligned}
&\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)\\
&\supseteq\frac{1}{2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Bigg[\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}\\
&\quad\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{y}\right)d\mathfrak{y}\\
&\quad+\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{y}-\mathfrak{v}_3)^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{y}\right)d\mathfrak{y}\Bigg],
\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}
&\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)\\
&\supseteq\frac{1}{2(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Bigg[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}
\end{aligned}$$

$$\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)\widehat{F}\left(\mathfrak{x},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\mathrm{d}\mathfrak{y}$$
$$+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\widehat{F}\left(\mathfrak{x},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\mathrm{d}\mathfrak{y}\Bigg]. \quad (3.17)$$

Multiplying both sides of (3.16) by $E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)$ and (3.17) by $E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}$ then by addition we get the first inclusion relation in (3.10).

Finally, by virtue of the second inclusion relation in (3.1), we have

$$\frac{1}{2(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Bigg[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)$$
$$\times\widehat{F}(\mathfrak{x},\mathfrak{v}_3)\mathrm{d}\mathfrak{x}+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{v}_3)\mathrm{d}\mathfrak{x}\Bigg]$$
$$\supseteq\frac{(\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3))}{2}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p), \quad (3.18)$$

$$\frac{1}{2(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Bigg[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)$$
$$\times\widehat{F}(\mathfrak{x},\mathfrak{v}_4)\mathrm{d}\mathfrak{x}+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\widehat{F}(\mathfrak{x},\mathfrak{v}_4)\mathrm{d}\mathfrak{x}\Bigg]$$
$$\supseteq\frac{(\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4))}{2}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p), \quad (3.19)$$

and

$$\frac{1}{2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Bigg[\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)$$
$$\widehat{F}(\mathfrak{v}_1,\mathfrak{y})\mathrm{d}\mathfrak{y}+\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{y}-\mathfrak{v}_3)^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{v}_1,\mathfrak{y})\mathrm{d}\mathfrak{y}\Bigg]$$
$$\supseteq\frac{(\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4))}{2}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p), \quad (3.20)$$

$$\frac{1}{2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Bigg[\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)$$
$$\widehat{F}(\mathfrak{v}_2,\mathfrak{y})\mathrm{d}\mathfrak{y}+\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{y}-\mathfrak{v}_3)^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{v}_2,\mathfrak{y})\mathrm{d}\mathfrak{y}\Bigg]$$
$$\supseteq\frac{(\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4))}{2}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p). \quad (3.21)$$

Multiplying both sides of (3.18) and (3.19) by $E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)$, (3.20)

and (3.21) by
$E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)$, then by combining the four inclusion relations, we achieve the last inclusion in (3.10). The proof is completed.

Remark 3.4: If we take $\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)=\mathfrak{v}_2-\mathfrak{v}_1$ and $\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)=\mathfrak{v}_4-\mathfrak{v}_3$ in Theorem 3.10, then

$$\begin{aligned}
&\widehat{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2},\frac{\mathfrak{v}_3+\mathfrak{v}_4}{2}\right)\\
&\supseteq \frac{B(\beta)}{4L_2\beta(\mathfrak{v}_4-\mathfrak{v}_3)^{\beta}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_3+}\right)\left\{\widehat{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2},\mathfrak{v}_4\right)\right\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_4-}\right)\left\{\widehat{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2},\mathfrak{v}_3\right)\right\}\right.\\
&\quad\left.-\frac{(1-\beta)}{B(\beta)}\left(\widehat{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2},\mathfrak{v}_4\right),\widehat{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2},\mathfrak{v}_3\right)\right)\right]\\
&+\frac{B(\alpha)}{4L_1\alpha(\mathfrak{v}_2-\mathfrak{v}_1)^{\alpha}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+}\right)\left\{\widehat{F}\left(\mathfrak{v}_2,\frac{\mathfrak{v}_3+\mathfrak{v}_4}{2}\right)\right\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2-}\right)\left\{\widehat{F}\left(\mathfrak{v}_1,\frac{\mathfrak{v}_3+\mathfrak{v}_4}{2}\right)\right\}\right.\\
&\quad\left.-\frac{(1-\alpha)}{B(\alpha)}\left(\widehat{F}\left(\mathfrak{v}_1,\frac{\mathfrak{v}_3+\mathfrak{v}_4}{2}\right),\widehat{F}\left(\mathfrak{v}_2,\frac{\mathfrak{v}_3+\mathfrak{v}_4}{2}\right)\right)\right]\\
&\supseteq \frac{B(\alpha)B(\beta)}{4L_1L_2\alpha\beta(\mathfrak{v}_2-\mathfrak{v}_1)^{\alpha}(\mathfrak{v}_4-\mathfrak{v}_3)^{\beta}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+,\mathfrak{v}_3+}\right)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4;p)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+,\mathfrak{v}_4-}\right)\right.\\
&\quad\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3;p)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2-,\mathfrak{v}_3+}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2-,\mathfrak{v}_4-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&\quad\left.-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\left(\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\right)\right]\\
&\supseteq \frac{B(\alpha)}{8L_1\alpha(\mathfrak{v}_2-\mathfrak{v}_1)^{\alpha}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)\}\right.\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+}\right)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+}\right)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)\}\\
&\quad\left.-\frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4))\right]\\
&\quad+\frac{B(\beta)}{8L_2\beta(\mathfrak{v}_4-\mathfrak{v}_3)^{\beta}}\left[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_4-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_4-}\right)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)\}\right.\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_3+}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_3+}\right)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)\}\\
&\quad\left.-\frac{(1-\beta)}{B(\beta)}(\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4))\right]\\
&\supseteq \frac{[\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)]}{4},
\end{aligned}$$

where

$$L_1:=E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_2-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)$$

and

$$L_2:=E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_4-\mathfrak{v}_3)^{\widetilde{\mu}_2};p).$$

Remark 3.5: By choosing the $\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)=\mathfrak{v}_2-\mathfrak{v}_1$, $\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)=\mathfrak{v}_4-\mathfrak{v}_3$, $p=0,\alpha_1=(0,0)$ and $\widehat{w}=(0,0)$ in Theorem 3.10, we get the Theorem 7 obtained in [46].

3.2.2. Generalized fractional Pachpatte type inclusions

The proceeding subsection is related to Hermite-Hadamard's type containments for the product of two interval valued coordinated pre-invex functions are termed as Pachpatte type inequalities.

Theorem 3.11: *Assume that the functions $\widehat{F},G:[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]\times[\mathfrak{v}_3,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)]\subset\mathbb{R}^2\to\mathbb{R}_I^+$ are both interval-valued co-ordinated pre-invex functions defined over $[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]\times[\mathfrak{v}_3,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)]$ along with $0\le\mathfrak{v}_1<\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)$, $0\le\mathfrak{v}_3<\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)$, and $\widehat{F}(\mathfrak{x},\mathfrak{y})=[\underline{\Pi}(\mathfrak{x},\mathfrak{y}),\overline{\Pi}(\mathfrak{x},\mathfrak{y})]$, and $G(\mathfrak{x},\mathfrak{y})=[\underline{g}(\mathfrak{x},\mathfrak{y}),\overline{g}(\mathfrak{x},\mathfrak{y})]$. Then, the following inclusion relation holds true:*

$$\begin{aligned}
&\frac{B(\alpha)B(\beta)}{\alpha\beta(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\\
&\times\Big[\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1{}^+,\mathfrak{v}_3{}^+}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1{}^+,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}\\
&+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,\mathfrak{v}_3{}^+}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)G(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)G(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\
&+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))G(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3))\Big]\\
&\supseteq 4E_1E_2[C(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)\\
&-D(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)-E(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)+\Psi(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)]\\
&-2E_2E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)[C(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)-D(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)]\\
&+2E_1E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)[-C(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)+D(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)]\\
&+E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)C(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4),
\end{aligned}$$

where

$$\begin{aligned}
E_1&:=\sum_{n=0}^{\infty}\frac{\beta_p(\widetilde{\gamma}+nk,\mathfrak{v}_3-\widetilde{\gamma})}{\beta(\widetilde{\gamma},\mathfrak{v}_3-\widetilde{\gamma})}\frac{(\mathfrak{v}_3)_{nk}}{(\widetilde{\mu}_1n+\alpha+1)(\widetilde{\mu}_1n+\alpha+2)\Gamma(\widetilde{\mu}_1n+\alpha)}\frac{\varrho^n}{(l)_{n\widetilde{\delta}}},\\
E_2&:=\sum_{n=0}^{\infty}\frac{\beta_p(\widetilde{\gamma}+nk,\mathfrak{v}_3-\widetilde{\gamma})}{\beta(\widetilde{\gamma},\mathfrak{v}_3-\widetilde{\gamma})}\frac{(\mathfrak{v}_3)_{nk}}{(\widetilde{\mu}_2n+\beta+1)(\widetilde{\mu}_2n+\beta+2)\Gamma(\widetilde{\mu}_2n+\beta)}\frac{\varrho^n}{(l)_{n\widetilde{\delta}}},
\end{aligned}$$

$$
\begin{aligned}
&C(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)\\
&=\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1\\
&+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3),\\
&D(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)\\
&=\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1\\
&+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_4),\\
&E(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)\\
&=\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3)\\
&+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Psi(\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3,\mathfrak{v}_4)\\
&=\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_4)\\
&\quad+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4).
\end{aligned}
$$

Proof. Since the functions $\widehat{F}$ and G are both interval-valued co-ordinated pre-invex defined over the rectangle $[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]\times[\mathfrak{v}_3,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)]$, we have that

$$
\widehat{F}_{\mathfrak{x}}(\mathfrak{y}):[\mathfrak{v}_3,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)]\to\mathbb{R}_I^+,\ \widehat{F}_{\mathfrak{x}}(\mathfrak{y})=\widehat{F}(\mathfrak{x},\mathfrak{y}),\ G_{\mathfrak{x}}(\mathfrak{y}):[\mathfrak{v}_3,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)]\to\mathbb{R}_I^+,\ G_{\mathfrak{x}}(\mathfrak{y})=G(\mathfrak{x},\mathfrak{y}),
$$

as well as

$$
\widehat{F}_{\mathfrak{y}}(\mathfrak{x}):[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]\to\mathbb{R}_I^+,\ \widehat{F}_{\mathfrak{y}}(\mathfrak{x})=\widehat{F}(\mathfrak{x},\mathfrak{y}),\ G_{\mathfrak{y}}(\mathfrak{x}):[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]\to\mathbb{R}_I^+,\ G_{\mathfrak{y}}(\mathfrak{x})=G(\mathfrak{x},\mathfrak{y}),
$$

are both the interval-valued pre-invex functions defined over the intervals $[\mathfrak{v}_3,\mathfrak{v}_4]$ and $[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]$, correspondingly, for every $\mathfrak{x}\in[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]$ along with $\mathfrak{y}\in[\mathfrak{v}_3,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)]$.

Now, in view of the inclusion relation in (3.5), which can be written as

$$
\begin{aligned}
&\frac{1}{(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Bigg[\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{y}-\mathfrak{v}_3)^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{x},\mathfrak{y})G(\mathfrak{x},\mathfrak{y})d\mathfrak{y}\\
&\quad+\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{x},\mathfrak{y})G(\mathfrak{x},\mathfrak{y})d\mathfrak{y}\Bigg]\\
&\supseteq 2E_2[-\widehat{F}(\mathfrak{x},\mathfrak{v}_3)G(\mathfrak{x},\mathfrak{v}_3)-\widehat{F}(\mathfrak{x},\mathfrak{v}_4)G(\mathfrak{x},\mathfrak{v}_4)+\widehat{F}(\mathfrak{x},\mathfrak{v}_3)G(\mathfrak{x},\mathfrak{v}_4)+\widehat{F}(\mathfrak{x},\mathfrak{v}_4)G(\mathfrak{x},\mathfrak{v}_3)]\\
&\quad+[\widehat{F}(\mathfrak{x},\mathfrak{v}_3)G(\mathfrak{x},\mathfrak{v}_3)+\widehat{F}(\mathfrak{x},\mathfrak{v}_4)G(\mathfrak{x},\mathfrak{v}_4)]E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p), \qquad (3.22)
\end{aligned}
$$

Multiplying both sides of (3.22) by $\frac{1}{(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\mathfrak{v}_2-\mathfrak{x})^{\widetilde{\mu}_1};p)$ and $\frac{1}{(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)$, respectively, and integrating the resulting inclusion relations regarding $\mathfrak{x}$ over $[\mathfrak{v}_1,\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)]$, we acquire that

$$\begin{aligned}
&\frac{1}{(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Bigg[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{y}-\mathfrak{v}_3)^{\beta-1}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}\\
&\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{x},\mathfrak{y})G(\mathfrak{x},\mathfrak{y})\mathrm{d}\mathfrak{y}\mathrm{d}\mathfrak{x}\\
&+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}\\
&\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{x},\mathfrak{y})G(\mathfrak{x},\mathfrak{y})\mathrm{d}\mathfrak{y}\mathrm{d}\mathfrak{x}\Bigg]\\
&\supseteq\frac{2E_2}{(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Bigg[-\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)\\
&\times[\widehat{F}(\mathfrak{x},\mathfrak{v}_3)G(\mathfrak{x},\mathfrak{v}_3)+\widehat{F}(\mathfrak{x},\mathfrak{v}_4)G(\mathfrak{x},\mathfrak{v}_4)]\mathrm{d}\mathfrak{x}\\
&+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)\\
&\times[\widehat{F}(\mathfrak{x},\mathfrak{v}_3)G(\mathfrak{x},\mathfrak{v}_4)+\widehat{F}(\mathfrak{x},\mathfrak{v}_4)G(\mathfrak{x},\mathfrak{v}_3)]\mathrm{d}\mathfrak{x}\Bigg]\\
&+\frac{E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)}{(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\alpha-1}\\
&\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)-\mathfrak{x})^{\widetilde{\mu}_1};p)[\widehat{F}(\mathfrak{x},\mathfrak{v}_3)G(\mathfrak{x},\mathfrak{v}_3)+\widehat{F}(\mathfrak{x},\mathfrak{v}_4)G(\mathfrak{x},\mathfrak{v}_4)]\mathrm{d}\mathfrak{x},
\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}
&\frac{1}{(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Bigg[\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{y}-\mathfrak{v}_3)^{\beta-1}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}\\
&\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{y}-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{x},\mathfrak{y})G(\mathfrak{x},\mathfrak{y})\mathrm{d}\mathfrak{y}\mathrm{d}\mathfrak{x}\\
&+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}\int_{\mathfrak{v}_3}^{\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\beta-1}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\\
&\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)-\mathfrak{y})^{\widetilde{\mu}_2};p)\widehat{F}(\mathfrak{x},\mathfrak{y})G(\mathfrak{x},\mathfrak{y})\mathrm{d}\mathfrak{y}\mathrm{d}\mathfrak{x}\Bigg]\\
&\supseteq\frac{2E_2}{(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Bigg[-\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\\
&\times[\widehat{F}(\mathfrak{x},\mathfrak{v}_3)G(\mathfrak{x},\mathfrak{v}_3)+\widehat{F}(\mathfrak{x},\mathfrak{v}_4)G(\mathfrak{x},\mathfrak{v}_4)]\mathrm{d}\mathfrak{x}\\
&+\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)[\widehat{F}(\mathfrak{x},\mathfrak{v}_3)G(\mathfrak{x},\mathfrak{v}_4)+\widehat{F}(\mathfrak{x},\mathfrak{v}_4)G(\mathfrak{x},\mathfrak{v}_3)]\mathrm{d}\mathfrak{x}\Bigg]\\
&+\frac{E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)}{(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\int_{\mathfrak{v}_1}^{\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}(\mathfrak{x}-\mathfrak{v}_1)^{\alpha-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{x}-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)\\
&[\widehat{F}(\mathfrak{x},\mathfrak{v}_3)G(\mathfrak{x},\mathfrak{v}_3)+\widehat{F}(\mathfrak{x},\mathfrak{v}_4)G(\mathfrak{x},\mathfrak{v}_4)]\mathrm{d}\mathfrak{x}.
\end{aligned}\tag{3.24}$$

Summing (3.23) and (3.24), we deduce that

$$
\begin{aligned}
&\frac{B(\alpha)B(\beta)}{\alpha\beta(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\\
&\quad\times\Big[\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,\mathfrak{v}_3^+}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,\mathfrak{v}_3^+}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)G(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)G(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&\quad-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\
&\quad+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\
&\quad G(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3))\Big]\\
&\supseteq\frac{2E_2B(\alpha)}{\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Big[-\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}\\
&\quad-\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4;p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4;p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}\\
&\quad-\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)G(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&\quad-\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)G(\mathfrak{v}_1,\mathfrak{v}_4;p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)G(\mathfrak{v}_1,\mathfrak{v}_4;p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)G(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&\quad+\frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)\\
&\quad+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3);p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)\\
&\quad+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_4))-\frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_4)\\
&\quad+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_3))\Big]\\
&\quad+\frac{B(\alpha)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2};p)}{\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Big[\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\Big)\\
&\quad\times\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4;p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)G(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)G(\mathfrak{v}_1,\mathfrak{v}_4;p)\}\\
&\quad-\frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))G(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))
\end{aligned}
$$

$$+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_4))\Big]. \tag{3.25}$$

Applying the inclusion relation (3.2) to every integral on the right hand side of the inclusion relation (3.25), it yields that

$$\begin{aligned}
&\frac{B(\alpha)}{\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Big[\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}\\
&+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)G(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&-\frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)G(\mathfrak{v}_1,\mathfrak{v}_3;p))\Big]\\
&\supseteq 2E_1\Big[-\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3)-\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)\\
&+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3)\Big]\\
&+E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)[\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3)],
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
&\frac{B(\alpha)}{\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Big[\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4;p)\}\\
&+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)G(\mathfrak{v}_1,\mathfrak{v}_4;p)\}\\
&-\frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)G(\mathfrak{v}_1,\mathfrak{v}_4;p))\Big]\\
&\supseteq 2E_1\Big[-\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_4)-\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)\\
&+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_4)\Big]\\
&+E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)[\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_4)],
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
&\frac{B(\alpha)}{\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Big[\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4;p)\}\\
&+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)G(\mathfrak{v}_1,\mathfrak{v}_4;p)\}\\
&-\frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)G(\mathfrak{v}_1,\mathfrak{v}_4;p))\Big]\\
&\supseteq 2E_1\Big[-\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_4)-\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)\\
&+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_4)\Big]\\
&+E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)[\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_4)],
\end{aligned} \tag{3.28}$$

and

$$\frac{B(\alpha)}{\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Big[\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}$$
$$+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)G(\mathfrak{v}_1,\mathfrak{v}_3;p)\}$$
$$-\frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)G(\mathfrak{v}_1,\mathfrak{v}_3;p))\Big]$$
$$\supseteq 2E_1\Big[-\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_3)-\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)$$
$$+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_3)\Big]$$
$$+E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1};p)[\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_3)]. \tag{3.29}$$

Substituting the inclusion relations (3.26)-(3.29) into the inclusion relation (3.25), we derive the desire inclusion. Thus proof is completed.

Remark 3.6: If we take $\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)=\mathfrak{v}_2-\mathfrak{v}_1$ and $\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)=\mathfrak{v}_4-\mathfrak{v}_3$ in (3.11), we get the result for interval-valued convex function.

$$\frac{B(\alpha)B(\beta)}{\alpha\beta(\mathfrak{v}_2-\mathfrak{v}_1)^{\alpha}(\mathfrak{v}_4-\mathfrak{v}_3)^{\beta}}\Big[\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+,\mathfrak{v}_3+}\Big)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4;p)G(\mathfrak{v}_2,\mathfrak{v}_4;p)\}$$
$$+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+,\mathfrak{v}_4-}\Big)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3;p)G(\mathfrak{v}_2,\mathfrak{v}_3;p)\}$$
$$+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2-,\mathfrak{v}_3+}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)G(\mathfrak{v}_1,\mathfrak{v}_4;p)\}+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2-,\mathfrak{v}_4-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)G(\mathfrak{v}_1,\mathfrak{v}_3;p)\}$$
$$-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\Big(\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)G(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_4)$$
$$+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)G(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3)\Big)\Big]$$
$$\supseteq 4E_1E_2[C(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)-D(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)-E(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)+\Psi(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)]$$
$$-2E_2E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_2-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)[C(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)-D(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)]$$
$$+2E_1E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_4-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)[-C(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)+D(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)]$$
$$+E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta+1,\widehat{w}_2,l}(\widehat{w}_2(\mathfrak{v}_4-\mathfrak{v}_3)^{\widetilde{\mu}_2};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha+1,\widehat{w}_1,l}(\widehat{w}_1(\mathfrak{v}_2-\mathfrak{v}_1)^{\widetilde{\mu}_1};p)C(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4).$$

where

$$C(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)=\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)G(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)G(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3),$$
$$D(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)=\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)G(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)G(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_4),$$
$$E(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)=\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)G(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)G(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_2,\mathfrak{v}_3),$$

and

$$\Psi(\mathfrak{v}_1, \mathfrak{v}_1 + \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4) = \widehat{F}(\mathfrak{v}_2, \mathfrak{v}_3)G(\mathfrak{v}_1, \mathfrak{v}_4) + \widehat{F}(\mathfrak{v}_1, \mathfrak{v}_4)G(\mathfrak{v}_2, \mathfrak{v}_3) + \widehat{F}(\mathfrak{v}_2, \mathfrak{v}_4)G(\mathfrak{v}_1, \mathfrak{v}_3) + \widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3)G(\mathfrak{v}_2, \mathfrak{v}_4).$$

Remark 3.7: If we take $\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1) = \mathfrak{v}_2 - \mathfrak{v}_1$, $\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3) = \mathfrak{v}_4 - \mathfrak{v}_3$, $p = 0, \alpha_1 = (0, 0)$ and $\widehat{w} = (0, 0)$ in (3.11), then we obtain Theorem 8 proved in [46].

3.3. Fractional Hermite-Hadamard-Fejer type inclusions

In this portion, we investigate new fractional Hermite-Hadamard-Fejer type inclusions for interval valued coordinated pre-invex functions.

Theorem 3.12: *Suppose that $\widehat{F} : K \times K \subset \mathbb{R}^2 \to \mathbb{R}_I^+$ is a given interval-valued co-ordinated pre-invex function defined over the rectangle $[\mathfrak{v}_1, \mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1)] \times [\mathfrak{v}_3, \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)]$ together with $0 \leq \mathfrak{v}_1 < \mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1)$, $0 \leq \mathfrak{v}_3 < \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)$, and $\widehat{F}(\mathfrak{x}, \mathfrak{y}) = [\underline{\Pi}(\mathfrak{x}, \mathfrak{y}), \overline{\Pi}(\mathfrak{x}, \mathfrak{y})]$. If the function $\Phi : [\mathfrak{v}_1, \mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1)] \times [\mathfrak{v}_3, \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)] \subset \mathbb{R}^2 \to \mathbb{R}$ is non negative integrable, as well as symmetric regarding two variable forms, i.e.*

$$\Phi(\mathfrak{x}, \mathfrak{y}) = \begin{cases} \Phi(\mathfrak{v}_1 + \mathfrak{v}_2 - \mathfrak{x}, \mathfrak{y}), \\ \\ \Phi(\mathfrak{x}, \mathfrak{v}_3 + \mathfrak{v}_4 - \mathfrak{y}), \\ \\ \Phi(\mathfrak{v}_1 + \mathfrak{v}_2 - \mathfrak{x}, \mathfrak{v}_3 + \mathfrak{v}_4 - \mathfrak{y}), \end{cases}$$

then we have

$$\begin{aligned}
&\widehat{F}\left(\frac{2\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1)}{2}, \frac{2\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)}{2}\right)\Bigg[\left(\epsilon^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}_{\widetilde{\mu}, \alpha, l, \widehat{w}, \mathfrak{v}_1{}^+, \mathfrak{v}_3{}^+}\right) \\
&\quad \{\Phi(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3); p)\} + \epsilon^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}_{\widetilde{\mu}, \alpha, l, \widehat{w}, \mathfrak{v}_1{}^+, (\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^-}\{\Phi(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3; p)\} \\
&\quad + \epsilon^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}_{\widetilde{\mu}, \alpha, l, \widehat{w}, (\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^-, \mathfrak{v}_3{}^+}\{\Phi(\mathfrak{v}_1, \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3); p)\} \\
&\quad + \epsilon^{\widetilde{\gamma}, \widetilde{\delta}, k, \mathfrak{v}_3}_{\widetilde{\mu}, \alpha, l, \widehat{w}, (\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^-, (\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^-}\{\Phi(\mathfrak{v}_1, \mathfrak{v}_3; p)\} \\
&\quad - \frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\left(\Phi(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)) + \Phi(\mathfrak{v}_1, \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\Phi(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\Phi(\mathfrak{v}_1,\mathfrak{v}_3))]\\
\supseteq &\Big[\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,\mathfrak{v}_3^+}\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\Phi(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\Phi(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}\\
&+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,\mathfrak{v}_3^+}\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\Phi(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)\Phi(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\Big(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\Phi(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\
&+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\Phi(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\
&+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)\Phi(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\Phi(\mathfrak{v}_1,\mathfrak{v}_3)\Big)\Big]\\
\supseteq &\left[\frac{(\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_3,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3))}{4}\right]\\
&\times\Big[\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,\mathfrak{v}_3^+}\{\Phi(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\{\Phi(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}\\
&+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,\mathfrak{v}_3^+}\{\Phi(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\{\Phi(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}(\Phi(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\
&+\Phi(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\Phi(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\Phi(\mathfrak{v}_1,\mathfrak{v}_3))].
\end{aligned}
$$

Proof. Proceeding from the relation (3.5) within the proof of Theorem 3.8, and multiplying both sides of it with $4\varrho^{\alpha-1}s^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1}\varrho^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}((\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2}s^{\widetilde{\mu}_2};p$ $\varrho\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+s\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))$, then by integrating the resulting inclusion with regard to (ϱ,s) on $[0,1]\times[0,1]$, we derive that

$$
\begin{aligned}
&4\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\int_0^1\int_0^1[\varrho^{\alpha-1}s^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1}\varrho^{\widetilde{\mu}_1};p)\\
&\times E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}((\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2}s^{\widetilde{\mu}_2};p)\Phi(\mathfrak{v}_1+\varrho\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+s\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))]\mathrm{d}s\mathrm{d}\varrho\\
\supseteq&\frac{4}{4}\Big[\int_0^1\int_0^1\varrho^{\alpha-1}s^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1}\varrho^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}((\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2}s^{\widetilde{\mu}_2};p)\\
&\times\Phi(\mathfrak{v}_1+\varrho\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+s\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))[\widehat{F}(\mathfrak{v}_1+\varrho\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+s\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\widehat{F}(\mathfrak{v}_1+\varrho\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3\\
&+(1-s)\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))]\mathrm{d}s\mathrm{d}\varrho\\
&+\int_0^1\int_0^1\varrho^{\alpha-1}s^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}((\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1}\varrho^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}((\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2}s^{\widetilde{\mu}_2};p)\\
&\times\Phi(\mathfrak{v}_1+\varrho\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+s\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))[\widehat{F}(\mathfrak{v}_1+(1-\varrho)\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+s\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\
&+\widehat{F}(\mathfrak{v}_1+(1-\varrho)\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+(1-s)\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))]\mathrm{d}s\mathrm{d}\varrho\Big].
\end{aligned}
$$

In accordance with the symmetry of the function $\Phi(\mathfrak{x},\mathfrak{y})$, it yields the proof of the first inclusion relation.

For the second inclusion relation, continuing from the inclusion relation (3.6) in the proof of Theorem 3.8, then multiplying both the sides of it with $\varrho^{\alpha-1}s^{\beta-1}E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_1,\alpha,\widehat{w}_1,l}(\widehat{w}_1(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\widetilde{\mu}_1}\varrho^{\widetilde{\mu}_1};p)E^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu}_2,\beta,\widehat{w}_2,l}(\widehat{w}_2(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\widetilde{\mu}_2}s^{\widetilde{\mu}_2};p)\Phi(\mathfrak{v}_1+\varrho\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+s\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))$, then by integrating the resulting inclusion with regard to (ϱ,s) on $[0,1]\times[0,1]$, In this way, we attain our required form.

Thus, the proof is completed.

Remark 3.8: If we take $\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)=\mathfrak{v}_2-\mathfrak{v}_1$ and $\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)=\mathfrak{v}_4-\mathfrak{v}_3$ in (3.12), we get the result for interval-valued convex function.

$$\begin{aligned}
&\widehat{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2},\frac{\mathfrak{v}_3+\mathfrak{v}_4}{2}\right)\\
&\quad\times\Bigg[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1{}^+,\mathfrak{v}_3{}^+}\right)\{\Phi(\mathfrak{v}_2,\mathfrak{v}_4;p)\}+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1{}^+,\mathfrak{v}_4{}^-}\{\Phi(\mathfrak{v}_2,\mathfrak{v}_3;p)\}+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2{}^-,\mathfrak{v}_3{}^+}\{\Phi(\mathfrak{v}_1,\mathfrak{v}_4;p)\}\\
&\quad+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2{}^-,\mathfrak{v}_4{}^-}\{\Phi(\mathfrak{v}_1,\mathfrak{v}_3;p)\}-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\left(\Phi(\mathfrak{v}_2,\mathfrak{v}_4)+\Phi(\mathfrak{v}_1,\mathfrak{v}_4)+\Phi(\mathfrak{v}_2,\mathfrak{v}_3)+\Phi(\mathfrak{v}_1,\mathfrak{v}_3)\right)\Bigg]\\
&\supseteq\Bigg[\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1{}^+,\mathfrak{v}_3{}^+}\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4;p)\Phi(\mathfrak{v}_2,\mathfrak{v}_4;p)\}+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1{}^+,\mathfrak{v}_4{}^-}\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3;p)\Phi(\mathfrak{v}_2,\mathfrak{v}_3;p)\}\\
&\quad+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2{}^-,\mathfrak{v}_3{}^+}\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4;p)\Phi(\mathfrak{v}_1,\mathfrak{v}_4;p)\}+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2{}^-,\mathfrak{v}_4{}^-}\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)\Phi(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&\quad-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\Big(\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)\Phi(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)\Phi(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)\Phi(\mathfrak{v}_2,\mathfrak{v}_3)\\
&\quad+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\Phi(\mathfrak{v}_1,\mathfrak{v}_3)\Big)\Bigg]\\
&\supseteq\left[\frac{(\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_3,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3))}{4}\right]\\
&\quad\times\Bigg[\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1{}^+,\mathfrak{v}_3{}^+}\{\Phi(\mathfrak{v}_2,\mathfrak{v}_4;p)\}+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1{}^+,\mathfrak{v}_4{}^-}\{\Phi(\mathfrak{v}_2,\mathfrak{v}_3;p)\}\\
&\quad+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2{}^-,\mathfrak{v}_3{}^+}\{\Phi(\mathfrak{v}_1,\mathfrak{v}_4;p)\}+\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_2{}^-,\mathfrak{v}_4{}^-}\{\Phi(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&\quad-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}\left(\Phi(\mathfrak{v}_2,\mathfrak{v}_4)+\Phi(\mathfrak{v}_1,\mathfrak{v}_4)+\Phi(\mathfrak{v}_2,\mathfrak{v}_3)+\Phi(\mathfrak{v}_1,\mathfrak{v}_3)\right)\Bigg].
\end{aligned}$$

Remark 3.9: From our main results, we can find some special cases for suitable choices of parameters involving generalized Mittag–Leffler functions. Moreover, if we take $\alpha\to 1^-$ some classical integrals can be derived. Finally, if we choose $\widetilde{\eta}(\mathfrak{y},\mathfrak{x})=\mathfrak{y}-\mathfrak{x}$, we can obtain new inequalities for interval-valued co-ordinated convex functions. We omit their proofs and the details are left to the interested readers.

3.4. Numerical examples with graphical analysis

In this section, we provide the numerical verification and graphical analysis of our main findings. As the graphs are the best way to compare any two quantities, so one can easily grasp the idea of the paper easily.

Example 3.1: *We set* $n = 0, p = 0, \alpha = 1, \beta = 1$ *in Theorem* 3.8*. If* $\widehat{F}(\mathfrak{x}, \mathfrak{y}) = [e^{2\mathfrak{x}}e^{2\mathfrak{y}}, (-2\mathfrak{x}^2 + 6\mathfrak{x} + 8)(-3\mathfrak{y}^2 + 6\mathfrak{y} + 5)]$ *and* $\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1) = \mathfrak{v}_2 - \mathfrak{v}_1$*, with* $\mathfrak{v}_1 = 0$*,* $\mathfrak{v}_2 = 1$*, and* $\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3) = \mathfrak{v}_4 - \mathfrak{v}_3$*, with* $\mathfrak{v}_3 = 0$ *and* $\mathfrak{v}_4 = 1$*, then*

$$\widehat{F}\left(\frac{2\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1)}{2}, \frac{2\mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)}{2}\right) = [7.39, 76.125],$$

$$\begin{aligned}
&\frac{B(\alpha)B(\beta)}{4\alpha\beta(\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3))^{\beta}}\Big[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,\mathfrak{v}_3^+}\right)\{\widehat{F}(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3); p)\}\\
&+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1^+,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\right)\{\widehat{F}(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3; p)\}\;.\\
&\quad\times\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,\mathfrak{v}_3^+}\right)\{\widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3); p)\}\\
&+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\right)\{\widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3; p)\}\\
&\quad-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}(\widehat{F}(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)))\\
&\quad+\widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3 + \widetilde{\eta}_2(\mathfrak{v}_4, \mathfrak{v}_3)) + \widehat{F}(\mathfrak{v}_1 + \widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1), \mathfrak{v}_3) + \widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3))\Big] = [10.21, 72.33]\\
&\frac{\widehat{F}(\mathfrak{v}_2, \mathfrak{v}_4) + \widehat{F}(\mathfrak{v}_1, \mathfrak{v}_4) + \widehat{F}(\mathfrak{v}_2, \mathfrak{v}_3) + \widehat{F}(\mathfrak{v}_1, \mathfrak{v}_3)}{4} = [17.60, 65].
\end{aligned}$$

which implies that

$$[7.39, 76.125] \supseteq [10.21, 72.33] \supseteq [17.60, 65].$$

which give the verification of Theorem 3.8.

- If we choose $n = 0, p = 0$, $\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1) = \mathfrak{v}_2 - \mathfrak{v}_1$, $\widetilde{\eta}_2(\mathfrak{v}_3, \mathfrak{v}_4) = \mathfrak{v}_4 - \mathfrak{v}_3$ with $\mathfrak{v}_1 = 0, \mathfrak{v}_2 = 2, \mathfrak{v}_3 = 0$, $\mathfrak{v}_4 = 1$ and If $\widehat{F}(\mathfrak{x}, \mathfrak{y}) = [e^{2\mathfrak{x}}e^{2\mathfrak{y}}, (-2\mathfrak{x}^2 + 6\mathfrak{x} + 8)(-3\mathfrak{y}^2 + 6\mathfrak{y} + 5)]$ and $\alpha, \beta \in (0, 1)$ in Theorem 3.8, then

Example 3.2: *We set* $n = 0, p = 0, \alpha = 1, \beta = 1$ *in Theorem* 3.10*. If* $\widehat{F}(\mathfrak{x}, \mathfrak{y}) = [e^{2\mathfrak{x}}e^{2\mathfrak{y}}, (-2\mathfrak{x}^2 + 6\mathfrak{x} + 8)(-3\mathfrak{y}^2 + 6\mathfrak{y} + 5)]$ *and* $\widetilde{\eta}_1(\mathfrak{v}_2, \mathfrak{v}_1) = \mathfrak{v}_2 - \mathfrak{v}_1$*, with* $\mathfrak{v}_1 = 0$*,* $\mathfrak{v}_2 = 1$*,*

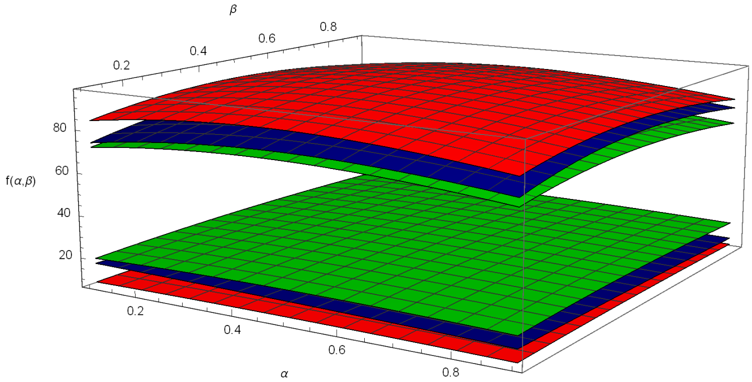

Figure 3.1: This is an image showing the comparison between left, middle and right sides of Theorem 3.8. Here red, blue and green colors show the left,middle and right intervals respectively.

and $\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)=\mathfrak{v}_4-\mathfrak{v}_3$, *with* $\mathfrak{v}_3=0$ *and* $\mathfrak{v}_4=1$, *then*

$$\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)=[7.39,76.13],$$

$$\begin{aligned}
&\frac{B(\beta)L_1}{4\beta(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Bigg[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_3+}\right)\left\{\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right)\right\}\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))-}\right)\left\{\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{v}_3\right)\right\}\\
&\quad-\frac{(1-\beta)}{B(\beta)}\left(\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)\right),\widehat{F}\left(\frac{2\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)}{2},\mathfrak{v}_3\right)\right)\Bigg]\\
&\quad+\frac{B(\alpha)L_2}{4\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Bigg[{}^{AB}I^{\sigma,\alpha}_{\mathfrak{v}_1+,\widehat{w},\rho}\left\{\widehat{F}\left(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\right\}\\
&\quad+{}^{AB}I^{\sigma,\alpha}_{\mathfrak{v}_2-,\widehat{w},\rho}\left\{\widehat{F}\left(\mathfrak{v}_1,\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\right\}\\
&\quad-\frac{(1-\alpha)}{B(\alpha)}\left(\widehat{F}\left(\mathfrak{v}_1,\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right),\widehat{F}\left(\mathfrak{v}_2,\frac{2\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)}{2}\right)\right)\Bigg]=[8.69,74.21]
\end{aligned}$$

$$\begin{aligned}
&\frac{B(\alpha)B(\beta)}{4\alpha\beta(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Big[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+,\mathfrak{v}_3+}\right)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))-}\right)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))-,\mathfrak{v}_3+}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))-,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&\quad-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\
&\quad+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3))\Big]=[10.205,72.33]
\end{aligned}$$

$$\begin{aligned}
&\frac{B(\alpha)L_2}{8\alpha(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}}\Big[\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))-}\right)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)\}\\
&\quad+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+}\right)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)\}+\left(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1+}\right)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4)\}\\
&\quad-\frac{(1-\alpha)}{B(\alpha)}(\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_4)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_4))\Big]
\end{aligned}$$

$$+\frac{B(\beta)L_1}{8\beta(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^\beta}\Big[\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)\}+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\Big)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)\}$$
$$+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_3^+}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\}+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_3^+}\Big)\{\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\}$$
$$-\frac{(1-\beta)}{B(\beta)}(\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_2,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)))\Big]$$
$$=[13.40,68.59]$$
$$\frac{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))}{4}$$
$$=[17.60,65].$$

which implies that

$$[7.39,76.13]\supseteq[8.69,74.92]\supseteq[10.205,72.33]\supseteq[13.40,68.59]\supseteq[17.60,65].$$

which give the verification of Theorem 3.10.

- If we choose $n=0,p=0$, $\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)=\mathfrak{v}_2-\mathfrak{v}_1$,$\widetilde{\eta}_2(,\mathfrak{v}_4)=\mathfrak{v}_4-\mathfrak{v}_3$ with $\mathfrak{v}_1=0,\mathfrak{v}_2=2,\mathfrak{v}_3=0$, $\mathfrak{v}_4=1$ and If $\widehat{F}(\mathfrak{x},\mathfrak{y})=[e^{2\mathfrak{x}}e^{2\mathfrak{y}},(-2\mathfrak{x}^2+6\mathfrak{x}+8)(-3\mathfrak{y}^2+6\mathfrak{y}+5)]$ and $\alpha,\beta\in(0,1)$ in Theorem 3.10, then

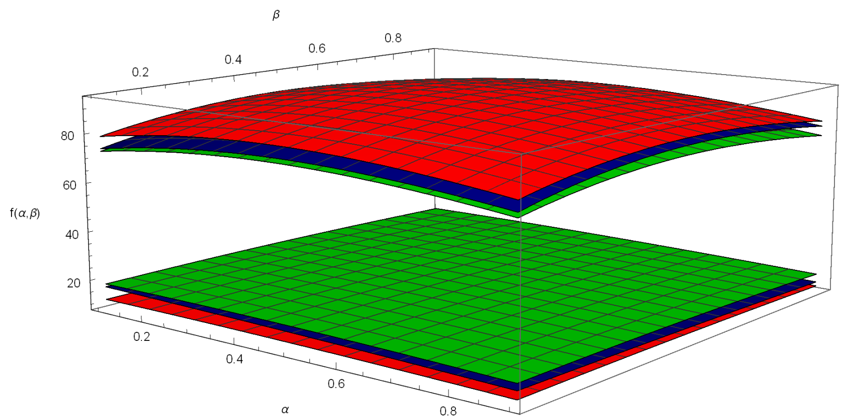

Figure 3.2: This is an image showing the comparison between left, middle and right sides of Theorem 3.10 where red , blue and green colors represents the left, middle and right intervals respectively. Clearly containments can be viewed.

Example 3.3: *We set $n=0,p=0,\alpha=1,\beta=1$ in Theorem 3.11. If $\widehat{F}(\mathfrak{x},\mathfrak{y})=[e^{x}e^{\mathfrak{y}},(-2\mathfrak{x}^2+6\mathfrak{x}+3)(-\mathfrak{y}^2+2\mathfrak{y}+1)]$ and $G(\mathfrak{x},\mathfrak{y})=[(\mathfrak{x}^2+1)(\mathfrak{y}^2+1),(-\mathfrak{x}^2+2\mathfrak{x}+2)(-\mathfrak{y}^2+4)]$ and $\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)=\mathfrak{v}_2-\mathfrak{v}_1$, with $\mathfrak{v}_1=0$ and $\mathfrak{v}_2=2$ and $\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3)=\mathfrak{v}_4-\mathfrak{v}_3$, with $\mathfrak{v}_3=0$*

and $\mathfrak{v}_4 = 1$*, then*

$$\begin{aligned}
&\frac{B(\alpha)B(\beta)}{\alpha\beta(\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^{\alpha}(\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^{\beta}}\Big[\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1{}^+,\mathfrak{v}_3{}^+}\Big)\\
&\quad\times\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},\mathfrak{v}_1{}^+,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\Big)\{\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3;p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,\mathfrak{v}_3{}^+}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)G(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3);p)\}\\
&\quad+\Big(\epsilon^{\widetilde{\gamma},\widetilde{\delta},k,\mathfrak{v}_3}_{\widetilde{\mu},\alpha,l,\widehat{w},(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1))^-,(\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))^-}\Big)\{\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3;p)G(\mathfrak{v}_1,\mathfrak{v}_3;p)\}\\
&\quad-\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}(\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))\\
&\quad+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))G(\mathfrak{v}_1,\mathfrak{v}_3+\widetilde{\eta}_2(\mathfrak{v}_4,\mathfrak{v}_3))+\widehat{F}(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)G(\mathfrak{v}_1+\widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1),\mathfrak{v}_3)\\
&\quad+\widehat{F}(\mathfrak{v}_1,\mathfrak{v}_3)G(\mathfrak{v}_1,\mathfrak{v}_3)\Big]=[40.48,351.56].
\end{aligned}$$

Also

$$\begin{aligned}
C(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)&=[65.36,270],\\
D(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)&=[47.91,297],\\
E(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)&=[46.15,248],\\
\Psi(\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)&=[39.81,253].
\end{aligned}$$

This implies that

$$[40.48,351.56]\supseteq[54.96,285.56].$$

This verifies Theorem 3.11.

- If we choose $n=0, p=0, \widetilde{\eta}_1(\mathfrak{v}_2,\mathfrak{v}_1)=\mathfrak{v}_2-\mathfrak{v}_1$, $\widetilde{\eta}_2(\mathfrak{v}_3,\mathfrak{v}_4)=\mathfrak{v}_4-\mathfrak{v}_3$ with $\mathfrak{v}_1=0, \mathfrak{v}_2=2, \mathfrak{v}_3=0$, $\mathfrak{v}_4=1$, $\alpha,\beta\in(0,1)$ and If $\widehat{F}(\mathfrak{x},\mathfrak{y})=[e^{x}e^{\mathfrak{y}},(-2\mathfrak{x}^2+6\mathfrak{x}+3)(-\mathfrak{y}^2+2\mathfrak{y}+1)]$, $G(\mathfrak{x},\mathfrak{y})=[(\mathfrak{x}^2+1)(\mathfrak{y}^2+1),(-\mathfrak{x}^2+2\mathfrak{x}+2)(-\mathfrak{y}^2+4)]$ in Theorem 3.11, then

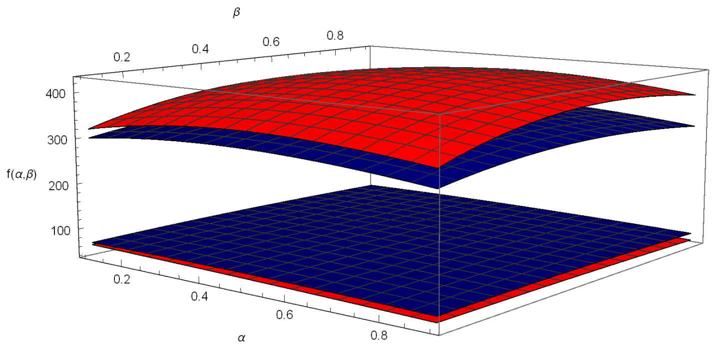

Figure 3.3: This is an image showing the comparison between left and right sides of Theorem 3.11 and red and blue colors indicate the left and right intervals.

Remark 3.10: Our main findings reveal that there are certain specific scenarios where generalized Mittag-Leffler functions can be used, depending on the choice of parameters. Additionally, if we let α approach 1^-, some classical integrals can be obtained. By selecting $\widetilde{\eta}(\mathfrak{y},\mathfrak{x})$ to be equal to $\mathfrak{y}-\mathfrak{x}$, we can derive novel inequalities for interval-valued coordinated convex functions. The proofs are not included in this discussion, but interested readers can refer to the details.

3.5. Conclusions

To summarize, our paper introduces new variations of Hermite-Hadamard's inequality by utilizing generalized interval-valued Mittag-Leffler fractional double AB-integrals and the class of interval-valued coordinated pre-invex functions. As far as we know, these results have not been previously explored in the literature. The application of convexity and pre-invex functions can have practical implications, such as maximizing the likelihood of multiple linear regressions that involve the Gauss-Laplace distribution. Additional information on this subject can be found in [48, 37, 3, 2, 8, 33]. We hope that our contributions and methodologies will inspire further research in this area.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Data Availability

No data were used to support this study

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Chapter 4

Analysis of Monkey Pox Transmission Dynamics in Society with Control Strategies Under Caputo-Fabrizio Fractal-Fractional Derivative

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Abstract

In this study, we apply both classical and fractional-order differential equations to construct a deterministic mathematical model of the monkey pox virus. The model accounts for every conceivable interaction that may play a role in the propagation of the disease throughout the population. In the current study, we investigated the corruption using a Caputo-fabrizio fractal-fractional derivative with exponentially decaying type kernel to examine the impact of vaccination and isolation. The model's fundamental mathematical characteristics have been thoroughly investigated. We have calculated the basic reproduction number and equilibrium points, as well as identified the feasible region for the model. The existence and stability of the model were proved using the banach fixed point theory and Picard's successive approximation method. The existence and uniqueness of the model's solution have also been established under appropriate conditions. In addition, the asymptotically local and global stability of the disease-free equilibrium states and endemic equilibrium states were investigated. Hyers-Ulam and Hyers-Ulam-Rassias consistency of the obsessive solution is also explored. In order to design effective infection control measures, we study the dynamic behavior of the system. The complex dynamics of monkey pox infection under the influence of different system input factors are explored by extensive numerical simulations of the proposed monkey pox model with varying input parameters. In this way, people can learn about the role that control parameters play in efforts to eradicate monkey pox. In order to design effective infection control measures, we study the dynamical behavior of the system. We presented a number of different parameters to the decision-makers in the community in order to control monkey pox.

Keywords:

Mathematical Modelling, Monkeypox, Vaccination, Model Fitted, Hyers-Ulam Stability, Hyers-Ulam-Rassias Stability, Analysis.

4.1. Introduction

The monkeypox virus is the infectious factor that is responsible for spreading this zoonotic disease from rodents to humans [11]. Although it has been seen in other parts of the world, Africa is where it is most commonly seen. Monkeypox was first identified in 1958 after two study groups of monkeys experienced pox-like outbreaks [8, 6]. The first instances of monkeypox were reported to the WHO in January 2022, and since then, several countries have reported the disease. There was one confirmed death among the 2103 cases reported to WHO as of June 15, 2022 [1, 9].

Most frequently, wild animals spread the virus from one person to another. Researchers are looking into the transmission of the disease via respiratory droplets, vaginal secretions, and semen [25, 5]. In most of the cases that have been reported, the virus has been passed from person to person. The virus can also spread by respiratory secretions during prolonged face-to-face contact or sexual activity or other intimate physical contact, as well as through direct contact with an infected rash, scabs, or body fluids [25, 5, 24, 30, 12]. Backaches, muscle aches, headache, fever, chills, fatigue, and swollen lymph nodes are some of the common symptoms of monkeypox. The majority of persons who contract monkeypox experience mild symptoms and make a full recovery within a few weeks. Immunocompromised individuals may experience more severe symptoms [1, 27]. Although a reliable method of treating monkeypox virus infection has yet to be discovered, the vaccine, antiviral drugs, and vaccine immune globulin originally developed to combat smallpox can be used to do the same. Since smallpox has been eliminated globally, the vaccine is not available at this time [2, 3].

Mathematical modeling can help solve technical, telecommunication, and physical challenges [22, 23, 35]. In addition, the current literature [34, 33, 37] demonstrates a wide range of practical applications of mathematical models. In the literature [28, 26, 17, 29, 4, 16, 38, 14, 21, 20, 15], various disease dynamics have been the subject of published mathematical models. While [26] explores HIV-1 infection's formulation of viral kinetics, [28] proposes a fractional order model for tuberculosis infection. A mathematical study using Indonesian HIV/AIDS data was proposed by the authors [38]. The seasonal element of mosquitoes is considered in [14], where the mathematical model of malaria is examined. A mathematical study was conducted to comprehend anxiety factors during COVID-19 outbreak [21], the mass action mechanism is used to design and simulate the epidemic model [20]. Recent uses of mathematical modeling include the study of the COVID-19 pandemic [17], the study of breast cancer [29], the study of the Lassa fever infection [4], analysis of the polio virus's dynamics [16], and the study of various infectious illnesses [15, 18, 13, 7]. Monkeypox has also been the subject of mathematical models. In [36], mathematical modeling is used to examine how the monkeypox virus spreads between humans and animals, while in [19], mathematical modeling is used to investigate how the virus is spread among humans. Without using any actual cases, the authors of [31] present a thorough examination of the monkeypox virus and its two modes of spreading, human-to-human and rodent-to-human. To simplify the complicated pathophysiological process, a mathematical model of compartments is proposed. By dividing human and animal populations into three groups, the authors of [6] were able to generate extensive mathematical results without resorting to actual data. Our study intends to bring the literature on monkeypox up to date by analyzing a novel mathematical model that uses real data to investigate the disease and takes into account both human-to-human and rodent-to-human transmission. Our research is intended to aid the United States' ongoing fight against disease.

The work is divided into the following sections: In Section 5.2, we detail the model's construction and provide an explanation of the various transmission channels and the relative contribution of each class to the overall flow rate. Data fitting with the real data of U.S.A and parameter estimates are handled in Section 4.3, and daily U.S. case reports are used for this purpose. In Section 4.4, we will examine the fundamental characteristics of the model and provide a qualitative evaluation of it. Parameters of the model are estimated, and the basic reproduction number's sensitivity is analyzed, in Section 4.4.3. The foundations of complex fractional calculus are discussed in Section 4.5. The H-U stability of the suggested fractal-fractional model is also discussed in detail in Section 5.5.2. For details on how to use numerical methods to solve the proposed model, (we reffer to see Section 5.6). In the section under Section 5.7, we present the findings of the analysis in graphical form. In Section 5.8, we summarize our findings and suggestions.

4.2. Model Formulation

Here, we assume the spread of monkey pox between humans and rodents and use that information to create a deterministic mathematical model of the virus. In the current model it is suppose that the susceptible human population are vaccinated against the virus. It is also suppose that identified suspected human are isolated against the virus. We considered ten compartmental model; seven human compartments and three rodent compartments (we reffer to see Fig. 5.1).

Due to recruitment, either from birth rate or immigration, the population of susceptible rodents can fluctuate with the rate Π_r. The rate of natural deaths per capita, μ_r, and the intensity of infections, $\alpha_1 I_r$, both have a negative impact on it. As a result, the growth or decline in the population of susceptible rodents is:

$$\frac{dS_r}{dt} = \Pi_r - (\alpha_1 I_r + \mu_r) S_r \tag{4.1}$$

The infectious force, modeled as $\alpha_1 I_r$, is applied to the number of exposed rodents. This group includes rodents that come into contact with the virus but do not become infected. The rate at which they contract the infection reduces it by β_1. This means that the rate of increase in the number of exposed rodents is:

$$\frac{dE_r}{dt} = \alpha_1 I_r S_r - (\beta_1 + \mu_r) E_r \tag{4.2}$$

As more and more rodents make the leap from exposed to confirmed infected, the total number of rodents with the infection grows at a rate of β_1 . This class comprises of rodents who are frequently infected with the virus and have ability to infect other rodents as well as human population. Consequently, they are dying at a rate of ∂_r due to infection and a rate of μ_r due to natural causes. So, the growth or decline in the rodent population is:

$$\frac{dI_r}{dt} = \beta_1 E_r - (\partial_r + \mu_r) I_r \tag{4.3}$$

The number of susceptible humans changes as a result of recruitment through birth rate or immigration Π_h. The susceptible class is created by daily recruitment of individuals born into homes. It's decreased by natural death per capita rate μ_h, rate of vaccination m and force of infection $\alpha_2 I_r + \alpha_3 I_h$. Hence, the rate of change in the number of susceptible humans is:

$$\frac{dS_h}{dt} = \Pi_h - (m + \alpha_2 I_r + \alpha_3 I_h + \mu_h) S_h \tag{4.4}$$

Vaccination of susceptible humans causes a change in the total number of vaccinated humans at a rate of m. This class comprises humans who are protected by vacinations as well as those peoples have been in contact with the vaccination. After vaccination, human exposure to the disease decreases at a rate of γ. Therefore, the rate of increase in human exposure is:

$$\frac{dV_h}{dt} = mS_h - (\gamma + \mu_h) V_h \tag{4.5}$$

Force of infection at rate $\alpha_2 I_r + \alpha_3 I_h$ and exposure rate γ after vaccination are used to model the number of exposed humans. These people have been in close contact with the virus and are at risk of infection, but have not yet contracted the virus themselves.

It is reduced by the infection rate β_3, the isolation rate β_2, and the recovery rate f_4, all of which are a result of vaccination. Therefore, the rate of increase in human exposure is:

$$\frac{dE_h}{dt} = \gamma V_h + (\alpha_2 I_r + \alpha_3 I_h) S_h - (\beta_2 + \beta_3 + f_4 + \mu_h) E_h \tag{4.6}$$

Those who are quarantined in an effort to contain the disease add to the population of people living in isolation at a rate of β_2. These people move from the open to the closed system at a rate of β_2. As a result, their numbers are lower because their infection rate is lower ω_1 and their recovery rate is lower f_3 as a result of being vaccinated for the disease. This means that the rate at which human changes occur in isolation can be expressed as:

$$\frac{dQ_h}{dt} = \beta_2 E_h - (\omega_1 + f_3 + \mu_h) Q_h \tag{4.7}$$

The progression from exposed to confirm infected humans results in an increase in the number of infected humans at a rate of β_3 . This category consists of individuals who are frequently infected by the virus. This decreases at the rate of f_2 for the rate at which they recover, ω_2 for the rate at which they move from the infected compartment to the clinically ill compartment, and ∂_h for the rate at which humans die from the disease. Therefore, the rate of increase in human infections is:

$$\frac{dI_h}{dt} = \beta_3 E_h + \omega_1 Q_h - (\omega_2 + f_2 + \partial_h + \mu_h) I_h \tag{4.8}$$

Transitioning from an infected compartment to calinically ill humans causes an increase in the number of calinically ill humans at a rate of ω_2. Those who fall into this category have been infected with a virus and require medical care. Thus, decreasing by the rate of recovery f_1 and the rate of human mortality ∂_h from the disease. Therefore, the rate of increase in human infections is:

$$\frac{dC_h}{dt} = \omega_2 I_h - (f_1 + \partial_h + \mu_h) C_h \tag{4.9}$$

The number of recovered human population increases as a result of the transitions from exposed, isolated, infected and calinicaly ill compartment at the rates of f_4, f_3, f_2 and f_1 respectively. The population then decreased due to natural per capita deaths. Therefore, the rate of transformation in the population is:

$$\frac{dR_h}{dt} = f_1C_h + f_2I_h + f_3Q_h + f_4E_h - \mu_h R_h \tag{4.10}$$

Consequently, a system of non-linear ordinary differential equations is given below based on the aforementioned descriptions:

$$\begin{aligned}
\frac{dS_r}{dt} &= \Pi_r - (\alpha_1 I_r + \mu_r) S_r, \\
\frac{dE_r}{dt} &= \alpha_1 I_r S_r - (\beta_1 + \mu_r) E_r, \\
\frac{dI_r}{dt} &= \beta_1 E_r - (\partial_r + \mu_r) I_r, \\
\frac{dS_h}{dt} &= \Pi_h - (m + \alpha_2 I_r + \alpha_3 I_h + \mu_h) S_h, \\
\frac{dV_h}{dt} &= mS_h - (\gamma + \mu_h) V_h, \\
\frac{dE_h}{dt} &= \gamma V_h + (\alpha_2 I_r + \alpha_3 I_h) S_h - (\beta_2 + \beta_3 + f_4 + \mu_h) E_h, \\
\frac{dQ_h}{dt} &= \beta_2 E_h - (\omega_1 + f_3 + \mu_h) Q_h, \\
\frac{dI_h}{dt} &= \beta_3 E_h + \omega_1 Q_h - (\omega_2 + f_2 + \partial_h + \mu_h) I_h, \\
\frac{dC_h}{dt} &= \omega_2 I_h - (f_1 + \partial_h + \mu_h) C_h, \\
\frac{dR_h}{dt} &= f_1 C_h + f_2 I_h + f_3 Q_h + f_4 E_h - \mu_h R_h,
\end{aligned} \tag{4.11}$$

Table 4.1: Discription of variables for the model 5.7.

Variables	Discription
$S_r(t)$	Class of susceptible rodent, those rodents who are potentially susceptible to disease.
$E_r(t)$	Class of exposed rodent, those rodents who have had direct or indirect contact with rodents who are afflicted with the disease.
$I_r(t)$	Class of infected rodent, those rodents who had been infected with the disease.

$S_h(t)$	Class of susceptible humans, those peoples who are potentially susceptible to disease.
$E_h(t)$	Class of exposed humans, those peoples who have had direct or indirect contact with patients who are afflicted with the disease.
$I_h(t)$	Class of infected humans, those people who had been infected with the disease.
$Q_h(t)$	Class of isolated humans, those people who are being kept in isolation to stop the disease spread.
$V_h(t)$	Class of vaccinated humans, those peoples who have been protected by vaccinations as well as those who have been in contact with the vaccine.
$C_h(t)$	Class of clinicaly ill humans, those peoples who need medical attention after being ill due to the disease.
$R_h(t)$	Class of recovered humans, .

4.3. Parameter estimation and model fitting

The purpose of this research is to identify the most appropriate model for the cases observed in the United States and to examine the impact of behavioral interventions on the spread of this novel infectious disease (see Figs. 4.4 & 4.3). We developed the most sophisticated epidemic model available to make sense of the data surrounding the monkey pox a global epidemic in the United States. Parameters and model selection findings for the best fit are displayed in Fig. 4.2. But it corresponds with the confirmed total number of infected cases reported in the United States (monkey pox cases were recorded in the United States in 2022). Finally, we demonstrate how the model's implausible assumptions promote healing for the injured and long-term health in the community. Preventive strategies have been proposed as a means of treating or eliminating viruses in the general population.

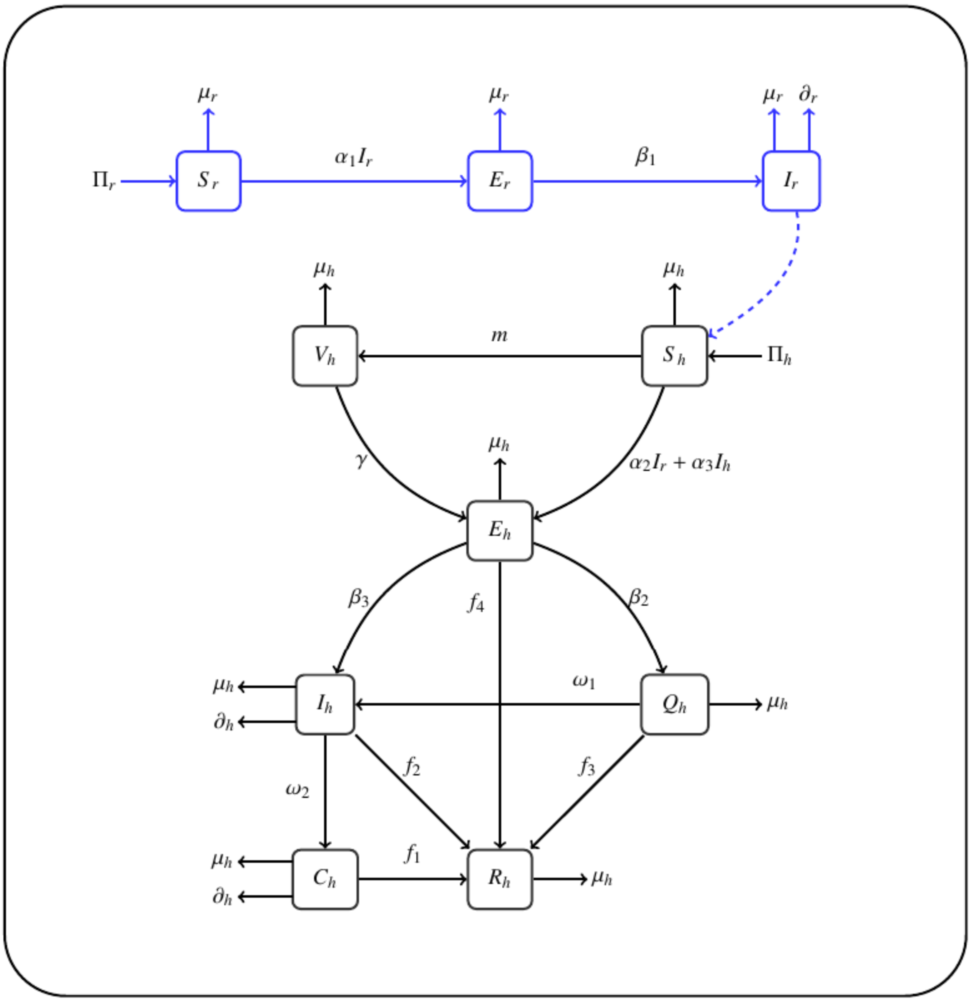


Figure 4.1: Dynamical Phase Diagram of Monkey Pox model, transmitting impact of vaccine.

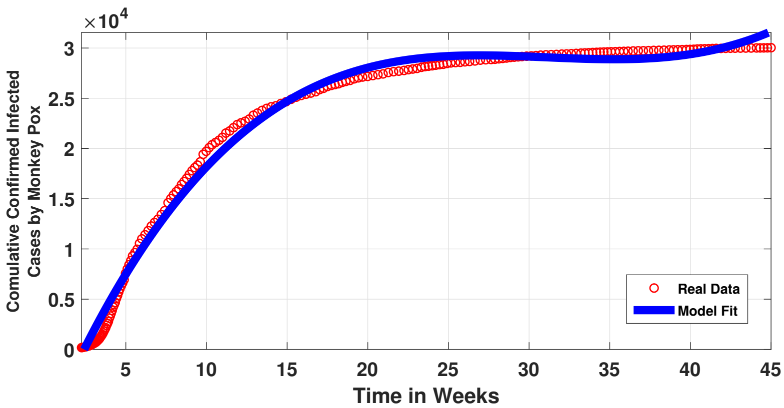


Figure 4.2: Model Fitting with real data of United State of America (U.S from 05/10/2022 to 01/01/2023).

Table 4.2: Discription of variables and parameters for the model 5.7.

Parameters	Discription	Value	Source
Π_h	Natural natality rate of humans.	64850	[32]
ω_2	Proportion of clinically recorded humans after conformation of infection.	0.94	Fitted
μ_h	Proportion of natural expiry rate.	0.000303	[32]
m	Proportion of suseptible human to be vaccinated.	0.1	Fitted
α_2	Proportion of exposed humans from susceptible class due to infected rodents.	0.00025	Fitted
α_3	Proportion of exposed humans from susceptible class due to infected human.	0.000062	Fitted
γ	Proportion of exposed humans after being vaccinated.	0.1	Estimated
f_1	Proportion of recovered humans from clinicaly recorded humans.	0.27	Fitted
Π_r	Natural natality rate of rodents.	0.2	[32]
∂_h	Disease related metality rate of humans.	0.003286	Estimated
f_4	Proportion of direct recovered humans after exposure.	0.088366	[32]
ω_1	Proportion of infected humans shifted from qurantined class.	0.1	Fitted
α_1	Proportion of exposed rodents from susceptible class.	0.027	Fitted
μ_r	Natural matelity rate of rodent.	0.02	[32]
∂_r	Disease related metality rate of rodents.	0.2	Fitted
f_2	Proportion of recovered humans after being infected.	0.83	Fitted
β_1	Proportion of infected rodents from susceptible class.	0.3	Fitted
f_3	Proportion of recovered humans from isolation.	0.52	Fitted

β_3	Proportion of infected humans from exposed class.	0.2	Fitted

4.4. Qualitative analysis of the model

4.4.1. Positivity and Boundedness of the Model Solution

Theorem 4.1:

Let $S_r(0) = S_{r_0}, E_r(0) = E_{r_0}, I_r(0) = I_{r_0}, S_h(0) = S_{h_0}, V_h(0) = V_{h_0}, E_h(0) = E_{h_0}, Q_h(0) = Q_{h_0}, I_h(0) = I_{h_0}, C_h(0) = C_{h_0}, R_h(0) = R_{h_0}$ *be the initial values of the state variables. If* $S_{r_0}, E_{r_0}, I_{r_0}, S_{h_0}, V_{h_0}, E_{h_0}, Q_{h_0}, I_{h_0}, C_{h_0}, R_{h_0}$ *are positive then this implies that* $S_r(t), E_r(t), I_r(t), S_h(t), V_h(t), E_h(t), Q_h(t), I_h(t), C_h(t), R_h(t)$ *are positive for all time* $t > 0$*. However*

$$\lim_{t\to\infty} \sup N_h(t) \leq \frac{\Pi_h}{\mu_h} \qquad \text{and} \qquad \lim_{t\to\infty} \sup N_r(t) \leq \frac{\Pi_r}{\mu_r} \tag{4.12}$$

Also, if $N_{h_0} \leq \frac{\Pi_h}{\mu_h}$*, then* $N_h \leq \frac{\Pi_h}{\mu_h}$*, and if* $N_{r_0} \leq \frac{\Pi_r}{\mu_r}$*, then* $N_r \leq \frac{\Pi_r}{\mu_r}$*, then the biological feasible feasible region for the differential equation are given by*

$$\mathtt{B}_h = \left\{(S_h, V_h, E_h, Q_h, I_h, C_h, R_h) \subset \mathbb{R}^7_+ : S_h + V_h + E_h + Q_h + I_h + C_h + R_h \leq \frac{\Pi_h}{\mu_h}\right\} \tag{4.13}$$

$$\mathtt{B}_r = \left\{(S_r, E_r, I_r) \subset \mathbb{R}^3_+ : S_r + E_r + I_r \leq \frac{\Pi_r}{\mu_r}\right\} \tag{4.14}$$

such that

$$\mathtt{B} = \mathtt{B}_h \times \mathtt{B}_r \subset \mathbb{R}^7_+ \times \mathbb{R}^3_+ \tag{4.15}$$

and we have $\mathtt{B}$ *is positive invariant region for the model solution.*

Proof. Let $S_{r_0}, E_{r_0}, I_{r_0}, S_{h_0}, V_{h_0}, E_{h_0}, Q_{h_0}, I_{h_0}, C_{h_0}, R_{h_0}$ be positive, and we'd like to demonstrate that the state variables are similarly positive. From system (5.7), we have

$$\frac{dS_r}{dt} = \Pi_r - (\alpha_1 I_r + \mu_r) S_r,$$
$$\frac{dS_r}{dt} + (\alpha_1 I_r + \mu_r) S_r = \Pi_r,$$

Since $\Pi_r \geq 0$ so that,

$$\begin{aligned} \frac{dS_r}{dt} + (\alpha_1 I_r + \mu_r) S_r &\geq 0, \\ \frac{dS_r}{dt} &\geq -(\alpha_1 I_r + \mu_r) S_r, \\ \frac{1}{S_r} dS_r &\geq -(\alpha_1 I_r + \mu_r) dt, \end{aligned} \tag{4.16}$$

by integration, we have,

$$\ln S_r \geq -\int (\alpha_1 I_r + \mu_r) dt + C,$$
$$\ln S_r \geq A(t) + C, \tag{4.17}$$

Here $A(t) = -\int (m + \mu_h + \phi_h) dt$, then at $t = 0$ we have

$$\ln S_{r_0} \geq A(0) + C, \tag{4.18}$$

subtract Eq. (5.15) from Eq. (5.16) and now we can write

$$\ln S_r - \ln S_{r_0} \geq A(t) - A(0),$$
$$\ln \frac{S_r}{S_{r_0}} \geq A(t) - A(0),$$

The exponential of each term yields,

$$\frac{S_r}{S_{r_0}} \geq \exp\{A(t) - A(0)\},$$
$$S_r \geq S_{r_0} \exp\{A(t) - A(0)\},$$

$$S_r \geq S_{r_0}, \qquad t \geq 0,$$

Since $S_r(0) = S_{r_0}$ is positive, that is, $S_{r_0} > 0$ for $t > 0$ it implies that

$$S_r \geq S_{r_0} > 0, \qquad t \geq 0,$$
$$S_r \geq 0, \quad \text{for all} \qquad t \geq 0.$$

For all $t > 0$, we get $S_{r_0} > 0$; hence, we know that $S_r(t)$ is positive. This means that $S_r > 0$ is true. Similar considerations apply to the remaining state variables. This shows that $E_r, I_r, S_h, V_h, E_h, Q_h, I_h, C_h$, and R_h are positive for all time $t > 0$.

The following theorem is used to characterize the solution's boundedness:

Theorem 4.2:

Given a positive set of solutions $(S_h(t), V_h(t), E_h(t), Q_h(t), I_h(t), C_h(t), R_h(t), S_r(t), E_r(t)\, I_r(t))$*, there exist a set* $\mathtt{B}$ *in which this solution set is contained and bounded.*

Proof. For the entire population, we have

$$N = N_r + N_h \tag{4.19}$$

Here N_r is the total rodent population and is given by:

$$N_r = S_r + E_r + I_r \tag{4.20}$$

implies that

$$\frac{dN_r}{dt} = \frac{dS_r}{dt} + \frac{dE_r}{dt} + \frac{dI_r}{dt},$$
$$\frac{dN_r}{dt} = \Pi_r - \mu_r N - \sigma_r I_r,$$

If we suppose there is zero rodents fatality rate due to disease, then we can write

$$\frac{dN_r}{dt} \leq \Pi_r - \mu_r N,$$

by keeping with the concept of integration and the exponential growth criterion, we can write

$$N_r(t) \leq \frac{\Pi_r}{\mu_r} - \left(\frac{\Pi_r}{\mu_r} - N_{r_0}\right) e^{-\mu_r t},$$

This can be shown by defining the upper bound of the inequality as $t \to \infty$, then $\sup N_r(t) = \frac{\Pi_r}{\mu_r}$ for all $N_r(t) \in \left[0, \frac{\Pi_r}{\mu_r}\right]$ and for $t = 0$ we have

$$N_r(t) \leq \frac{\Pi_r}{\mu_r}. \tag{4.21}$$

Which implies that $N_r(t)$ is bounded. Also N_h is the total number of people and is given by:

$$N_h = S_h + V_h + E_h + Q_h + I_h + C_h + R_h,$$

implies that

$$\frac{dN_h}{dt} = \frac{dS_h}{dt} + \frac{dV_h}{dt} + \frac{dE_h}{dt} + \frac{dQ_h}{dt} + \frac{dI_h}{dt} + \frac{dC_h}{dt} + \frac{dR_h}{dt}.$$

$$\frac{dN_h}{dt} = \Pi_h - \mu_h N_h - \sigma_h(I_h + C_h),$$

If we suppose there is zero human fatality rate due to disease, then we can write

$$\frac{dN_h}{dt} \leq \Pi_h - \mu_h N,$$

by keeping with the concept of integration and the exponential growth criterion, we can write

$$N_h(t) \leq \frac{\Pi_h}{\mu_h} - \left(\frac{\Pi_h}{\mu_h} - N_{h_0}\right) e^{-\mu_h t},$$

This can be shown by defining the upper bound of the inequality as $t \to \infty$, then $\sup N_h(t) = \frac{\Pi_h}{\mu_h}$ for all $N_h(t) \in \left[0, \frac{\Pi_h}{\mu_h}\right]$ and for $t = 0$ we have

$$N_h(t) \leq \frac{\Pi_h}{\mu_h} \tag{4.22}$$

which implies that $N_h(t)$ is bounded.

This proves, as expected, that the model's equations have a positive invariant solution, and thus establishes the concept of $\mathtt{B}$. This leads us to the conclusion that the model is both epidemiologically feasible and well-posed in $\mathtt{B}$.

4.4.2. The Equilibrium State of the system

In mathematical biology, an equilibrium state is a steady condition in which all of a system's constituent parts are in appropriate proportions and undergo no net change over time. This condition typically exists when there are no external forces or disturbances acting on the system. The study of biological processes and the ability to predict how they might react to changes in environmental conditions or external interventions require an appreciation of the equilibrium state.

4.4.2.1. Monkey Pox Free Equilibrium State

The variables $E_r = I_r = E_h = Q_h = I_h = C_h = R_h = 0$ must stay in equilibrium to prevent an outbreak of monkey pox. In this particular scenario, the system (5.7) demonstrates that the monkey pox free equilibrium point is indicated by the notation $\Omega_{MFE} = \left(S_r^{MFE}(t), E_r^{MFE}(t), I_r^{MFE}(t), S_h^{MFE}(t), V_h^{MFE}(t), E_h^{MFE}(t), Q_h^{MFE}(t), I_h^{MFE}(t), C_h^{MFE}(t), R_h^{MFE}(t)\right)$ and is the point at which there are no longer any cases of disease in the community. The number of cases in each group will be 0. So, the monkey pox-free equilibrium meets the conditions.

$$\Omega_{MFE} = \left(\frac{\Pi_r}{\mu_r}, 0, 0, \frac{\Pi_h}{m+\mu_h}, \frac{m\Pi_h}{(\gamma+\mu_h)(m+\mu_h)}, 0, 0, 0, 0, 0\right). \tag{4.23}$$

4.4.2.2. Existance of Monkey Pox Endemic Equilibrium State

The potential existence of an endemic equilibrium point, represented by the notation $\Omega_{MEE}(\cdot)$, will be investigated below, and denoted as:

$$\Omega_{MEE}\left(S_r^*(t), E_r^*(t), I_r^*(t), S_h^*(t), V_h^*(t), E_h^*(t), Q_h^*(t), I_h^*(t), C_h^*(t), R_h^*(t)\right)$$

. So, if for $\phi_h = \alpha_2 I_r + \alpha_3 I_h$ we are looking for a second equilibrium point, we will find the endemic equilibrium point as follows:

$$S_r^*(t) = \frac{(\beta_1+\mu_r)(\sigma_r+\mu_r)}{\alpha_1\beta_1}, \qquad E_r^*(t) = \frac{\Pi_r\alpha_1\beta_1 - \mu_r(\beta_1+\mu_r)(\sigma_r+\mu_r)}{\alpha_1\beta_1(\beta_1+\mu_r)},$$

$$I_r^*(t) = \frac{\Pi_r\alpha_1\beta_1 - \mu_r(\beta_1+\mu_r)(\sigma_r+\mu_r)}{\alpha_1(\beta_1+\mu_r)(\sigma_r+\mu_r)},$$

$$S_h^*(t) = \frac{\Pi_h}{m+\mu_h+\phi_h}, \qquad V_h^*(t) = \frac{m\Pi_h}{(\gamma+\mu_h)(m+\mu_h+\phi_h)},$$

$$E_h^*(t) = \frac{\Pi_h(m\gamma+(\gamma+\mu_h)\phi_h)}{(\gamma+\mu_h)(\beta_2+\beta_3+f_4+\mu_h)(m+\mu_h+\phi_h)},$$

$$Q_h^*(t) = \frac{\beta_2\Pi_h(m\gamma+(\gamma+\mu_h)\phi_h)}{(\gamma+\mu_h)(\beta_2+\beta_3+f_4+\mu_h)(\omega_1+f_3+\mu_h)(m+\mu_h+\phi_h)},$$

$$I_h^*(t) = \frac{\Pi_h(\beta_3(\omega_1+f_3+\mu_h)+\beta_2\omega_1)(m\gamma+(\gamma+\mu_h)\phi_h)}{(\gamma+\mu_h)(\beta_2+\beta_3+f_4+\mu_h)(\omega_1+f_3+\mu_h)(f_2+\omega_2+\sigma_h+\mu_h)(m+\mu_h+\phi_h)},$$

$$C_h^*(t) = \frac{\Pi_h\omega_2(\beta_3(\omega_1+f_3+\mu_h)+\beta_2\omega_1)(m\gamma+(\gamma+\mu_h)\phi_h)}{(\gamma+\mu_h)(\beta_2+\beta_3+f_4+\mu_h)(\omega_1+f_3+\mu_h)(f_2+\omega_2+\sigma_h+\mu_h)(m+\mu_h+\phi_h)},$$

$$R_h^*(t) = \frac{\begin{array}{c}\Pi_h(m\gamma+(\gamma+\mu_h)\phi_h)(f_1\omega_2(\beta_3(\omega_1+f_3+\mu_h)+\beta_2\omega_1)+f_2(f_1+\sigma_h+\mu_h)\\ (\beta_3(\omega_1+f_3+\mu_h)+\beta_2\omega_1)+f_3\beta_2(f_2+\omega_2+\sigma_h+\mu_h)(f_1+\sigma_h+\mu_h)+\\ f_4(\omega_1+f_3+\mu_h)(f_2+\omega_2+\sigma_h+\mu_h)(f_1+\sigma_h+\mu_h))\end{array}}{\begin{array}{c}\mu_h(\gamma+\mu_h)(\beta_2+\beta_3+f_4+\mu_h)(\omega_1+f_3+\mu_h)\\ (f_2+\omega_2+\sigma_h+\mu_h)(f_1+\sigma_h+\mu_h)(m+\mu_h+\phi_h)\end{array}}.$$

4.4.3. Basic Reproduction Number

The basic reproduction number indicated by $\mathcal{R}_0$, is a quantitative measure of the average reproduction rate of virus during a certain time frame. People/rodents who are infected with the virus and have the ability to effect others, making it a contagious problem. In this section, we construct two new system (4.24) and (4.27) based on the previously proposed system and is written as

$$\begin{cases} \frac{dE_r}{dt} = \alpha_1 I_r S_r - (\beta_1+\mu_r)E_r, \\ \frac{dI_r}{dt} = \beta_1 E_r - (\partial_r+\mu_r)I_r, \end{cases} \tag{4.24}$$

To get the basic reproduction number for the system (4.24), a next-generation matrix approach has been applied here to the system (4.24). This study generates matrix $\boldsymbol{F_r}$

and $\boldsymbol{V_r}$, i.e.

$$\boldsymbol{F_r} = \begin{pmatrix} \alpha_1 I_r S_r \\ \\ -\partial_r I_r \end{pmatrix} \qquad \boldsymbol{V_r} = \begin{pmatrix} (\beta_1 + \mu_r) E_r \\ \\ \mu_r I_r - \beta_1 E_r \end{pmatrix}$$

The Jacobian matrix of $\boldsymbol{F_r}$ and $\boldsymbol{V_r}$ at DFE, denoted by $\mathtt{F_r}$ and $\mathtt{V_r}$ are given as follows:

$$\mathtt{F}_r = \begin{pmatrix} 0 & \frac{\alpha_1 \Pi_r}{\mu_r} \\ \\ 0 & -\partial_r \end{pmatrix} \qquad \mathtt{V}_r = \begin{pmatrix} \beta_1 + \mu_r & 0 \\ \\ -\beta_1 & \mu_r \end{pmatrix}$$

This means that the matrix $\mathtt{F_r V_r}^{-1}$ is the model structure's next-generation matrix (4.24). This means that the next-generation matrix's spectral radius can be calculated as follows: $\mathcal{R}_r = \varrho\left(\mathtt{F}_r \mathtt{V}_r^{-1}\right)$. Thus,

$$\mathtt{F}_r \mathtt{V}_r^{-1} = \begin{pmatrix} \frac{\alpha_1 \beta_1 \Pi_r}{\mu_r^2 (\beta_1 + \mu_r)} & \frac{\alpha_1 \Pi_r}{\mu_r^2} \\ \\ \frac{-\beta_1 \partial_r}{\mu_r (\beta_1 + \mu_r)} & \frac{-\partial_r}{\mu_r} \end{pmatrix} \tag{4.25}$$

So, $\varrho\left(\mathtt{F}_r \mathtt{V}_r^{-1}\right) = \frac{\alpha_1 \beta_1 \Pi_r}{\mu_r^2 (\beta_1 + \mu_r)}$. Therefore

$$\mathcal{R}_r = \frac{\alpha_1 \beta_1 \Pi_r}{\mu_r^2 (\beta_1 + \mu_r)} \tag{4.26}$$

The basic reproduction number is a crucial metric in epidemiology as it helps determine the potential for disease spread within a population. By understanding how many new infections can arise from a single infected individual, public health officials can better assess the risk and implement appropriate control measures. Through comprehensive data collection and mathematical modeling, researchers strive to obtain accurate estimates of N for different diseases, aiding in effective public health strategies to mitigate outbreaks and protect communities. The human population is the second group of populations. For the second group, we obtain the following system to determine the fundamental reproduction number:

$$\begin{cases} \dfrac{dE_h}{dt} = \gamma V_h + (\alpha_2 I_r + \alpha_3 I_h) S_h - (\beta_2 + \beta_3 + f_4 + \mu_h) E_h, \\ \dfrac{dQ_h}{dt} = \beta_2 E_h - (\omega_1 + f_3 + \mu_h) Q_h, \\ \dfrac{dI_h}{dt} = \beta_3 E_h + \omega_1 Q_h - (\omega_2 + f_2 + \partial_h + \mu_h) I_h, \\ \dfrac{dC_h}{dt} = \omega_2 I_h - (f_1 + \partial_h + \mu_h) C_h, \end{cases} \tag{4.27}$$

The Jacobian matrix of $\boldsymbol{F_h}$ and $\boldsymbol{V_h}$ at DFE, denoted by $\mathtt{F_h}$ and $\mathtt{V_h}$ are given as follows:

$$\mathtt{F}_h = \begin{pmatrix} 0 & 0 & \frac{\alpha_3 \Pi_h}{m+\mu_h} & 0 \\ \beta_2 & 0 & 0 & 0 \\ \beta_3 & \omega_1 & 0 & 0 \\ 0 & 0 & \omega_2 & 0 \end{pmatrix},$$

$$\mathtt{V}_h = \begin{pmatrix} \beta_2 + \beta_3 + f_4 + \mu_h & 0 & 0 & 0 \\ 0 & \omega_1 + f_3 + \mu_h & 0 & 0 \\ 0 & 0 & \omega_2 + f_2 + \partial_h + \mu_h & 0 \\ 0 & 0 & 0 & f_1 + \partial_h + \mu_h \end{pmatrix}.$$

This means that the matrix $\mathtt{F_r V_r}^{-1}$ is the model structure's next-generation matrix (4.24). This means that the next-generation matrix's spectral radius can be calculated as follows: $\mathcal{R}_h = \varrho\left(\mathtt{F}_h \mathtt{V}_h^{-1}\right)$.

So, $\varrho\left(\mathtt{F}_h \mathtt{V}_h^{-1}\right) = \frac{\alpha_3 \Pi_h}{(m+\mu_h)(\beta_2+\beta_3+f_4+\mu_h)(\omega_1+f_3+\mu_h)(\omega_2+f_2+\partial_h+\mu_h)^2 (f_1+\partial_h+\mu_h)}$. Therefore

$$\mathcal{R}_h = \frac{\alpha_3 \Pi_h}{(m+\mu_h)(\beta_2+\beta_3+f_4+\mu_h)(\omega_1+f_3+\mu_h)(\omega_2+f_2+\partial_h+\mu_h)^2 (f_1+\partial_h+\mu_h)} \tag{4.28}$$

Hence,

$$\mathcal{R}_0 = \max\{\mathcal{R}_h, \mathcal{R}_r\} \tag{4.29}$$

We consider the following cases:

1. If $\alpha_3\Pi_h > (m+\mu_h)(\beta_2+\beta_3+f_4+\mu_h)(\omega_1+f_3+\mu_h)(\omega_2+f_2+\partial_h+\mu_h)^2 \times (f_1+\partial_h+\mu_h)$ and $\alpha_1\beta_1\Pi_r > \mu_r^2(\beta_1+\mu_r)$, then $\mathcal{R}_0 > 1$.
2. If $\alpha_3\Pi_h > (m+\mu_h)(\beta_2+\beta_3+f_4+\mu_h)(\omega_1+f_3+\mu_h)(\omega_2+f_2+\partial_h+\mu_h)^2 \times (f_1+\partial_h+\mu_h)$ and $\alpha_1\beta_1\Pi_r < \mu_r^2(\beta_1+\mu_r)$, then $\mathcal{R}_0 > 1$.
3. If $\alpha_3\Pi_h < (m+\mu_h)(\beta_2+\beta_3+f_4+\mu_h)(\omega_1+f_3+\mu_h)(\omega_2+f_2+\partial_h+\mu_h)^2 \times (f_1+\partial_h+\mu_h)$ and $\alpha_1\beta_1\Pi_r > \mu_r^2(\beta_1+\mu_r)$, then $\mathcal{R}_0 > 1$.
4. If $\alpha_3\Pi_h < (m+\mu_h)(\beta_2+\beta_3+f_4+\mu_h)(\omega_1+f_3+\mu_h)(\omega_2+f_2+\partial_h+\mu_h)^2 \times (f_1+\partial_h+\mu_h)$ and $\alpha_1\beta_1\Pi_r < \mu_r^2(\beta_1+\mu_r)$, then $\mathcal{R}_0 < 1$.

If $\mathcal{R}_0 < 1$, then the Monkey Pox free equilibrium (MFE) is locally asymptotically stable; otherwise, it is unstable.

4.4.4. Stability of Monkey Pox Free Equilibrium (MFE)

We have used the method described in [10] to determine the requirements for global stability for MFE, which states that if the model system can be expressed as follows:

$$\frac{dX}{dt} = F(X,Z), \tag{4.30}$$

$$\frac{dZ}{dt} = G(X,Z), \quad G(X,0) = 0, \tag{4.31}$$

Here, the uninfected people are $X \in \mathcal{R}^n$ and the infected people are $Z \in \mathcal{R}^m$. The disease-free equilibrium is represented by the notation $Q_0 = (X_0, 0)$. The global stability of the disease-free equilibrium is now assured by the next two conditions.

K_1 : For $\frac{dX}{dt} = F(X,0)$, X_0 is globally asymptotically stable.

K_2 : $G(X,Z) = BZ - \hat{G}(X,Z)$ where $\hat{G}(X,Z) \geq 0$ for $X, Z \in \mathtt{B}$.

In this case, the model's feasibility is denoted by the expression $\mathtt{B}$, where $B = D_Z G(X_0, 0)$ is an M-matrix. After that, we can define Ω_{MFE}'s global stability with the following theorem.

Lemma 4.1: *When $\mathcal{R}_0 < 1$ and assumptions K_1 and K_2 are met, the equilibrium point $Q_0 = (X_0, 0)$ is a globally asymptotically stable.*

Here, we prove that our proposed model system's disease-free equilibrium, Ω_{MFE}, is globally stable with the help of the following theorem.

Theorem 4.3: *The MFE point Ω_{MFE} is globally asymptotically stable provided $\mathcal{R}_0 \leq 1$.*

Proof. To begin, we will demonstrate that K_1 is true by showing that:

$$F(X,0) = \begin{pmatrix} \Pi_r - \mu_r S_r \\ -(\beta_1 + \mu_r) E_r \\ \Pi_h - (m + \mu_h) S_h \\ -\mu_h R_h \end{pmatrix} \tag{4.32}$$

If $F(X,0)$ is a function, then its characteristic polynomial is

$$(\lambda + \mu_r)(\lambda + \beta_1 + \mu_r)(\lambda + m + \mu_h)(\lambda + \mu_h), \tag{4.33}$$

which implies that $\lambda_1 = -\mu_r, \lambda_2 = -\beta_1 - \mu_r, \lambda_3 = -m - \mu_h, \lambda_4 = -\mu_h$. Hence, $X = X_0$ is globally asymptotically stable.

Now, we have:

$$G(X,Z) = BZ - \hat{G}(X,Z), \tag{4.34}$$

$$G(X,Z) = \begin{pmatrix} -a_1 & 0 & \alpha_3 S_h^0 & 0 & \alpha_2 S_h^0 \\ \beta_2 & -a_2 & 0 & 0 & 0 \\ \beta_3 & \omega_1 & -a_3 & 0 & 0 \\ 0 & 0 & \omega_2 & -a_4 & 0 \\ 0 & 0 & 0 & 0 & -a_5 \end{pmatrix} \begin{pmatrix} E_h \\ Q_h \\ I_h \\ C_h \\ I_r \end{pmatrix} - \begin{pmatrix} \beta_1 \left(S_h^0 - S_h\right) + \beta_2 \left(S_h^0 - S_h\right) \\ 0 \\ 0 \\ 0 \\ \beta_1 E_r \end{pmatrix} \tag{4.35}$$

Where $a_1 = \beta_2 + \beta_3 + f_4 + \mu_h, a_2 = \omega_1 + f_3 + \mu_h, a_3 = \omega_2 + f_2 + \sigma_h + \mu_h, a_4 = f_1 + \sigma_h + \mu_h, a_5 = \sigma_r + \mu_r$, and it is easy to see that B satisfies all of the requirements laid out in K_2.

4.5. Preliminaries

Here, for the convenience of our readers, we will quickly go over a few key concepts in fractal-fractional calculus. More information on this innovative use of calculus can be found in [29].

Definition 4.1: The Riemann-Liouville fractal-fractional derivative of $f(t)$ with fractional order ϵ and an exponentially decaying kernel with the assumption that $f(t)$ is fractal differentiable and continuous on some open interval (a,b) with fractal order τ, is defined as:

$$^{FFE}D_{0,t}^{\epsilon,\tau}(f(t)) = \frac{N(\epsilon)}{1-\epsilon}\frac{d}{dt^{\tau}}\int_0^t \exp\left(-\frac{\epsilon}{1-\epsilon}(t-s)\right) f(s)\,ds, \tag{4.36}$$

where $\epsilon > 0, \tau \leq m \in \mathbb{N}$ and $N(0) = N(1) = 1$.

Definition 4.2: The fractal-fractional integral of $f(t)$ with fractional order ϵ and an exponentially decaying type kernel with the assumption that $f(t)$ is fractal integrable and continuous on some open interval (a,b) with fractal order τ and is defined as follows,

$$^{FFE}J_{0,t}^{\epsilon,\tau}(f(t)) = \frac{\epsilon\tau}{M(\epsilon)}\int_0^t s^{\epsilon-1} f(s)\,ds + \frac{\tau(1-\epsilon)\,t^{\tau-1} f(t)}{M(\epsilon)}. \tag{4.37}$$

are simply referred to fractal-fractional integral operators. These operators are mathematical tools used to extend the concept of differentiation and integration to non-integer orders. Fractal-fractional integral operators provide a powerful framework for analyzing complex systems with self-similarity or long-range dependencies. By incorporating fractal geometry into the traditional calculus framework, these operators offer a more comprehensive understanding of phenomena that exhibit intricate patterns at different scales. In summary, these operators bridge the gap between classical calculus and the intricate dynamics observed in many real-world systems, opening up new avenues for analysis and interpretation. Therefore, the following is the proposed nonlinear fractional model, which is based on fractal-fractional operators:

$$^{FFE}D_{0,t}^{\epsilon,\tau}(S_r(t)) = \Pi_r - (\alpha_1 I_r + \mu_r) S_r,$$
$$^{FFE}D_{0,t}^{\epsilon,\tau}(E_r(t)) = \alpha_1 I_r S_r - (\beta_1 + \mu_r) E_r,$$

$$
\begin{aligned}
{}^{FFE}D_{0,t}^{\epsilon,\tau}\left(I_r\left(t\right)\right) &= \beta_1 E_r - \left(\partial_r + \mu_r\right) I_r,\\
{}^{FFE}D_{0,t}^{\epsilon,\tau}\left(S_h\left(t\right)\right) &= \Pi_h - \left(m + \alpha_2 I_r + \alpha_3 I_h + \mu_h\right) S_h,\\
{}^{FFE}D_{0,t}^{\epsilon,\tau}\left(V_h\left(t\right)\right) &= m S_h - \left(\gamma + \mu_h\right) V_h,\\
{}^{FFE}D_{0,t}^{\epsilon,\tau}\left(E_h\left(t\right)\right) &= \gamma V_h + \left(\alpha_2 I_r + \alpha_3 I_h\right) S_h - \left(\beta_2 + \beta_3 + f_4 + \mu_h\right) E_h,\\
{}^{FFE}D_{0,t}^{\epsilon,\tau}\left(Q_h\left(t\right)\right) &= \beta_2 E_h - \left(\omega_1 + f_3 + \mu_h\right) Q_h,\\
{}^{FFE}D_{0,t}^{\epsilon,\tau}\left(I_h\left(t\right)\right) &= \beta_3 E_h + \omega_1 Q_h - \left(\omega_2 + f_2 + \partial_h + \mu_h\right) I_h,\\
{}^{FFE}D_{0,t}^{\epsilon,\tau}\left(C_h\left(t\right)\right) &= \omega_2 I_h - \left(f_1 + \partial_h + \mu_h\right) C_h,\\
{}^{FFE}D_{0,t}^{\epsilon,\tau}\left(R_h\left(t\right)\right) &= f_1 C_h + f_2 I_h + f_3 Q_h + f_4 E_h - \mu_h R_h,
\end{aligned} \tag{4.38}
$$

where ${}^{FFE}D_{0,t}^{\alpha,\tau}\left(\cdot\right)$ is the fractal-fractional derivative of order $0 < \epsilon \leq 1$ and fractal dimension $0 < \tau \leq 1$ in caputo fabrizio sense with exponential law and the variables with the appropriate initial conditions are supposed to be non-negative.

Lemma 4.2: *Let f is continuous on any open interval (a, b), then the following fractal-fractional derivative*

$$
{}^{FFE}D_{0,t}^{\epsilon,\tau}\left(f\left(t\right)\right) = Y\left(t\right) \tag{4.39}
$$

has a unique solution

$$
f\left(t\right) = f\left(0\right) + \frac{\epsilon\tau}{M\left(\epsilon\right)} \int_0^t s^{\epsilon-1} f\left(s\right) ds + \frac{\tau\left(1-\epsilon\right) t^{\tau-1} f\left(t\right)}{M\left(\epsilon\right)}. \tag{4.40}
$$

For $Z = C\left(\mathfrak{T}, \mathbb{R}^{10}\right)$, under the norm for $0 \leq t \leq T < \infty$ the Banach space can be represented by $B = Z \times Z \times Z \times Z \times Z \times Z$ under the norm given by

$$
\|W\| = \sup_{t\in\mathfrak{T}} \left|W\left(t\right)\right|, \qquad \text{for} \quad W \in Z,
$$

where $\left|W\left(t\right)\right| = \left|S\left(t\right) + R\left(t\right) + C\left(t\right) + P\left(t\right) + T\left(t\right) + I\left(t\right)\right\|$, and $S, R, C, P, T, I \in C\left(\mathfrak{T}, \mathbb{R}\right)$.

Considered here is the model (5.7), in which the fractional differential operators have a caputo-fabrizio fractal fractional form. This means that the initial (say) model

(5.7) may be transformed into the following

$$
\begin{aligned}
{}^{CF}D_{0,t}^{\epsilon}\left\{S_r(t)\right\} &= \tau t^{\tau-1}\Phi_1\left(t, S_r(t)\right),\\
{}^{CF}D_{0,t}^{\epsilon}\left\{E_r(t)\right\} &= \tau t^{\tau-1}\Phi_2\left(t, E_r(t)\right),\\
{}^{CF}D_{0,t}^{\epsilon}\left\{I_r(t)\right\} &= \tau t^{\tau-1}\Phi_3\left(t, I_r(t)\right),\\
{}^{CF}D_{0,t}^{\epsilon}\left\{S_h(t)\right\} &= \tau t^{\tau-1}\Phi_4\left(t, S_h(t)\right),\\
{}^{CF}D_{0,t}^{\epsilon}\left\{V_h(t)\right\} &= \tau t^{\tau-1}\Phi_5\left(t, V_h(t)\right),\\
{}^{CF}D_{0,t}^{\epsilon}\left\{E_h(t)\right\} &= \tau t^{\tau-1}\Phi_6\left(t, E_h(t)\right),\\
{}^{CF}D_{0,t}^{\epsilon}\left\{Q_h(t)\right\} &= \tau t^{\tau-1}\Phi_7\left(t, Q_h(t)\right),\\
{}^{CF}D_{0,t}^{\epsilon}\left\{I_h(t)\right\} &= \tau t^{\tau-1}\Phi_8\left(t, I_h(t)\right),\\
{}^{CF}D_{0,t}^{\epsilon}\left\{C_h(t)\right\} &= \tau t^{\tau-1}\Phi_9\left(t, C_h(t)\right),\\
{}^{CF}D_{0,t}^{\epsilon}\left\{R_h(t)\right\} &= \tau t^{\tau-1}\Phi_{10}\left(t, R_h(t)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_1\left(t, S_r(t)\right) &= \Pi_r - \left(\alpha_1 I_r + \mu_r\right) S_r,\\
\Phi_2\left(t, E_r(t)\right) &= \alpha_1 I_r S_r - \left(\beta_1 + \mu_r\right) E_r,\\
\Phi_3\left(t, I_r(t)\right) &= \beta_1 E_r - \left(\partial_r + \mu_r\right) I_r,\\
\Phi_4\left(t, S_h(t)\right) &= \Pi_h - \left(m + \alpha_2 I_r + \alpha_3 I_h + \mu_h\right) S_h,\\
\Phi_5\left(t, V_h(t)\right) &= m S_h - \left(\gamma + \mu_h\right) V_h,\\
\Phi_6\left(t, E_h(t)\right) &= \gamma V_h + \left(\alpha_2 I_r + \alpha_3 I_h\right) S_h - \left(\beta_2 + \beta_3 + f_4 + \mu_h\right) E_h,\\
\Phi_7\left(t, Q_h(t)\right) &= \beta_2 E_h - \left(\omega_1 + f_3 + \mu_h\right) Q_h,\\
\Phi_8\left(t, I_h(t)\right) &= \beta_3 E_h + \omega_1 Q_h - \left(\omega_2 + f_2 + \partial_h + \mu_h\right) I_h,\\
\Phi_9\left(t, C_h(t)\right) &= \omega_2 I_h - \left(f_1 + \partial_h + \mu_h\right) C_h,\\
\Phi_{10}\left(t, R_h(t)\right) &= f_1 C_h + f_2 I_h + f_3 Q_h + f_4 E_h - \mu_h R_h,
\end{aligned}
$$

Caputo-Fabrizio integral is used, and we get

$$
S_r(t) = S_r(0) + \frac{\tau t^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_1\left(t, S_r(t)\right) + \frac{\epsilon\tau}{M(\epsilon)}\int_0^t \xi^{\tau-1}\Phi_1\left(\xi, S_r(\xi)\right)d\xi,
$$

$$E_r(t) = E_r(0) + \frac{\tau t^{\tau-1}(1-\epsilon)}{M(\epsilon)} \Phi_2(t, E_r(t)) + \frac{\epsilon\tau}{M(\epsilon)} \int_0^t \xi^{\tau-1} \Phi_2(\xi, E_r(\xi))\, d\xi,$$
$$I_r(t) = I_r(0) + \frac{\tau t^{\tau-1}(1-\epsilon)}{M(\epsilon)} \Phi_3(t, I_r(t)) + \frac{\epsilon\tau}{M(\epsilon)} \int_0^t \xi^{\tau-1} \Phi_3(\xi, I_r(\xi))\, d\xi,$$
$$S_h(t) = S_h(0) + \frac{\tau t^{\tau-1}(1-\epsilon)}{M(\epsilon)} \Phi_4(t, S_h(t)) + \frac{\epsilon\tau}{M(\epsilon)} \int_0^t \xi^{\tau-1} \Phi_4(\xi, S_h(\xi))\, d\xi,$$
$$V_h(t) = V_h(0) + \frac{\tau t^{\tau-1}(1-\epsilon)}{M(\epsilon)} \Phi_5(t, V_h(t)) + \frac{\epsilon\tau}{M(\epsilon)} \int_0^t \xi^{\tau-1} \Phi_5(\xi, V_h(\xi))\, d\xi,$$
$$E_h(t) = E_h(0) + \frac{\tau t^{\tau-1}(1-\epsilon)}{M(\epsilon)} \Phi_6(t, E_h(t)) + \frac{\epsilon\tau}{M(\epsilon)} \int_0^t \xi^{\tau-1} \Phi_6(\xi, E_h(\xi))\, d\xi,$$
$$Q_h(t) = Q_h(0) + \frac{\tau t^{\tau-1}(1-\epsilon)}{M(\epsilon)} \Phi_7(t, Q_h(t)) + \frac{\epsilon\tau}{M(\epsilon)} \int_0^t \xi^{\tau-1} \Phi_7(\xi, Q_h(\xi))\, d\xi,$$
$$I_h(t) = I_h(0) + \frac{\tau t^{\tau-1}(1-\epsilon)}{M(\epsilon)} \Phi_8(t, I_h(t)) + \frac{\epsilon\tau}{M(\epsilon)} \int_0^t \xi^{\tau-1} \Phi_8(\xi, I_h(\xi))\, d\xi,$$
$$C_h(t) = C_h(0) + \frac{\tau t^{\tau-1}(1-\epsilon)}{M(\epsilon)} \Phi_9(t, C_h(t)) + \frac{\epsilon\tau}{M(\epsilon)} \int_0^t \xi^{\tau-1} \Phi_9(\xi, C_h(\xi))\, d\xi,$$
$$R_h(t) = R_h(0) + \frac{\tau t^{\tau-1}(1-\epsilon)}{M(\epsilon)} \Phi_{10}(t, R_h(t)) + \frac{\epsilon\tau}{M(\epsilon)} \int_0^t \xi^{\tau-1} \Phi_{10}(\xi, R_h(\xi))\, d\xi,$$

4.5.1. Existence and Uniqueness

To illustrate the qualitative characteristics of the solution for model (5.22), we used the Picard-Lindel'f method and the fixed point theory. We'll start by trying to rewrite the model (5.22), which now looks like this:

$$\begin{cases} {}^{FFE}D_{0,t}^{\epsilon,\tau} W(t) = \Phi(t, W(t)), & 0 < \epsilon, \quad \tau \leq 1, \\ W(0) = W_0 \geq 0, & t \in [0, T] \quad \text{and} \quad (T < \infty) \end{cases} \tag{4.41}$$

where

$$W(t)=\begin{pmatrix} S_r(t)\\ E_r(t)\\ I_r(t)\\ S_h(t)\\ V_h(t)\\ E_h(t)\\ Q_h(t)\\ I_h(t)\\ C_h(t)\\ R_h(t)\end{pmatrix},\qquad W(0)=\begin{pmatrix} S_r(0)=S_{r_0}\\ E_r(0)=E_{r_0}\\ I_r(0)=I_{r_0}\\ S_h(0)=S_{h_0}\\ V_h(0)=V_{h_0}\\ E_h(0)=E_{h_0}\\ Q_h(0)=Q_{h_0}\\ I_h(0)=I_{h_0}\\ C_h(0)=C_{h_0}\\ R_h(0)=R_{h_0}\end{pmatrix}=W_0,\qquad \Phi(t,W(t))=\begin{pmatrix} \Phi_1(t,S_r(t))\\ \Phi_2(t,E_r(t))\\ \Phi_3(t,I_r(t))\\ \Phi_4(t,S_h(t))\\ \Phi_5(t,V_h(t))\\ \Phi_6(t,E_h(t))\\ \Phi_7(t,Q_h(t))\\ \Phi_8(t,I_h(t))\\ \Phi_9(t,C_h(t))\\ \Phi_{10}(t,R_h(t))\end{pmatrix},$$

In the view of Lemma 5.2, the system (5.23) yields

$$W(t)=W(0)+\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\Phi(t,W(t))+\frac{\epsilon\tau}{N(\epsilon)}\int_0^t\xi^{\epsilon-1}\Phi(\xi,W(\xi))\,d\xi.$$

Furthermore, Φ satisfies

$$\|\Phi(\xi,W_1(\xi))-\Phi(\xi,W_2(\xi))\|\le L_\Phi\,|W_1(\xi)-W_2(\xi)|\,,\qquad L_\Phi>0. \tag{4.42}$$

Theorem 4.4:

Assume that the system (5.23) *has a unique solution if the assumption* (5.24) *holds with*

$$\mathcal{P}:=\left(\frac{\tau(1-\epsilon)T^{\tau-1}}{N(\epsilon)}+\frac{\epsilon\tau T^{\epsilon}}{N(\epsilon)}\right)L_\Phi\ \le 1.$$

Proof. Consider the Picard operator, which is represented by the symbol $\Lambda: Z\to Z$ and has the definition

$$\Lambda W(t)=W(0)+\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\Phi(t,W(t))+\frac{\epsilon\tau}{N(\epsilon)}\int_0^t\xi^{\epsilon-1}\Phi(\xi,W(\xi))\,d\xi. \tag{4.43}$$

and set $\sup_{t\in\mathfrak{T}}\Phi(\xi,0)=\Phi_0$. It is essential to note that the solution to the system

(5.23) is constrained, i.e.,

$$\begin{aligned}\|\Lambda W - W_0\| &= \sup_{t\in\mathfrak{T}} |\Lambda W(t) - W(0)|, \\ &= \sup_{t\in\mathfrak{T}} \left| \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)} \Phi(t, W(t)) + \frac{\epsilon\tau}{N(\epsilon)} \int_0^t \xi^{\epsilon-1}\Phi(\xi, W(\xi))\, d\xi \right|, \\ &\le \sup_{t\in\mathfrak{T}} \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)} |\Phi(t, W(t))| + \frac{\epsilon\tau}{N(\epsilon)} \int_0^t \xi^{\epsilon-1} |\Phi(\xi, W(\xi))|\, d\xi, \\ &\le \sup_{t\in\mathfrak{T}} \left(\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)} + \frac{\epsilon\tau t^{\epsilon}}{N(\epsilon)} \right) L_\Phi, \\ &\le \left(\frac{\tau(1-\epsilon)T^{\tau-1}}{N(\epsilon)} + \frac{\epsilon\tau T^{\epsilon}}{N(\epsilon)} \right) L_\Phi \le \beta L_\Phi,\end{aligned}$$

where $\beta L_\Phi < 1$. Now, according to the definition of the Picard operator (4.43) for any $W_1, W_2 \in Z$, we get

$$\begin{aligned}\|\Lambda W_1 - \Lambda W_2\| &= \sup_{t\in\mathfrak{T}} \|\Lambda W_1(t) - \Lambda W_2(t)\|, \\ &\le \sup_{t\in\mathfrak{T}} \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)} \|\Phi(t, W_1(t)) - \Phi(t, W_2(t))\| + \\ &\qquad \frac{\epsilon\tau}{N(\epsilon)} \int_0^t \xi^{\epsilon-1} \|\Phi(\xi, W_1(\xi)) - \Phi(\xi, W_2(\xi))\|\, d\xi, \\ &\le \sup_{t\in\mathfrak{T}} \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)} L_\Phi |W_1(t) - W_2(t)| + \\ &\qquad \frac{\epsilon\tau}{N(\epsilon)} \int_0^t \xi^{\epsilon-1} L_\Phi |W_1(\xi) - W_2(\xi)|\, d\xi, \\ &\le \sup_{t\in\mathfrak{T}} \left(\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)} + \frac{\epsilon\tau t^{\epsilon}}{N(\epsilon)} \right) L_\Phi, \\ &\le \left(\frac{\tau(1-\epsilon)T^{\tau-1}}{N(\epsilon)} + \frac{\epsilon\tau T^{\epsilon}}{N(\epsilon)} \right) L_\Phi \le \beta L_\Phi,\end{aligned}$$

which implies that $\|\Lambda W_1 - \Lambda W_2\| \le \mathcal{P}\|W_1 - W_2\|$. As $\mathcal{P}$ is a contraction, model (5.23) has a unique solution in accordance with the Banach contraction principle.

Definition 4.3: Suppose that $\Phi \in C\left(\mathfrak{T} \times \mathbb{R}^{10},\ \mathbb{R}\right)$, then for $0 < \epsilon, \tau \le 1$ the system (5.20) is said to be Hyers–Ulam stable if there exists $\delta, C_\Phi > 0$ such that, for each solution $\bar{Y} \in Z$ satisfies

$$\left\| {}^{FFE}D_{0,t}^{\epsilon,\tau} \bar{Y}(t) - \Phi\left(t, \bar{Y}(t)\right) \right\| \le \delta, \qquad \forall \quad t \in \mathfrak{T}, \tag{4.44}$$

there exist a solution $Y \in Z$ of (5.20) with

$$\left|\bar{Y}(t) - Y(t)\right| \leq \delta C_{\Phi}, \qquad \forall \quad t \in \mathfrak{T}, \tag{4.45}$$

where $\delta = \max(\delta_j)^T$ and $C_{\Phi} = \max\left(C_{\Phi_j}\right), \quad j = 1,2,3,4,5,6.$

Definition 4.4: Suppose $F \in C(\mathfrak{T}, \mathbb{R}^+)$ and $\Phi \in C\left(\mathfrak{T} \times \mathbb{R}^{10}, \mathbb{R}\right)$, then for $0 < \epsilon, \tau \leq 1$ the system (5.20) is said to be Hyers–Ulam–Rassias stable if there exist $C_{\Phi,F} > 0$ such that, for each solution $\bar{Y} \in Z$ satisfies

$$\left\|{}^{FFE}D_{0,t}^{\epsilon,\tau}\bar{Y}(t) - \Phi\left(t, \bar{Y}(t)\right)\right\| \leq \delta F(t)^T, \qquad \forall \quad t \in \mathfrak{T}, \tag{4.46}$$

there exist a solution $Y \in Z$ of (5.20) with

$$\left|\bar{Y}(t) - Y(t)\right| \leq \delta C_{\Phi,F} F(t), \qquad \forall \quad t \in \mathfrak{T}, \tag{4.47}$$

where $C_{\Phi,F} = \max\left(C_{\Phi_j,F_j}\right)^T$ and $F = \max(F_j)^T, \quad j = 1,2,3,4,5,6.$

Lemma 4.3: *Let $\delta > 0$, and if there exist a function $g(t) \in Z$, which satisfies*

(i) $g(t) \leq \delta, \qquad \forall \quad t \in \mathfrak{T}$

(ii) ${}^{FFE}D_{0,t}^{\epsilon,\tau}\bar{Y}(t) = \Phi\left(t, \bar{Y}(t)\right) + g(t), \quad \forall t \in \mathfrak{T} \quad g = \max(g_j)^T, \quad j = 1,2,3,4,5,6.$

then the function $\bar{Y} \in Z$ satisfies (5.28).

Lemma 4.4: *Let $F \in C(\mathfrak{T}, \mathbb{R})$, and if there exist a function $g^*(t) \in Z$, which satisfies*

(i) $g^*(t) \leq \delta, \qquad \forall \quad t \in \mathfrak{T}$

(ii) ${}^{FFE}D_{0,t}^{\epsilon,\tau}\bar{Y}(t) = \Phi\left(t, \bar{Y}(t)\right) + g^*(t), \qquad \forall \quad t \in \mathfrak{T} \qquad g^* = \max\left(g_j^*\right)^T, \quad j = 1,2,3,4,5,6.$

then the function $\bar{Y} \in Z$ satisfies (5.30).

4.5.2. Hyers-Ulam Stability Analysis

Here we examine the Hyers-Ulam and Hyers-Ulam-Rassias stability of the suggested model (5.23). The importance of stability in approximating a solution motivates us to perform a nonlinear functional analysis of the many forms of stability in the present model.

Lemma 4.5: *If W satisfies the integral inequality given by* (5.32) *then $W \in Z$ satisfies by the system* (5.23)

$$\left| W(t) - W_0 - \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\Phi(\xi, W(\xi)) - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\Phi(\xi, W(\xi))\,d\xi \right| \leq \Upsilon\delta, \quad (4.48)$$

where $\Upsilon := \left(\frac{\tau(1-\epsilon)T^{\tau-1}}{N(\epsilon)} + \frac{\epsilon\tau T^{\epsilon}}{N(\epsilon)}\right)$.

Proof. According to Theorem 5.27 with (ii) of Lemma 5.3, the system exist

$$\begin{aligned} {}^{FFE}D_{0,t}^{\epsilon,\tau} W(t) &= \Phi(t, W(t)) + f(t), \qquad t \in \mathfrak{T} \\ W(0) &= W_0 \geq 0, \end{aligned}$$

and has a unique solution

$$W(t) = W_0 + \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\left(\Phi(\xi, W(\xi)) + f(\xi)\right) + \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\left(\Phi(\xi, W(\xi)) + f(\xi)\right)d\xi.$$

It continues to follow from (i) of Lemma 5.3 that

$$\begin{aligned} &\left| W(t) - W_0 - \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\Phi(\xi, W(\xi)) - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\Phi(\xi, W(\xi))\,d\xi \right|, \\ &= \sup_{t\in\mathfrak{T}} \left| \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)} f(\xi) + \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1} f(\xi)\,d\xi \right|, \\ &\leq \sup_{t\in\mathfrak{T}} \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}|f(\xi)| + \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}|f(\xi)|\,d\xi, \\ &\leq \sup_{t\in\mathfrak{T}} \left(\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)} + \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}d\xi\right)\delta, \\ &\leq \left(\frac{\tau(1-\epsilon)T^{\tau-1}}{N(\epsilon)} + \frac{\epsilon\tau T^{\epsilon}}{N(\epsilon)}\right)\delta, \\ &\leq \Upsilon\delta. \end{aligned}$$

Theorem 4.5:

Suppose that $\Phi \in C\left(\mathfrak{T} \times \mathbb{R}^{10},\ \mathbb{R}\right)$ and the system (5.24) *satisfies the condition $1 - \Upsilon L_{\Phi} > 0$, then the system* (5.23) *is said to be Hyers-Ulam stable.*

Proof. Let $W \in Z$ be a unique solution for the system (5.23), and $\bar{W} \in Z$ satisfies by (5.30). Then, by taking the Lemma 5.3 into consideration, for any $\delta > 0$, $t \in \mathfrak{T}$ we have

$$
\begin{aligned}
\left\|\bar{W} - W\right\| &= \sup_{t\in\mathfrak{T}} \left|\bar{W} - W\right|, \\
&= \sup_{t\in\mathfrak{T}} \left|\bar{W} - W_0 - \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\Phi(\xi, W(\xi)) - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\Phi(\xi, W(\xi))\, d\xi\right|, \\
&\leq \sup_{t\in\mathfrak{T}} \left|\bar{W} - W_0 - \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\Phi\left(\xi, \bar{W}(\xi)\right) - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\Phi\left(\xi, \bar{W}(\xi)\right) d\xi\right| \\
&\qquad + \sup_{t\in\mathfrak{T}} \left|\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\left(\Phi\left(\xi, \bar{W}(\xi)\right) - \Phi(\xi, W(\xi))\right)\right. \\
&\qquad\qquad \left. - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\left(\bar{W}(\xi) - \Phi(\xi, W(\xi))\right) d\xi\right|, \\
&\leq \Upsilon\delta + \sup_{t\in\mathfrak{T}} \left(\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\left|\Phi\left(\xi, \bar{W}(\xi)\right) - \Phi(\xi, W(\xi))\right|\right. \\
&\qquad\qquad \left. - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\left|\bar{W}(\xi) - \Phi(\xi, W(\xi))\right| d\xi\right), \\
&\leq \Upsilon\delta + \sup_{t\in\mathfrak{T}} \left(\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}L_\Phi\left|\bar{W}(\xi) - W(\xi)\right|\right. \\
&\qquad\qquad \left. - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}L_\Phi\left|\bar{W}(\xi) - W(\xi)\right| d\xi\right), \\
&\leq \Upsilon\delta + \sup_{t\in\mathfrak{T}} \left(\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)} - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1} d\xi\right) L_\Phi\left|\bar{W}(\xi) - W(\xi)\right|, \\
&\leq \Upsilon\delta + \left(\frac{\tau(1-\epsilon)T^{\tau-1}}{N(\epsilon)} - \frac{\epsilon\tau T^{\epsilon}}{N(\epsilon)}\right) L_\Phi\left|\bar{W}(\xi) - W(\xi)\right|, \\
&\leq \Upsilon\delta + \Upsilon L_\Phi\left\|\bar{W} - W\right\|,
\end{aligned}
$$

which implies that

$$\left\|\bar{W} - W\right\| \leq C_\Phi\delta$$

where $C_\Phi := \dfrac{\Upsilon}{1 - \Upsilon L_\Phi}$.

Theorem 4.6:

Suppose that $\Phi \in C\left(\mathfrak{T} \times \mathbb{R}^{10},\ \mathbb{R}\right)$ satisfies (5.24) and $F \in C\left(\mathfrak{T}, \mathbb{R}^{+}\right)$ be an increasing function implies that the system (5.23) is Hyers-Ulam–Rassias stable with respect to F

on $\mathfrak{T}$, such that

$$^{FFE}J_{0,t}^{\epsilon,\tau}F(t) \leq C_F F(t), \qquad C_F > 0. \tag{4.49}$$

provided that $1 - \Upsilon L_\Phi > 0$.

Proof. Let us assume that there exists exactly one solution $W \in Z$ to the system (5.23). So, considering Theorem 5.27, we arrive at

$$W(t) = W(0) + \tfrac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\Phi(t, W(t)) + \tfrac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\Phi(\xi, W(\xi))\,d\xi.$$

On consideration of (5.28) and (5.30), we have

$$\left|W(t) - W_0 - \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\Phi(t, W(t)) - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\Phi(\xi, W(\xi))\,d\xi\right| \leq \delta C_F F(t)$$

Hence

$$\begin{aligned}
\left|\bar{W} - W\right| &= \sup_{t\in\mathfrak{T}}\left|\bar{W} - W_0 - \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\Phi(t, W(t)) - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\Phi(\xi, W(\xi))\,d\xi\right|, \\
&\leq \sup_{t\in\mathfrak{T}}\left|\bar{W} - W_0 - \frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\Phi\left(t, \bar{W}(t)\right) - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\Phi\left(\xi, \bar{W}(\xi)\right)d\xi\right| \\
&\quad + \sup_{t\in\mathfrak{T}}\left|\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\left(\Phi\left(t, \bar{W}(t)\right) - \Phi(t, W(t))\right)\right. \\
&\quad \left. - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\left(\Phi\left(\xi, \bar{W}(\xi)\right) - \Phi(t, W(t))\right)d\xi\right|, \\
&\leq \delta C_F F(t) + \sup_{t\in\mathfrak{T}}\left(\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}\left|\Phi\left(\xi, \bar{W}(\xi)\right) - \Phi(\xi, W(\xi))\right|\right. \\
&\quad \left. - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}\left|\bar{W}(\xi) - \Phi(\xi, W(\xi))\right|d\xi\right), \\
&\leq \delta C_F F(t) + \sup_{t\in\mathfrak{T}}\left(\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)}L_\Phi\left|\bar{W}(\xi) - W(\xi)\right|\right. \\
&\quad \left. - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}L_\Phi\left|\bar{W}(\xi) - W(\xi)\right|d\xi\right), \\
&\leq \delta C_F F(t) + \sup_{t\in\mathfrak{T}}\left(\frac{\tau(1-\epsilon)t^{\tau-1}}{N(\epsilon)} - \frac{\epsilon\tau}{N(\epsilon)}\int_0^t \xi^{\epsilon-1}d\xi\right)L_\Phi\left|\bar{W}(\xi) - W(\xi)\right|, \\
&\leq \delta C_F F(t) + \left(\frac{\tau(1-\epsilon)T^{\tau-1}}{N(\epsilon)} - \frac{\epsilon\tau T^\epsilon}{N(\epsilon)}\right)L_\Phi\left|\bar{W}(\xi) - W(\xi)\right|, \\
&\leq \delta C_F F(t) + \Upsilon L_\Phi\left|\bar{W} - W\right|,
\end{aligned}$$

which implies that

$$\left|\bar{W}-W\right| \leq \delta C_{\Phi,F}F\left(t\right) \tag{4.50}$$

where $C_{\Phi,F} := \dfrac{C_F}{1-\Upsilon L_\Phi}$.

4.6. Numerical scheme

In this section, we give a thorough explanation of how the numerical approach was developed. Thus, at t_{n+1} we have

$$\begin{aligned}
S_r\left(t_{n+1}\right) &= S_r\left(t_0\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_1\left(t_n,S_r\left(t_n\right)\right)+\frac{\epsilon\tau}{M\left(\epsilon\right)}\int_{t_0}^{t_{n+1}}\xi^{\tau-1}\Phi_1\left(\xi,S_r\left(\xi\right)\right)d\xi,\\
E_r\left(t_{n+1}\right) &= E_r\left(t_0\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_2\left(t_n,E_r\left(t_n\right)\right)+\frac{\epsilon\tau}{M\left(\epsilon\right)}\int_{t_0}^{t_{n+1}}\xi^{\tau-1}\Phi_2\left(\xi,E_r\left(\xi\right)\right)d\xi,\\
I_r\left(t_{n+1}\right) &= I_r\left(t_0\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_3\left(t_n,I_r\left(t_n\right)\right)+\frac{\epsilon\tau}{M\left(\epsilon\right)}\int_{t_0}^{t_{n+1}}\xi^{\tau-1}\Phi_3\left(\xi,I_r\left(\xi\right)\right)d\xi,\\
S_h\left(t_{n+1}\right) &= S_h\left(t_0\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_4\left(t_n,S_h\left(t_n\right)\right)+\frac{\epsilon\tau}{M\left(\epsilon\right)}\int_{t_0}^{t_{n+1}}\xi^{\tau-1}\Phi_4\left(\xi,S_h\left(\xi\right)\right)d\xi,\\
V_h\left(t_{n+1}\right) &= V_h\left(t_0\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_5\left(t_n,V_h\left(t_n\right)\right)+\frac{\epsilon\tau}{M\left(\epsilon\right)}\int_{t_0}^{t_{n+1}}\xi^{\tau-1}\Phi_5\left(\xi,V_h\left(\xi\right)\right)d\xi,\\
E_h\left(t_{n+1}\right) &= E_h\left(t_0\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_6\left(t_n,E_h\left(t_n\right)\right)+\frac{\epsilon\tau}{M\left(\epsilon\right)}\int_{t_0}^{t_{n+1}}\xi^{\tau-1}\Phi_6\left(\xi,E_h\left(\xi\right)\right)d\xi,\\
Q_h\left(t_{n+1}\right) &= Q_h\left(t_0\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_7\left(t_n,Q_h\left(t_n\right)\right)+\frac{\epsilon\tau}{M\left(\epsilon\right)}\int_{t_0}^{t_{n+1}}\xi^{\tau-1}\Phi_7\left(\xi,Q_h\left(\xi\right)\right)d\xi,\\
I_h\left(t_{n+1}\right) &= I_h\left(t_0\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_8\left(t_n,I_h\left(t_n\right)\right)+\frac{\epsilon\tau}{M\left(\epsilon\right)}\int_{t_0}^{t_{n+1}}\xi^{\tau-1}\Phi_8\left(\xi,I_h\left(\xi\right)\right)d\xi,\\
C_h\left(t_{n+1}\right) &= C_h\left(t_0\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_9\left(t_n,C_h\left(t_n\right)\right)+\frac{\epsilon\tau}{M\left(\epsilon\right)}\int_{t_0}^{t_{n+1}}\xi^{\tau-1}\Phi_9\left(\xi,C_h\left(\xi\right)\right)d\xi,\\
R_h\left(t_{n+1}\right) &= R_h\left(t_0\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_{10}\left(t_n,R_h\left(t_n\right)\right)+\frac{\epsilon\tau}{M\left(\epsilon\right)}\int_{t_0}^{t_{n+1}}\xi^{\tau-1}\Phi_{10}\left(\xi,R_h\left(\xi\right)\right)d\xi,
\end{aligned}$$

By taking the gap between the two successive terms as our starting point, we have

$$\begin{aligned}
S_r\left(t_{n+1}\right) &= S_r\left(t_n\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_1\left(t_n,S_r\left(t_n\right)\right)-\frac{\tau t_{n-1}^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_1\left(t_{n-1},S_r\left(t_{n-1}\right)\right)\\
&\quad+\frac{\epsilon\tau}{M\left(\epsilon\right)}\int_{t_n}^{t_{n+1}}\xi^{\tau-1}\Phi_1\left(\xi,S_r\left(\xi\right)\right)d\xi,\\
E_r\left(t_{n+1}\right) &= E_r\left(t_n\right)+\frac{\tau t_n^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_2\left(t_n,E_r\left(t_n\right)\right)-\frac{\tau t_{n-1}^{\tau-1}\left(1-\epsilon\right)}{M\left(\epsilon\right)}\Phi_2\left(t_{n-1},E_r\left(t_{n-1}\right)\right)
\end{aligned}$$

$$+\frac{\epsilon\tau}{M(\epsilon)}\int_{t_n}^{t_{n+1}}\xi^{\tau-1}\Phi_2(\xi,E_r(\xi))\,d\xi,$$

$$I_r(t_{n+1})=I_r(t_n)+\frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_3(t_n,I_r(t_n))-\frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_3(t_{n-1},I_r(t_{n-1}))$$
$$+\frac{\epsilon\tau}{M(\epsilon)}\int_{t_n}^{t_{n+1}}\xi^{\tau-1}\Phi_3(\xi,I_r(\xi))\,d\xi,$$

$$S_h(t_{n+1})=S_h(t_n)+\frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_4(t_n,S_h(t_n))-\frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_4(t_{n-1},S_h(t_{n-1}))$$
$$+\frac{\epsilon\tau}{M(\epsilon)}\int_{t_n}^{t_{n+1}}\xi^{\tau-1}\Phi_4(\xi,S_h(\xi))\,d\xi,$$

$$V_h(t_{n+1})=V_h(t_n)+\frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_5(t_n,V_h(t_n))-\frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_5(t_{n-1},V_h(t_{n-1}))$$
$$+\frac{\epsilon\tau}{M(\epsilon)}\int_{t_n}^{t_{n+1}}\xi^{\tau-1}\Phi_5(\xi,V_h(\xi))\,d\xi,$$

$$E_h(t_{n+1})=E_h(t_n)+\frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_6(t_n,E_h(t_n))-\frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_6(t_{n-1},E_h(t_{n-1}))$$
$$+\frac{\epsilon\tau}{M(\epsilon)}\int_{t_n}^{t_{n+1}}\xi^{\tau-1}\Phi_6(\xi,E_h(\xi))\,d\xi,$$

$$Q_h(t_{n+1})=Q_h(t_n)+\frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_7(t_n,Q_h(t_n))-\frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_7(t_{n-1},Q_h(t_{n-1}))$$
$$+\frac{\epsilon\tau}{M(\epsilon)}\int_{t_n}^{t_{n+1}}\xi^{\tau-1}\Phi_7(\xi,Q_h(\xi))\,d\xi,$$

$$I_h(t_{n+1})=I_h(t_n)+\frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_8(t_n,I_h(t_n))-\frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_8(t_{n-1},I_h(t_{n-1}))$$
$$+\frac{\epsilon\tau}{M(\epsilon)}\int_{t_n}^{t_{n+1}}\xi^{\tau-1}\Phi_8(\xi,I_h(\xi))\,d\xi,$$

$$C_h(t_{n+1})=C_h(t_n)+\frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_9(t_n,C_h(t_n))-\frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_9(t_{n-1},C_h(t_{n-1}))$$
$$+\frac{\epsilon\tau}{M(\epsilon)}\int_{t_n}^{t_{n+1}}\xi^{\tau-1}\Phi_9(\xi,C_h(\xi))\,d\xi,$$

$$R_h(t_{n+1})=R_h(t_n)+\frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_{10}(t_n,R_h(t_n))-\frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_{10}(t_{n-1},R_h(t_{n-1}))$$
$$+\frac{\epsilon\tau}{M(\epsilon)}\int_{t_n}^{t_{n+1}}\xi^{\tau-1}\Phi_{10}(\xi,R_h(\xi))\,d\xi,$$

If the function $\xi^{\tau-1}\Phi(\xi,W(\xi))$ is approximated over the finite interval $[t_j,t_{j+1}]$ for $j=1,2,3,\ldots,10$ and $W=[S_r,S_r,E_r,I_h,S_h,V_h,E_h,Q_h,I_h,C_h,R_h]^T$ using the Lagrangian piece-wise interpolation such that

$$S_r(t_{n+1})=S_r(t_n)+\frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_1(t_n,S_r(t_n))-\frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_1(t_{n-1},S_r(t_{n-1}))+$$
$$\frac{\epsilon\tau}{M(\epsilon)}\times\left[\frac{3}{2}(\Delta t)\,t_n^{\tau-1}\Phi_1(t_n,S_r(t_n))-\frac{\Delta t}{2}t_{n-1}^{\tau-1}\Phi_1(t_{n-1},S_r(t_{n-1}))\right],$$

$$E_r(t_{n+1}) = E_r(t_n) + \frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_2(t_n, E_r(t_n)) - \frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_2(t_{n-1}, E_r(t_{n-1})) + \frac{\epsilon\tau}{M(\epsilon)} \times \left[\frac{3}{2}(\Delta t)\, t_n^{\tau-1}\Phi_2(t_n, E_r(t_n)) - \frac{\Delta t}{2} t_{n-1}^{\tau-1}\Phi_2(t_{n-1}, E_r(t_{n-1}))\right],$$

$$I_r(t_{n+1}) = I_r(t_n) + \frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_3(t_n, I_r(t_n)) - \frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_3(t_{n-1}, I_r(t_{n-1})) + \frac{\epsilon\tau}{M(\epsilon)} \times \left[\frac{3}{2}(\Delta t)\, t_n^{\tau-1}\Phi_3(t_n, I_r(t_n)) - \frac{\Delta t}{2} t_{n-1}^{\tau-1}\Phi_3(t_{n-1}, I_r(t_{n-1}))\right],$$

$$S_h(t_{n+1}) = S_h(t_n) + \frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_4(t_n, S_h(t_n)) - \frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_4(t_{n-1}, S_h(t_{n-1})) + \frac{\epsilon\tau}{M(\epsilon)} \times \left[\frac{3}{2}(\Delta t)\, t_n^{\tau-1}\Phi_4(t_n, S_h(t_n)) - \frac{\Delta t}{2} t_{n-1}^{\tau-1}\Phi_4(t_{n-1}, S_h(t_{n-1}))\right],$$

$$V_h(t_{n+1}) = V_h(t_n) + \frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_5(t_n, V_h(t_n)) - \frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_5(t_{n-1}, V_h(t_{n-1})) + \frac{\epsilon\tau}{M(\epsilon)} \times \left[\frac{3}{2}(\Delta t)\, t_n^{\tau-1}\Phi_5(t_n, V_h(t_n)) - \frac{\Delta t}{2} t_{n-1}^{\tau-1}\Phi_5(t_{n-1}, V_h(t_{n-1}))\right],$$

$$E_h(t_{n+1}) = E_h(t_n) + \frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_6(t_n, E_h(t_n)) - \frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_6(t_{n-1}, E_h(t_{n-1})) + \frac{\epsilon\tau}{M(\epsilon)} \times \left[\frac{3}{2}(\Delta t)\, t_n^{\tau-1}\Phi_6(t_n, E_h(t_n)) - \frac{\Delta t}{2} t_{n-1}^{\tau-1}\Phi_6(t_{n-1}, E_h(t_{n-1}))\right],$$

$$Q_h(t_{n+1}) = Q_h(t_n) + \frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_7(t_n, Q_h(t_n)) - \frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_7(t_{n-1}, Q_h(t_{n-1})) + \frac{\epsilon\tau}{M(\epsilon)} \times \left[\frac{3}{2}(\Delta t)\, t_n^{\tau-1}\Phi_7(t_n, Q_h(t_n)) - \frac{\Delta t}{2} t_{n-1}^{\tau-1}\Phi_7(t_{n-1}, Q_h(t_{n-1}))\right],$$

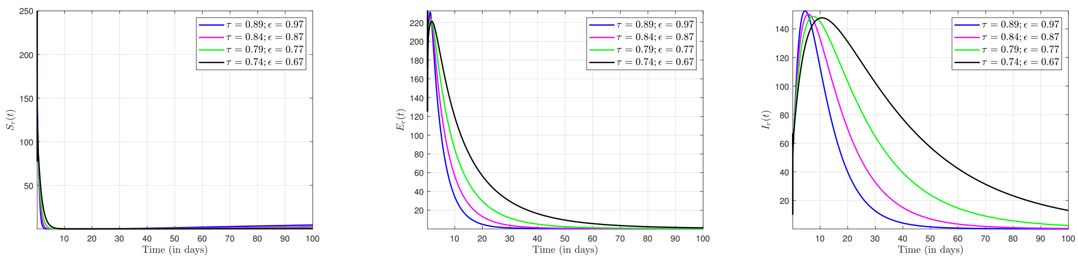
$$I_h(t_{n+1}) = I_h(t_n) + \frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_8(t_n, I_h(t_n)) - \frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_8(t_{n-1}, I_h(t_{n-1})) + \frac{\epsilon\tau}{M(\epsilon)} \times \left[\frac{3}{2}(\Delta t)\, t_n^{\tau-1}\Phi_8(t_n, I_h(t_n)) - \frac{\Delta t}{2} t_{n-1}^{\tau-1}\Phi_8(t_{n-1}, I_h(t_{n-1}))\right],$$

$$C_h(t_{n+1}) = C_h(t_n) + \frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_9(t_n, C_h(t_n)) - \frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_9(t_{n-1}, C_h(t_{n-1})) + \frac{\epsilon\tau}{M(\epsilon)} \times \left[\frac{3}{2}(\Delta t)\, t_n^{\tau-1}\Phi_9(t_n, C_h(t_n)) - \frac{\Delta t}{2} t_{n-1}^{\tau-1}\Phi_9(t_{n-1}, C_h(t_{n-1}))\right],$$

$$R_h(t_{n+1}) = R_h(t_n) + \frac{\tau t_n^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_{10}(t_n, R_h(t_n)) - \frac{\tau t_{n-1}^{\tau-1}(1-\epsilon)}{M(\epsilon)}\Phi_{10}(t_{n-1}, R_h(t_{n-1})) + \frac{\epsilon\tau}{M(\epsilon)} \times \left[\frac{3}{2}(\Delta t)\, t_n^{\tau-1}\Phi_{10}(t_n, R_h(t_n)) - \frac{\Delta t}{2} t_{n-1}^{\tau-1}\Phi_{10}(t_{n-1}, R_h(t_{n-1}))\right],$$

4.7. Results and Discussion

In the following part, we are going to investigate the quantitative behavior of the model. The various vaccination possibilities in the suggested model (5.7) are considered, and it explores how the numerical results are important for the model. For our simulation, we computed some of the parameter values shown in Fig. 4.2, which presents the model's solutions for a range of values for the relevant parameters taken from the literature. Here, we take into account a wide range of starting point population densities over a number of different compartments, including $S_r = 250, E_r = 125, I_r = 10, S_h = 8000, V_h = 5000, E_h = 3000, Q_h = 500, I_h = 1000, C_h = 1500$, and $R_h = 2000$. The simulations allowed us to observe the dynamics of the disease spread over time, as well as the impact of different parameter values on the final results. The results showed that increasing the initial population density of infected individuals led to a faster spread of the disease, while increasing the initial population density of susceptible individuals slowed it down. Additionally, varying other parameters such as transmission rates and recovery rates had significant effects on the overall spread and severity of the disease. Overall, these simulations provide valuable insights into how different factors can influence the spread and control of infectious diseases and can help inform public health policies and interventions aimed at mitigating their impact.



(a) (b) (c)

Figure 4.3: Population densities of rodents acquired from the Caputo-Fabrizio derivative with an exponentially decaying kernel under a fractal fractional operator for varying values of ϵ and τ.

In conclusion, the simulations conducted on the spread and control of infectious diseases have revealed that a number of factors can influence the severity and rate of transmission. For instance, increasing the initial population density of susceptible individuals can slow down the spread of the disease. Furthermore, varying transmission rates and recovery rates can have significant effects on the overall spread and severity of the disease. These insights are crucial in informing public health policies and interventions aimed at mitigating the impact of infectious diseases. By understanding how different factors interact to influence disease transmission, public health officials can develop more

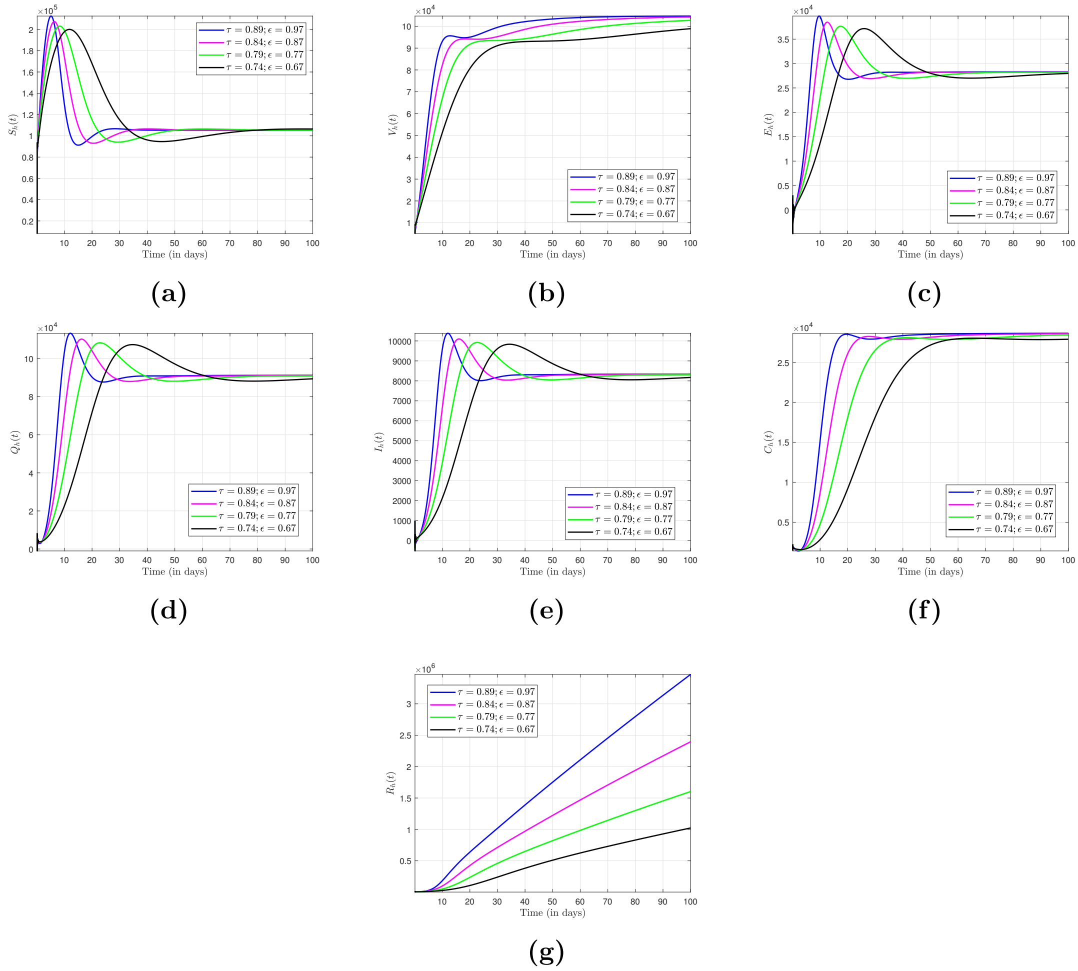


Figure 4.4: Human Population densities with Caputo-Fabrizio derivative having exponentially decay type kernel under fractal fractional operator for differet values of ϵ and τ.

effective strategies for controlling outbreaks and preventing future epidemics. Ultimately, these simulations provide significant tools for improving our understanding of infectious diseases and developing more effective approaches to managing them.

4.7.1. Analysis

The human population density is a critical factor in understanding the dynamics of human societies. It is determined by various factors, including the availability of assets, facilities, and possibilities for growth. Human population density for varying values of m is shown in Fig. 4.5. The impact of the parameter γ for varying values is shown in Fig. 4.6. This is a crucial consideration for public health officials when implementing vaccination programs. The relationship between human population density and disease exposure is a complex one, as illustrated in Fig. 4.7. The impact of the parameter α_3 is shown in Fig. 4.7. The graph shows that as the value of α_3 increases, so does the population density and the exposure rate of diseases. The impact of the parameter α_2 is shown in Fig. 4.8 Which shows the exposure of diseases in humans due to the infected rodents. As the graphs in Fig. 4.8 demonstrates, human population density has a significant impact on the spread of diseases from infected rodents. The higher the population density, the greater the risk of exposure to these illnesses. By analyzing the population density for different values of β_2, we can gain insight into how isolation rates affect disease susceptibility. As we can see from Fig. 4.9, exposure to diseases leads to an increase in susceptibility. However, vaccination and isolation rates have the potential to reduce exposure, as well as the number of infected and clinically recorded cases. The impact of parameter β_3 is shown in Fig. 4.10. This is due to the fact that the chance of transmission rises in proportion to the density of a population because more people will come into contact with one another. The effect of parameter ω_1 for varying values is shown in Fig. 4.11. Which shown by increases the value of ω_1 causes to increases the risk of infection in society even being isolated from the outside world. This is because a higher population density means that there are more people living in a smaller area, which makes it easier for diseases to spread from person to person. The data in Fig. 4.12 highlights the impact of confirmed cases on human population density at different values of ω_2. The graph suggests that as the number of confirmed cases increases, population density tends to decrease. This could be due to people taking precautions and avoiding crowded areas in order to minimize their risk of contracting the infection. The effect of parameter f_1 on calinically recorded cases is shown in Fig. 4.13. The impact of parameter f_2 for varying values is shown in Fig. 4.14. Which shows the increase in recovery rate of infected individuals has on population density. As the recovery rate increases, the population density decreases. The impact of the parameter f_3 is shown in Fig. 4.15. Which shows the increase the recovery rate of isolated individuals has on population density. As the recovery rate of isolated individuals increases, the population density also increases. This is evident from the data presented in Fig. 4.15, which clearly shows a positive correlation between these two variables. It is important to note that this trend may not hold true for all populations and environments, as various factors such as resource availability and migration patterns can also impact population density. The impact of the parameter f_4 is shown in Fig. 4.16. Which shows the increase the recovery rate afer the exposure of the diseases. As the recovery rate increases after exposure to diseases, the human population density varies for different values of f_4, as depicted in Fig. 4.16. This highlights the importance of effective disease control measures to prevent overcrowding and potential outbreaks.

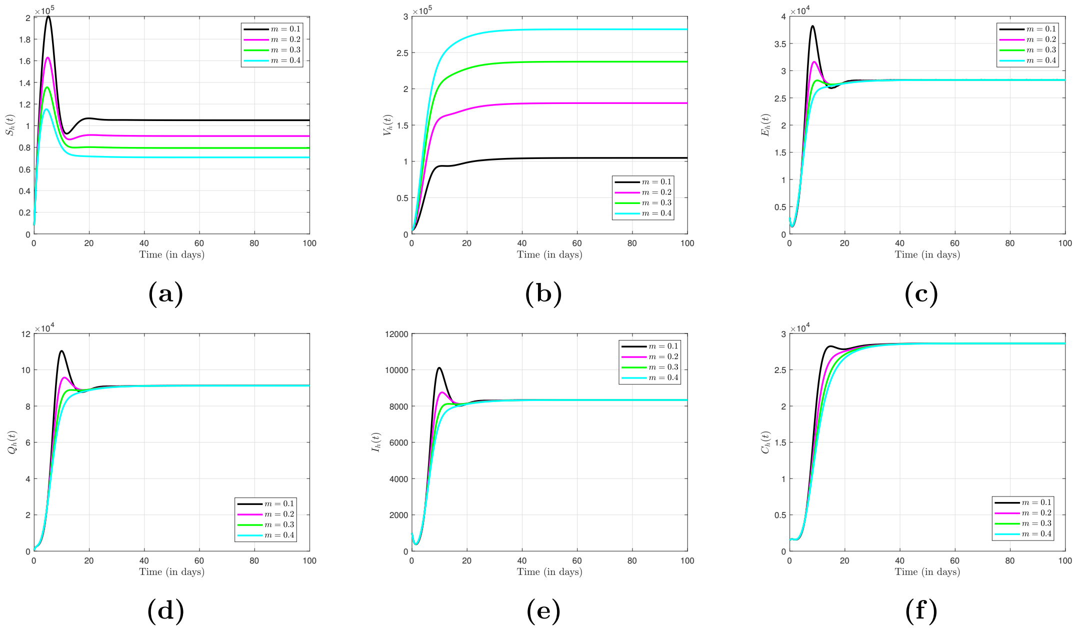


Figure 4.5: Density of the human population for various m values.

4.8. Conclusion

This study investigated monkeypox viral transmission dynamics through a fractal-fractional differential system model. Humans and rodents were the two largest classes in the population. The developed model was parameterized using cumulative reported data in U.S using CDC data. The results demonstrate that the suggested model provides a good fit to the data and may be used to reliably forecast the development of this disease in the United States. We show that the disease-free state of the model is locally asymptotically stable if the threshold quantity $\mathcal{R}_0 < 1$, but unstable otherwise. Also we have developed the Hyers-Ullam and Hyers-Ullam-Rassias stability results for the considered

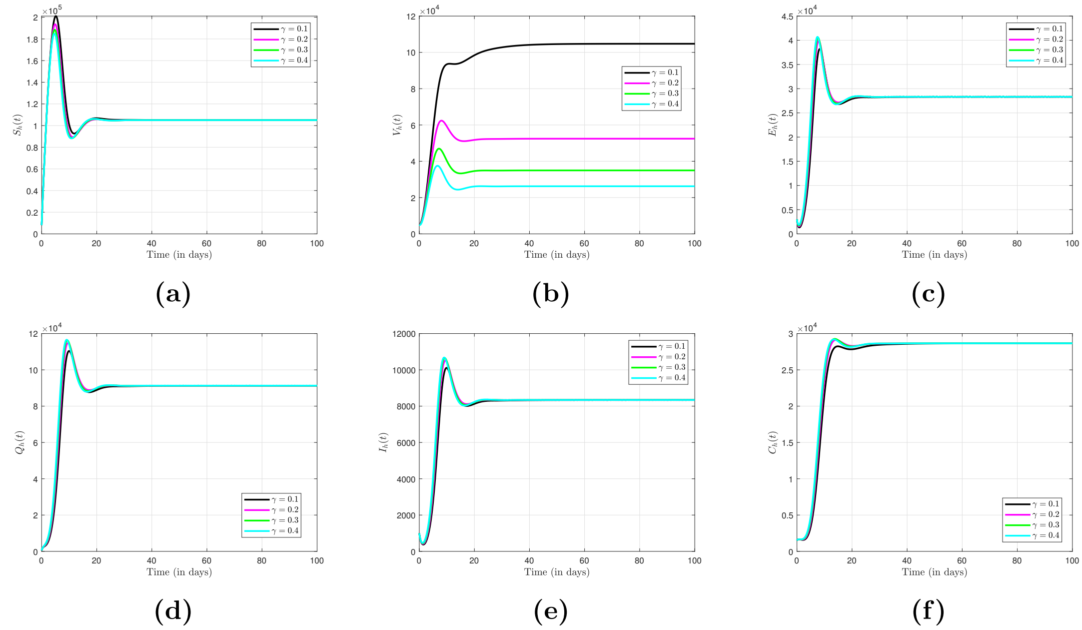


Figure 4.6: Density of the human population for various γ values.

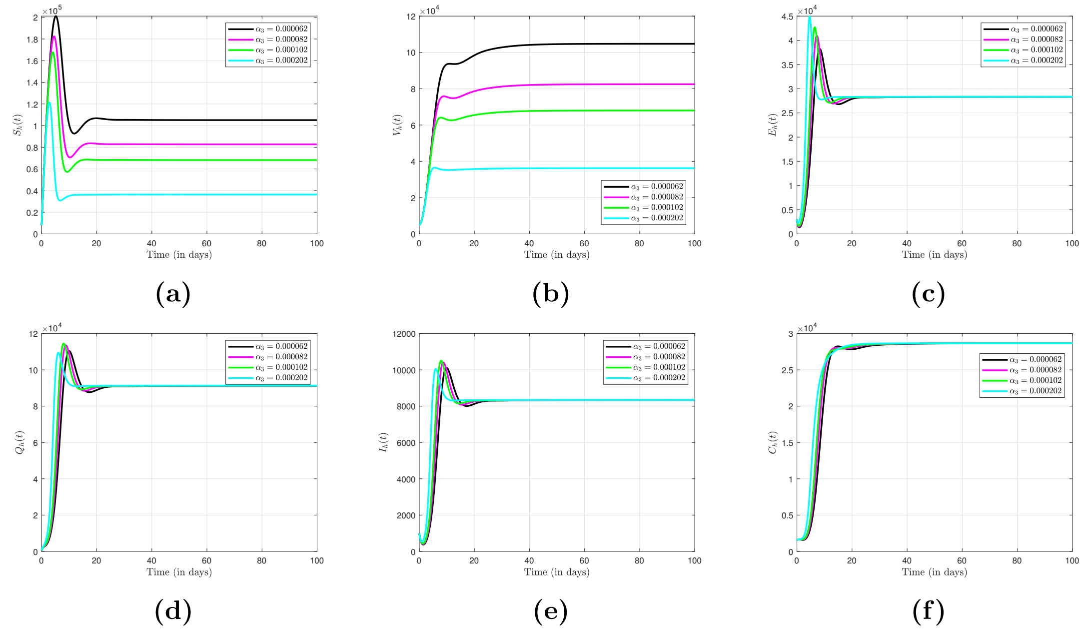


Figure 4.7: Density of the human population for various α_3 values.

model. The dynamical behaviour of the system has been established using numerical simulations with varying input parameters. Different input factors' impacts on monkeypox disease dynamics were examined.

In order to comprehend the dynamics of human societies, it is essential to consider the density of the human population. It depends on a number of factors, such as the

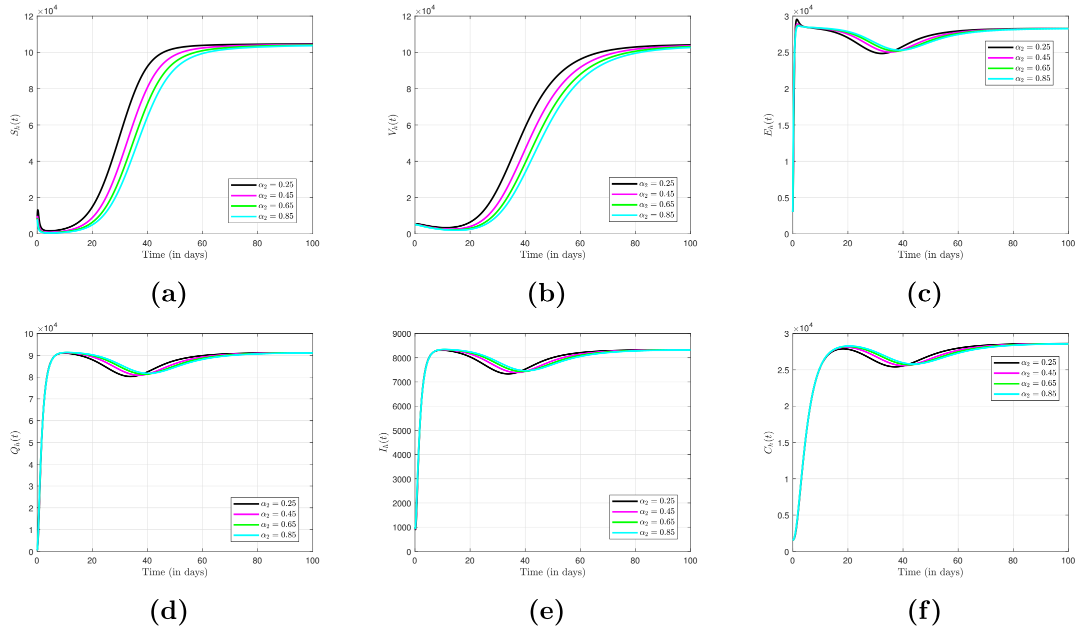


Figure 4.8: Density of the human population for various α_2 values.

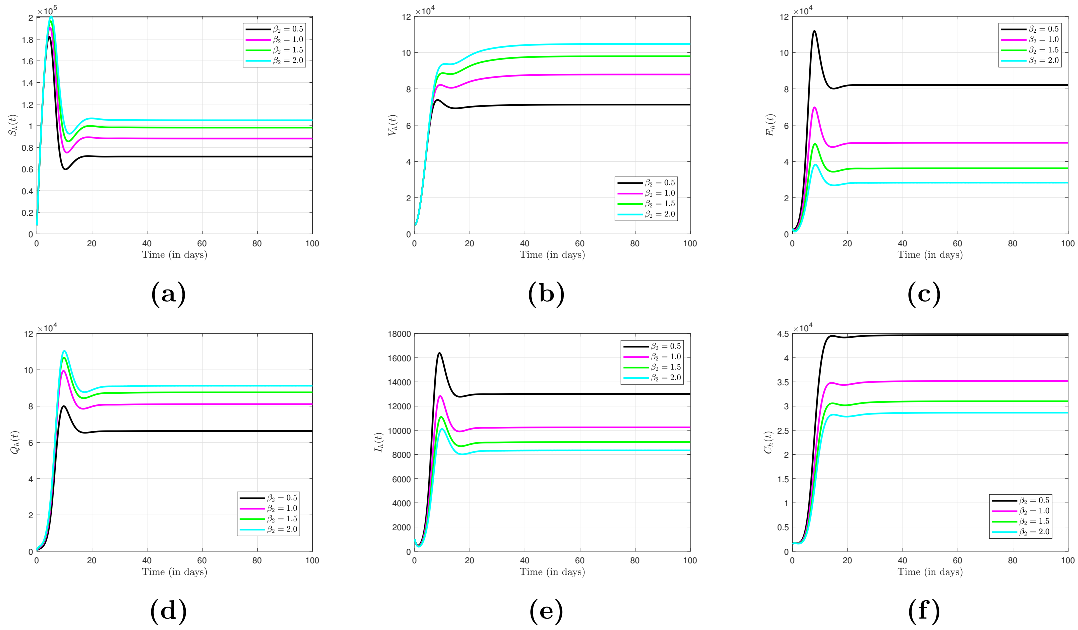


Figure 4.9: Density of the human population for various β_2 values.

vaccination rate, isolation rate, exposure of virus in society, recovery rate. Ultimately, understanding the complex interplay between these various factors is crucial for predicting and managing population growth and ensuring sustainable development for future generations. The density of the human population for different values of m is illustrated in Fig. 4.5. The impact of the parameter γ for varying values is shown in Fig. 4.6. When

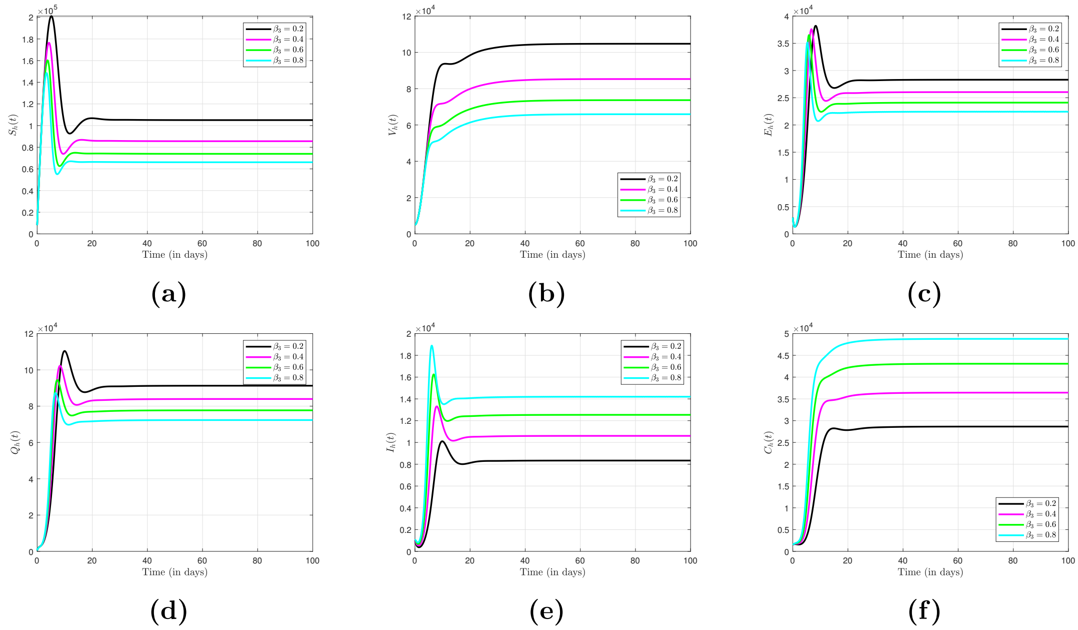


Figure 4.10: Density of the human population for various β_3 values.

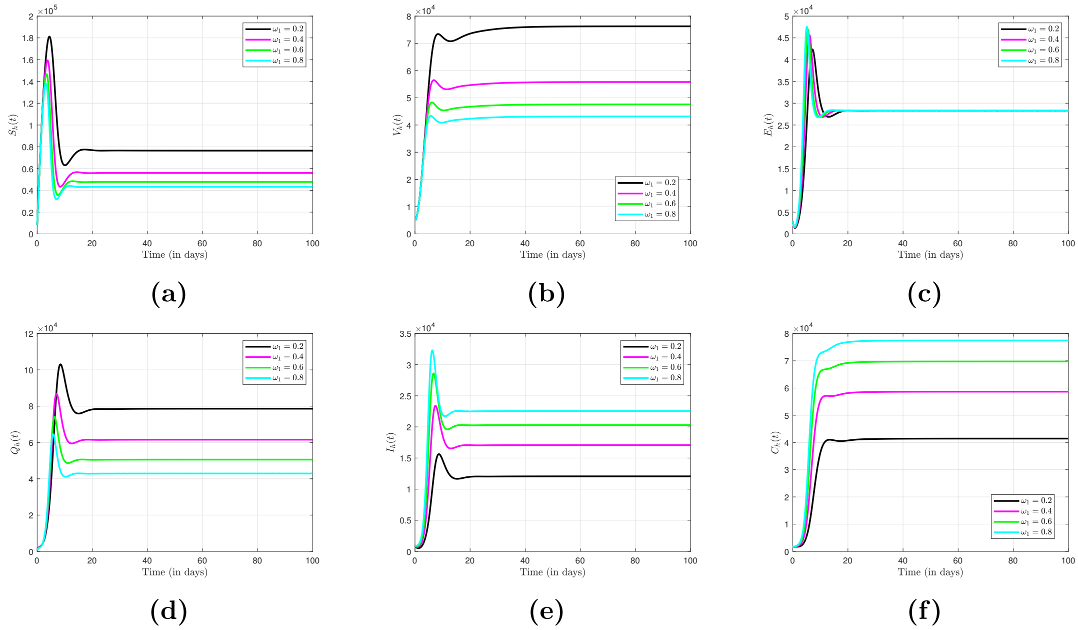


Figure 4.11: Density of the human population for various ω_1 values.

putting vaccination programs into place, public health officials must take this into account critically. As shown in Fig. 4.7, there is a complicated relationship between the density of the human population and the exposure to disease. The influence of the parameter α_3 is depicted in Fig. 4.7. The graph depicts an increase in population density and disease exposure with an increase in α_3. Exposure of diseases in humans because of

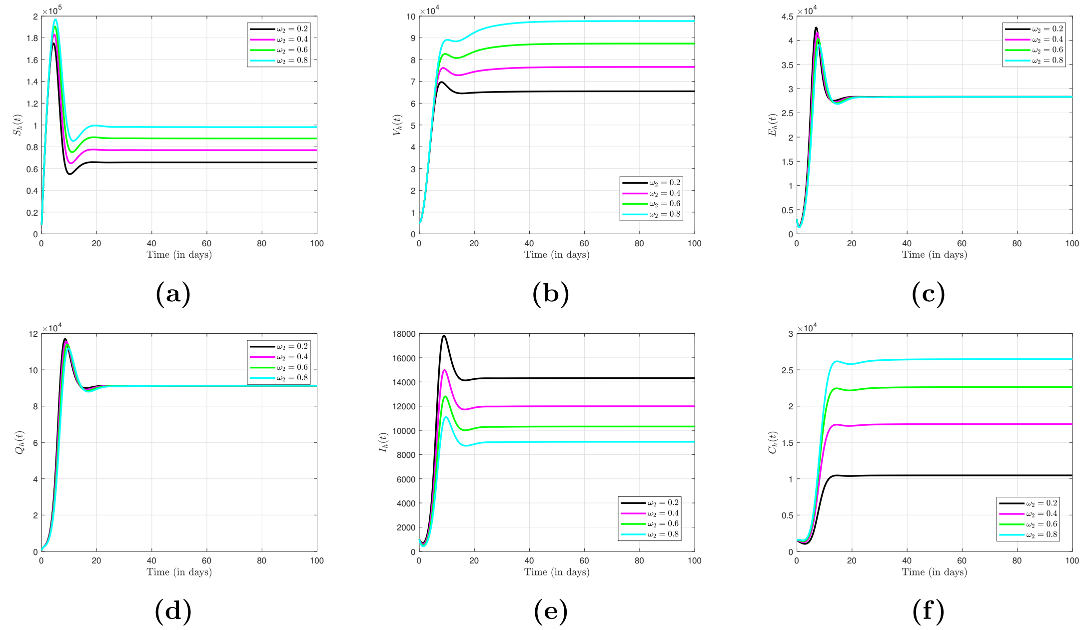


Figure 4.12: Density of the human population for various ω_2 values.

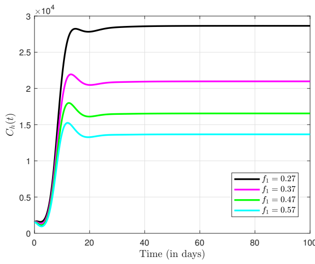


Figure 4.13: Density of the human population for various f_1 values.

infected rodents is displayed in Fig. 4.8, which demonstrates the effect of the parameter α_2. We can learn about the relationship between isolation and disease susceptibility by examining the population density for varying values of β_2. The results of the Fig. 4.9 show that susceptibility to disease increases after exposure to the disease. The effect of parameter β_3 is shown in Fig. 4.10. This is because the chance of transmission goes up as the population density goes up, since more people will be in contact with each other. The effect of parameter ω_1 for varying values is shown in Fig. 4.11. Which shown by increases the value of ω_1 causes to increases the risk of infection in society. The data in Fig. 4.12 highlights the impact of confirmed cases on human population density at different values of ω_2. In Fig. 4.13, we see the impact of parameter f_1 on calinically recorded cases. The results of using different values for the parameter f_2 are displayed in

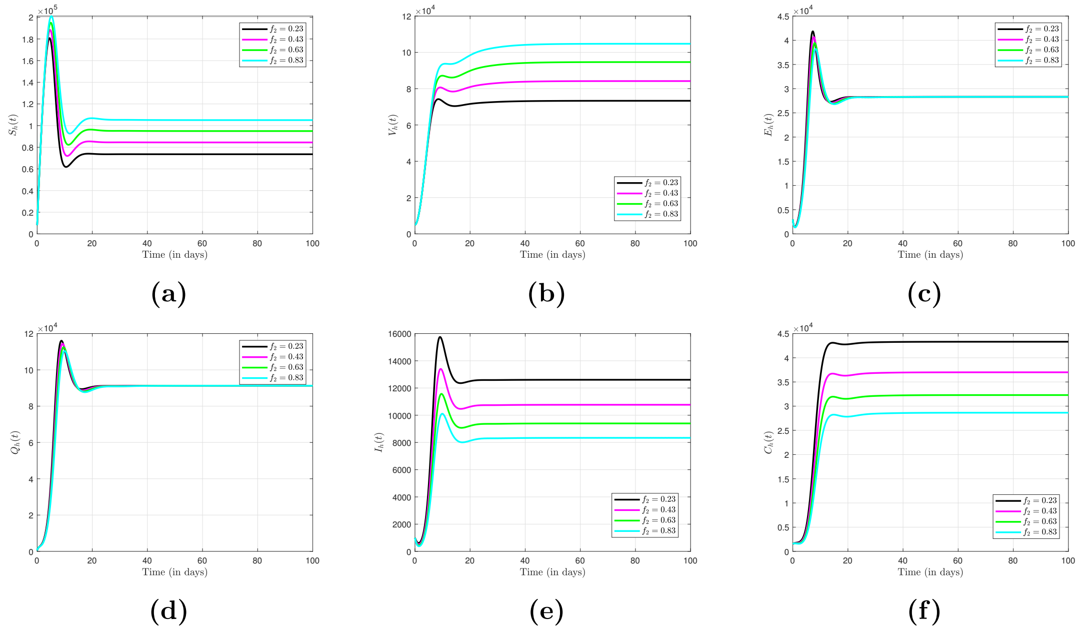


Figure 4.14: Density of the human population for various f_2 values.

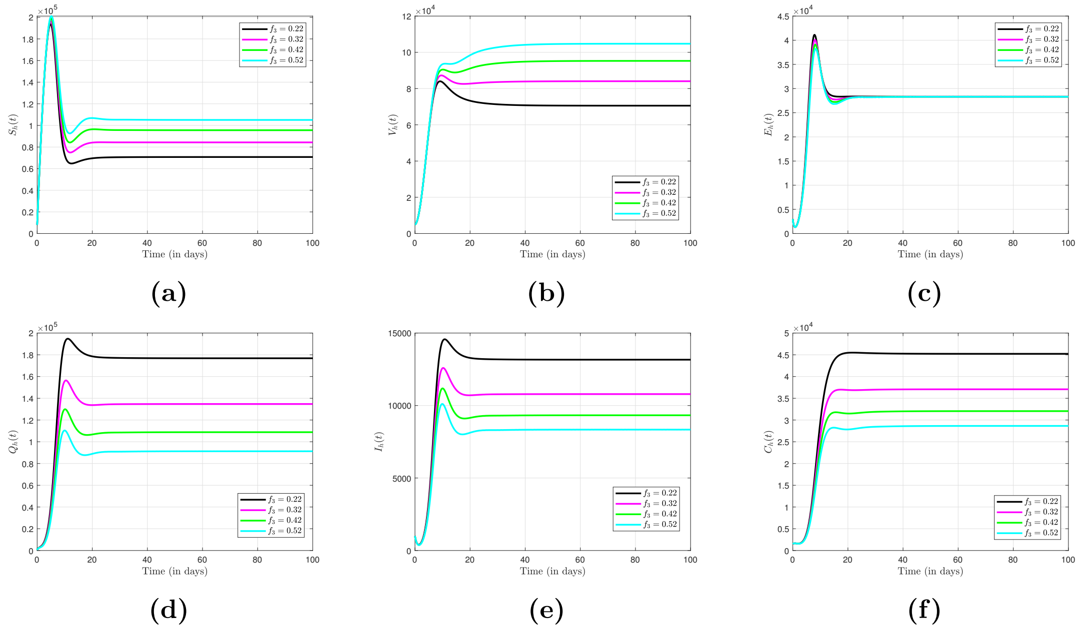


Figure 4.15: Density of the human population for various f_3 values.

Fig. 4.14. which demonstrates the effect on population size of a rise in the proportion of infected people who make a full recovery. The population density decreases as the recovery rate rises. To see how f_3 plays a role, check out Fig. 4.15. Which demonstrates the positive impact that reuniting previously isolated individuals has on overall population growth. The population density grows in step with the rate at which formerly isolated

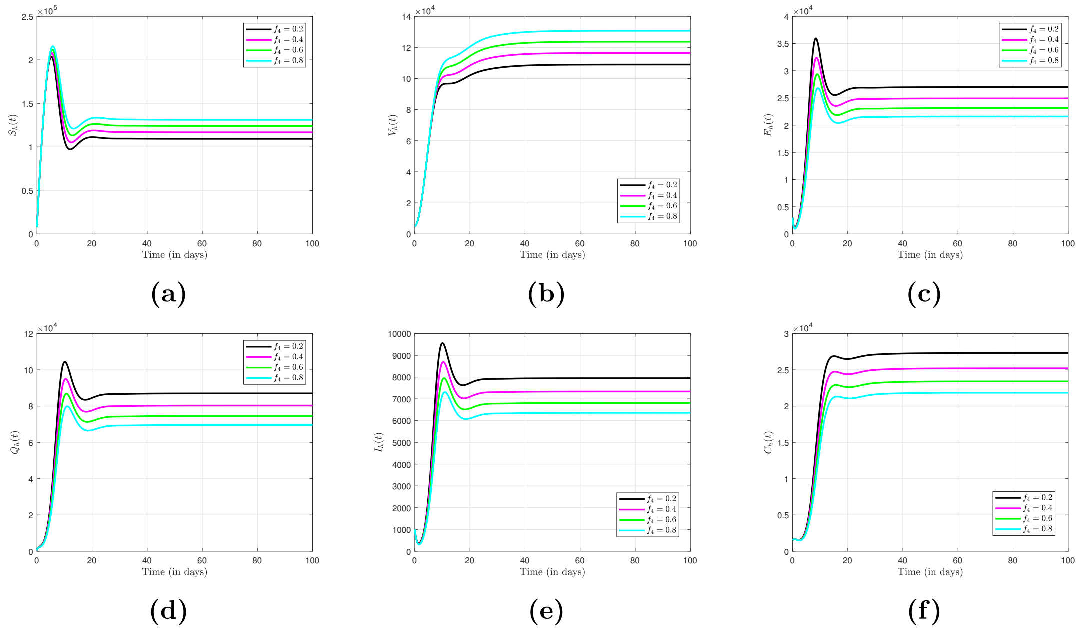


Figure 4.16: Density of the human population for various f_4 values.

individuals are reintegrated into society. This is evident from the data presented in Fig. 4.15, which clearly shows a positive correlation between these two variables. It is important to note that this trend may not hold true for all populations and environments, as various factors such as resource availability and migration patterns can also impact population density. The impact of the parameter f_4 is shown in Fig. 4.16. Which shows the increase the recovery rate afer the exposure of the diseases. As the recovery rate increases after exposure to diseases, the human population density varies for different values of f_4, as depicted in Fig. 4.16. This highlights the importance of effective disease control measures to prevent overcrowding and potential outbreaks.

Furthermore, our findings suggest that the implementation of effective control measures such as vaccination and quarantine can significantly reduce the spread of monkeypox in U.S. These results have important implications for public health policy and underscore the need for continued surveillance and monitoring of this disease. It is important to consider the potential ethical concerns and cultural factors that may affect the implementation of vaccination and quarantine measures in U.S, as these can impact their effectiveness and acceptance by the population.

It is hoped that this publication would aid scientists in their efforts to better understand and foresee epidemics throughout the world. The primary objective of this study is to determine the impact of different vaccination regimens on the occurrence, severity, and resolution of future instances of monkey pox. This puts us in a better position to make choices in the future that will offer us greater agency over the aforementioned scenarios or significantly mitigate their negative effects.

Availability of data and materials

All data used in this analysis is provided in CDC recorded monkey pox cases since the epidemic began, according to the national center for emerging and zoonotic infectious diseases *Source: United State of America.*

Authors contribution

The authors contributed equally in this paper. All authors have read and approved the final version of manuscript.

Declaration of Competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Chapter 5

Modeling and Analysis of Corruption Dynamics in Society under Fractal-Fractional Derivative in Caputo Sense with Power-Law

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Abstract

In this article, the social dynamics of corruption are investigated using a six-compartmental model. It is formed by six independent parts, all of which combine to form a system of ordinary differential equations for six different variables. In this study, we investigated the corruption using Caputo fractal-fractional derivative with power-law type kernel. The basic mathematical properties of the model have been extensively studied. We have determined the model's feasible region, and calculating its corruption transmission generation number and equilibrium points. Picard's successive approximation approach and the banach fixed point theory were used to demonstrate the model's existence and stability conditions. In addition, we have examined the locally and globally asymptotically stability of the corruption free and corruption persistent equilibrium states. The consistency of the obsessed solution in Hyers-Ulam and Hyers-Ulam-Rassias sense is also explored. For presenting numerical data, Matlab software is the preferred tool. Different

fractional orders and parameter values are represented graphically.

Keywords:

Mathematical Modelling, Corruption Analysis, Hyers-Ulam Stability, Hyers-Ulam-Rassias Stability, Sensitivity Analysis.

5.1. Introduction

Corruption refers to official wrongdoing in which governmental (public) or private (business) officeholders benefit financially by abusing their position of trust for personal gain [6]. In the literature, different types of corruption are described and include arbitrary corruption, and pervasive corruption [14], private corruption [27], and public corruption [12]. Corruption may arise from whether the supply or the demand side [15]. In general, corruption poses a significant challenge to the maintenance of law and order, democratic republic, human dignity, justice, and social equality. It also slows down economic growth and puts the fairness and proper functioning of market economies at risk [12, 22]. It is a significant concern in every country in the world, but particularly in developing countries [22]. Corruption is still rampant in modern society, despite the fact that most societies have anticorruption regulations or methods in action.

Now a days, corruption is a significant problem on a worldwide scale; nevertheless, the countries most severely impacted by it are those with weak economies and are located mostly in the Sahara Desert's southern region [7, 1, 21, 24, 11, 10, 13]. The corrupted person infects the next susceptible member of society, just as an epidemic does. Corruption hinders development because it weakens both the domestic economy and the nation's ability to maintain peace and order on the global stage [13, 2, 29].

Mathematical models that include optimal control assessments can be helpful in gaining an understanding of the dynamics of the transmission of corruption as well as in deciding how best to intervene in corrupt systems. Several authors have written about the epidemiological corruption modelling approach, including [1, 9, 17, 21, 13].By focusing on people who benefit from corrupt officials and politicians, Nathan and Jackob created a compartmentalised epidemiological model of corruption in Kenya [24]. For the dynamics of corruption, the authors in [5], used a SIR model. Moreover, a single optimum control method was added as part of the model's expansion. A model for preventing corruption was developed in [16], which demonstrated that it is feasible to eliminate the corruption entirely if the termination rate is equal to the corruption rate. In [31], the level of corruption was measured using a model based on difference equations. To prevent corruption, the author of [25] explored a game-theoretical strategy. Corruption growth and decay models were developed using differential equations in the study cited in [31]. The study [19] did come up with a mathematical model for corruption by looking at how anti-corruption campaigns and counseling in jail raise people's awareness. The fundamental reproduction number was calculated, and the existence of corruption-free and endemic equilibrium points was examined. In [12], a mathematical model of corruption's dynamics was developed by the authors, and their findings are discussed. A deterministic model of corruption within a population was developed and studied in [1]. They determined the BRN, as well as the points of equilibrium for noncorrupt and corrupt environments. Numerical simulations were done, and the results showed that corruption can be reduced to a limited significant degree that can be dealt with, but not completely removed.

Compression models that use fractional derivatives are more accurate and more closely reflect the reality [18, 3, 28, 30]. Motivated by the work of Mokaya [23], we have created a new mathematical model on curruption dynamics in society. This work is made up of six different sections. In Section 5.2, we give a condensed description of the mathematical modelling of the corruption dynamics in society. In Section 5.4, we are going to evaluate the model's qualitative analysis and basic properties of that as well. In Section 5.5, we talk about the basics of advanced fractional calculus. Section 5.5.1 represents the solution's existence and uniqueness. In Section 5.5.2, Furthermore, we discuss about Hyers Ulam stability for the suggested model. To solve the proposed model numerically, refer to the scheme described in Section 5.6. Finally, results are plotted graphically in Section 5.7. At the end in Section 5.8, concluding remarks are offered.

5.2. Mathematical Model and Formulation

In this research study, the prevalence of corruption in society is investigated using a deterministic mathematical model. Our approach is predicated on the idea that corruption in a society may spread throughout a population like a virus. The whole populations, denoted by $W(t)$, and is divided into six different groups, namely susceptible, risk of corruption, confirmed corrupt, punished, transformed and immune Individuals (for further information see Table (5.1)). Considering this, the total population is $W(t) = S(t) + R(t) + C(t) + P(t) + T(t) + I(t)$. If the natural natality and death rates of a population are Π and μ respectively, then Fig. 5.1 is where you might find the dynamic flow transfer from one compartment to the other.

Every day, people from high-moral-standard families are recruited into the susceptible class, where they face a Π rate of corruption. This group represents law-abiding citizens who have never participated in corrupt activities that affect the country's economy and growth, but who are nonetheless susceptible to the harmful effects of corruption. The corruption rate increased by a factor of ω, whereas the natural death rate declined by a factor of μ. Hence, the rate of change in the number of susceptible individuals is:

$$\frac{dS}{dt} = v\Pi + c\alpha T - (\omega C + \mu) S, \tag{5.1}$$

The proportion of exposed individuals is represented by the strength of corruption at rate ω, and certain corrupted people are classified as being at risk by the factor f_2 as a consequence of aspects like inspection and evaluation. This group is made up of individuals that surround corrupt individuals who have a possibility of becoming corrupted themselves, but don't. It's reduced significantly by the rate $f_1$at because they are becoming corrupt, by per capita rate $p$for which they were becoming transferred and the natural mortality rate per capita μ. Hence, the rate of change in the number of exposed individuals is:

$$\frac{dR}{dt} = \omega CS + f_2 RC - (p + \mu) R, \tag{5.2}$$

The proportion of corrupted individuals tends to increase with the rate of f_1 as a direct consequence of the transformation from risk of corruption to confirm corrupted individuals. This group is made up of individuals who are often associated with corruption as well as having power to affect susceptible as per capita rate $(1 - v)$ and people at threat of being corrupt. Hence, decreasing by natural mortality rate per capita μ and the corruption induced punished rate γ. So that the rate of change in the number of corrupted individuals is:

$$\frac{dC}{dt} = f_1 pR - (f_2 R + \gamma + \mu) C, \tag{5.3}$$

Those who have been corrupt and require assistance grow the population of people who have been punished at a rate of γ. With a rate of γ, these people are transferred from the corrupted compartment to the punished one. This group contains individuals individuals who were found to be guilty of or punished for engaging in corrupt activities, and who have been tried and convicted to a period of time throughout which they are criminalized from engaging in corrupt activities and are unable to exert any influence over others. Thus, faded by natural death per capita rate μ and is converted as per capita rate θ by punished for corrupt practices. Hence, the rate of change at which the number of punished individuals changes is given as:

$$\frac{dP}{dt} = \gamma C - (\theta + \mu) P, \tag{5.4}$$

The population that includes those that are at risk for corruption yet do not directly participate in corrupt activities rises at a per capita rate of p, while the population of those who have been converted increases at a rate of θas a consequence of the transition from punishment to rehabilitative compartments. This classification contains formerly convicted individuals who have undergone rehabilitation as part of their sentence and who have either become susceptible to or resistant to the effects of corruption as a result of their exposure. Those that retrieved from becoming corrupt due to constitutional protections with the rate α, and decreased by natural per capita death μ. The proportion of transformed individuals who transfer into the susceptible class is per capita rate c. Hence, the rate of change in the transformed population is:

$$\frac{dT}{dt} = \theta P + (1 - f_1) pR - (\alpha + \mu) T, \tag{5.5}$$

As a result of the change from transformed to recovered compartments, the number of immune humans grows at a rate of α. This category includes those who, no matter what the circumstances may be, will never participate in dishonest or illegal practices. Afterwards it went down by μdue to natural deaths per capita. Thus, the rate of change in the population of immune individuals is:

$$\frac{dI}{dt} = (1-c)\,\alpha T + (1-v)\,\Pi - \mu I, \tag{5.6}$$

The following system of non-linear ordinary differential equations represents the relevant mathematical expressions based on the aforementioned explanations.

$$\begin{cases} \dfrac{dS}{dt} = v\Pi + c\alpha T - (\omega C + \mu)\,S, \\ \dfrac{dR}{dt} = \omega CS + f_2 RC - (p+\mu)\,R, \\ \dfrac{dC}{dt} = f_1 pR - (f_2 R + \gamma + \mu)\,C, \\ \dfrac{dP}{dt} = \gamma C - (\theta + \mu)\,P, \\ \dfrac{dT}{dt} = \theta P + (1-f_1)\,pR - (\alpha+\mu)\,T, \\ \dfrac{dI}{dt} = (1-c)\,\alpha T + (1-v)\,\Pi - \mu I, \end{cases} \tag{5.7}$$

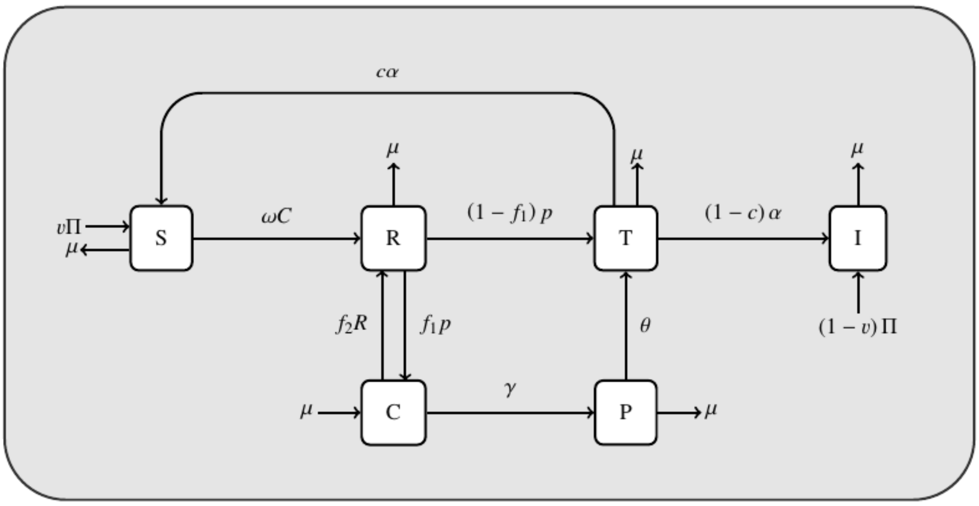


Figure 5.1: Dynamical Phase Diagram for transmission of corruption in society.

Table 5.1: Discription of variables and parameters for the model (5.7).

Variables	Discription	Value
$S(t)$	Group of susceptible people.	300
$R(t)$	Group of risk people.	100
$C(t)$	Group of corrupt people.	50
$P(t)$	Group of punished people.	83
$T(t)$	Group of transformed people.	70
$I(t)$	Group of immune people.	228
Parameters	**Discription**	
θ	The time rate of change of punished populations who enter transformed group	0.21
γ	The time rate of change of corrupt populations who enter punished group	0.082
μ	Natural death rate of population	0.0163
α	Out flow of the time rate of change of transformed populations	0.45
f_2	The transmission coefficient from corrupt group to risk group	0.003
f_1	The time rate of change of populations in risk group moving to corrupt group	0.51
Π	Recruitment of populations into the population	70
ω	Corruption transmission probability per contact	0.01025
p	Outflow of the rate of change in the population at risk.	0.0023
c	The time rate of change of populations in transformed group moving to susceptible group	0.043
v	The time rate of change of populations not recruit immune	0.431

5.3. Positivity and Invariant region of the model solution

That perhaps the associated variables of the model (5.7) remain nonnegative is a necessary condition for establishing the epidemiological relevance of the deterministic mathematical model (given by Section 5.2). By starting with nonnegative values for time, it is also possible to explain why the model's solution will continue to be nonnegative indefinitely beyond zero. The following lemma holds.

Lemma 5.1: *The region $\mathfrak{T} \subset \mathcal{R}_{+}^{6}$ represents the boundedness and biologically feasible region for the system* (5.7) *and are given by*

$$\mathfrak{T} = \left\{ (S(t),\ R(t),\ C(t),\ P(t),\ T(t),\ I(t)) \in \mathcal{R}_{+}^{6} \;\middle|\; 0 \leq \sup_{t\to+\infty} W(t) \leq \frac{\Pi}{\mu} \right\} \tag{5.8}$$

Proof. The proposed approach (5.7) is assessed in order to investigate the first equation.

$$\frac{dS}{dt} = v\Pi + c\alpha T - (\omega C + \mu) S,$$

it leads us to the assumption that

$$\frac{dS}{dt} \leq -(\omega C + \mu) S,$$

in keeping with the concept of integration and the exponential growth criterion

$$\frac{dS}{dt} \leq S(0)\, e^{-(\omega C + \mu)t},$$

which implies that

$$S(t) \geq 0.$$

Likewise, we have $R(t) \geq 0, C(t) \geq 0, P(t) \geq 0, T(t) \geq 0$ and $I(t) \geq 0$ for all $t \geq 0$. Thus that the planes $S = R = C = P = T = I = 0$ are not violated by the model solution. We can write the system (5.7) as: if we assume that the entire population can be represented by the function $W(t)$.

$$\frac{dW(t)}{dt} \le \Pi - \mu W(t). \tag{5.9}$$

So that, we can write

$$W(t) \le \frac{\Pi}{\mu} - \left(\frac{\Pi}{\mu} - W_0\right) e^{-\mu t}. \tag{5.10}$$

Going to benefit [8] we could really state that, if $W_0 < \frac{\Pi}{\mu}$, so that as $t \to \infty$, asymptotically, $W(t) \to \frac{\Pi}{\mu}$ tends to imply that $0 \le W(t) \le \frac{\Pi}{\mu}$. Thus, all solutions to the current model will converge to the reliable region $\mathfrak{B}$ [26].

5.4. Qualitative analysis of the model

The corruption transmission generation number is a very important part of the analysis of the mathematical model (5.7). This section will compute and explain the invariant region and corruption transmission generation number for the given model (5.7), and it will also investigate

- Local stability of its CFE point (see Theorem 5.1).
- Global stability of its CFE point (see Theorem 5.2).
- Existence and uniqueness of its CPE point (see Theorem 5.3).
- Global stability of its CPE point (see Theorem 5.5).

5.4.1. The CFE Point

The variables $R = C = P = T = 0$ are set in order to preserve a CFE point. In this case, system (5.7) shows that the CFE point is as follows:

$$\text{CFE} = \left(\frac{v\Pi}{\mu}, 0, 0, 0, 0, \frac{(1-v)\Pi}{\mu}\right). \tag{5.11}$$

5.4.2. Corruption Transmission Generation Number

The corruption transmission generation number indicated by $\mathcal{R}_c$, is a quantitative measure of the average rate of corruption during a certain time frame. People who are involve in corruption and have the ability to effect others, making it a contagious problem. In this section, we construct a new system (5.12) based on the previously suggested system, which includes just the risk and corrupt population classes and is written as

$$\begin{cases} \dfrac{dR}{dt} = \omega CS + f_2 RC - (p+\mu) R, \\ \dfrac{dC}{dt} = f_1 pR - (f_2 R + \gamma + \mu) C, \end{cases} \tag{5.12}$$

To get the corruption transmission generation number for the system (5.7), a next-generation matrix approach has been applied here to the system (5.12). This study generates matrix $\mathbf{F}$ and $\mathbf{V}$, i.e.

$$\mathbf{F} = \begin{pmatrix} \omega CS + f_2 RC \\ \\ -f_2 RC \end{pmatrix} \qquad \mathbf{V} = \begin{pmatrix} (p+\mu) R \\ \\ (\gamma+\mu) C - f_1 pR \end{pmatrix}$$

The Jacobian matrix of $\mathbf{F}$ and $\mathbf{V}$ at CFE, denoted by $\mathtt{F}$ and $\mathtt{V}$ are given as follows:

$$\mathtt{F} = \begin{pmatrix} 0 & \frac{\omega v \Pi}{\mu} \\ \\ 0 & 0 \end{pmatrix} \qquad \mathtt{V} = \begin{pmatrix} p+\mu & 0 \\ \\ -f_1 p & \gamma+\mu \end{pmatrix}$$

Therefore, $\mathtt{FV}^{-1}$ is the next generation matrix of the model structure (5.12). So, as described in [5], $\mathcal{R}_c = \varrho\left(\mathtt{FV}^{-1}\right)$ where ϱ stands for spectral radius of the next-generation matrix $\mathtt{FV}^{-1}$. Thus,

$$\mathtt{FV}^{-1} = \begin{pmatrix} \frac{\omega v p f_1 \Pi}{\mu(p+\mu)(\gamma+mu)} & \frac{\omega v \Pi (p+\mu)}{\mu} \\ \\ 0 & 0 \end{pmatrix}$$

So, $\varrho\left(\mathtt{FV}^{-1}\right) = \frac{\omega v p f_1 \Pi}{\mu(p+\mu)(\gamma+mu)} = \mathcal{R}_c$, where $k_1 = p+\mu$ and $k_2 = \gamma+\mu$. Therefore

$$\mathcal{R}_c = \frac{\omega v p f_1 \Pi}{\mu (p+\mu)(\gamma+\mu)} \tag{5.13}$$

Now, we use $\mathcal{R}_c$ to find out if the system (5.7) really had any equilibria. Our next step is to investigate the CFE stability of the model at the local stage.

Theorem 5.1: *The CFE is locally asymptotically stable if* $\mathcal{R}_c < 1$.

Proof. Taking the system (5.7) into account, we have the Jacobian matrix $\mathcal{J}$ (CFE).

$$\mathcal{J}\left(\text{CFE}\right) = \begin{pmatrix} -\mu & 0 & \frac{\omega v \Pi}{\mu} & 0 & c\alpha & 0 \\ 0 & -\left(p+\mu\right) & \frac{\omega v \Pi}{\mu} & 0 & 0 & 0 \\ 0 & pf_1 & -\left(\gamma+\mu\right) & 0 & 0 & 0 \\ 0 & 0 & \gamma & -\left(\theta+\mu\right) & 0 & 0 \\ 0 & -\left(1-f_1\right)p & 0 & 0 & -\left(\alpha+\mu\right) & 0 \\ 0 & 0 & 0 & 0 & \left(1-c\right)\alpha & -\mu \end{pmatrix} \tag{5.14}$$

the characteristic equation for the system (5.7) is obtained as:

$$\begin{aligned} \mathcal{J}\left(\text{CFE}\right) = & \left(-\lambda^{1,6}-\mu\right)\left(-\lambda^{2,3}-\lambda^{2,3}\left(\gamma+p+2\mu\right)+\left(p+\mu\right)\left(\gamma+\mu\right)\left(1-\mathcal{R}_c\right)\right) \\ & \left(-\lambda^{4}-\left(\theta+\mu\right)\right)\left(-\lambda^{5}-\left(\alpha+\mu\right)\right), \end{aligned} \tag{5.15}$$

From (5.15) we can says that, if $\mathcal{J}$ (CFE), meaning that all eigenvalues have to be negative. Then, at CFE, the system (5.7) shows local asymptotic stability. The proof is now complete.

Next, we use the method described in [20] to investigate the global asymptotic stability of the model (5.7) at the CFE point. The global asymptotic stability of the epidemic model can be demonstrated by constructing a Lyapunov function satisfying two conditions: (i) the function constructed must be positive and definite, and (ii) The model's solution needs certain criteria $\mathtt{L}' \leq -c\mathtt{L}$ to be met by the time derivative of the function, where c is a positive constant.

Theorem 5.2: *The corruption model* (5.7) *is globally asymtotically stable at CFE if*

$\mathcal{R}_c < 1$.

Proof. Consider the following Lyapunov function for the creation of the proof of global stability,

$$\mathtt{L} = \mathtt{L}_1 R + \mathtt{L}_2 C, \tag{5.16}$$

where, $\mathtt{L}j > 0$ for $j = 1, 2$ are constants that may be calculated later. Also, by taking the derivative of (5.16) and using the equations from (5.7), we have

$$\begin{aligned}
\frac{d\mathtt{L}}{dt} &= \mathtt{L}_1 \frac{dR}{dt} + \mathtt{L}_2 \frac{dC}{dt}, \\
&= \mathtt{L}_1 \left(\omega CS + f_2 RC - (p+\mu) R\right) + \mathtt{L}_2 \left(f_1 pR - (f_2 R + \gamma + \mu) C\right), \\
&= \left(pf_1 \mathtt{L}_2 - (p+\mu) \mathtt{L}_1\right) R + \left(\frac{v\omega\Pi}{\mu} \mathtt{L}_1 - (\gamma+\mu) \mathtt{L}_2\right) C,
\end{aligned}$$

Now let's determine the value for each of the constants $\mathtt{L}_1 = \frac{pf_1}{p+\mu} \mathtt{L}_2$ and using into the above equation we get the following:

$$\frac{d\mathtt{L}}{dt} \leq \left(pf_1 - \frac{pf_1 (p+\mu)}{p+\mu}\right) \mathtt{L}_2 R + \left(\frac{v\omega pf_1 \Pi}{\mu (p+\mu)} - (\gamma+\mu)\right) \mathtt{L}_2 C,$$

by taking $\mathtt{L}_2 = 1$, and can be written as

$$\frac{d\mathtt{L}}{dt} \leq (\gamma+\mu)(\mathcal{R}_c - 1) C,$$

Here $\frac{d\mathtt{L}}{dt} \leq 0$, whenever $\mathcal{R}_c \leq 1$ and $\frac{d\mathtt{L}}{dt} = 0$ iff $C = 0$. Using $C = 0$ in the equations of the model (5.7), we can get (S, R, C, P, T, I) approaches to the CFE for $t \to \infty$. Thus, the CFE point is the largest persistent group $\mathfrak{T}$. According to LaSalle's Invariance Principle, if $\mathcal{R}_c \leq 1$, the system (5.7) is globally asymptotically stable in $\mathfrak{T}$.

5.4.3. The CPE Point

In this section, we will investigate whether or not there is a CPE point, which will be represented by the notation $(S^*, R^*, C^*, P^*, T^*, I^*)$. Consequently, the second equilibrium point is the CPE point, which is found as follows:

$$
\begin{aligned}
v\Pi + c\alpha T - (\omega C + \mu) S &= 0,\\
\omega CS + f_2 RC - k_1 R &= 0,\\
f_1 pR - (f_2 R + k_2) C &= 0,\\
\gamma C - k_3 P &= 0,\\
\theta P + (1 - f_1) pR - k_4 T &= 0,\\
(1 - c) \alpha T + (1 - v) \Pi - \mu I &= 0,
\end{aligned}
$$

where $k_3 = \theta + \mu$ and $k_4 = \alpha + \mu$. As a result, we may draw the following conclusion:

Theorem 5.3: *The unique CPE point for the system* (5.7) *is given by*

$$
\begin{aligned}
S^* &= \frac{k_1 k_2 (\phi_k - k_3 \phi_c) - p f_1 f_2 k_2 k_3 (\alpha + \mu) (k_1 k_2 \mu - v p f_1 \Pi)}{\omega p f_1 (\phi_k - k_3 (f_2 (\alpha + \mu) (k_1 k_2 \mu - v p f_1 \Pi) - \phi_c))},\\
R^* &= \frac{k_1 k_2 (\alpha + \mu) (k_1 k_2 \mu - v p f_1 \Pi)}{\phi_k - k_3 (\phi_c + f_2 (\alpha + \mu) (k_1 k_2 \mu - v p f_1 \Pi))},\\
C^* &= \frac{p f_1 k_3 (\alpha + \mu) (k_1 k_2 \mu - v p f_1 \Pi)}{\phi_k - k_3 \phi_c}, \qquad P^* = \frac{\gamma p f_1 k_3 (\alpha + \mu) (k_1 k_2 \mu - v p f_1 \Pi)}{k_3 (\phi_k - k_3 \phi_c)},\\
T^* &= \frac{\begin{array}{c}(k_1 k_2 \mu - v p f_1 \Pi) (p f_1 ((\phi_k - k_3 \phi_c) - f_2 k_3 (\alpha + \mu)\\ (k_1 k_2 \mu - v p f_1 \Pi)) \theta \gamma + (1 - f_1) p k_2 k_3)\end{array}}{(\phi_k - k_3 \phi_c) ((\phi_k - k_3 \phi_c) - f_2 k_3 (\alpha + \mu) (k_1 k_2 \mu - v p f_1 \Pi))},\\
I^* &= \frac{\begin{array}{c}(1 - c) k_2 \phi_h + c \omega p f_1 \Pi (1 - v) (\phi_k - k_3 \phi_c)\\ ((\phi_k - k_3 \phi_c) - p f_2 k_3 (\alpha + \mu) (k_1 k_2 \mu - v p f_1 \Pi))\end{array}}{c \omega \mu p f_1 (\phi_k - k_3 \phi_c) ((\phi_k - k_3 \phi_c) - p f_2 k_3 (\alpha + \mu) (k_1 k_2 \mu - v p f_1 \Pi))},
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_c &= (\alpha + \mu) (f_2 (k_1 k_2 \mu - v p f_1 \Pi) + p f_1 (k_1 k_2 \omega + v (f_2 \Pi - k_2 \mu))),\\
\phi_k &= ((f_1 (\theta \gamma - k_2 k_3) + k_2 k_3) p) c \omega \alpha p f_1,\\
\phi_h &= (k_1 (\phi_k - k_3 \phi_c) - p f_1 f_2 k_3 (\alpha + \mu) (k_1 k_2 \mu - v p f_1 \Pi)) (\omega p f_1 k_3 (\alpha + \mu)\\
&\quad (k_1 k_2 \mu - v p f_1 \Pi) + \mu (\phi_k - k_3 \phi_c)) - v p f_1 \Pi (\phi_k - k_3 \phi_c) ((\phi_k - k_3 \phi_c)\\
&\quad - f_2 k_3 (\alpha + \mu) (k_1 k_2 \mu - v p f_1 \Pi)),
\end{aligned}
$$

5.4.3.1. Global Stability Analysis

Theorem 5.4: *The system* (5.7) *does not contain any periodic orbits.*

Proof. We use Dulac's criteria to get the desired results. Well, so let's say $X = (S, R, C, P, T, I)$. Using Dulac's function

$$G = \frac{1}{SR},$$

we are able to obtain

$$\begin{aligned}
G\frac{dS}{dt} &= \frac{v\Pi}{SR} + \frac{c\alpha T}{SR} - \frac{(\omega C + \mu)}{R},\\
G\frac{dR}{dt} &= \frac{\omega C}{R} + \frac{f_2 C}{S} - \frac{k_1}{S},\\
G\frac{dC}{dt} &= \frac{f_1 p}{S} - \frac{(f_2 R + k_2)\, C}{SR},\\
G\frac{dP}{dt} &= \frac{\gamma C}{SR} - \frac{k_3 P}{SR},\\
G\frac{dT}{dt} &= \frac{\theta P}{SR} + \frac{(1-f_1)\, p}{S} - \frac{k_4 T}{SR},\\
G\frac{dI}{dt} &= \frac{(1-c)\,\alpha T}{SR} + \frac{(1-v)\,\Pi}{SR} - \frac{\mu I}{SR},
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{dGX}{dt} &= \frac{\partial}{\partial S}\left(G\frac{dS}{dt}\right) + \frac{\partial}{\partial R}\left(G\frac{dR}{dt}\right) + \frac{\partial}{\partial C}\left(G\frac{dC}{dt}\right) + \frac{\partial}{\partial P}\left(G\frac{dP}{dt}\right) + \frac{\partial}{\partial T}\left(G\frac{dT}{dt}\right)\\
&\quad + \frac{\partial}{\partial I}\left(G\frac{dI}{dt}\right),\\
&= \frac{\partial}{\partial S}\left(\frac{v\Pi}{SR} + \frac{c\alpha T}{SR} - \frac{(\omega C + \mu)}{R}\right) + \frac{\partial}{\partial R}\left(\frac{\omega C}{R} + \frac{f_2 C}{S} - \frac{k_1}{S}\right) + \frac{\partial}{\partial C}\left(\frac{f_1 p}{S} - \right.\\
&\qquad \left.\frac{(f_2 R + k_2)\, C}{SR}\right) + \frac{\partial}{\partial P}\left(\frac{\gamma C}{SR} - \frac{k_3 P}{SR}\right) + \frac{\partial}{\partial T}\left(\frac{\theta P}{SR} + \frac{(1-f_1)\, p}{S} - \frac{k_4 T}{SR}\right) +\\
&\qquad \frac{\partial}{\partial I}\left(\frac{(1-c)\,\alpha T}{SR} + \frac{(1-v)\,\Pi}{SR} - \frac{\mu I}{SR}\right),\\
&= -\frac{v\Pi}{S^2 R} - \frac{c\alpha T}{S^2 R} - \frac{\omega C}{R^2} - \frac{f_2 R + k_2}{SR} - \frac{k_3}{SR} - \frac{k_4}{SR} - \frac{\mu}{SR},\\
&= -\left(\frac{v\Pi + c\alpha T}{S^2 R} + \frac{\omega C}{R^2} + \frac{f_2 R + k_2 + k_3 + k_4 + \mu}{SR}\right),\\
&< 0,
\end{aligned}$$

Hence, a periodic orbit cannot exist in the system (5.7). Finally, the proof is done with it.

The Poincaré-Bendixson theory states that if $\mathfrak{B}$ is positively consistent, all solutions to the system (5.7) originate from $\mathfrak{B}$ and remain there for all t. As a final step in this discussion, consider the following theorem.

Theorem 5.5: *For CPE point the system* (5.7) *is globally asymptotically stable if* $\mathcal{R}_0 > 1$*, and unstable otherwise.*

5.5. Preliminaries

For the convenience of our readers, we briefly go over a few key concepts from fractal-fractional calculus in this section. More information on this innovative use of calculus can be found in [4].

Definition 5.1: Suppose that $y(t)$ be continuous and fractal differentiable on (a, b) with order τ then the fractal–fractional derivative of $y(t)$ with order ϵ in the Riemann–Liouville sense having power law type kernel is defined as follows:

$$^{FFP}D_{0,t}^{\epsilon,\tau}\left(y\left(t\right)\right) = \frac{1}{\Gamma\left(m-\epsilon\right)}\frac{d}{dt^{\tau}}\int_{0}^{t}\left(t-s\right)^{m-\epsilon-1}y\left(s\right)ds, \tag{5.17}$$

where $m-1 < \epsilon, \tau \leq m \in \mathbb{N}$ and $\frac{dy(s)}{ds^{\tau}} = \lim_{t\to s}\frac{y(t)-y(s)}{t^{\tau}-s^{\tau}}$.

Definition 5.2: Suppose that $y(t)$ be continuous on an open interval (a, b) then the fractal–fractional integral of $y(t)$ with order ϵ having power law type kernel is defined as follows:

$$^{FFP}J_{0,t}^{\epsilon,\tau}\left(y\left(t\right)\right) = \frac{\tau}{\Gamma\left(\epsilon\right)}\int_{0}^{t}\left(t-s\right)^{\epsilon-1}s^{\tau-1}y\left(s\right)ds. \tag{5.18}$$

are simply referred to fractal-fractional integral operators. As a result, the suggested fractal-fractional operator-based nonlinear fractional model is as follows:

$$\begin{cases} {}^{FFP}D_{0,t}^{\epsilon,\tau}(S(t)) = v\Pi + c\alpha T - (\omega C + \mu) S, \\ {}^{FFP}D_{0,t}^{\epsilon,\tau}(R(t)) = \omega CS + f_2 RC - (p + \mu) R, \\ {}^{FFP}D_{0,t}^{\epsilon,\tau}(C(t)) = f_1 pR - (f_2 R + \gamma + \mu) C, \\ {}^{FFP}D_{0,t}^{\epsilon,\tau}(P(t)) = \gamma C - (\theta + \mu) P, \\ {}^{FFP}D_{0,t}^{\epsilon,\tau}(T(t)) = \theta P + (1 - f_1) pR - (\alpha + \mu) T, \\ {}^{FFP}D_{0,t}^{\epsilon,\tau}(I(t)) = (1 - c)\alpha T + (1 - v)\Pi - \mu I, \end{cases} \tag{5.19}$$

where ${}^{FFP}D_{0,t}^{\alpha,\tau}(\cdot)$ is the fractal-fractional derivative of order $0 < \epsilon \leq 1$ and fractal dimension $0 < \tau \leq 1$ in caputo sense with power law and the variables with the appropriate initial conditions are supposed to be non-negative.

Lemma 5.2: *Let f is continuous on any open interval (a, b), then the following fractal-fractional derivative*

$$ {}^{FFP}D_{0,t}^{\epsilon,\tau}(f(t)) = Y(t) \tag{5.20}$$

has a unique solution

$$f(t) = f(0) + \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t - s)^{\epsilon - 1} s^{\tau - 1} y(s)\, ds. \tag{5.21}$$

For $Z = C(\mathfrak{T}, \mathbb{R}^6)$, under the norm for $0 \leq t \leq T < \infty$ the Banach space can be represented by $B = Z \times Z \times Z \times Z \times Z \times Z$ under the norm given by

$$\|W\| = \sup_{t \in \mathfrak{T}} |W(t)|, \qquad \text{for} \quad W \in Z,$$

where $|W(t)| = |S(t) + R(t) + C(t) + P(t) + T(t) + I(t)\|$, and $S, R, C, P, T, I \in C(\mathfrak{T}, \mathbb{R})$.

Considered here is the model (5.7), in which the fractional differential operators have a caputo fractal fractional form. This means that the initial (say) model (5.7) may be transformed into the following

$$ {}^{CF}D_{0,t}^{\epsilon,\tau}\{S(t)\} = \tau t^{\tau - 1} \Phi_1(t, S(t)),$$

$$
\begin{aligned}
{}^{CF}D_{0,t}^{\epsilon,\tau}\left\{R\left(t\right)\right\} &= \tau t^{\tau-1}\Phi_{2}\left(t,R\left(t\right)\right),\\
{}^{CF}D_{0,t}^{\epsilon,\tau}\left\{C\left(t\right)\right\} &= \tau t^{\tau-1}\Phi_{3}\left(t,C\left(t\right)\right),\\
{}^{CF}D_{0,t}^{\epsilon,\tau}\left\{P\left(t\right)\right\} &= \tau t^{\tau-1}\Phi_{4}\left(t,P\left(t\right)\right),\\
{}^{CF}D_{0,t}^{\epsilon,\tau}\left\{T\left(t\right)\right\} &= \tau t^{\tau-1}\Phi_{5}\left(t,T\left(t\right)\right),\\
{}^{CF}D_{0,t}^{\epsilon,\tau}\left\{I\left(t\right)\right\} &= \tau t^{\tau-1}\Phi_{6}\left(t,I\left(t\right)\right),
\end{aligned} \tag{5.22}
$$

where

$$
\begin{aligned}
\Phi_{1}\left(t,I\left(t\right)\right) &= v\Pi+c\alpha T-\left(\omega C+\mu\right)S,\\
\Phi_{2}\left(t,I\left(t\right)\right) &= \omega CS+f_{2}RC-\left(p+\mu\right)R,\\
\Phi_{3}\left(t,I\left(t\right)\right) &= f_{1}pR-\left(f_{2}R+\gamma+\mu\right)C,\\
\Phi_{4}\left(t,I\left(t\right)\right) &= \gamma C-\left(\theta+\mu\right)P,\\
\Phi_{5}\left(t,I\left(t\right)\right) &= \theta P+\left(1-f_{1}\right)pR-\left(\alpha+\mu\right)T,\\
\Phi_{6}\left(t,I\left(t\right)\right) &= \left(1-c\right)\alpha T+\left(1-v\right)\Pi-\mu I,
\end{aligned}
$$

Applying the caputo integral, we obtain

$$
\begin{aligned}
S\left(t\right) &= S\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_{0}^{t}\left(t-\xi\right)^{\epsilon-1}\xi^{\tau-1}\Phi_{1}\left(\xi,S\left(\xi\right)\right)d\xi,\\
R\left(t\right) &= R\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_{0}^{t}\left(t-\xi\right)^{\epsilon-1}\xi^{\tau-1}\Phi_{2}\left(\xi,R\left(\xi\right)\right)d\xi,\\
C\left(t\right) &= C\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_{0}^{t}\left(t-\xi\right)^{\epsilon-1}\xi^{\tau-1}\Phi_{3}\left(\xi,C\left(\xi\right)\right)d\xi,\\
P\left(t\right) &= P\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_{0}^{t}\left(t-\xi\right)^{\epsilon-1}\xi^{\tau-1}\Phi_{4}\left(\xi,P\left(\xi\right)\right)d\xi,\\
T\left(t\right) &= T\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_{0}^{t}\left(t-\xi\right)^{\epsilon-1}\xi^{\tau-1}\Phi_{5}\left(\xi,T\left(\xi\right)\right)d\xi,\\
I\left(t\right) &= I\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_{0}^{t}\left(t-\xi\right)^{\epsilon-1}\xi^{\tau-1}\Phi_{6}\left(\xi,I\left(\xi\right)\right)d\xi,
\end{aligned}
$$

5.5.1. Existence and uniqueness under Caputo (power law) case

To illustrate the qualitative characteristics of the solution for model (5.22), we used the Picard-Lindel'f method and the fixed point theory. We'll start by trying to rewrite the model (5.22), which now looks like this:

$$\begin{cases} {}^{FFP}D_{0,t}^{\epsilon,\tau} W(t) = \Phi(t, W(t)), & 0 < \epsilon, \quad \tau \leq 1, \\ W(0) = W_0 \geq 0, & t \in [0,T] \quad \text{and} \quad (T < \infty) \end{cases} \tag{5.23}$$

where

$$W(t) = \begin{pmatrix} S(t) \\ R(t) \\ C(t) \\ P(t) \\ T(t) \\ I(t) \end{pmatrix}, \qquad W(0) = \begin{pmatrix} S(0) = S_0 \\ R(0) = R_0 \\ C(0) = C_0 \\ P(0) = P_0 \\ T(0) = T_0 \\ I(0) = I_0 \end{pmatrix} = W_0, \qquad \Phi(t, W(t)) = \begin{pmatrix} \Phi_1(t, S(t)) \\ \Phi_2(t, R(t)) \\ \Phi_3(t, C(t)) \\ \Phi_4(t, P(t)) \\ \Phi_5(t, T(t)) \\ \Phi_6(t, I(t)) \end{pmatrix},$$

In the view of Lemma 5.2, the system (5.23) yields

$$W(t) = W(0) + \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} \Phi(\xi, W(\xi)) \, d\xi.$$

Furthermore, Φ satisfies

$$\|\Phi(\xi, W_1(\xi)) - \Phi(\xi, W_2(\xi))\| \leq L_\Phi |W_1(\xi) - W_2(\xi)|, \qquad L_\Phi > 0. \tag{5.24}$$

We define again the following mapping

$$\begin{aligned} \Lambda W(t) &= W(0) + \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} \Phi(\xi, W(\xi)) \, d\xi, \\ \|\Lambda W(t) - W(0)\| &< \Upsilon \Rightarrow \delta, \end{aligned}$$

which is the $\sup_{\prod_b^a}|\Phi| = \delta$ and $\delta < \frac{\Upsilon\Gamma(\epsilon)}{\epsilon\tau a^{\tau+\epsilon-3}\mathcal{B}(\epsilon,\tau)}$. Also by taking $W_1, W_2 \in C\left[I_n(t_n), A_b(t_n)\right]$, we evaluate the following

$$\|\Lambda W_1 - \Lambda W_2\| < \frac{\epsilon L\tau}{\Gamma(\epsilon)}\mathcal{B}(\epsilon,\tau)a^{\tau+\epsilon-3} \tag{5.25}$$

Therefore, the contractive characteristic can be achieved if

$$L < \frac{\Gamma(\epsilon)}{\epsilon\tau a^{\tau+\epsilon-3}\mathcal{B}(\epsilon,\tau)} \tag{5.26}$$

Consequently, if the preceding condition holds and

$$K < \frac{\Upsilon\Gamma(v)}{\epsilon\tau a^{\tau+\epsilon-3}\mathcal{B}(\epsilon,\tau)} \tag{5.27}$$

If this is the case, then we have proved that there is a unique solution to the problem under the power law.

Definition 5.3: Suppose that $\Phi \in C\left(\mathfrak{T}\times\mathbb{R}^6,\ \mathbb{R}\right)$, then for $0 < \epsilon, \tau \le 1$ the system (5.20) is said to be Hyers–Ulam stable if there exists $\delta, C_\Phi > 0$ such that, for each solution $\bar{Y} \in Z$ satisfies

$$\left\|{}^{FFP}D_{0,t}^{\epsilon,\tau}\bar{Y}(t) - \Phi\left(t,\bar{Y}(t)\right)\right\| \le \delta, \qquad \forall \quad t \in \mathfrak{F}, \tag{5.28}$$

there exist a solution $Y \in Z$ of (5.20) with

$$\left|\bar{Y}(t) - Y(t)\right| \le \delta C_\Phi, \qquad \forall \quad t \in \mathfrak{F}, \tag{5.29}$$

where $\delta = \max(\delta_j)^T$ and $C_\Phi = \max\left(C_{\Phi_j}\right), \quad j = 1,2,3,4,5,6.$

Definition 5.4: Suppose $F \in C(\mathfrak{T},\ \mathbb{R}^+)$ and $\Phi \in C\left(\mathfrak{T}\times\mathbb{R}^6,\ \mathbb{R}\right)$, then for $0 < \epsilon, \tau \le 1$ the system (5.20) is said to be Hyers–Ulam–Rassias stable if there exist $C_{\Phi,F} > 0$ such that, for each solution $\bar{Y} \in Z$ satisfies

$$\left\|{}^{FFP}D_{0,t}^{\epsilon,\tau}\bar{Y}(t) - \Phi\left(t,\bar{Y}(t)\right)\right\| \le \delta G(t)^T, \qquad \forall \quad t \in \mathfrak{F}, \tag{5.30}$$

there exist a solution $Y \in Z$ of (5.20) with

$$\left|\bar{Y}(t) - Y(t)\right| \leq \delta C_{\Phi,G} G(t), \qquad \forall \quad t \in \mathfrak{F}, \tag{5.31}$$

where $C_{\Phi,G} = \max\left(C_{\Phi_j,G_j}\right)^T$ and $G = \max(G_j)^T$, $\quad j = 1,2,3,4,5,6.$

Lemma 5.3: *Let $\delta > 0$, and if there exist a function $g(t) \in Z$, which satisfies*

(i) $f(t) \leq \delta, \qquad \forall \quad t \in \mathfrak{F}$

(ii) ${}^{FFP}D_{0,t}^{\epsilon,\tau}\bar{Y}(t) = \Phi\left(t, \bar{Y}(t)\right) + f(t), \quad \forall t \in \mathfrak{F} \quad f = \max(f_j)^T, \quad j = 1,2,3,4,5,6.$

then the function $\bar{Y} \in Z$ satisfies (5.28).

Lemma 5.4: *Let $F \in C(\mathfrak{T}, \mathbb{R})$, and if there exist a function $g^*(t) \in Z$, which satisfies*

(i) $f^*(t) \leq \delta, \qquad \forall \quad t \in \mathfrak{F}$

(ii) ${}^{FFP}D_{0,t}^{\epsilon,\tau}\bar{Y}(t) = \Phi\left(t, \bar{Y}(t)\right) + f^*(t), \quad \forall t \in \mathfrak{F} \quad f^* = \max\left(f_j^*\right)^T, \quad j = 1,2,3,4,5,6.$

then the function $\bar{Y} \in Z$ satisfies (5.30).

5.5.2. Hyers-Ulam Stability

Here we examine the Hyers-Ulam and Hyers-Ulam-Rassias stability of the suggested model (5.23). The importance of stability in approximating a solution motivates us to perform a nonlinear functional analysis of the many forms of stability in the present model.

Lemma 5.5: *If W satisfies the integral inequality given by* (5.32) *then $W \in Z$ satisfies by the system* (5.23)

$$\left|W(t) - W_0 - \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} \Phi(\xi, W(\xi))\, d\xi\right| \leq \Upsilon\delta, \tag{5.32}$$

where $\Upsilon := \left(\frac{\mathcal{B}(\epsilon,\tau)\tau}{\Gamma(\epsilon)}\right)$.

Proof. According to Theorem 5.27 with (ii) of Lemma 5.3, the system exist

$${}^{FFP}D_{0,t}^{\alpha,\tau} W(t) = \Phi(t, W(t)) + f(t), \qquad t \in \mathfrak{F}$$

$$W(0) = W_0 \geq 0,$$

and has a unique solution

$$W(t) = W_0 + \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} (\Phi(\xi, W(\xi)) + f(\xi))\, d\xi.$$

It continues to follow from (i) of Lemma 5.3 that

$$\begin{aligned}
&\left| W(t) - W_0 - \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} \Phi(\xi, W(\xi))\, d\xi \right|, \\
=& \sup_{t\in\mathfrak{T}} \left| \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} f(\xi)\, d\xi \right|, \\
\leq& \sup_{t\in\mathfrak{T}} \left(\frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} |f(\xi)|\, d\xi \right), \\
\leq& \sup_{t\in\mathfrak{T}} \left(\frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} d\xi \right) \delta, \\
\leq& \frac{\mathcal{B}(\epsilon, \tau)\tau\delta}{\Gamma(\epsilon)}, \\
\leq& \Upsilon\delta.
\end{aligned}$$

Theorem 5.6:

Suppose that $\Phi \in C\left(\mathfrak{T} \times \mathbb{R}^6,\ \mathbb{R}\right)$ *and the system* (5.24) *satisfies the condition* $1 - \Upsilon L_\Phi > 0$*, then the system* (5.23) *is said to be Hyers-Ulam stable.*

Proof. Let $W \in Z$ be a unique solution for the system (5.23), and $\bar{W} \in Z$ satisfies by (5.30). Then, by taking the Lemma 5.3 into consideration, for any $\delta > 0$, $t \in \mathfrak{T}$ we have

$$\begin{aligned}
\left\| \bar{W} - W \right\| =& \sup_{t\in\mathfrak{T}} \left| \bar{W} - W \right|, \\
=& \sup_{t\in\mathfrak{T}} \left| \bar{W} - W_0 - \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} \Phi(\xi, W(\xi))\, d\xi \right|, \\
\leq& \sup_{t\in\mathfrak{T}} \left| \bar{W} - W_0 - \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} \Phi\left(\xi, \bar{W}(\xi)\right) d\xi \right| \\
&+ \sup_{t\in\mathfrak{T}} \left| \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} \left(\Phi\left(\xi, \bar{W}(\xi)\right) - \Phi(\xi, W(\xi)) \right) d\xi \right|, \\
\leq& \Upsilon\delta + \sup_{t\in\mathfrak{T}} \left(\frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} \left(\Phi\left(\xi, \bar{W}(\xi)\right) - \Phi(\xi, W(\xi)) \right) d\xi \right),
\end{aligned}$$

$$
\begin{aligned}
&\leq \Upsilon\delta + \sup_{t\in\mathfrak{F}} \left(\frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} L_\Phi \left| \bar{W}(\xi) - W(\xi) \right| d\xi \right), \\
&\leq \Upsilon\delta + \sup_{t\in\mathfrak{F}} \left(\frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} d\xi \right) L_\Phi \left| \bar{W}(\xi) - W(\xi) \right|, \\
&\leq \Upsilon\delta + \left(\frac{\mathcal{B}(\epsilon,\tau)\tau}{\Gamma(\epsilon)} \right) L_\Phi \left| \bar{W}(\xi) - W(\xi) \right|, \\
&\leq \Upsilon\delta + \Upsilon L_\Phi \left\| \bar{W} - W \right\|,
\end{aligned}
$$

which implies that

$$\left\| \bar{W} - W \right\| \leq C_\Phi \delta$$

where $C_\Phi := \dfrac{\Upsilon}{1-\Upsilon L_\Phi}$.

Theorem 5.7:

Suppose that $\Phi \in C\left(\mathfrak{T} \times \mathbb{R}^6,\ \mathbb{R}\right)$ *satisfies* (5.24) *and* $G \in C\left(\mathfrak{T}, \mathbb{R}^+\right)$ *be an increasing function implies that the system* (5.23) *is Hyers-Ulam–Rassias stable with respect to* G *on* $\mathfrak{T}$, *such that*

$$^{FFP}J_{0,t}^{\epsilon,\tau} G(t) \leq C_F G(t), \qquad C_F > 0. \tag{5.33}$$

provided that $1 - \Upsilon L_\Phi > 0$.

Proof. Let us assume that there exists exactly one solution $W \in Z$ to the system (5.23). So, considering Theorem 5.27, we arrive at

$$W(t) = W(0) + \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} \Phi(\xi, W(\xi))\, d\xi.$$

On consideration of (5.28) and (5.30), we have

$$\left| W(t) - W_0 - \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} \Phi(\xi, W(\xi))\, d\xi \right| \leq \delta C_G G(t)$$

Hence

$$\left| \bar{W} - W \right| = \sup_{t\in\mathfrak{F}} \left| \bar{W} - W_0 - \frac{\tau}{\Gamma(\epsilon)} \int_0^t (t-\xi)^{\epsilon-1} \xi^{\tau-1} \Phi(\xi, W(\xi))\, d\xi \right|,$$

$$
\begin{aligned}
&\leq \sup_{t\in\mathfrak{F}}\left|\bar{W}-W_0-\frac{\tau}{\Gamma(\epsilon)}\int_0^t (t-\xi)^{\epsilon-1}\xi^{\tau-1}\Phi\left(\xi,\bar{W}(\xi)\right)d\xi\right| \\
&\quad +\sup_{t\in\mathfrak{F}}\left|\frac{\tau}{\Gamma(\epsilon)}\int_0^t (t-\xi)^{\epsilon-1}\xi^{\tau-1}\left(\Phi\left(\xi,\bar{W}(\xi)\right)-\Phi\left(\xi,W(\xi)\right)\right)d\xi\right|, \\
&\leq \delta C_G G(t)+\sup_{t\in\mathfrak{F}}\left(\frac{\tau}{\Gamma(\epsilon)}\int_0^t (t-\xi)^{\epsilon-1}\xi^{\tau-1}\left(\Phi\left(\xi,\bar{W}(\xi)\right)-\Phi\left(\xi,W(\xi)\right)\right)d\xi\right), \\
&\leq \delta C_G G(t)+\sup_{t\in\mathfrak{F}}\left(\frac{\tau}{\Gamma(\epsilon)}\int_0^t (t-\xi)^{\epsilon-1}\xi^{\tau-1}L_\Phi\left|\bar{W}(\xi)-W(\xi)\right|d\xi\right), \\
&\leq \delta C_G G(t)+\sup_{t\in\mathfrak{F}}\left(\frac{\tau}{\Gamma(\epsilon)}\int_0^t (t-\xi)^{\epsilon-1}\xi^{\tau-1}d\xi\right)L_\Phi\left|\bar{W}(\xi)-W(\xi)\right|, \\
&\leq \delta C_G G(t)+\left(\frac{\mathcal{B}(\epsilon,\tau)\tau}{\Gamma(\epsilon)}\right)L_\Phi\left|\bar{W}(\xi)-W(\xi)\right|, \\
&\leq \delta C_G G(t)+\Upsilon L_\Phi\left\|\bar{W}-W\right\|,
\end{aligned}
$$

which implies that

$$\left\|\bar{W}-W\right\| \leq \delta C_{\Phi,G}G(t) \tag{5.34}$$

where $C_{\Phi,G} := \dfrac{C_G}{1-\Upsilon L_\Phi}$.

5.6. Numerical scheme

In this part, we'll talk about the fractal fractional model (ref:sys:model FFP), which Caputo defines as the differentiation operator. Here, we add the fractal dimension to the aforementioned mathematical model sys:model FFP in order to capture self-similarities. It is possible to modify the aforementioned equations to apply to the Volterra type integral because the fractional integral is differentiable. Thus, the Riemann-Liouville fractal-fractional derivative is written as

$$\frac{1}{\Gamma(1-\epsilon)}\frac{d}{dt}\int_0^t (t-\tau)^{\epsilon}f(\tau)\,d\tau\frac{1}{\tau t^{\tau-1}},$$

such that system sys:model FFP becomes

$${}^{RL}D_{0,t}^{\epsilon,\tau}\left(S(t)\right) = \tau t^{\tau-1}\left[v\Pi+c\alpha T-(\omega C+\mu)S\right],$$

$$\begin{aligned}
{}^{RL}D_{0,t}^{\epsilon,\tau}\left(R\left(t\right)\right) &= \tau t^{\tau-1}\left[\omega CS+f_2RC-\left(p+\mu\right)R\right],\\
{}^{RL}D_{0,t}^{\epsilon,\tau}\left(C\left(t\right)\right) &= \tau t^{\tau-1}\left[f_1pR-\left(f_2R+\gamma+\mu\right)C\right],\\
{}^{RL}D_{0,t}^{\epsilon,\tau}\left(P\left(t\right)\right) &= \tau t^{\tau-1}\left[\gamma C-\left(\theta+\mu\right)P\right],\\
{}^{RL}D_{0,t}^{\epsilon,\tau}\left(T\left(t\right)\right) &= \tau t^{\tau-1}\left[\theta P+\left(1-f_1\right)pR-\left(\alpha+\mu\right)T\right],\\
{}^{RL}D_{0,t}^{\epsilon,\tau}\left(I\left(t\right)\right) &= \tau t^{\tau-1}\left[\left(1-c\right)\alpha T+\left(1-v\right)\Pi-\mu I\right],
\end{aligned}$$

Now, the Caputo derivative stands in for the Riemann-Liouville derivative such that the integer-order beginning conditions can be used. Then we integrate both sides using the Riemann-Liouville fractional integral, leading to the following

$$\begin{aligned}
S\left(t\right) &= S\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^t\xi^{\tau-1}\left(t-\xi\right)^{\epsilon-1}\Phi_1\left(\xi,S\left(\xi\right)\right)d\xi,\\
R\left(t\right) &= R\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^t\xi^{\tau-1}\left(t-\xi\right)^{\epsilon-1}\Phi_2\left(\xi,R\left(\xi\right)\right)d\xi,\\
C\left(t\right) &= C\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^t\xi^{\tau-1}\left(t-\xi\right)^{\epsilon-1}\Phi_3\left(\xi,C\left(\xi\right)\right)d\xi,\\
P\left(t\right) &= P\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^t\xi^{\tau-1}\left(t-\xi\right)^{\epsilon-1}\Phi_4\left(\xi,P\left(\xi\right)\right)d\xi,\\
T\left(t\right) &= T\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^t\xi^{\tau-1}\left(t-\xi\right)^{\epsilon-1}\Phi_5\left(\xi,T\left(\xi\right)\right)d\xi,\\
I\left(t\right) &= I\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^t\xi^{\tau-1}\left(t-\xi\right)^{\epsilon-1}\Phi_6\left(\xi,I\left(\xi\right)\right)d\xi,
\end{aligned}$$

Here, we give a thorough explanation of how the numerical approach was developed. Thus, at t_{n+1} we have

$$\begin{aligned}
S\left(t_{n+1}\right) &= S\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^{t_{n+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_1\left(\xi,S\left(\xi\right)\right)d\xi,\\
R\left(t_{n+1}\right) &= R\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^{t_{n+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_2\left(\xi,R\left(\xi\right)\right)d\xi,\\
C\left(t_{n+1}\right) &= C\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^{t_{n+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_3\left(\xi,C\left(\xi\right)\right)d\xi,\\
P\left(t_{n+1}\right) &= P\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^{t_{n+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_4\left(\xi,P\left(\xi\right)\right)d\xi,\\
T\left(t_{n+1}\right) &= T\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^{t_{n+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_5\left(\xi,T\left(\xi\right)\right)d\xi,\\
I\left(t_{n+1}\right) &= I\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\int_0^{t_{n+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_6\left(\xi,I\left(\xi\right)\right)d\xi,
\end{aligned}$$

Taking the difference between the consecutive terms, we obtain

$$\begin{aligned} S\left(t_{n+1}\right) &= S\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_1\left(\xi,S\left(\xi\right)\right)d\xi,\\ R\left(t_{n+1}\right) &= R\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_2\left(\xi,R\left(\xi\right)\right)d\xi,\\ C\left(t_{n+1}\right) &= C\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_3\left(\xi,C\left(\xi\right)\right)d\xi,\\ P\left(t_{n+1}\right) &= P\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_4\left(\xi,P\left(\xi\right)\right)d\xi,\\ T\left(t_{n+1}\right) &= T\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_5\left(\xi,T\left(\xi\right)\right)d\xi,\\ I\left(t_{n+1}\right) &= I\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\Phi_6\left(\xi,I\left(\xi\right)\right)d\xi, \end{aligned}$$

If the function $\xi^{\tau-1}\Phi\left(\xi,W\left(\xi\right)\right)$ is approximated over the finite interval $[t_j,t_{j+1}]$ for $j=1,2,3,4,5,6$ and $W=[S,R,C,P,T,I]^T$ using the Lagrangian piece-wise interpolation such that

$$\begin{aligned} \mathcal{P}_j^1\left(\xi\right) &= \frac{\xi-t_{j-1}}{t_j-t_{j-1}}t_j^{\tau-1}\Phi_1\left(t_j,S\left(t_j\right)\right)-\frac{\xi-t_j}{t_j-t_{j-1}}t_{j-1}^{\tau-1}\Phi_1\left(t_{j-1},S\left(t_{j-1}\right)\right),\\ \mathcal{P}_j^2\left(\xi\right) &= \frac{\xi-t_{j-1}}{t_j-t_{j-1}}t_j^{\tau-1}\Phi_2\left(t_j,R\left(t_j\right)\right)-\frac{\xi-t_j}{t_j-t_{j-1}}t_{j-1}^{\tau-1}\Phi_2\left(t_{j-1},R\left(t_{j-1}\right)\right),\\ \mathcal{P}_j^3\left(\xi\right) &= \frac{\xi-t_{j-1}}{t_j-t_{j-1}}t_j^{\tau-1}\Phi_3\left(t_j,C\left(t_j\right)\right)-\frac{\xi-t_j}{t_j-t_{j-1}}t_{j-1}^{\tau-1}\Phi_3\left(t_{j-1},C\left(t_{j-1}\right)\right),\\ \mathcal{P}_j^4\left(\xi\right) &= \frac{\xi-t_{j-1}}{t_j-t_{j-1}}t_j^{\tau-1}\Phi_4\left(t_j,P\left(t_j\right)\right)-\frac{\xi-t_j}{t_j-t_{j-1}}t_{j-1}^{\tau-1}\Phi_4\left(t_{j-1},P\left(t_{j-1}\right)\right),\\ \mathcal{P}_j^5\left(\xi\right) &= \frac{\xi-t_{j-1}}{t_j-t_{j-1}}t_j^{\tau-1}\Phi_5\left(t_j,T\left(t_j\right)\right)-\frac{\xi-t_j}{t_j-t_{j-1}}t_{j-1}^{\tau-1}\Phi_5\left(t_{j-1},T\left(t_{j-1}\right)\right),\\ \mathcal{P}_j^6\left(\xi\right) &= \frac{\xi-t_{j-1}}{t_j-t_{j-1}}t_j^{\tau-1}\Phi_6\left(t_j,I\left(t_j\right)\right)-\frac{\xi-t_j}{t_j-t_{j-1}}t_{j-1}^{\tau-1}\Phi_6\left(t_{j-1},I\left(t_{j-1}\right)\right), \end{aligned}$$

Thus, we obtain

$$\begin{aligned} S\left(t_{n+1}\right) &= S\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\mathcal{P}_k^1\left(\xi\right)d\xi,\\ R\left(t_{n+1}\right) &= R\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\mathcal{P}_k^2\left(\xi\right)d\xi,\\ C\left(t_{n+1}\right) &= C\left(0\right)+\frac{\tau}{\Gamma\left(\epsilon\right)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1}\mathcal{P}_k^3\left(\xi\right)d\xi, \end{aligned}$$

$$
\begin{aligned}
P\left(t_{n+1}\right) &= P(0)+\frac{\tau}{\Gamma(\epsilon)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1} \mathcal{P}_k^4(\xi)\, d\xi, \\
T\left(t_{n+1}\right) &= T(0)+\frac{\tau}{\Gamma(\epsilon)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1} \mathcal{P}_k^5(\xi)\, d\xi, \\
I\left(t_{n+1}\right) &= I(0)+\frac{\tau}{\Gamma(\epsilon)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \xi^{\tau-1}\left(t_{n+1}-\xi\right)^{\epsilon-1} \mathcal{P}_k^6(\xi)\, d\xi,
\end{aligned}
$$

The following numerical scheme is obtained by solving the integrals of the right-hand sides

$$
\begin{aligned}
S\left(t_{n+1}\right) &= S(0)+\frac{\tau(\Delta t)^{\epsilon}}{\Gamma(\epsilon+2)} \sum_{k=0}^{n}\left[t_k^{\tau-1} \Phi_1\left(t_k, S\left(t_k\right)\right) \times\left((n+1-k)^{\epsilon} \times\right.\right. \\
&\quad (n-k+2+\epsilon)-(n-k)^{\epsilon}(n-k+2+2\epsilon))-t_{k-1}^{\tau-1} \times \\
&\quad \left.\Phi_1\left(t_{k-1}, S\left(t_{k-1}\right)\right)\left((n+1-k)^{\epsilon+1}-(n-k)^{\epsilon}(n-k+1+\epsilon)\right)\right], \\
R\left(t_{n+1}\right) &= R(0)+\frac{\tau(\Delta t)^{\epsilon}}{\Gamma(\epsilon+2)} \sum_{k=0}^{n}\left[t_k^{\tau-1} \Phi_2\left(t_k, R\left(t_k\right)\right) \times\left((n+1-k)^{\epsilon} \times\right.\right. \\
&\quad (n-k+2+\epsilon)-(n-k)^{\epsilon}(n-k+2+2\epsilon))-t_{k-1}^{\tau-1} \times \\
&\quad \left.\Phi_2\left(t_{k-1}, R\left(t_{k-1}\right)\right) \times\left((n+1-k)^{\epsilon+1}-(n-k)^{\epsilon}(n-k+1+\epsilon)\right)\right], \\
C\left(t_{n+1}\right) &= C(0)+\frac{\tau(\Delta t)^{\epsilon}}{\Gamma(\epsilon+2)} \sum_{k=0}^{n}\left[t_k^{\tau-1} \Phi_3\left(t_k, C\left(t_k\right)\right) \times\left((n+1-k)^{\epsilon} \times\right.\right. \\
&\quad (n-k+2+\epsilon)-(n-k)^{\epsilon}(n-k+2+2\epsilon))-t_{k-1}^{\tau-1} \times \\
&\quad \left.\Phi_3\left(t_{k-1}, C\left(t_{k-1}\right)\right) \times\left((n+1-k)^{\epsilon+1}-(n-k)^{\epsilon}(n-k+1+\epsilon)\right)\right], \\
P\left(t_{n+1}\right) &= P(0)+\frac{\tau(\Delta t)^{\epsilon}}{\Gamma(\epsilon+2)} \sum_{k=0}^{n}\left[t_k^{\tau-1} \Phi_4\left(t_k, P\left(t_k\right)\right) \times\left((n+1-k)^{\epsilon} \times\right.\right. \\
&\quad (n-k+2+\epsilon)-(n-k)^{\epsilon}(n-k+2+2\epsilon))-t_{k-1}^{\tau-1} \times \\
&\quad \left.\Phi_4\left(t_{k-1}, P\left(t_{k-1}\right)\right) \times\left((n+1-k)^{\epsilon+1}-(n-k)^{\epsilon}(n-k+1+\epsilon)\right)\right], \\
T\left(t_{n+1}\right) &= T(0)+\frac{\tau(\Delta t)^{\epsilon}}{\Gamma(\epsilon+2)} \sum_{k=0}^{n}\left[t_k^{\tau-1} \Phi_5\left(t_k, T\left(t_k\right)\right) \times\left((n+1-k)^{\epsilon} \times\right.\right. \\
&\quad (n-k+2+\epsilon)-(n-k)^{\epsilon}(n-k+2+2\epsilon))-t_{k-1}^{\tau-1} \times \\
&\quad \left.\Phi_5\left(t_{k-1}, T\left(t_{k-1}\right)\right) \times\left((n+1-k)^{\epsilon+1}-(n-k)^{\epsilon}(n-k+1+\epsilon)\right)\right], \\
I\left(t_{n+1}\right) &= I(0)+\frac{\tau(\Delta t)^{\epsilon}}{\Gamma(\epsilon+2)} \sum_{k=0}^{n}\left[t_k^{\tau-1} \Phi_6\left(t_k, I\left(t_k\right)\right) \times\left((n+1-k)^{\epsilon} \times\right.\right. \\
&\quad (n-k+2+\epsilon)-(n-k)^{\epsilon}(n-k+2+2\epsilon))-t_{k-1}^{\tau-1} \times \\
&\quad \left.\Phi_6\left(t_{k-1}, I\left(t_{k-1}\right)\right) \times\left((n+1-k)^{\epsilon+1}-(n-k)^{\epsilon}(n-k+1+\epsilon)\right)\right],
\end{aligned}
$$

5.7. Results and Simulations

The dynamics of the population densities for the proposed model with different fractal-fractional order derivatives are plotted in Fig. 5.2. These simulations provide valuable insights into the effects of varying parameters on population density. By validating the analytic conclusions from previous sections, these numerical results confirm the accuracy and reliability of the model (5.7). The importance of these numerical findings cannot be overstated, as they contribute to a deeper understanding of population dynamics and their relationship with fractal-fractional order derivatives. By exploring a range of parameter values, we can observe how different factors influence population density over time.

In Fig. 5.2a, the population that is exposed to corruption continues to grow, leading to a rapid spread of corruption, as depicted in Fig. 5.2b. Interestingly, the density of the corrupted population decreases with an increase in the fractal-fractional order, as observed in Fig. 5.2c. On the other hand, both the punished and transformed population densities show an increase with the increase in fractal-fractional order, as illustrated in Figs. 5.2d & 5.2e respectively. Moreover, there is a noticeable rise in the immune population density as the fractal-fractional order increases, as evidenced by Fig. 5.2f. This suggests that higher levels of fractal-fractional order may contribute to a more effective immune response against corruption.

The increase in immune individuals is a positive sign, as it indicates that the corruption is losing its grip on the population. The efforts to control and mitigate the spread of the corruption seem to be paying off. However, it is crucial to remain vigilant and continue practicing preventive measures to ensure that the progress made so far is not undone. By maintaining a high population density of immune individuals, we can hope to achieve herd immunity and protect those who are still susceptible. It is crucial to prioritize the implementation of anti-corruption policies and to encourage people to uphold the law in order to further increase immunity levels against corruption. With collective efforts and perseverance, we can overcome this challenging period and restore normalcy in our communities. This can be achieved by implementing strict anti-corruption measures and promoting transparency in all sectors of society. Additionally, educating the public about the detrimental effects of corruption and the importance of reporting any

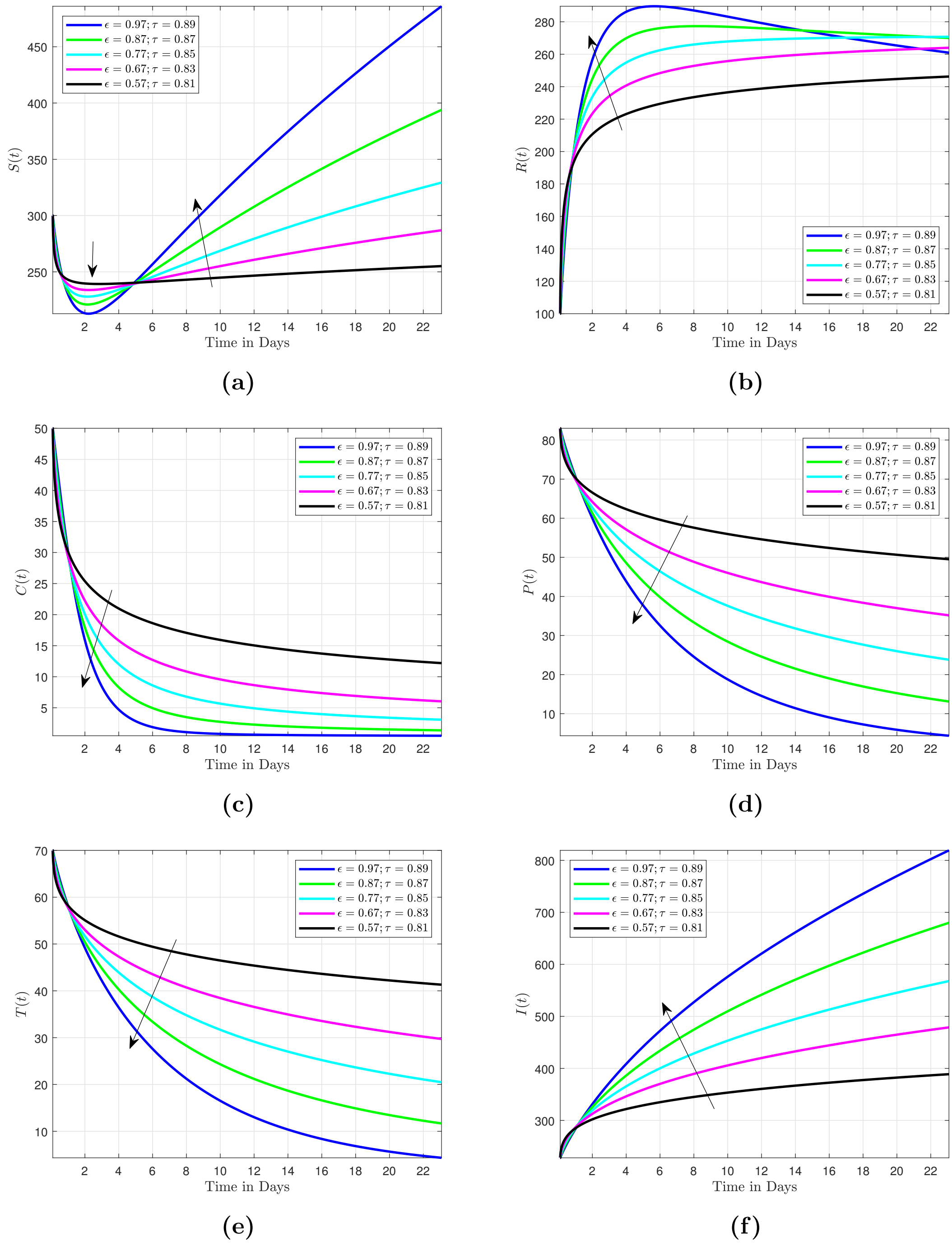


Figure 5.2: Population densities with Atangana Beleanu Caputo derivativs under fractal fractional operator for differet values of ϵ and τ.

suspicious activities can help reduce the contact rate between corrupt individuals and the susceptible population. It is crucial to strengthen law enforcement agencies and ensure that they have the necessary resources to effectively investigate and prosecute corruption cases. Furthermore, fostering a culture of integrity and ethical behavior within institutions can create a deterrent for individuals tempted to engage in corrupt practices. By taking these proactive steps, we can gradually diminish the influence of corruption and protect our communities from its harmful consequences. Together, we can build a society where honesty and fairness prevail, ensuring a brighter future for all.

5.7.1. Discussion/Recommendations

In Fig. 5.3, it is possible to observe how the parameter ω affects the most sensitive people. As the S population decreases as ω rises from 0.2 to 1.0, this shows that the population's susceptibility to corruption decreases as ω rises. This decrease in susceptibility can be attributed to the fact that higher values of ω lead to a stronger emphasis on individual values and beliefs, making it less likely for individuals to be influenced by corrupt practices. Additionally, as ω increases, individuals become more aware of the consequences of corruption and are more inclined to resist it. This can be seen in the decreasing trend of corruption cases as ω rises. Therefore, it can be concluded that increasing ω has a positive impact on reducing corruption and promoting integrity within the population.

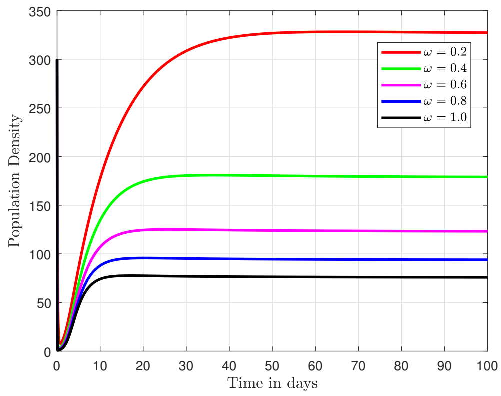
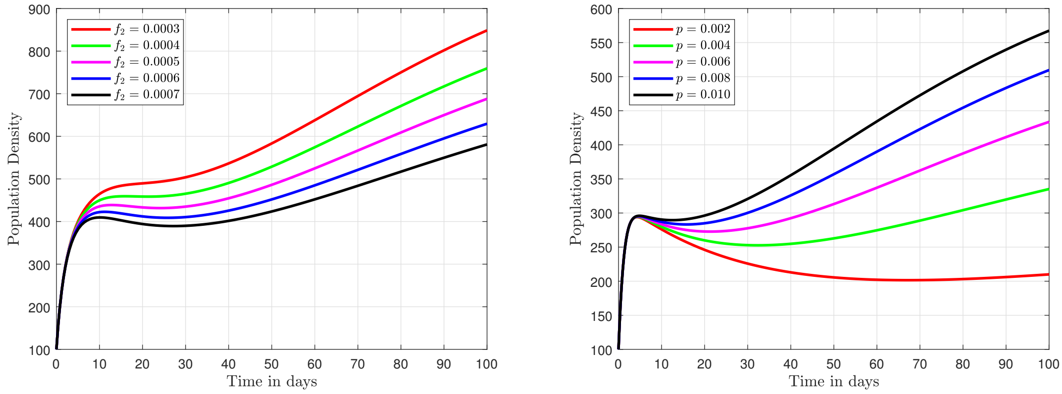


Figure 5.3: Susceptible population density for different parameteric values.

In Fig. 5.4, we see how the parameters f_2 and p influence the possibility that a person will become corrupted and the probability that they will become corrupted themselves. The risk population falls as the value of the parameter f_2 rises from 0.0003 to 0.0007 as shown in Fig. 5.4a. The effect of the parameter p is displayed in Fig. 5.4b, which reveals that the risk population grows as the value of p rises from 0.002 to 0.010. These findings highlight the importance of considering both parameters in understanding and addressing corruption risks. By carefully examining their impact, policymakers can develop effective strategies to mitigate corruption and protect individuals from its detrimental effects.

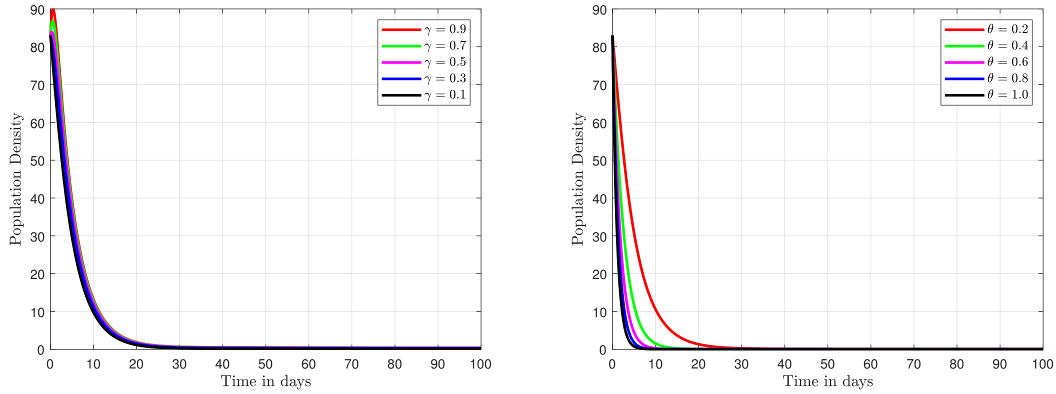


(a) (b)

Figure 5.4: Risk population density for different parameteric values.

In Fig. 5.5, we see how the parameters γ and θ affect persons who have been penalised for either direct or indirect participation in corrupt actions. As shown in Fig. 5.5a, the penalized population increases as γ increases from 0.1 to 0.9. The effect of parameter θ is seen in Fig. 5.5b. When θ values grow from 0.2 to 1.0, the penalized population declines. This suggests that a higher value of γ leads to a greater number of individuals being penalized for their involvement in corrupt actions. On the other hand, increasing the value of θ has the opposite effect, resulting in a decrease in the number of individuals who are penalized. These findings highlight the importance of carefully selecting and fine-tuning these parameters when implementing policies aimed at combating corruption. By understanding how different values of γ and θ impact the penalized population, policymakers can make informed decisions to effectively address corruption and its consequences. Ultimately, finding the right combination of parameters can help create a fair and just system that discourages corruption and promotes integrity within society.

In Fig. 5.6, we see how the parameters f_2, p, and γ affect those who are either directly or indirectly involved in corrupt activities. As can be seen in Fig. 5.6a, as the value of

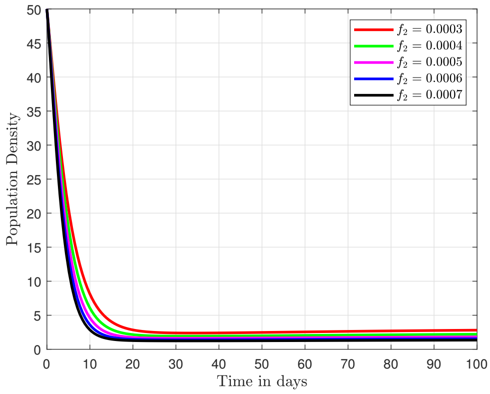


(a) **(b)**

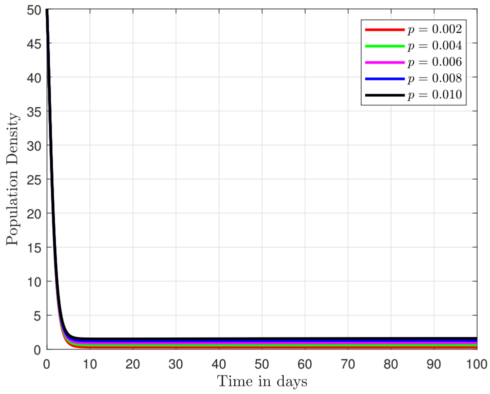
Figure 5.5: Punished population density for different parameteric values.

the parameter f_2 rises from 0.0003 to 0.0007, the corrupt population decreases. The risk population decreases as p decreases from 0.002 to 0.010, as shown in Fig. 5.6b. It is also observed that, for a given γ, increasing it from 0.1 to 0.9 results in a corresponding decrease in population density. This suggests that stricter enforcement measures and higher penalties for corruption can effectively deter individuals from engaging in corrupt activities. Additionally, reducing the overall risk of detection and punishment by implementing comprehensive anti-corruption policies can also contribute to a decrease in the corrupt population. It is important to note that these findings hold true for both direct participants in corruption and those indirectly involved, emphasizing the need for a comprehensive approach to tackling corruption. Furthermore, the impact of these factors on population density highlights the interconnected nature of corruption and its effects on society as a whole. By addressing these key variables, policymakers can work towards creating an environment that discourages corrupt behavior and promotes transparency and accountability.

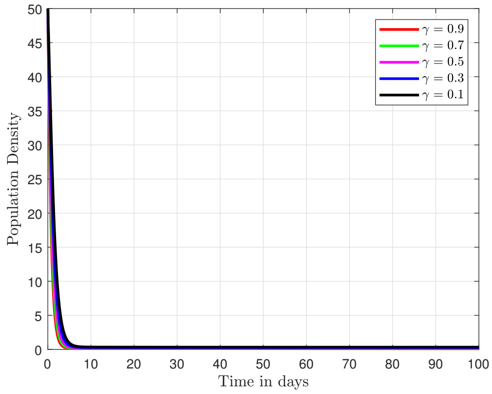
Parameters α, p and θ have been shown in Fig. 5.7 to illustrate the after-effects of corruption-related punishments for those who have been transposed. As can be seen in Fig. 5.7a, the transpose population grows as the parameter α decreases from 0.8 to 0.4. In Fig. 5.7b, the effect of parameter p is shown. This shows that as p goes from 0.002 to 0.010, the risk population goes up. In Fig. 5.5b, we see the impact of changing θ from 0.2 to 1.0. The risk population experiences a noticeable increase as θ increases, indicating a higher susceptibility to corruption-related punishments. This finding further



(a)



(b)



(c)

Figure 5.6: Corrupted population density for different parameteric values.

emphasizes the need for stringent measures and effective deterrents against corruption. By understanding the interplay between different parameters, such as α, p, and θ, policymakers can devise strategies that effectively combat corruption and its detrimental consequences. It is crucial to address this issue comprehensively and implement measures that not only punish wrongdoers but also prevent corruption from taking root in society.

5.8. Conclusion

In this study, we provide a fractional order model of corruption dynamics in society that has six different compartments. A fractal-fractional derivative in the caputo sense, with power-law, was used to develop a mathematical model of corruption dynamics. We have also shown that the solutions are positive and bounded in the biologically feasible region.

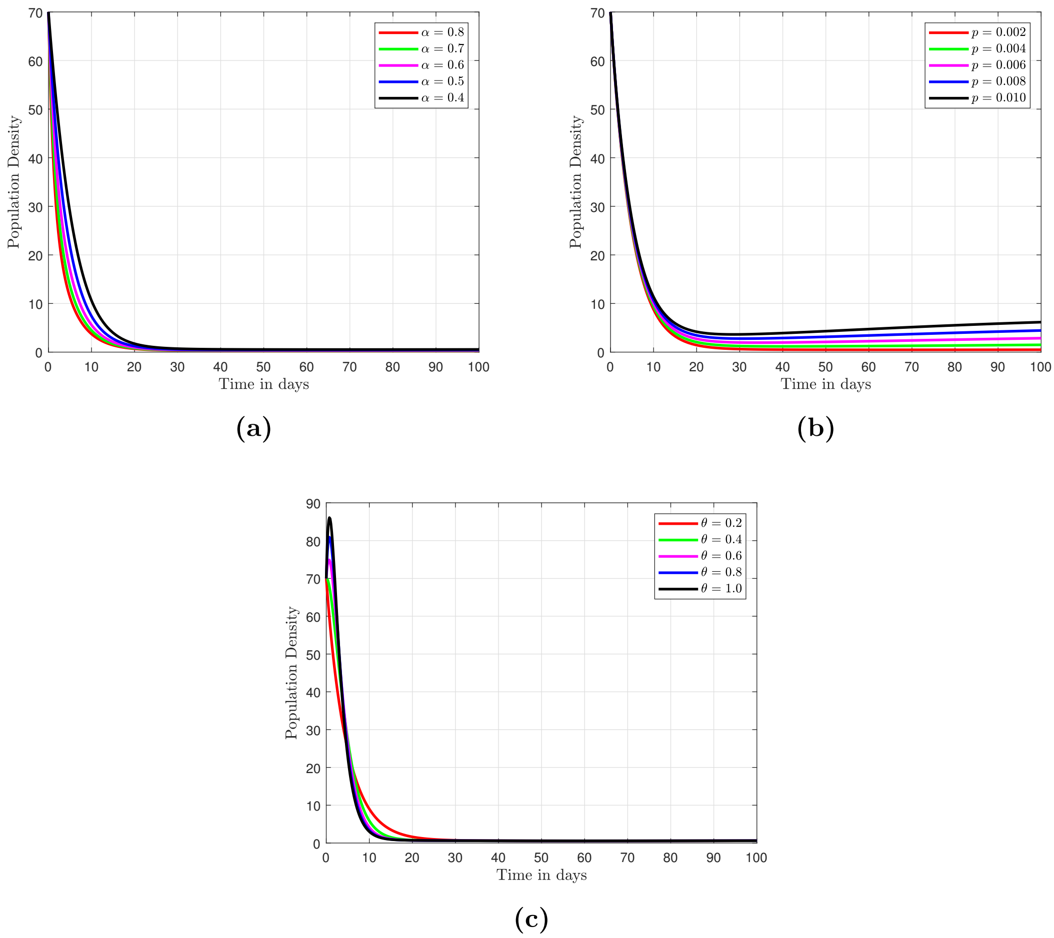


Figure 5.7: Transposed population density for different parameteric values.

Existence and uniqueness of solutions are established using the fixed point technique and the Picard-Lindelf approach. In addition, we have provided the Hyers-Ullam as well as Hyers-Ullam-Rassias stability results for the model under consideration. At the end, various different graphical representations of the findings that were acquired from the calculation are presented.

The corruption-free and CPE points had also been examined. The corruption transmission generation number is calculated by using the next-generation matrix approach, and with the help of jacobian matrix it is shown the CFE point is locally asymtotially stable when $\mathcal{R}_0 < 1$ unstable otherwise. The morally corruption-free and the CPE points were shown to be globally asymptotically stable by Lasselle's invariance principle of Lyapunov functions wherever $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$, respectively. We then perform numerical simulations to verify the theoretical analysis and investigate the existence and stability of equilibria.

By examining the concentrations and behaviors illustrated in Fig. 5.2, we gain valuable insights into how the model accounts for these variations. The impact of the different factors on the population's trend, as can be seen in Figs. 5.3,5.5,5.4,5.7, & 5.6, is crucial in understanding the dynamics of changing populations. This comprehensive approach allows us to analyze and interpret the complex interplay. Through a comprehensive appreciation of these dynamics, we can work toward a balance between human prerequisites and those of existing eco-systems. The following policies are made to stop people from getting involved in or spreading corrupt practices:

1. Strengthening the Legal Framework: It is crucial to enhance the legal framework to combat corrupt practices effectively. This includes enacting stricter laws, increasing penalties for corruption offenses, and ensuring swift and fair trials. By doing so, we can create a deterrent effect and discourage individuals from engaging in corrupt activities.
2. Promoting Transparency and Accountability: Transparency and accountability are essential in preventing corruption. Governments should implement measures such as mandatory financial disclosures for public officials, regular audits of government agencies, and the establishment of independent anti-corruption bodies. These initiatives will help identify and address corrupt practices promptly.
3. Strengthening Transparency and Accountability: Implementing robust mechanisms to ensure transparency and accountability in all sectors of society is crucial. This includes promoting open access to information, enforcing strict financial regulations, and establishing independent oversight bodies to monitor and investigate corrupt practices.
4. Enhancing Education and Awareness: Investing in education and awareness programs that highlight the detrimental effects of corruption is essential. By educating individuals from a young age about the importance of integrity, ethics, and the rule of law, we can foster a culture that rejects corrupt practices.
5. Empowering Whistleblowers: Creating a safe environment for whistleblowers to come forward with information about corruption is vital. Offering legal protection, anonymity, and incentives for reporting corruption can encourage individuals to

expose wrongdoing without fear of retaliation.

6. Strengthening Law Enforcement: Equipping law enforcement agencies with the necessary resources, training, and technology is crucial in combating corruption effectively. This includes improving investigative techniques, enhancing cooperation between different agencies, and ensuring swift prosecution.
7. Enhancing International Cooperation: Corruption often transcends national borders, making international cooperation crucial in tackling this issue effectively. Sharing information, best practices, and intelligence with other countries can help identify and disrupt transnational criminal networks involved in corruption schemes. Collaborating on extradition treaties and mutual legal assistance agreements can facilitate the prosecution of corrupt individuals who seek refuge abroad. Additionally, participating in international anti-corruption initiatives like the United Nations Convention against Corruption (UNCAC) can demonstrate a country's commitment to combating corruption on a global scale.
8. Empowering Civil Society: Engaging civil society organizations, including non-governmental organizations (NGOs) and community-based groups, is vital in the fight. These organizations play a significant role in raising awareness, monitoring government activities, and advocating for transparency and accountability. By empowering civil society, countries can create a more inclusive and participatory approach to combating corruption, ultimately leading to a more effective and sustainable anti-corruption effort.

It is hoped that this work would aid scholars in their quest to define and foresee global corruption. The overarching purpose of this study is to learn whether effective policies will influence subsequent corruption actions, punishments, and recoveries. Because of this, we may move forward with confidence in our ability to solve the aforementioned problems or at least mitigate their impact.

Availability of data and materials

The facts that support this model come from articles that have already been published and are cited in the right places.

Authors contribution

The authors contributed equally in this paper. All authors have read and approved the final version of manuscript.

Declaration of Competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Abbreviations

CFE	Corruption Free Equilibrium Point
CPE	Corruption Persistence Equilibrium Point

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Recently fractional calculus has gained much attention of the scientists due to its application in the fields of science and engineering like fluid dynamics, bio engineering, heat transform Fuzzy analysis. Modelling of dynamical problems with fractional order differential equations are the base of different existing systems, solving these fraction order differential systems are challenging. It is worth mentioning that various aspects of fractional order (singular/non-singular kernels) modelling that may include deterministic or uncertain (viz. fuzzy or interval or stochastic) scenarios are also important to understand the behaviour of the physical systems. As such, the aim of this book will be to include computation and modelling for obtaining exact and/or numerical solutions for fractional order systems.

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