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The Jensen-Mercer Inequality

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Preface

One of the most profound and widely applicable inequalities in modern mathematical analysis is Jensen's inequality for convex functions. Its utility spans a remarkable range of disciplines, from information theory and economics to functional analysis and numerical approximations. In 2003, A. McD. Mercer obtained a seminal variant of this inequality, now famously known as the Jensen–Mercer inequality:

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i),$$

for a convex function $f : [a,b] \to \mathbb{R}$, points $x_1, \ldots, x_n \in [a,b]$, and positive weights $w_1, \ldots, w_n$ with $W_n = \sum_{i=1}^{n} w_i$. This elegant result reframes the classical inequality by providing a bound that incorporates the endpoints of the interval, offering a new perspective and powerful tool for establishing other inequalities.

This book is born from the vibrant and extensive research activity that Mercer's initial result has inspired over the past two decades. Our primary aim is to compile, systematize, and present the vast array of developments, refinements, generalizations, and applications stemming from the Jensen–Mercer inequality and its integral analogue, the Jessen–Mercer inequality. We have endeavored to create a comprehensive reference that gathers scattered results from the literature into a single, structured volume, making it an invaluable resource for researchers and graduate students in mathematical analysis and its applications.

Chapter 1 lays the foundation by presenting the classical Jensen–Mercer and Jessen–Mercer inequalities. It discusses their essential bounds and establishes the fundamental framework upon which the entire book is built.

Chapter 2 is dedicated to the refinements of these core inequalities. We explore various approaches to obtaining sharper bounds, including those associated with positive n-tuples and the use of index set functions. The chapter also demonstrates the immediate applications of these refinements to the theory of power means and other classical means, highlighting the practical implications of the theoretical advances.

In **Chapter 3**, we turn our attention to the converses of the Jensen–Mercer and Jessen–Mercer inequalities. This chapter provides refinements of these converse inequalities and ventures into a significant generalization by examining these concepts on convex hulls in $\mathbb{R}^k$, extending the theory to higher dimensions.

Chapter 4 investigates means of Mercer type, exploring their properties and relationships. A key focus here is on the concept of exponential convexity. We present mean value theorems associated with the Jensen–Mercer functional and delve into n-exponential and logarithmic convexity, providing a robust analytical framework for these new classes of means.

Chapter 5 showcases how the Mercer concept can be used to derive variants of other well-known inequalities. This includes novel forms of the Jensen-Steffensen, Čebyšev, and Pečarić inequalities, as well as an inequality of Milne, demonstrating the unifying and generative power of the Mercer framework.

The concept of the normalized Jensen–Mercer functional is introduced and studied in **Chapter 6**. We establish bounds for this functional and investigate its various properties, particularly in the discrete case, providing a deeper understanding of its behavior.

Chapter 7 extends the theory beyond convexity to the realm of superquadratic functions. We present variants of the Jensen-Steffensen and Jessen inequalities of Mercer's type for this broader class of functions, showing the adaptability and scope of the initial idea.

Chapter 8 generalizes the Jensen–Mercer inequality to multivariate functions. This includes results for P-convex functions and functions with nondecreasing increments. The chapter also contains generalizations and refinements of celebrated inequalities like those of Čebyšev and

Hölder, derived within this multivariate Mercer context.

Chapter 9 elevates the discussion to the operator level, presenting the Jensen–Mercer inequality in the setting of Hilbert spaces. This operator-theoretic perspective allows for applications to various operator means, bridging the gap between classical inequality theory and modern operator theory.

Chapters 10–20 delve into more specialized and advanced topics. These include:

- **Fractional Integral Inequalities:** Exploring Hermite-Hadamard-Mercer type inequalities via various fractional operators.
- **Information-Theoretic Applications:** Deriving new estimates for Csiszár divergence and Zipf-Mandelbrot entropy.
- **Numerical Analysis:** Developing Simpson-Mercer-type inequalities with applications to special functions.
- **Ostrowski-Mercer Inequalities:** Establishing new bounds and generalizations of Ostrowski-type inequalities within the Mercer framework, connecting pointwise estimates with integral means.
- **Dynamic Calculus:** Generalizing the inequality on time scales for both the Δ- and $\diamond_\alpha$-integrals, with applications to models in economics.

The book concludes with a dedicated chapter of **Concluding Remarks**, which serves a dual purpose. It provides a reflective synthesis of the most significant recent advances from the years 2024 and 2025—covering trends like refinements via strong convexity, fractional extensions, and information-theoretic applications—while also formally marking the end of this edition's coverage. This final chapter outlines the rationale for concluding the volume at this juncture and offers a perspective on open questions and promising future research directions, providing a natural bridge for subsequent research and potential future editions.

Finally, the **Appendix** provides a concise overview of essential concepts related to convexity, operator convexity, and classical inequalities, making the book more self-contained.

This volume is the culmination of a concerted effort to map the extensive landscape shaped by the Jensen–Mercer inequality. We believe it not only serves as a definitive reference for existing results but also as a catalyst for future research, pointing towards uncharted territories and open problems. We are deeply grateful to the global community of analysts whose brilliant work made this compilation possible.

Asif R. Khan, Inam Ullah Khan Kundi and Josip Pečarić

CHAPTER 1

Jensen–Mercer and Jessen–Mercer Inequalities for Convex Functions

The Jensen–Mercer and Jessen–Mercer inequalities constitute significant variants of the classical Jensen inequality for convex functions. These inequalities provide sharper bounds and structural flexibility, making them valuable tools in modern convex analysis and inequality theory. In this chapter, we present the Jensen–Mercer inequality, derive key bounds, and examine the Jessen–Mercer extension as a related but distinct generalization. The results established here form the analytical basis for further advanced developments in subsequent chapters.

1.1 Jensen–Mercer Inequality

In 2003, Mercer [174] obtained a new variant of Jensen's inequality which we would refer to as Jensen–Mercer inequality. Throughout this book, when using the interval $[a, b]$, we assume that $-\infty < a < b < \infty$.

To prove the Jensen–Mercer inequality, we shall need the following lemma from [174]. In this lemma, the inequality (1.1) is, in fact, a special case of Szegö's inequality; for details, see [204, p. 161].

Lemma 1.1: *If $f : [a, b] \to \mathbb{R}$ is a convex function, then for any $x \in [a, b]$, the following inequality holds*

$$f(a + b - x) \leq f(a) + f(b) - f(x). \tag{1.1}$$

Proof: For every $x \in [a, b]$, there exists a unique $\lambda \in [0, 1]$ such that $x = \lambda a + (1 - \lambda) b$.

Since f is convex, we have

$$\begin{aligned} f(a+b-x) &= f(a+b-\lambda a-(1-\lambda)b) \\ &= f((1-\lambda)a+\lambda b) \\ &\leq (1-\lambda)f(a)+\lambda f(b) \\ &= f(a)+f(b)-(\lambda f(a)+(1-\lambda)f(b)) \\ &\leq f(a)+f(b)-f(\lambda a+(1-\lambda)b) \\ &= f(a)+f(b)-f(x). \end{aligned}$$

□

Remark 1.1: It is easy to check that for a *real* n*-tuple* $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ *satisfying* $W_n = \sum_{i=1}^{n} w_i = 1$ and for arbitrary n-tuple $\boldsymbol{x} \in [a, b]^n$ inequalities

$$a \leq \sum_{i=1}^{n} w_i x_i \leq b \tag{1.2}$$

hold. Furthermore, (1.2) implies $a + b - \sum_{i=1}^{n} w_i x_i \in [a, b]$.

Theorem 1.1 ***(Jensen–Mercer Inequality)*:** *If f is a convex function on an interval $[a, b]$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in [a, b]^n$ and if $\boldsymbol{w} = (w_1, \ldots, w_n)$ is a nonnegative n-tuple with $\sum_{i=1}^{n} w_i = 1$, then following inequality holds*

$$f\left(a+b-\sum_{i=1}^{n} w_i x_i\right) \leq f(a)+f(b)-\sum_{i=1}^{n} w_i f(x_i). \tag{1.3}$$

Proof: By Jensen's inequality (*A*.7) and by inequality (1.1), we have

$$\begin{aligned} f\left(a+b-\sum_{i=1}^{n} w_i x_i\right) &= f\left(\sum_{i=1}^{n} w_i (a+b-x_i)\right) \\ &\leq \sum_{i=1}^{n} w_i f(a+b-x_i) \\ &\leq \sum_{i=1}^{n} w_i (f(a)+f(b)-f(x_i)) \\ &= f(a)+f(b)-\sum_{i=1}^{n} w_i f(x_i). \end{aligned}$$

□

Remark 1.2:

1. If in Theorem 1.1 the condition $W_n = 1$ is replaced with $W_n > 0$, then the inequality
$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right) \le f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i)$$
holds. It is another form of Jensen–Mercer inequality.
2. In [174], author assumed that $0 < x_1 < x_2 < \cdots < x_n$. In our opinion this supposition is extra as positivity and order of elements are not necessary.
3. Instead of considering end points of the interval a and b, we can also consider $m = \min\limits_{i\in\{1,2,\ldots,n\}}\{x_i\}$ and $M = \max\limits_{i\in\{1,2,\ldots,n\}}\{x_i\}$, see for example [137].
4. An interesting alternative proof of Theorem 1.1 is based on Theorem $A.29$ can be found in [8] using $\boldsymbol{\xi}$ in place of $\boldsymbol{x}$, $\boldsymbol{\eta}$ in place of $\boldsymbol{y}$ and $\boldsymbol{p}$ in place of $\boldsymbol{w}$.

Moreover, the Jensen–Mercer inequality remains valid for the weights satisfying the conditions as for the reverse Jensen's inequality (see Theorem A.7).

Theorem 1.2: *Let $w_1, \ldots, w_n$ be real numbers such that*

$$w_1 > 0, \quad w_i \le 0 \quad (i \in \{2,\ldots,n\}), \quad W_n > 0. \tag{1.4}$$

Let $[a,b]$ be an interval in $\mathbb{R}$, and $x_1,\ldots,x_n \in [a,b]$ such that $\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i \in [a,b]$. If f is a convex function on $[a,b]$, then (1.3) holds.

Proof: Weights $w_1,\ldots,w_n$ satisfy the conditions (1.3) and $\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i \in [a,b]$, so by inequality (1.1) and by the reversed Jensen's inequality ($A.10$), we have

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right) \le f(a)+f(b)-f\left(\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right) \le f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i).$$

□

In [242], Witkowski gave another proof of Theorem 1.1 stated as under:

Theorem 1.3: *Under the assumptions of Theorem 1.1, the inequality (1.3) holds. For concave function the inequality (1.3) reverses.*

Proof: For every $x_i \in [a,b]$, we choose $\lambda_i \in [0,1]$ such that $x_i = (1-\lambda_i)a+\lambda_ib$.

By using Jensen's inequality three times, we have

$$\begin{aligned} f\left(a+b-\sum_{i=1}^{n} w_i x_i\right) &= f\left(\sum_{i=1}^{n} w_i\left((1-\lambda_i)a+\lambda_i b\right)\right) \\ &\leq \sum_{i=1}^{n} w_i f\left((1-\lambda_i)a+\lambda_i b\right) \\ &\leq \sum_{i=1}^{n} w_i\left[(1-\lambda_i) f(a)+\lambda_i f(b)\right] \\ &= \sum_{i=1}^{n} w_i\left[f(a)-\lambda_i f(a)+f(b)-(1-\lambda_i) f(b)\right] \\ &= f(a)+f(b)-\sum_{i=1}^{n} w_i\left[\lambda_i f(a)+(1-\lambda_i) f(b)\right] \\ &\leq f(a)+f(b)-\sum_{i=1}^{n} w_i f(x_i). \end{aligned}$$

□

In 2005, in paper [8] was proved that the Jensen–Mercer inequality remains valid for the weights satisfying the conditions as for the Jensen-Steffensen inequality (see Theorem A.6).

Theorem 1.4: *Let $[a, b]$ be an interval in $\mathbb{R}$, let $\boldsymbol{x}=(x_1, \ldots, x_n)$ be a monotone n-tuple from $[a, b]^n$ and let $\boldsymbol{w}=(w_1, \ldots, w_n)$ be a real n-tuple satisfying (A.9). If a function $f:[a, b] \rightarrow \mathbb{R}$ is convex on $[a, b]$, then inequality (1.3) holds.*

Proof: First assume that $\boldsymbol{x}$ is increasing. Set $m=n+2$ and define the m-tuples $\boldsymbol{y}$ and $\boldsymbol{v}$ as

$$\begin{aligned} & y_1=a, \quad y_2=x_1, \quad y_3=x_2, \quad \ldots, \quad y_{n+1}=x_n, \quad y_{n+2}=b, \\ & v_1=1, \quad v_2=-\frac{w_1}{W_n}, \quad v_3=-\frac{w_2}{W_n}, \quad \ldots, \quad v_{n+1}=-\frac{w_n}{W_n}, \quad v_{n+2}=1. \end{aligned}$$

Then, for $j \in\{1, \ldots n\}$, $V_j=\sum_{i=1}^{j} v_i=1-\frac{1}{W_n} \sum_{i=1}^{j-1} w_i=\frac{1}{W_n} \sum_{i=j}^{n} w_i$ and $V_{n+1}=0$, $V_{n+2}=1$. Hence, the m-tuples $\boldsymbol{y}$ and $\boldsymbol{v}$ satisfy the conditions of Theorem *A.6* and we can apply the Jensen-Steffensen inequality on them to get inequality (1.3) . In case $\boldsymbol{x}$ is decreasing we simply replace $\boldsymbol{x}$ and $\boldsymbol{w}$ with $\widetilde{\boldsymbol{x}}=(x_n, \ldots, x_1)$ and $\widetilde{\boldsymbol{w}}=(w_1, \ldots, w_n)$ respectively, and then argue in the same way. □

Remark 1.3: In article [7], the authors have also considered the equality case. They stated that:

If f is strictly convex, then the equality holds in (1.3) $\Longleftrightarrow$ one of the following two cases occurs:

(1) either $\overline{x} = a$ or $\overline{x} = b$,

(2) there exist $l \in \{2, \dots, n-1\}$ such that $\overline{x} = a + b - x_l$ and

$$\begin{cases} x_1 = a, \quad x_n = b, \quad or \quad x_1 = b, \quad x_n = a, \\ \overline{W_j}(x_{j-1} - x_j) = 0, \quad j \in \{2, \dots, l\}, \\ W_j(x_j - x_{j-1}) = 0, \quad j \in \{l, \dots, n-1\}, \end{cases}$$

where $\overline{W_j} = \sum_{i=j}^{n} w_i$, $j \in \{1, \dots, n\}$ and $\overline{x} = \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i$.

In the special case where $w_i > 0$ for $i \in \{1, \dots, n\}$ and f is strictly convex, the equality holds in (1.3) iff $x_i = a$ or $x_i = b$ for $i \in \{1, \dots, n\}$. For proof, we refer the reader to [7].

1.1.1 Bounds of Jensen–Mercer Inequality

In [117], the authors made a remark about the lower bound of the Jensen–Mercer inequality.

Remark 1.4: Using Jensen's and Jensen-Steffensen's inequality, it is easy to prove the following inequalities

$$f\left(\tfrac{a+b}{2}\right) - \frac{1}{W_n} \sum_{i=1}^{n} w_i f(x_i) \le f\left(a + b - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right) \le f(a) + f(b) - \frac{1}{W_n} \sum_{i=1}^{n} w_i f(x_i) \quad (1.5)$$

where f is a convex function on $(a - \epsilon, b + \epsilon)$, $\epsilon > 0$, and $x \in (a, b)$, $w_i > 0$ for $i \in \{1, \dots, n\}$. If f is concave, the reverse inequalities hold in (1.5).

Now, suppose that the conditions in Theorem *A*.5 are fulfilled, except that the function f satisfies $f(x) + f(a + b - x) = 2f((a + b)/2)$. It is immediate (consider the function $f(x) - f((a + b)/2)$) that the inequality (*A*.7) still holds. Using $f(x) = 2f((a + b)/2) - f(a + b - x)$, the inequality (*A*.7) gives: $2f\left(\frac{a+b}{2}\right) - \frac{1}{W_n} \sum_{i=1}^{n} w_i f(x_i) \le f\left(a + b - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right)$ so the left hand side of the inequality (1.5) is valid also in this case. On the other hand, if $f((a+b)/2) = 0$ (so $f(a) + f(b) = 0$), the previous inequality can be written as

$$f\left(a + b - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right) \ge f(a) + f(b) - \frac{1}{W_n} \sum_{i=1}^{n} w_i f(x_i)$$

which is reverse of the right hand side inequality of (1.5), so the concavity properties of the function f are prevailing in this case.

1.2 Jessen–Mercer Inequality

Throughout this section, we let E be a nonempty set and L be a subspace of the vector space $\mathbb{R}^E$ over $\mathbb{R}$ that contains $\mathbf{1}$, and L that have all the properties defined in $L1$, $L2$, $L3$ and $L4$ as defined in Section $A.3$ along with the Positive (Isotonic) Linear Functional defined in $A1$, $A2$ in the same section.

Let $\varphi : [a, b] \to \mathbb{R}$ be a continuous function. In [76], under the assumptions of Theorem $A.14$, the following variant of the Jessen inequality is proved, if φ is convex, then

$$\varphi(a + b - A(f)) \leq \varphi(a) + \varphi(b) - A(\varphi(f)); \tag{1.6}$$

if φ is concave, then the inequality (1.6) is reversed.

In 2004, Gavrea [102] has obtained the following generalization of the Jensen–Mercer inequality (1.3).

Theorem 1.5: *Let A be an isotonic linear functional defined on $C[a, b]$ such that $A(1) = 1$. Then for any convex function φ on $[a, b]$,*

$$\varphi(a + b - a_1) \leq A(\psi) \leq \varphi(a) + \varphi(b) - \varphi(a)\frac{b - a_1}{b - a} - \varphi(b)\frac{a_1 - a}{b - a} \leq \varphi(a) + \varphi(b) - A(\varphi),$$

where $\psi(t) = \varphi(a + b - t)$ and $a_1 = A(id)$.

Remark 1.5: Although it is not explicitly stated above, it is obvious that function φ needs to be continuous on $[a, b]$.

We give an extension of Theorem 1.5 as under:

Theorem 1.6: *Let L satisfy properties $L1$, $L2$ on a nonempty set E, and let φ be a continuous convex function on an interval $I = [a, b]$. If A is an isotonic linear functional on L with $A(\mathbf{1}) = 1$, then for all $f \in L$ such that $\varphi(f), \varphi(a + b - f) \in L$ (so that $a \leq f(t) \leq b$ for all $t \in E$), we have the following variant of Jessen's inequality*

$$\varphi\left(a + b - A\left(f\right)\right) \leq \varphi\left(a\right) + \varphi\left(b\right) - A\left(\varphi\left(f\right)\right). \tag{1.7}$$

In fact, to be more specific we have the following series of inequalities

$$\varphi\left(a+b-A\left(f\right)\right) \leq A\left(\varphi\left(a+b-f\right)\right) \leq \frac{b-A\left(f\right)}{b-a}\varphi(b)+\frac{A\left(f\right)-a}{b-a}\varphi(a)$$
$$\leq \varphi\left(a\right)+\varphi\left(b\right)-A\left(\varphi\left(f\right)\right). \quad (1.8)$$

If the function φ is concave, inequalities (1.7) and (1.8) are reversed.

Proof: First, notice that if $\varphi\left(f\right) \in L$, then $a \leq f(t) \leq b$ for all $t \in E$. Hence, $a = A\left(a\mathbf{1}\right) \leq A\left(f\right) \leq A\left(b\mathbf{1}\right) = b$, i.e., $A\left(f\right) \in I$.

Since φ is continuous and convex, the same is also true for the function $\psi : [a,b] \to \mathbb{R}$ defined by $\psi(t) = \varphi(a+b-t), \quad t \in [a,b]$.

By Theorem A.14. $\psi(A\left(f\right)) \leq A(\psi\left(f\right))$, i.e., $\varphi\left(a+b-A\left(f\right)\right) \leq A\left(\varphi(a+b-f)\right)$. Applying Theorem A.15 to ψ and then to φ, we have

$$\begin{aligned} A\left(\varphi(a+b-f)\right) &\leq \frac{b-A\left(f\right)}{b-a}\psi\left(a\right)+\frac{A\left(f\right)-a}{b-a}\psi\left(b\right) = \frac{b-A\left(f\right)}{b-a}\varphi\left(b\right)+\frac{A\left(f\right)-a}{b-a}\varphi\left(a\right) \\ &= \varphi\left(a\right)+\varphi\left(b\right)-\left[\frac{b-A\left(f\right)}{b-a}\varphi\left(a\right)+\frac{A\left(f\right)-a}{b-a}\varphi\left(b\right)\right] \leq \varphi\left(a\right)+\varphi\left(b\right)-A\left(\varphi\left(f\right)\right). \end{aligned}$$

The last statement follows immediately from the facts that if φ is concave then $-\varphi$ is convex, and that A is linear on L. □

Remark 1.6: In Theorem 1.6, taking $L = C\left[a,b\right]$ and $g = id$, we obtain the results of Theorem 1.5. On the other hand, the results of Theorem 1.5 for the functional B defined on L by $B(\varphi) = A(\varphi(f))$, for which $B(\mathbf{1}) = 1$ and $B(id) = A(f)$, become the results of Theorem 1.6. Hence, these results are equivalent.

Corollary 1.1: *Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space, and let $f : \Omega \to [a,b]$ be a measurable function. Then for any continuous convex function $\varphi : [a,b] \to \mathbb{R}$,*

$$\begin{aligned} \varphi\left(a+b-\int_{\Omega} f \mathrm{d}\mu\right) &\leq \int_{\Omega} \varphi\left(a+b-f\right) \mathrm{d}\mu \\ &\leq \frac{b-\int_{\Omega} f \mathrm{d}\mu}{b-a}\varphi\left(b\right)+\frac{\int_{\Omega} f \mathrm{d}\mu - a}{b-a}\varphi\left(a\right) \leq \varphi\left(a\right)+\varphi\left(b\right)-\int_{\Omega}\varphi\left(f\right)\mathrm{d}\mu. \end{aligned}$$

Proof: This is a special case of Theorem 1.6 for the functional A defined on class $L^1\left(\mu\right)$ as $A(f) = \int_{\Omega} f \mathrm{d}\mu$. □

CHAPTER 2

Refinements of Jensen–Mercer and Jessen–Mercer Inequalities

This chapter is devoted to several refinements of the Jensen–Mercer and Jessen–Mercer inequalities, developed through both classical and modern techniques. We first introduce refinements based on the D-functional and a hierarchy of means including arithmetic, geometric, harmonic, power, and quasi-arithmetic means. These yield improved bounds and structural sharpenings over the original inequalities. We then extend the analysis to refinements involving positive n-tuples, followed by a further generalization via index set functions, offering a unified framework for comparison and interpolation of Mercer-type inequalities. The results obtained here serve as foundational tools for subsequent advancements addressed in later chapters. Most of the topics in this chapter can be found in the papers [24, 32, 76, 167].

2.1 Refinements of the Jensen–Mercer Inequality

Throughout this chapter, we will use some assumptions stated as below:

A: Let $f : [a,b] \to \mathbb{R}$ be a convex function. If $x_i \in [a,b]$ and $w_i > 0$, $i \in \{1,2,\ldots,n\}$ with $\sum_{i=1}^{n} w_i = 1$, then for any nonempty subset I of $\{1,2,\ldots,n\}$ we put $\overline{I} := \{1,2,\ldots,n\} \setminus I \neq \emptyset$ and define $W_I = \sum_{i\in I} w_i$ and $W_{\overline{I}} = 1 - \sum_{i\in I} w_i$. For the convex function f and the n-tuple $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{w} = (w_1,\ldots,w_n)$, we can define the following functional

$$D(\mathbf{w},\mathbf{x},f;I) := W_I f\left(a+b-\frac{1}{W_I}\sum_{i\in I} w_i x_i\right) + W_{\overline{I}} f\left(a+b-\frac{1}{W_{\overline{I}}}\sum_{i\in \overline{I}} w_i x_i\right). \tag{2.1}$$

Throughout the chapter we further assume that $I \subset \{1,2,\ldots,n\}$ with $I \neq \emptyset$ and $I \neq \{1,2,\ldots,n\}$ unless stated.

Along with the assumptions as stated in **A** we further assume that:

B: Let all the assumptions of **A** be valid. Further let for $\emptyset \neq I \subset \{1,\ldots,n\}$ let A_I, G_I, H_I and $M_I^{[r]}$ be the arithmetic, geometric, harmonic means, and power mean of order $r \in \mathbb{R}$, respectively of $x_i \in [a,b]$, where $0 < a < b$, formed with the positive weights w_i, $i \in I$. For $I = \{1,\ldots,n\}$ we denote the arithmetic, geometric, harmonic and power means by A_n, G_n, H_n and $M_n^{[r]}$ respectively, we define

$$\begin{aligned}
\tilde{A}_I :&= a+b-\frac{1}{W_I}\sum_{i\in I} w_i x_i = a+b-A_I\\
\tilde{G}_I :&= \frac{ab}{\left(\prod_{i\in I} x_i^{w_i}\right)^{\frac{1}{W_I}}} = \frac{ab}{G_I}\\
\tilde{H}_I :&= \left(a^{-1}+b^{-1}-\frac{1}{W_I}\sum_{i\in I} w_i x_i^{-1}\right)^{-1} = \left(a^{-1}+b^{-1}-H_I^{-1}\right)^{-1}\\
\tilde{M}_I^{[r]} :&= \begin{cases} \left(a^r+b^r-\left(M_I^{[r]}\right)^r\right)^{\frac{1}{r}}, & r \neq 0;\\ \tilde{G}_I, & r = 0,\end{cases}
\end{aligned}$$

where

$$M_I^{[r]} := \begin{cases} \left(\frac{1}{W_I}\sum_{i\in I} w_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0;\\ \left(\prod_{i\in I} x_i^{w_i}\right)^{\frac{1}{W_I}}, & r = 0.\end{cases}$$

Moreover we define quasi-arithmetic mean as unbder:

C: Let $\varphi : [a,b] \to \mathbb{R}$ be a strictly monotonic and continuous function. Then for a given n- tuple $\mathbf{x} = (x_1,\ldots,x_n) \in [a,b]^n$ and positive n- tuple $\mathbf{w} = (w_1,\ldots,w_n)$ with $\sum_{i=1}^n w_i = 1$, the value $M_\varphi^{[n]} = \varphi^{-1}\left(\sum_{i=1}^n w_i\varphi(x_i)\right)$ is well defined and is called quasi–arithmetic mean of $\mathbf{x}$ with wight $\mathbf{w}$ (see for example [47, p. 215]), we define $\tilde{M}_\varphi^{[n]} = \varphi^{-1}\left(\varphi(a)+\varphi(b)-\sum_{i=1}^n w_i\varphi(x_i)\right)$.

The following refinement of (1.3) is valid [151].

Theorem 2.1: *Under the assumptions stated in* **A***, we have*

$$f\left(a+b-\sum_{i=1}^n w_i x_i\right) \leq D(\mathbf{w},\mathbf{x},f;I) \leq f(a)+f(b)-\sum_{i=1}^n w_i f(x_i). \tag{2.2}$$

Proof: By the convexity of the function f we have

$$\begin{aligned}
& f\left(a+b-\sum_{i=1}^{n} w_i x_i\right)=f\left(\sum_{i=1}^{n} w_i\left(a+b-x_i\right)\right) \\
& =f\left(W_I\left(\frac{1}{W_I} \sum_{i \in I} w_i\left(a+b-x_i\right)\right)+W_{\bar{I}}\left(\frac{1}{W_{\bar{I}}} \sum_{i \in \bar{I}} w_i\left(a+b-x_i\right)\right)\right) \\
& \leq W_I f\left(\frac{1}{W_I} \sum_{i \in I} w_i\left(a+b-x_i\right)\right)+W_{\bar{I}} f\left(\frac{1}{W_{\bar{I}}} \sum_{i \in \bar{I}} w_i\left(a+b-x_i\right)\right) \\
& =D(\mathbf{w}, \mathbf{x}, f ; I)
\end{aligned}$$

for any I, which proves the first inequality in (2.2).

By the Jensen–Mercer inequality (1.3), we also have

$$\begin{aligned}
& D(\mathbf{w}, \mathbf{x}, f ; I)=W_I f\left(a+b-\frac{1}{W_I} \sum_{i \in I} w_i x_i\right)+W_{\bar{I}} f\left(a+b-\frac{1}{W_{\bar{I}}} \sum_{i \in \bar{I}} w_i x_i\right) \\
& \leq W_I\left(f(a)+f(b)-\frac{1}{W_I} \sum_{i \in I} w_i f\left(x_i\right)\right)+W_{\bar{I}}\left(f(a)+f(b)-\frac{1}{W_{\bar{I}}} \sum_{i \in \bar{I}} w_i f\left(x_i\right)\right) \\
& =f(a)+f(b)-\sum_{i=1}^{n} w_i f\left(x_i\right) .
\end{aligned}$$

for any I, which proves the second inequality in (2.2). □

Remark 2.1: In [167] from the proof of Theorem 2.3 we have left inequality of (2.2).

Remark 2.2: Note that the inequality (2.2) can be written in an equivalent form as

$$f\left(a+b-\sum_{i=1}^{n} w_i x_i\right) \leq \min_I D(\mathbf{w}, \mathbf{x}, f ; I), \tag{2.3}$$

$$\max_I D(\mathbf{w}, \mathbf{x}, f ; I) \leq f(a)+f(b)-\sum_{i=1}^{n} w_i f\left(x_i\right). \tag{2.4}$$

The following special cases of (2.3) and (2.4) can be given:

$$\begin{aligned}
f\left(a+b-\sum_{i=1}^{n} w_i x_i\right) & \leq \min_{k \in\{1, \ldots, n\}} D_k(\mathbf{w}, \mathbf{x}, f), \\
\max_{k \in\{1, \ldots, n\}} D_k(\mathbf{w}, \mathbf{x}, f) & \leq f(a)+f(b)-\sum_{i=1}^{n} w_i f\left(x_i\right).
\end{aligned}$$

For further remarks and related corollary see [151].

2.1.1 Applications to Power Means

Theorem 2.2: *Under the assuptions given in* **B** *we have:*

$$(i)\quad \tilde{G}_n \le \min_I \tilde{A}_I^{W_I}\tilde{A}_{\bar{I}}^{W_{\bar{I}}} \quad \textit{and}\ \tilde{A}_n \ge \max_I \tilde{A}_I^{W_I}\tilde{A}_{\bar{I}}^{W_{\bar{I}}}. \tag{2.5}$$

$$(ii)\quad \tilde{G}_n \le \min_I \left[W_I\tilde{G}_I + W_{\bar{I}}\tilde{G}_{\bar{I}}\right] \quad \textit{and}\ \tilde{A}_n \ge \max_I \left[W_I\tilde{G}_I + W_{\bar{I}}\tilde{G}_{\bar{I}}\right]. \tag{2.6}$$

Proof: (i) Applying Theorem 2.1 to the convex function $f(x) = -\ln x$, we obtain

$$-\ln \tilde{A}_n \le -W_I \ln \tilde{A}_I - W_{\bar{I}} \ln \tilde{A}_{\bar{I}} \le -\ln \tilde{G}_n. \tag{2.7}$$

Now (2.5) follows from Remark 2.2 and (2.7).

(ii) Applying Theorem 2.1 to the convex function $f(x) = \exp x$, and replacing a, b, and x_i with $\ln a, \ln b$, and $\ln x_i$ respectively and using Remark 2.2, we obtain (2.6). □

The following particular case of Theorem 2.2 is of interest.

Corollary 2.1: *Under the assumptions of Theorem 2.2, we have*

$$(i)\quad \frac{1}{\tilde{G}_n} \le \min_I \frac{1}{\tilde{H}_I^{W_I}\tilde{H}_{\bar{I}}^{W_{\bar{I}}}} \quad \textit{and} \quad \frac{1}{\tilde{H}_n} \ge \max_I \frac{1}{\tilde{H}_I^{W_I}\tilde{H}_{\bar{I}}^{W_{\bar{I}}}}.$$

$$(ii)\quad \frac{1}{\tilde{G}_n} \le \min_I \left[\frac{W_I}{\tilde{G}_I} + \frac{W_{\bar{I}}}{\tilde{G}_{\bar{I}}}\right] \quad \textit{and} \quad \frac{1}{\tilde{H}_n} \ge \max_I \left[\frac{W_I}{\tilde{G}_I} + \frac{W_{\bar{I}}}{\tilde{G}_{\bar{I}}}\right].$$

Proof: Directly from Theorem 2.2 by the substitutions $a \to \frac{1}{a}, b \to \frac{1}{b}, x_i \to \frac{1}{x_i}$. □

Theorem 2.3: *Under the assumptions given in* **B**, *for $r \le 1$, we have*

$$\tilde{M}_n^{[r]} \le \min_I \left[W_I\tilde{M}_I^{[r]} + W_{\bar{I}}\tilde{M}_{\bar{I}}^{[r]}\right], \qquad \tilde{A}_n \ge \max_I \left[W_I\tilde{M}_I^{[r]} + W_{\bar{I}}\tilde{M}_{\bar{I}}^{[r]}\right] \tag{2.8}$$

For $r \ge 1$, the inequalities in (2.8) are reversed.

Proof: For $r \le 1, r \ne 0$, use Theorem 2.1 for the convex function $f(x) = x^{\frac{1}{r}}$, and replacing a, b, and x_i with a^r, b^r, and x_i^r respectively and for $r = 0$ use Theorem 2.1 for the convex function $f(x) = \exp x$; replacing a, b, and x_i with $\ln a, \ln b$, and $\ln x_i$ respectively, we obtain (2.8) by Remark 2.2.

If $r \ge 1$, then the function $f(x) = x^{\frac{1}{r}}$, is concave, so the inequalities in (2.8) are reversed. □

Corollary 2.2: *Under the assumptions of Theorem 2.3, we have*

$$\tilde{H}_n \leq \min_I \left[W_I \tilde{H}_I + W_{\bar{I}} \tilde{H}_{\bar{I}}\right], \qquad \tilde{A}_n \geq \max_I \left[W_I \tilde{H}_I + W_{\bar{I}} \tilde{H}_{\bar{I}}\right].$$

Remark 2.3: Part (ii) of Theorem 2.2 is also direct consequences of Theorem 2.3.

Theorem 2.4: *Let $r, s \in \mathbb{R}$, $r \leq s$. Under the assumptions given in* **B** *we have: (i) If $s \geq 0$, then*

$$\begin{aligned} \left(\tilde{M}_n^{[r]}\right)^s &\leq \min_I \left[W_I \left(\tilde{M}_I^{[r]}\right)^s + W_{\bar{I}} \left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right], \\ \left(\tilde{M}_n^{[r]}\right)^s &\geq \max_I \left[W_I \left(\tilde{M}_I^{[r]}\right)^s + W_{\bar{I}} \left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right] \end{aligned} \tag{2.9}$$

(ii) *If $s < 0$, then inequalities in (2.9) are reversed.*

Proof: Let $s \geq 0$. Using Theorem 2.1 and Remark 2.2 to the convex function $f(x) = x^{\frac{s}{r}}$, and replacing a, b, and x_i with a^r, b^r, and x_i^r respectively, we obtain (2.9).

If $s < 0$, then the function $f(x) = x^{\frac{s}{r}}$, is concave so inequalities in (2.9) are reversed. □

Theorem 2.5: *Let all the assuptions given in* **A** *be valid. Let $\varphi, \psi : [a, b] \to \mathbb{R}$ be strictly monotonic and continuous functions. If $\psi \circ \varphi^{-1}$ is convex on $[a, b]$, then*

$$\begin{aligned} \psi\left(\tilde{M}_\varphi^{[n]}\right) &\leq \min_I \left[W_I \psi\left(\tilde{M}_\varphi^{[I]}\right) + W_{\bar{I}} \psi\left(\tilde{M}_\varphi^{[\bar{I}]}\right)\right], \\ \psi\left(\tilde{M}_\varphi^{[n]}\right) &\geq \max_I \left[W_I \psi\left(\tilde{M}_\varphi^{[I]}\right) + W_{\bar{I}} \psi\left(\tilde{M}_\varphi^{[\bar{I}]}\right)\right] \end{aligned} \tag{2.10}$$

$$\textit{where} \quad \tilde{M}_\varphi^{[J]} = \varphi^{-1}\left(\varphi(a) + \varphi(b) - \frac{1}{W_J} \sum_{i \in J} w_i \varphi(x_i)\right).$$

Proof: Applying Theorem 2.1 to the convex function $f = \psi \circ \varphi^{-1}$ and replacing a, b, and x_i with $\varphi(a), \varphi(b)$, and $\varphi(x_i)$ respectively and then using Remark 2.2, we obtain (2.10). □

Remark 2.4: Theorem 2.2, 2.3 and 2.4 follow from Theorem 2.5, by choosing adequate functions φ,ψ and appropriate substitutions.

2.2 Further Refinements of the Jensen–Mercer Inequality

In present section we state some refinements of Jensen–Mercer type results which are extracted from [152].

D: Let all the assumptions as stared in **A** be valid, further for fixed values of k, we assume the (non-empty) subsets $J_1^k, \ldots, J_l^k$ of $\{1, 2, \ldots, n\}$, $1 \leq l \leq k \leq n$ such that $J_1^k \cup \ldots \cup J_l^k = \{1, 2, \ldots, n\}$ and $J_i^k \cap I_j^k = \emptyset$ for $i \neq j$, where $i, j = 1, \ldots, n$. Consider the functional

$$\overset{J_1^k,\ldots,J_l^k}{D_k}(\phi, \mathbf{w}, \mathbf{x}) := \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} \phi(a + b - \frac{1}{W_{J_j^k}} \sum_{i \in J_j^k} w_i x_i),$$

for the function $\phi : [a, b] \to \mathbb{R}$, $x_i \in [a, b]$, $w_i > 0$ and $W_J = \sum_{i \in J} w_i$, $J \subset \{1, \ldots, n\}$.

Theorem 2.6: *Assume* **D** *holds. Then*

$$\phi(a) + \phi(b) - \frac{1}{W_n} \sum_{i=1}^{n} w_i \phi(x_i) \geq \max_{J_1^3, J_2^3, J_3^3} \left[\overset{J_1^3,J_2^3,J_3^3}{D_3}(\phi, \mathbf{w}, \mathbf{x}) \right] \geq \max_{J_1^2, J_2^2} \left[\overset{J_1^2,J_2^2}{D_2}(\phi, \mathbf{w}, \mathbf{x}) \right]$$
$$\geq \phi\left(a + b - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right).$$

Theorem 2.7: *Assume* **D** *holds. Then*

$$\phi(a) + \phi(b) - \frac{1}{W_n} \sum_{i=1}^{n} w_i \phi(x_i) \geq \min_{J_1^3, J_2^3, J_3^3} [\overset{J_1^3,J_2^3,J_3^3}{D_3}(\phi, \mathbf{w}, \mathbf{x})]$$
$$\geq \min_{J_1^2, J_2^2} [\overset{J_1^2,J_2^2}{D_2}(\phi, \mathbf{w}, \mathbf{x})] \geq \phi(a + b - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i).$$

We conclude that Theorem 2.6 can be further generalized to k indexed set, and by observing the proof, we can work out that generalization enable us to tighten the results. In order to present the generalization, we start with some notations.

$$\mathcal{M}_2 := \max_{J_1^2, J_2^2} [\overset{J_1^2,J_2^2}{D_2}(\phi, \mathbf{w}, \mathbf{x})],$$
$$\vdots$$
$$\mathcal{M}_{k-1} := \max_{J_1^{k-1}, \ldots, J_{l-1}^{k-1}} [\overset{J_1^{k-1},\ldots,J_{l-1}^{k-1}}{D_{k-1}}(\phi, \mathbf{w}, \mathbf{x})],$$

where $J_1^{k-1}, \ldots, J_{l-1}^{k-1}$ are the (non-empty) subsets of $\{1, \ldots, n\}$. Using these notations, we can rewrite the inequality from Theorem 2.6 as follows:

$$\phi(a) + \phi(b) - \frac{1}{W_n} \sum_{i=1}^{n} w_i \phi(x_i) \geq \mathcal{M}_3 \geq \mathcal{M}_2 \geq \phi(a + b - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i).$$

Similarly we denote $\mathcal{H}_{k-1} := \min_{J_1^{k-1},\ldots,J_l^{k-1}} \left[D_{k-1}^{J_1^{k-1},\ldots,J_l^{k-1}}(\phi, \mathbf{w}, \mathbf{x}) \right]$.

Theorem 2.8: *Assume* **D** *holds. Then*

$$\phi(a) + \phi(b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(x_i) \geq \mathcal{M}_k \geq \mathcal{M}_{k-1} \geq \ldots \geq \mathcal{M}_3 \geq \mathcal{M}_2 \geq \phi(a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i).$$

Proof: See [152]. □

Theorem 2.9: *Assume* **D** *holds. Then*

$$\phi(a) + \phi(b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(x_i) \geq \mathcal{H}_k \geq \mathcal{H}_{k-1} \geq \ldots$$
$$\geq \mathcal{H}_3 \geq \mathcal{H}_2 \geq \phi(a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i). \quad (2.11)$$

Proof: The proof is similar to the proof of Theorem 2.8, but we use the minimum of the functional for l index sets instead functionals for $l-1$ index sets. □

Remark 2.5: The operator extensions of Theorem 2.8 and Theorem 2.11 can also be given by using the idea given in [118, 154].

Theorem 2.10: *Let $\phi : J \to \mathbb{R}$ be a convex function on the open interval J and $[a,b] \subset J$. If $x_i \in [a,b]$ and $w_1, \ldots, w_n \geq 0$ with $\sum_{i=1}^{n} w_i = 1$, then*

$$\begin{aligned} &\phi(a) + \phi(b) - \sum_{i=1}^{n} w_i\phi(x_i) - \phi(\overline{y}) \\ &\geq \left| \sum_{i=1}^{n} w_i \left| \phi(a+b-x_i) - \phi(\overline{y}) \right| - |\phi'_+(\overline{y})| \sum_{i=1}^{n} w_i \left| \sum_{i=1}^{n} w_i x_i - x_i \right| \right| \end{aligned} \quad (2.12)$$

where $\overline{y} = a + b - \sum_{i=1}^{n} w_i x_i$.

If the function ϕ is concave, then the left-hand side of (2.12) is

$$\phi(\overline{y}) - \phi(a) - \phi(b) + \sum_{i=1}^{n} w_i\phi(x_i).$$

Theorem 2.11: *Let $\phi : J \to \mathbb{R}$ be a monotone convex function on the open interval J and $[a,b] \subset J$. If $x_i \in [a,b]$ and $w_1, w_2, \ldots, w_n \geq 0$ with $\sum_{i=1}^{n} w_i = 1$. Let $I = \{i \in \{1, 2, \ldots, n\} =$*

$I_n : x_i \leq \sum_{i=1}^{n} w_i x_i\}$, *then*

$$\phi(a)+\phi(b)-\sum_{i=1}^{n} w_i\phi(x_i)-\phi(\overline{y}) \geq \Bigg|\sum_{i=1}^{n} w_i sgn\Big(\sum_{i=1}^{n} w_i x_i - x_i\Big)\Big[\phi(a+b-x_i)+\phi'_{+}(\overline{y})x_i\Big] - \Big(1-2W_{I_n\setminus I}\Big)\Big[\phi(\overline{y})+\phi'_{+}(\overline{y})\sum_{i=1}^{n} w_i x_i\Big]\Bigg|, \quad (2.13)$$

where $\overline{y} = a+b-\sum_{i=1}^{n} w_i x_i$.

If the function f is monotone concave, then the left-hand side of (2.13) is

$$\phi(\overline{y})-\phi(a)-\phi(b)+\sum_{i=1}^{n} w_i\phi(x_i).$$

2.2.1 Applications to Power Meanss

Theorem 2.12: *Under the assumptions stated in Theorems 2.10 and **B**, the following inequalities are valid:*

(i)
$$\tilde{A_n}-\tilde{G_n} \geq \Bigg|\sum_{i=1}^{n} w_i\Big|\frac{ab}{x_i}-\overline{y}\Big| - \overline{y}\sum_{i=1}^{n} w_i\Big|\ln\frac{G_n}{x_i}\Big|\Bigg|.$$

(ii)
$$\frac{1}{\tilde{H_n}}-\frac{1}{\tilde{G_n}} \geq \Bigg|\sum_{i=1}^{n} w_i\Big|\frac{x_i}{ab}-\frac{1}{\tilde{H_n}}\Big| - \frac{1}{\tilde{H_n}}\sum_{i=1}^{n} w_i\Big|\ln\frac{x_i}{G_n}\Big|\Bigg|.$$

(iii)
$$\frac{\tilde{A_n}}{\tilde{G_n}} \geq exp\Bigg(\Bigg|\sum_{i=1}^{n} w_i\Big|-\ln\Big((a+b-x_i)\overline{y}\Big)\Big| - \frac{1}{\overline{y}}\sum_{i=1}^{n} w_i\Big|\sum_{i=1}^{n} w_i x_i - x_i\Big|\Bigg|\Bigg).$$

(iv) *For $x>0, r\neq 0$ and $r\leq 1$, we have*

$$\tilde{A_n}-\tilde{M_n}^{[r]} \geq \Bigg|\sum_{i=1}^{n} w_i\Big|(a^r+b^r-x_i^r)^{\frac{1}{r}}-\tilde{M_n^{[r]}}\Big| - \frac{1}{r}\Big(a^r+b^r-\sum_{i=1}^{n} w_i x_i^r\Big)^{\frac{1-r}{r}}\sum_{i=1}^{n} w_i\Big|\sum_{i=1}^{n} w_i x_i^r - x_i^r\Big|\Bigg|.$$

(v) *For $0<r\leq s$ and $r,s\in\mathbb{R}$ we have*

$$\tilde{M}_n^{[s]}-\Big(\tilde{M}_n^{[r]}\Big)^s \geq \Bigg|\sum_{i=1}^{n} w_i\Big|(a^r+b^r-x_i^r)^{\frac{s}{r}}-\Big(a^r+b^r-\sum_{i=1}^{n} w_i x_i^r\Big)^{\frac{s}{r}}\Big| - \frac{s}{r}\Big(a^r+b^r-\sum_{i=1}^{n} w_i x_i^r\Big)^{\frac{s-r}{r}}\sum_{i=1}^{n} w_i\Big|\sum_{i=1}^{n} w_i x_i^r - x_i^r\Big|\Bigg|.$$

Theorem 2.13: *Under the assumptions stated in Theorems 2.11 and* **B**, *the following inequalities are valid:*

(*i*)

$$\tilde{A_n} - \tilde{G_n} \geq \left|\sum_{i=1}^{n} w_i sgn\left(\ln \frac{G_n}{x_i}\right)\left[\frac{ab}{x_i} - \overline{y}\ln x_i\right] - \left(1 - 2W_{I_n\setminus I}\right)\left[\overline{y}(1 - \sum_{i=1}^{n} w_i x_i)\right]\right|.$$

(*ii*)

$$\frac{1}{\tilde{H_n}} - \frac{1}{\tilde{G_n}} \geq \left|\sum_{i=1}^{n} w_i sgn\left(\ln \frac{x_i}{G_n}\right)\left[\frac{x_i}{ab} + \frac{\ln x_i}{\tilde{H_n}}\right] - \left(1 - 2W_{I_n\setminus I}\right)\left[\frac{1}{\tilde{H_n}} - \frac{1}{\tilde{H}_n H_n}\right]\right|.$$

(*iii*)

$$\frac{\tilde{A_n}}{\tilde{G_n}} \geq exp\Bigg(\sum_{i=1}^{n} sgn\left(\sum_{i=1}^{n} w_i x_i - x_i\right)\left[\ln\left(\frac{1}{a+b-x_i}\right) + \frac{x_i}{\overline{y}}\right] - (1 - 2W_{I_n\setminus I})\left[\ln\left(\frac{1}{\overline{y}}\right) + \frac{1}{\overline{y}}\sum_{i=1}^{n} w_i x_i\right]\Bigg).$$

(*iv*) *For* $x > 0, r \neq 0$ *and* $r \leq 1$, *we have*

$$\begin{aligned}\tilde{A_n} - \tilde{M_n}^{[r]} \geq \Bigg|&\sum_{i=1}^{n} w_i sgn\left(\sum_{i-1}^{n} w_i x_i^r - x_i^r\right)\\ &\times\left[\left(a^r + b^r - x_i^r\right)^{\frac{1}{r}} - \frac{1}{r}\left(a^r + b^r - \sum_{i=1}^{n} w_i x_i^r\right)^{\frac{1-r}{r}} x_i^r\right]\\ &- \left(1 - 2W_{I_n\setminus I}\right)\left[\tilde{M_n}^{[r]} - \frac{1}{r}\left(a^r + b^r - \sum_{i=1}^{n} w_i x_i^r\right)^{\frac{1-r}{r}}\sum_{i=1}^{n} w_i x_i^r\right]\Bigg|.\end{aligned}$$

(*v*) *For* $0 < r \leq s$ *and* $r, s \in \mathbb{R}$ *we have*

$$\begin{aligned}\tilde{M}_n^{[s]} - \left(\tilde{M}_n^{[r]}\right)^s \geq \Bigg|&\sum_{i=1}^{n} w_i sgn\left(\sum_{i=1}^{n} w_i x_i^r - x_i^r\right)\Bigg[\left(a^r + b^r - x_i^r\right)^{\frac{s}{r}}\\ &- \frac{s}{r}\left(a^r + b^r - \sum_{i=1}^{n} w_i x_i^r\right)^{\frac{s-r}{r}} x_i^r\Bigg] - \left(1 - 2W_{I_n\setminus I}\right)\Bigg[\left(a^r + b^r - \sum_{i=1}^{n} w_i x_i^r\right)^{\frac{s}{r}}\\ &- \frac{s}{r}\left(a^r + b^r - \sum_{i=1}^{n} w_i x_i^r\right)^{\frac{s-r}{r}}\sum_{i=1}^{n} w_i x_i^r\Bigg]\Bigg|.\end{aligned}$$

Remark 2.6: Similar to last section, we can proof the above stated results by suitable substitutions of functions and variables, see also [152] for the details.

Now, we generalize the given applications for quasi–arithmetic mean.

Theorem 2.14: *Under the assumptions stated in Theorems 2.10 and* **C***, the following inequality holds*

$$\psi\left(\tilde{M_\psi}^{[n]}\right) - \psi\left(\tilde{M_\varphi}^{[n]}\right) \geq \left| \sum_{i=1}^{n} w_i \middle| \psi\circ\varphi^{-1}\left(\varphi(a)+\varphi(b)-\varphi(x_i)-\psi\left(\tilde{M_\varphi}^{[n]}\right)\right)\right| \\ - \left|(\psi\circ\varphi^{-1})'\varphi\left(\tilde{M_\varphi}^{[n]}\right)\right| \sum_{i=1}^{n} w_i \left| \sum_{i=1}^{n} w_i\varphi(x_i)-\varphi(x_i)\right| \Bigg|, \quad (2.14)$$

provided that $\psi\circ\varphi^{-1}$ *is convex and* ψ *is strictly increasing or* $\psi\circ\varphi^{-1}$ *is concave and* ψ *is strictly decreasing.*

Theorem 2.15: *Under the assumptions stated in Theorems 2.11 and* **C***, the following inequality holds:*

$$\psi\left(\tilde{M_\psi}^{[n]}\right) - \psi\left(\tilde{M_\varphi}^{[n]}\right) \geq \left| \sum_{i=1}^{n} w_i sgn\left(\sum_{i=1}^{n} w_i\varphi(x_i)-\varphi(x_i)\right)\left[\psi\circ\varphi^{-1}\left(\varphi(a)+\varphi(b)-\varphi(x_i)\right)\right.\right. \\ \left.\left. + (\psi\circ\varphi^{-1})'\varphi\left(\tilde{M_\varphi}^{[n]}\right)\varphi(x_i)\right] - \left(1-2W_{I_n\setminus I}\right)\left[\psi\left(\tilde{M_\varphi}^{[n]}\right)+(\psi\circ\varphi^{-1})'\varphi\left(\tilde{M_\varphi}^{[n]}\right)\sum_{i=1}^{n} w_i x_i\right]\right|, \quad (2.15)$$

provided that $\psi\circ\varphi^{-1}$ *is convex and* ψ *is strictly increasing or* $\psi\circ\varphi^{-1}$ *is concave and* ψ *is strictly decreasing.*

Theorem 2.16: *Under the assumptions stated in Theorems 2.6, the following inequalities are valid:*

(*i*)
$$\tilde{A_n} \geq \max\left[\frac{W_{J_1^k}}{W_n}\left(\tilde{G_{J_1^k}}\right)+\cdots+\frac{W_{J_l^k}}{W_n}\left(\tilde{G_{J_l^k}}\right)\right] \geq \tilde{G_n}.$$

(*ii*)
$$\frac{1}{\tilde{H_n}} \geq \max\left[\frac{W_{J_1^k}}{W_n}\left(\frac{1}{\tilde{G_{J_1^k}}}\right)+\cdots+\frac{W_{J_l^k}}{W_n}\left(\frac{1}{\tilde{G_{J_l^k}}}\right)\right] \geq \frac{1}{\tilde{G_n}}.$$

(*iii*)
$$-\ln\tilde{G_n} \geq \max\left[-\frac{W_{J_1^k}}{W_n}\ln\tilde{A_{J_1^k}}-\cdots-\frac{W_{J_l^k}}{W_n}\ln\tilde{A_{J_l^k}}\right] \geq -\ln\tilde{A_n}.$$

(iv)

$$\tilde{A}_n \geq \max\left[\frac{W_{J_1^k}}{W_n}\left(\tilde{M^{[r]}_{J_1^k}}\right)+\cdots+\frac{W_{J_l^k}}{W_n}\left(\tilde{M^{[r]}_{J_l^k}}\right)\right] \geq \tilde{M^{[r]}_n}.$$

(v)

$$\left(\tilde{M^{[s]}_n}\right)^s \geq \max\left[\frac{W_{J_1^k}}{W_n}\left(\tilde{M^{[r]}_{J_1^k}}\right)^s+\cdots+\frac{W_{J_l^k}}{W_n}\left(\tilde{M^{[r]}_{J_l^k}}\right)^s\right] \geq \left(\tilde{M^{[r]}_n}\right)^s.$$

(vi)

$$\varphi\left(\tilde{M^{[n]}_\varphi}\right) \geq \max\left[\frac{W_{J_1^k}}{W_n}\varphi\left(\tilde{M^{J_1^k}_\phi}\right)+\cdots+\frac{W_{J_l^k}}{W_n}\varphi\left(\tilde{M^{J_l^k}_\phi}\right)\right] \geq \varphi\left(\tilde{M^{[n]}_\phi}\right).$$

where in last result (vi) we assumed that $\varphi\circ\phi^{-1}$ is convex on $[a,b]$.

Remark 2.7: Similar results as stated in above theorem can also be stated for Theorem 2.6 by using inequality (2.11).

2.3 New Refinements of the Jensen–Mercer Inequality Associated to Positive n–Tuples

The results of this section are extracted from [153]:

Theorem 2.17: *Let $f:[a,b]\to\mathbb{R}$ be a convex function and let $x_i\in[a,b]$, $w_i,p_i,q_i\in\mathbb{R}^+$ such that $W_n=\sum_{i=1}^n w_i$ and $p_i+q_i=1$ for each $i\in\{1,2,\ldots,n\}$, then*

$$\begin{aligned} f\left(a+b-\frac{1}{W_n}\sum_{i=1}^n w_ix_i\right) &\leq \sum_{i=1}^n w_ip_if\left(\frac{\sum_{i=1}^n w_ip_i(a+b-x_i)}{\sum_{i=1}^n w_ip_i}\right)\\ &+\sum_{i=1}^n w_iq_if\left(\frac{\sum_{i=1}^n w_iq_i(a+b-x_i)}{\sum_{i=1}^n w_iq_i}\right) \leq f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^n w_if(x_i). \end{aligned} \tag{2.16}$$

In the following theorem we present integral version of the above theorem.

Theorem 2.18: *Let $f:[a,b]\to\mathbb{R}$ be a convex function defined on the interval $[a,b]$. Let $p,u,v,g:[\alpha,\beta]\to\mathbb{R}$ be integrable functions such that $g(\omega)\in[a,b], u(\omega),v(\omega),p(\omega)\in\mathbb{R}^+$ for all $\omega\in[\alpha,\beta]$ and $v(\omega)+u(\omega)=1$, $P=\int_\alpha^\beta p(\omega)d\omega$. Then*

$$f\left(a+b-\frac{1}{P}\int_\alpha^\beta p(\omega)g(\omega)d\omega\right) \leq \frac{1}{P}\int_\alpha^\beta u(\omega)p(\omega)d\omega f\left(\frac{\int_\alpha^\beta p(\omega)u(\omega)(a+b-g(\omega))d\omega}{\int_\alpha^\beta p(\omega)u(\omega)d\omega}\right) \tag{2.17}$$

$$+\frac{1}{P}\int_{\alpha}^{\beta}p(\omega)v(\omega)d\omega f\left(\frac{\int_{\alpha}^{\beta}p(\omega)v(\omega)(a+b-g(\omega))d\omega}{\int_{\alpha}^{\beta}p(\omega)v(\omega)d\omega}\right)\leq f(a)+f(b)-\frac{1}{P}\int_{\alpha}^{\beta}p(\omega)f(g(\omega))d\omega.$$

If the function f is concave then the reverse inequalities hold in (2.17).

In the following theorem we present a refinement of variant of Jensen-Steffensen inequality:

Theorem 2.19: *Let $f:[a,b]\rightarrow\mathbb{R}$ be a convex function. Let $x_i\in[a,b]$, $w_i,p_i,q_i\in\mathbb{R}, w_ip_i, w_iq_i\neq 0$ for $i\in\{1,2,\ldots,n\}$, such that $p_i+q_i=1$ for all $i\in\{1,2,\ldots,n\}$ and $W_n=\sum_{i=1}^{n}w_i$. If $x_1\leq x_2\leq\cdots\leq x_n$ or $x_1\geq x_2\geq\cdots\geq x_n$ and*

$$0\leq\sum_{i=1}^{k}w_ip_i\leq\sum_{i=1}^{n}w_ip_i,\ \ k\in\{1,2,\ldots,n\},\quad \sum_{i=1}^{n}w_ip_i>0\quad \textit{and} \tag{2.18}$$

$$0\leq\sum_{i=1}^{k}w_iq_i\leq\sum_{i=1}^{n}w_iq_i,\ \ k\in\{1,2,\ldots,n\},\quad \sum_{i=1}^{n}w_iq_i>0. \tag{2.19}$$

Then

$$\begin{aligned}f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right)&\leq\frac{1}{W_n}\sum_{i=1}^{n}p_iw_if\left(a+b-\frac{\sum_{i=1}^{n}w_ip_i(a+b-x_i)}{\sum_{i=1}^{n}w_ip_i}\right)\\&+\frac{1}{W_n}\sum_{i=1}^{n}w_iq_if\left(\frac{\sum_{i=1}^{n}w_iq_i(a+b-x_i)}{\sum_{i=1}^{n}w_iq_i}\right)\leq f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i).\end{aligned} \tag{2.20}$$

If the function f is concave then the reverse inequalities hold in (2.20).

Proof: Since $p_i+q_i=1$ for all $i\in\{1,2,\ldots,n\}$, therefore we have

$$\begin{aligned}&f\Big(a+b-\tfrac{1}{W_n}\sum_{i=1}^{n}w_ix_i\Big)=f\Big(\tfrac{1}{W_n}\sum_{i=1}^{n}w_i(a+b-x_i)\Big)\\&=f\Big(\tfrac{1}{W_n}\sum_{i=1}^{n}w_ip_i(a+b-x_i)+\tfrac{1}{W_n}\sum_{i=1}^{n}w_iq_i(a+b-x_i)\Big)\\&=f\left(\frac{\sum_{i=1}^{n}w_ip_i}{W_n}\frac{\sum_{i=1}^{n}w_ip_i(a+b-x_i)}{\sum_{i=1}^{n}w_ip_i}+\frac{\sum_{i=1}^{n}w_iq_i}{W_n}\frac{\sum_{i=1}^{n}w_iq_i(a+b-x_i)}{\sum_{i=1}^{n}w_iq_i}\right)\end{aligned} \tag{2.21}$$

Since $\sum_{i=1}^{n}w_ip_i>0$ and $\sum_{i=1}^{n}w_iq_i>0$, therefore $\sum_{i=1}^{n}w_ip_i+\sum_{i=1}^{n}w_iq_i>0$. Which gives that $W_n>0$.

Now by applying convexity of f on the right side of (2.21) one may obtain

$$
\begin{aligned}
& f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i(a+b-x_i)\right) \\
& \le \frac{\sum_{i=1}^{n} w_i p_i}{W_n} f\left(\frac{\sum_{i=1}^{n} w_i p_i(a+b-x_i)}{\sum_{i=1}^{n} w_i p_i}\right) + \frac{\sum_{i=1}^{n} w_i q_i}{W_n} f\left(\frac{\sum_{i=1}^{n} w_i q_i(a+b-x_i)}{\sum_{i=1}^{n} w_i q_i}\right) \\
& \le \frac{1}{W_n}\sum_{i=1}^{n} w_i p_i\left(f(a)+f(b)-\frac{\sum_{i=1}^{n} w_i p_i f(x_i)}{\sum_{i=1}^{n} w_i p_i}\right) \\
& + \frac{1}{W_n}\sum_{i=1}^{n} w_i q_i\left(f(a)+f(b)-\frac{\sum_{i=1}^{n} w_i q_i f(x_i)}{\sum_{i=1}^{n} w_i q_i}\right) \\
& = f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i).
\end{aligned}
$$

The above inequality holds by using Theorem *A*.6. □

Remark 2.8: If we add (2.18) and (2.19), then the variant of Jensen- Steffensen inequality conditions (*A*.9) will be obtained.

2.3.1 Applications to Power Means

Under the assumptions of Theorem 2.17 and **B**, we give the following results.

Corollary 2.3: *The following inequality is valid*

$$\tilde{G}_n(\boldsymbol{x};\boldsymbol{w}) \le \sum_{i=1}^{n} w_i p_i \tilde{G}_n(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{p}) + \sum_{i=1}^{n} w_i q_i \tilde{G}_n(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{q}) \le \tilde{A}_n(\boldsymbol{x};\boldsymbol{w}). \tag{2.22}$$

Proof: Use the function $f(x)=\exp(x)$ and replacing $a, b,$ and x_i by $\ln a, \ln b,$ and $\ln x_i$ in (2.16) we will get (2.22). □

Corollary 2.4: *By taking* $a \to \frac{1}{a}, b \to \frac{1}{b}, x_i \to \frac{1}{x_i}$, *in* (2.22) *we have*

$$\frac{1}{\tilde{G}_n(\boldsymbol{x};\boldsymbol{w})} \le \frac{\sum_{i=1}^{n} w_i p_i}{\tilde{G}_n(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{p})} + \frac{\sum_{i=1}^{n} w_i q_i}{\tilde{G}_n(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{q})} \le \frac{1}{\tilde{H}_n(\boldsymbol{x};\boldsymbol{w})}.$$

Corollary 2.5: *The following inequality holds*

$$\tilde{G}_n(\boldsymbol{x};\boldsymbol{w}) \le \left(\tilde{A}_n(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{p})\right)^{\sum_{i=1}^{n} w_i p_i}\left(\tilde{A}_n(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{q})\right)^{\sum_{i=1}^{n} w_i q_i} \le \tilde{A}_n(\boldsymbol{x};\boldsymbol{w}). \tag{2.23}$$

Proof: Using the function $f(x)=-\ln x$ in (2.16) and then applying exponential function, we will get (2.23). □

Corollary 2.6: *For $s \neq 0$ and $s \leq 1$, we have*

$$\tilde{M}_n^{[s]}(\boldsymbol{x};\boldsymbol{w}) \leq \sum_{i=1}^{n} w_i p_i \tilde{M}_n^{[s]}(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{p}) + \sum_{i=1}^{n} w_i q_i \tilde{M}_n^{[s]}(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{q}) \leq \tilde{A}_n(\boldsymbol{x};\boldsymbol{w}). \tag{2.24}$$

Proof: Use the function $f(x) = x^{\frac{1}{s}}$ and replace a, b, and x_i by a^s, b^s, and x_i^s respectively in (2.16) to get (2.24). □

Corollary 2.7: *For $t, s \in \mathbb{R}$ with $0 < t \leq s$, we have*

$$\left(\tilde{M}_n^{[t]}(\boldsymbol{x};\boldsymbol{w})\right)^s \leq \sum_{i=1}^{n} w_i p_i \left(\tilde{M}_n^{[t]}(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{p})\right)^s + \sum_{i=1}^{n} w_i q_i \left(\tilde{M}_n^{[t]}(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{q})\right)^s \leq \tilde{M}_n^{[s]}(\boldsymbol{x};\boldsymbol{w}). \tag{2.25}$$

Proof: Use the function $f(x) = x^{\frac{s}{t}}$ and replace a, b, and x_i by a^t, b^t, and x_i^t respectively in (2.16) to get (2.25). □

Now, we generalize the given applications for quasi–arithmetic mean.

Corollary 2.8: *The following inequality holds*

$$f\left(\tilde{M}_\phi^{[n]}(\boldsymbol{x};\boldsymbol{w})\right) \leq \sum_{i=1}^{n} w_i p_i f\left(\tilde{M}_\phi^{[n]}(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{p})\right) + \sum_{i=1}^{n} w_i q_i f\left(\tilde{M}_\phi^{[n]}(\boldsymbol{x};\boldsymbol{w}\cdot\boldsymbol{q})\right)$$
$$\leq f\left(\tilde{M}_\phi^{[n]}(\boldsymbol{x};\boldsymbol{w})\right). \tag{2.26}$$

provided that $f \circ \phi^{-1}$ is convex and f is strictly increasing.

Proof: Replacing $f(x)$ by $f \circ \phi^{-1}(x)$ and $a,\ b, x_i$ by $\phi(a), \phi(b), \phi(x_i)$ respectively in (2.16), then apply f^{-1} to get (2.26). □

2.3.2 Further Generalizations

In this section, we present further refinement of the Jensen–Mercer as well as variant of the Jensen–Mercer inequalities concerning to m sequences whose sum is equal to unity.

Theorem 2.20: *Let $f : [a, b] \to \mathbb{R}$ be a convex function defined on the interval $[a, b]$. Let $x_i \in [a, b]$, $w_i, q_i^l \in \mathbb{R}^+$ $(i \in \{1, 2, \ldots, n\}, l \in \{1, 2, \ldots, m\})$ such that $\sum_{l=1}^{m} q_i^l = 1$ for each $i \in \{1, 2, \ldots, n\}$, $W_n = \sum_{i=1}^{n} w_i$. Assume that L_1 and L_2 be non empty disjoint subsets of $\{1, 2, \ldots, m\}$ such that $L_1 \cup L_2 = \{1, 2, \ldots, m\}$. Then*

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n}\sum_{l\in L_1} q_i^l w_i f\left(\frac{\sum_{i=1}^{n}\sum_{l\in L_1} q_i^l w_i (a+b-x_i)}{\sum_{i=1}^{n}\sum_{l\in L_1} q_i^l w_i}\right)$$
$$+\frac{1}{W_n}\sum_{i=1}^{n}\sum_{l\in L_2} q_i^l w_i f\left(\frac{\sum_{i=1}^{n}\sum_{l\in L_2} q_i^l w_i (a+b-x_i)}{\sum_{i=1}^{n}\sum_{l\in L_2} q_i^l w_i}\right) \leq f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i). \quad (2.27)$$

If the function f is concave, then the reverse inequalities hold in (2.27).

Remark 2.9: We can also give applications of Theorem 2.20 as given in previous sections.

The following theorem is the integral analogue of Theorem 2.20.

Theorem 2.21: *Let $f : I \to \mathbb{R}$ be a convex function defined on the interval I. Let $p, g, u_l \in L[a,b]$ such that $g(\omega) \in I, p(\omega), u_l(\omega) \in \mathbb{R}^+$ for all $\omega \in [a,b]$ $(l \in \{1,2,\ldots,n\})$ and $\sum\limits_{l=1}^{n} u_l(\omega) = 1$, $P = \int_a^b p(\omega)d\omega$. Assume that L_1 and L_2 be non empty disjoint subsets of $\{1,2,\ldots,n\}$ such that $L_1 \cup L_2 = \{1,2,\ldots,n\}$. Then*

$$\frac{1}{P}\int_a^b p(\omega)f(g(\omega))\,d\omega \geq \frac{1}{P}\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)\,d\omega\, f\left(\frac{\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)g(\omega)\,d\omega}{\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)\,d\omega}\right)$$
$$+\frac{1}{P}\int_a^b \sum_{l\in L_2} u_l(\omega)p(\omega)\,d\omega\, f\left(\frac{\int_a^b \sum_{l\in L_2} u_l(\omega)p(\omega)g(\omega)\,d\omega}{\int_a^b \sum_{l\in L_2} u_l(\omega)p(\omega)\,d\omega}\right) \geq f\left(\frac{1}{P}\int_a^b p(\omega)g(\omega)\,d\omega\right). \quad (2.28)$$

If the function f is concave then the reverse inequalities hold in (2.28).

In the following theorem, we present further refinement of the variant of Jensen-Steffensen's inequality associated to m certain sequences.

Theorem 2.22: *Let $f : [a,b] \to \mathbb{R}$ be a convex function defined on the interval $[a,b]$. Let $x_i \in I$, $w_i, q_i^l \in \mathbb{R}, w_i p_i, w_i q_i^l \neq 0$ $(i \in \{1,2,\ldots,n\}, l \in \{1,2,\ldots,m\})$ such that $\sum_{l=1}^{m} q_i^l = 1$ for each $i \in \{1,2,\ldots,n\}$ and $W_n = \sum_{i=1}^{n} w_i$. Assume that L_1 and L_2 be non empty disjoint subsets of $\{1,2,\ldots,m\}$ such that $L_1 \cup L_2 = \{1,2,\ldots,m\}$. If $x_1 \leq x_2 \leq \cdots \leq x_n$ or $x_1 \geq x_2 \geq \cdots \geq x_n$, and $\sum_{i=1}^{n} w_i q_i^l > 0$ for each $l \in \{1,2,\ldots,m\}$ and*

$$0 \leq \sum_{i=1}^{k} w_i q_i^l \leq \sum_{i=1}^{n} w_i q_i^l, \ \ k \in \{1,2,\ldots,n\},$$

then

$$\begin{aligned} f\Big(a+b-\tfrac{1}{W_n}\sum_{i=1}^{n}w_ix_i\Big) &\le \frac{1}{W_n}\sum_{i=1}^{n}\sum_{l\in L_1}q_i^lw_if\left(\frac{\sum_{i=1}^{n}\sum_{l\in L_1}q_i^lw_i(a+b-x_i)}{\sum_{i=1}^{n}\sum_{l\in L_1}q_i^lw_i}\right)\\ &+\frac{1}{W_n}\sum_{i=1}^{n}\sum_{l\in L_2}q_i^lw_if\left(\frac{\sum_{i=1}^{n}\sum_{l\in L_2}q_i^lw_i(a+b-x_i)}{\sum_{i=1}^{n}\sum_{l\in L_2}q_i^lw_i}\right)\le f(a)+f(b)-\tfrac{1}{W_n}\sum_{i=1}^{n}w_if(x_i) \end{aligned} \tag{2.29}$$

If the function f is concave then the reverse inequalities hold in (2.29).

Proof: Since $\sum_{i=1}^{n}w_iq_i^l>0$ for each $l\in\{1,2,\ldots,m\}$, therefore $\sum_{l=1}^{m}\sum_{i=1}^{n}\times w_iq_i^l>0$. Also, $\sum_{l=1}^{m}q_i^l=1$. Hence we conclude that $W_n>0$.

Now proceeding in the same way as in the proof of Theorem 2.20 but use variant of Jensen-Steffensen's inequality instead of Jensen-Steffensem's inequality, we will obtain (2.29). □

2.4 Refinements of the Jessen–Mercer Inequality

The following refinement of (1.6) holds.

Theorem 2.23: *Under the assumptions of Theorem A.14, we have*

$$\varphi(a+b-A(f))\le\overline{D}(A,f,\varphi;E_1)\le\varphi(a)+\varphi(b)-A(\varphi(f)); \tag{2.30}$$

where

$$\overline{D}(A,f,\varphi;E_1)=A(\chi_{E_1})\,\varphi\bigg(a+b-\frac{A(f\cdot\chi_{E_1})}{A(\chi_{E_1})}\bigg)+A(\chi_{E\setminus E_1})\,\varphi\bigg(a+b-\frac{A(f\cdot\chi_{E\setminus E_1})}{A(\chi_{E\setminus E_1})}\bigg) \tag{2.31}$$

for all $E_1\in\mathfrak{A}$ such that $0<A(\chi_{E_1})<1$.

Proof: The first inequality follows by using the definition of a convex function and the second inequality follows by using (1.6) for $A_1(f)$ instead of $A(f)$. □

Remark 2.10: In [167] from the proof of Theorem 4.1 we have left the inequality of (2.31).

Remark 2.11: Note that the inequality (2.30) can be written in an equivalent form as

$$\varphi(a+b-A(f))\le\min_{E_1\in\mathfrak{A}}\overline{D}(A,f,\varphi;E_1),\qquad \varphi(a)+\varphi(b)-A(\varphi(f))\ge\max_{E_1\in\mathfrak{A}}\overline{D}(A,f,\varphi;E_1).$$

The following particular case of Theorem 2.23 is of interest:

Corollary 2.9: *Let (Ω, P, μ) be a probability measure space, and let $f : \Omega \to [a, b]$ be a measurable function. Then for any continuous convex function $\varphi : [a, b] \to \mathbb{R}$, and for any set E_1 in P with $\mu(E_1) \neq 0$ we have*

$$\begin{aligned}
\varphi\left(a+b-\int_{\Omega} f d\mu\right) &\leq \min_{E_1 \in P}\left[\mu(E_1)\varphi\left(a+b-\frac{1}{\mu(E_1)}\int_{E_1} f d\mu\right)\right. \\
&\quad \left. +\mu(\Omega\setminus E_1)\varphi\left(a+b-\frac{1}{\mu(\Omega\setminus E_1)}\int_{\Omega\setminus E_1} f d\mu\right)\right] \\
\text{and} \qquad \varphi(a)+\varphi(b)-\int_{\Omega}\varphi(f)d\mu &\geq \max_{E_1 \in P}\left[\mu(E_1)\varphi\left(a+b-\frac{1}{\mu(E_1)}\int_{E_1} f d\mu\right)\right. \\
&\quad \left. +\mu(\Omega\setminus E_1)\varphi\left(a+b-\frac{1}{\mu(\Omega\setminus E_1)}\int_{\Omega\setminus E_1} f d\mu\right)\right].
\end{aligned}$$

Proof: This is a special case of Theorem 2.23 for the functional A defined on the class $L^1(\mu)$ as $A(f) = \int_\Omega f d\mu$. □

Remark 2.12: We may also obtain similar results as in Theorem 2.5 for the generalized quasi-arithmetic means of Mercer's type defined in [76], as

$$\tilde{M}_\varphi(f, A) = \varphi^{-1}(\varphi(a) + \varphi(b) - A(\varphi(f))).$$

Remark 2.13: For recent and new refinements of Jessen–Mercer inequality for convex functions, we refer the reader to [148]. Interested readers can also see [59] for Jessen–Mercer type inequalities for coordinated convex functions via fractional integrals.

2.5 Index Set Function and Refinements of the Jensen–Mercer Inequality

In this section we give some refinements of inequality (1.4) using the index set functions. From these refinements we obtain some refinements of the well known inequalities among arithmetic, geometric, harmonic, power and quasi-arithmetic means of the Mercer's type [167].

Let I be a finite nonempty set of positive integers, and let $f : [a, b] \to \mathbb{R}$. Let $\mathbf{w} = \{w_i\}_{i\in I}$, $\mathbf{x} = \{x_i\}_{i\in I}$ be real sequences such that $x_i \in [a, b]$ for all $i \in I$, and $A_I(\mathbf{x}, \mathbf{w}) = \frac{1}{W_I}\sum_{i\in I} w_i x_i \in [a, b]$, where $W_I = \sum_{i\in I} w_i$. If we define the index set function F as

$$F(I) = W_I\left[f(a) + f(b) - \frac{1}{W_I}\sum_{i\in I} w_i f(x_i) - f\left(a+b-\frac{1}{W_I}\sum_{i\in I} w_i x_i\right)\right],$$

then the following theorem is valid.

Theorem 2.24: *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Let I and J be finite nonempty sets of positive integers such that $I\cap J=\varnothing$. Let $\mathbf{w}=\{w_i\}_{i\in I\cup J}$, $\mathbf{x}=\{x_i\}_{i\in I\cup J}$ be real sequences such that $x_i\in[a,b]$ $(i\in I\cup J)$, $W_{I\cup J}>0$, and $A_S(\mathbf{x},\mathbf{w})\in[a,b]$ $(S=I,J,I\cup J)$. If $W_I>0$ and $W_J>0$, then*

$$F(I\cup J)\geq F(I)+F(J). \tag{2.32}$$

If $W_I\cdot W_J<0$, then the inequality in (2.32) is reversed.

Proof: See [167] for idea of the proof. □

Corollary 2.10: *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Let $I_1,\dots,I_k$ be finite nonempty sets of positive integers such that $I_i\cap I_j=\varnothing$, for all $i\neq j\in\{1,\dots,k\}$. Let $\mathbf{w}=\{w_i\}_{i\in\cup_{j=1}^k I_j}$, $\mathbf{x}=\{x_i\}_{i\in\cup_{j=1}^k I_j}$ be real sequences such that $x_i\in[a,b]$ $\left(i\in\cup_{j=1}^k I_j\right)$, $W_{\cup_{j=1}^k I_j}>0$, and $A_S(\mathbf{x},\mathbf{w})\in[a,b]$ $\left(S=\{I_1,\dots,I_k,\cup_{j=1}^l I_j\},\right.$
$(l=2,\dots,n)\big)$. If $W_{I_j}>0$ $(j\in\{1,\dots,k\})$, then*

$$F\left(\bigcup_{j=1}^k I_j\right)\geq\sum_{j=1}^k F(I_j). \tag{2.33}$$

If $W_{I_1}>0$ and $W_{I_j}<0$ for $j\in\{2,\dots,k\}$, then the inequality in (2.33) is reversed.

Proof: Directly from Theorem 2.24 by induction. □

The following corollaries give refinements of the inequality (1.4).

Corollary 2.11: *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $I_k=\{1,\dots,k\}$ for $k\in\{1,\dots,n\}$. Let $\mathbf{w}=\{w_i\}_{i\in I_n}$, $\mathbf{x}=\{x_i\}_{i\in I_n}$ be real sequences such that $x_i\in[a,b]$ $(i\in I_n)$, and $w_1>0$. If $w_i\geq 0$ for $i\in\{2,\dots,n\}$, then*

$$F(I_n)\geq F(I_{n-1})\geq\cdots\geq F(I_2)\geq F(I_1)\geq 0.$$

If $w_i\leq 0$ for $i\in\{2,\dots,n\}$, $W_{I_n}>0$ and $A_{I_n}(\mathbf{x},\mathbf{w})\in[a,b]$, then

$$0\leq F(I_n)\leq F(I_{n-1})\leq\cdots\leq F(I_2)\leq F(I_1).$$

Corollary 2.12: *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $I_k=\{1,\dots,k\}$ $(k=1,\dots,n)$.*

Let $\mathbf{w} = \{w_i\}_{i\in I_n}$, $\mathbf{x} = \{x_i\}_{i\in I_n}$ *be real sequences such that* $x_i \in [a, b]$ $(i \in I_n)$.

If $w_i > 0$ *for all* $i \in \{1, \ldots, n\}$, *then*

$$F(I_n) \geq \max_{1\leq i<j\leq n} \left\{ (w_i + w_j) \left[f(a) + f(b) - \frac{w_i f(x_i) + w_j f(x_j)}{w_i + w_j} \right.\right.$$
$$\left.\left. - f\left(a + b - \frac{w_i x_i + w_j x_j}{w_i + w_j} \right) \right] \right\}, \tag{2.34}$$

and

$$F(I_n) \geq \max_{1\leq i\leq n} \{ w_i [f(a) + f(b) - f(x_i) - f(a + b - x_i)] \}. \tag{2.35}$$

If $\mathbf{w}$ *satisfy* (1.4) *and* $A_{I_n}(\mathbf{x}, \mathbf{w}) \in [a, b]$, *then*

$$F(I_n) \leq \min_{2\leq j\leq n} \left\{ (w_1 + w_j) \left[f(a) + f(b) - \frac{w_1 f(x_1) + w_j f(x_j)}{w_1 + w_j} \right.\right.$$
$$\left.\left. - f\left(a + b - \frac{w_1 x_1 + w_j x_j}{w_1 + w_j} \right) \right] \right\}. \tag{2.36}$$

Proof: Suppose that $w_i > 0$ for all $i \in \{1, \ldots, n\}$. Similarly as $F(I_n) \geq F(I_2)$ in Corollary 2.11, we may conclude that

$$F(I_n) \geq F(\{i, j\}) \text{ for all } i \neq j \in \{1, \ldots, n\}, \tag{2.37}$$

so the inequality (2.34) immediately follows. From (2.37) we have $F(I_n) \geq F(\{i\})$ for all $i \in \{1, \ldots, n\}$, so the inequality (2.35) is also proved.

The inequality (2.36) can be proved in the similar way. □

2.5.1 Index Set Function and Jessen–Mercer Inequality

Let all the assumptions of Theorem 2.23 be valid. We define the function $F_f : \mathcal{A} \to \mathbb{R}$ with

$$F_f(E_1) = A(\chi_{E_1}) \left[\varphi(a) + \varphi(b) - \frac{A(\varphi(f) \cdot \chi_{E_1})}{A(\chi_{E_1})} - \varphi\left(a + b - \frac{A(f \cdot \chi_{E_1})}{A(\chi_{E_1})} \right) \right].$$

Theorem 2.25: *Under the above assumptions, we have*

$$F_f(E) \geq F_f(E_1) + F_f(E \setminus E_1) \geq F_f(E_1) \geq 0 \tag{2.38}$$

for all $E_1 \in \mathcal{A}$ *such that* $0 < A(\chi_{E_1}) < 1$.

Proof: See [167]. □

Corollary 2.13: *Let* $\{E_1, \ldots, E_n\}$ *be a partition of* E *(i.e.,* $E = \bigcup_{j=1}^{n} E_j, E_i \cap E_j = \emptyset$ *for all* $i \neq j \in \{1, \ldots, n\}$*) such that* $0 < A\left(\chi_{E_j}\right) < 1$ *for all* $j \in \{1, \ldots, n\}$*. If* φ *is convex, then*

$$F_f(E) \geq \sum_{j=1}^{n} F_f(E_j), \tag{2.39}$$

$$F_f(E) \geq F_f\left(\bigcup_{j=1}^{n-1} E_j\right) \geq F_f\left(\bigcup_{j=1}^{n-2} E_j\right) \geq \cdots \geq F_f(E_1 \cup E_2) \geq F_f(E_1) \geq 0. \tag{2.40}$$

Proof: Directly from Theorem 2.25 by induction. □

Remark 2.14: If φ is concave, then the inequalities in (2.38), (2.39) and (2.40) are reversed.

Corollary 2.14: *Let* $\{E_1, \ldots, E_n\}$ *be a partition of* E *such that* $0 < A\left(\chi_{E_j}\right) < 1$ *for all* $j \in \{1, \ldots, n\}$*. If* φ *is convex, then*

$$F_f(E) \geq \max_{1 \leq i < j \leq n} \{F_f(E_i \cup E_j)\}, \qquad F_f(E) \geq \max_{1 \leq j \leq n} \{F_f(E_j)\}. \tag{2.41}$$

If φ *is concave, then the inequalities in* (2.41)*, with* max *replaced by* min*, are reversed.*

Remark 2.15: Also, we can get similar results as in Theorem 2.29 and Corollary 2.17 for generalized quasi-arithmetic means of Mercer's type.

Remark 2.16: Analogous assertions can be formulated for concave functions using the fact that f is concave iff $-f$ is convex.

Remark 2.17: Analogous results as stated in application part of Subsection 2.1 can also be stated for this section as well.

2.6 Refinements of Some Inequalities Among Means

Let all the assumptions of Corollary 2.11 and **B** be valid. For the various properties of these means and relations among them we refer the reader to [47]. For example, it is well known that

$$\left(\frac{A_n}{G_n}\right)^{W_n} \geq \left(\frac{A_{n-1}}{G_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \left(\frac{A_1}{G_1}\right)^{W_1} \geq 1,$$

$$W_n(A_n - G_n) \geq W_{n-1}(A_{n-1} - G_{n-1}) \geq \cdots \geq W_1(A_1 - G_1) \geq 0.$$

Theorem 2.26: *Under the above stated assumptions, we have*

(*i*)

$$\left(\frac{\widetilde{A}_n}{\widetilde{G}_n}\right)^{W_n} \geq \left(\frac{\widetilde{A}_{n-1}}{\widetilde{G}_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \left(\frac{\widetilde{A}_1}{\widetilde{G}_1}\right)^{W_1} \geq 1, \tag{2.42}$$

(*ii*)

$$W_n\left(\widetilde{A}_n - \widetilde{G}_n\right) \geq W_{n-1}\left(\widetilde{A}_{n-1} - \widetilde{G}_{n-1}\right) \geq \cdots \geq W_1\left(\widetilde{A}_1 - \widetilde{G}_1\right) \geq 0.$$

Proof: (*i*) Applying Corollary 2.11 to the convex function $f(x) = -\ln x$, we obtain

$$\ln\left(\frac{\widetilde{A}_n}{\widetilde{G}_n}\right)^{W_n} \geq \ln\left(\frac{\widetilde{A}_{n-1}}{\widetilde{G}_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \ln\left(\frac{\widetilde{A}_1}{\widetilde{G}_1}\right)^{W_1} \geq 0,$$

from which (2.42) follows. (*ii*) Applying Corollary 2.11 to the convex function $f(x) = \exp x$, and replacing a, b, and x_i with $\ln a$, $\ln b$, and $\ln x_i$ respectively, we obtain

$$W_n\left(\widetilde{A}_n - \widetilde{G}_n\right) \geq W_{n-1}\left(\widetilde{A}_{n-1} - \widetilde{G}_{n-1}\right) \geq \cdots \geq W_1\left(\widetilde{A}_1 - \widetilde{G}_1\right) \geq 0,$$

since in this case

$$F(I_k) = W_k\left(a + b - \frac{1}{W_k}\sum_{i=1}^{k} w_i x_i - \exp\left(\ln a + \ln b - \frac{1}{W_k}\sum_{i=1}^{k} w_i \ln x_i\right)\right) = W_k\left(\widetilde{A}_k - \widetilde{G}_k\right).$$

□

Corollary 2.15: *Under the above stated assumptions, we have*

(*i*)

$$\left(\frac{\widetilde{G}_n}{\widetilde{H}_n}\right)^{W_n} \geq \left(\frac{\widetilde{G}_{n-1}}{\widetilde{H}_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \left(\frac{\widetilde{G}_1}{\widetilde{H}_1}\right)^{W_1} \geq 1,$$

(*ii*)

$$W_n\left(\frac{1}{\widetilde{H}_n} - \frac{1}{\widetilde{G}_n}\right) \geq W_n\left(\frac{1}{\widetilde{H}_{n-1}} - \frac{1}{\widetilde{G}_{n-1}}\right) \geq \cdots \geq W_1\left(\frac{1}{\widetilde{H}_1} - \frac{1}{\widetilde{G}_1}\right) \geq 0.$$

Proof: Directly from Theorem 2.26 by the substitutions $a \to \frac{1}{a}$, $b \to \frac{1}{b}$, $x_i \to \frac{1}{x_i}$. □

Theorem 2.27: *Let all the assumptions given in Theorem 2.26 be valid. For $r \leq 1$, we have*

$$W_n\left(\widetilde{A}_n - \widetilde{M}_n^{[r]}\right) \geq W_{n-1}\left(\widetilde{A}_{n-1} - \widetilde{M}_{n-1}^{[r]}\right) \geq \cdots \geq W_1\left(\widetilde{A}_1 - \widetilde{M}_1^{[r]}\right) \geq 0. \tag{2.43}$$

For $r \geq 1$, the inequalities in (2.43) are reversed.

Proof: Suppose that $r \leq 1$. Applying Corollary 2.11 to the convex function $f(x) = x^{\frac{1}{r}}$, and replacing a, b, and x_i with a^r, b^r, and x_i^r respectively, we obtain (2.43) since in this case

$$F(I_k) = W_k \left(a + b - \frac{1}{W_k} \sum_{i=1}^{k} w_i x_i - \left(a^r + b^r - \frac{1}{W_k} \sum_{i=1}^{k} w_i x_i^r \right)^{\frac{1}{r}} \right) = W_k \left(\widetilde{A}_k - \widetilde{M}_k^{[r]} \right).$$

If $r \geq 1$, then the function $f(x) = x^{\frac{1}{r}}$ is concave, so the inequalities in (2.43) are reversed. □

Corollary 2.16: *Under the assumptions given in* **B**, *we have*

$$W_n \left(\widetilde{A}_n - \widetilde{H}_n \right) \geq W_{n-1} \left(\widetilde{A}_{n-1} - \widetilde{H}_{n-1} \right) \geq \cdots \geq W_1 \left(\widetilde{A}_1 - \widetilde{H}_1 \right) \geq 0.$$

Remark 2.18: Obviously, the assertion (ii) from Theorem 2.26 is also direct consequence of Theorem 2.27.

Theorem 2.28: *Let all the assumptions given in Theorem 2.26 be valid. Let $r, s \in \mathbb{R}$, $r \leq s$.*

(i) *If $s > 0$, then*

$$\begin{aligned} W_n \left(\left(\widetilde{M}_n^{[s]} \right)^s - \left(\widetilde{M}_n^{[r]} \right)^s \right) &\geq W_{n-1} \left(\left(\widetilde{M}_{n-1}^{[s]} \right)^s - \left(\widetilde{M}_{n-1}^{[r]} \right)^s \right) \\ &\geq \cdots \geq W_1 \left(\left(\widetilde{M}_1^{[s]} \right)^s - \left(\widetilde{M}_1^{[r]} \right)^s \right) \geq 0, \end{aligned} \tag{2.44}$$

(ii) *If $s < 0$, then the inequalities in (2.44) are reversed.*

Proof: Suppose that $s > 0$. Applying Corollary 2.11 to the convex function $f(x) = x^{\frac{s}{r}}$, and replacing a, b, and x_i with a^r, b^r, and x_i^r respectively, we obtain (2.44) since

$$\begin{aligned} F(I_k) &= W_k \left(a^s + b^s - \frac{1}{W_k} \sum_{i=1}^{k} w_i x_i^s - \left(a^r + b^r - \frac{1}{W_k} \sum_{i=1}^{k} w_i x_i^r \right)^{\frac{s}{r}} \right) \\ &= W_k \left(\left(\widetilde{M}_k^{[s]} \right)^s - \left(\widetilde{M}_k^{[r]} \right)^s \right). \end{aligned}$$

If $s < 0$, then the function $f(x) = x^{\frac{s}{r}}$ is concave, so *the* inequalities in (2.44) are reversed. □

Theorem 2.29: *Let all the assumptions given in Theorem 2.26 and* **D** *be valid. Then*

$$\begin{aligned} W_n \left(\psi \left(\widetilde{M}_\psi^{[n]} \right) - \psi \left(\widetilde{M}_\varphi^{[n]} \right) \right) &\geq W_{n-1} \left(\psi \left(\widetilde{M}_\psi^{[n-1]} \right) - \psi \left(\widetilde{M}_\varphi^{[n-1]} \right) \right) \\ &\geq \cdots \geq W_1 \left(\psi \left(\widetilde{M}_\psi^{[1]} \right) - \psi \left(\widetilde{M}_\varphi^{[1]} \right) \right) \geq 0. \end{aligned} \tag{2.45}$$

If $\psi \circ \varphi^{-1}$ is concave on $[a,b]$, then the inequalities in (2.45) are reversed.

Proof: Applying Corollary 2.11 to the convex function $f=\psi\circ\varphi^{-1}$, and replacing a, b, and x_i with $\varphi(a)$, $\varphi(b)$, and $\varphi(x_i)$ respectively, we obtain (2.45) since in this case

$$F(I_k)=W_k\left(\psi(a)+\psi(b)-\frac{1}{W_k}\sum_{i=1}^{k}w_i\psi(x_i)\right.$$
$$\left.-\left(\psi\circ\varphi^{-1}\right)\left(\varphi(a)+\varphi(b)-\frac{1}{W_k}\sum_{i=1}^{k}w_i\varphi(x_i)\right)\right)=W_k\left(\psi\left(\widetilde{M}_{\psi}^{[k]}\right)-\psi\left(\widetilde{M}_{\varphi}^{[k]}\right)\right).$$

□

Remark 2.19: Theorems 2.26, 2.27 and 2.28 follow from Theorem 2.29, by choosing adequate functions φ and ψ, and appropriate substitutions.

Corollary 2.17: *Let all the assumptions given in Theorem 2.29 be valid. Then*

$$W_n\left(\psi\left(\widetilde{M}_{\psi}^{[n]}\right)-\psi\left(\widetilde{M}_{\varphi}^{[n]}\right)\right)\geq\max_{1\leq i<j\leq n}\left\{(w_i+w_j)\left[\psi(a)+\psi(b)-\frac{w_i\psi(x_i)+w_j\psi(x_j)}{w_i+w_j}\right.\right.$$
$$\left.\left.-\left(\psi\circ\varphi^{-1}\right)\left(\varphi(a)+\varphi(b)-\frac{w_i\varphi(x_i)+w_j\varphi(x_j)}{w_i+w_j}\right)\right]\right\},\tag{2.46}$$

and

$$W_n\left(\psi\left(\widetilde{M}_{\psi}^{[n]}\right)-\psi\left(\widetilde{M}_{\varphi}^{[n]}\right)\right)\tag{2.47}$$
$$\geq\max_{1\leq i\leq n}\left\{w_i\left[\psi(a)+\psi(b)-\psi(x_i)-\left(\psi\circ\varphi^{-1}\right)\left(\varphi(a)+\varphi(b)-\varphi(x_i)\right)\right]\right\}.$$

If $\psi\circ\varphi^{-1}$ is concave on $[a,b]$, then the inequalities in (2.46) and (2.47), with max *replaced by* min, *are reversed.*

Remark 2.20: Analogous assertions can be formulated for the means of Mercer's type formed with weights satisfying (1.4).

CHAPTER 3

Refinements of the Converse–Jensen–Mercer and Converse–Jessen–Mercer Inequalities and A Generalization on Convex Hulls in $\mathbb{R}^k$

This chapter focuses on refinements of the converse forms of the Jensen–Mercer and Jessen–Mercer inequalities in both discrete and integral settings. We explore sharper bounds and structural enhancements that strengthen the classical converse results, and we extend these inequalities to a more general framework on convex hulls in $\mathbb{R}^k$. The developments presented here provide a unified perspective on converse-type Mercer inequalities and set the stage for further generalizations in operator and functional contexts in subsequent chapters.

3.1 Refinements of Converse–Jensen–Mercer Inequalities

The following refinement related to converse of Jensen's inequality (as stated in Theorem A.8) is obtained in [29]. Throughout this chapter we assume that $\delta_f = f(a) + f(b) - 2f(\frac{a+b}{2})$.

Theorem 3.1: *If $f : I \longrightarrow \mathbb{R}$ is a convex function, $x_1, \ldots, x_n \in I$, $w_1, \ldots, w_n$ be non-negative real numbers and $W_n = \sum_{i=1}^{n} w_i > 0$, then*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i) \leq \frac{b-\check{x}}{b-a}f(a) + \frac{\check{x}-a}{b-a}f(b) - \left(\frac{1}{W_n}\sum_{i=1}^{n} w_i(\tilde{x}_i)\right)\delta_f$$

where, $\tilde{x}_i = \frac{1}{2} - \frac{1}{b-a}|x_i - \frac{a+b}{2}|$ and $\check{x} = \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i$.

3.1.1 Refinements of Discrete Jensen–Mercer Inequality

Theorem 3.2: *Under the assumptions of the Theorem 1.1, the following series of inequalities is valid*

$$
\begin{aligned}
&f(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i) \le \frac{1}{W_n}\sum_{i=1}^{n}w_if\left(a+b-x_i\right)\\
&\quad \le \frac{b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i}{b-a}f(b)+\frac{\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i-a}{b-a}f(a) \le f\left(a\right)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i).
\end{aligned}
$$

Theorem 3.3: *Let $I=[a,b]$ be an interval in $\mathbb{R}$. If $f:I\longrightarrow\mathbb{R}$ is a convex function, $x_1,\ldots,x_n\in I$ and $w_1,\ldots,w_n$ be non-negative real numbers $W_n=\sum_{i=1}^{n}w_i>0$, then*

$$
\begin{aligned}
&f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right) \le \frac{1}{W_n}\sum_{i=1}^{n}w_if(a+b-x_i)\\
&\le \frac{b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i}{b-a}f(b)+\frac{\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i-a}{b-a}f(a)\\
&-\left[\frac{1}{W_n}\sum_{i=1}^{n}w_i\left(\frac{1}{2}-\frac{1}{b-a}\left|x_i-\frac{a+b}{2}\right|\right)\right]\delta_f\\
&\le f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i)-\left(1-\frac{2}{b-a}\frac{1}{W_n}\sum_{i=1}^{n}w_i\left|x_i-\frac{a+b}{2}\right|\right)\delta_f\\
&\le f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i).
\end{aligned}
$$

Now here we state and prove further result which provides another generalized refinement of discrete type Jensen–Mercer inequality.

Theorem 3.4: *Under the same assumptions of Theorem 3.2, we have*

$$
\begin{aligned}
&f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right)\\
&\le \frac{b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i}{b-a}f(b)+\frac{\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i-a}{b-a}f(a)-\left(\frac{1}{2}-\frac{1}{b-a}\left|\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i-\frac{a+b}{2}\right|\right)\delta_f\\
&\le f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i)\\
&-\left[1-\frac{1}{b-a}\left(\frac{1}{W_n}\sum_{i=1}^{n}w_i\left|x_i-\frac{a+b}{2}\right|+\left|\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i-\frac{a+b}{2}\right|\right)\right]\delta_f\\
&\le f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i)-\left(1-\frac{2}{b-a}\frac{1}{W_n}\sum_{i=1}^{n}w_i\left|x_i-\frac{a+b}{2}\right|\right)\delta_f
\end{aligned}
$$

$$\leq f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i).$$

Using Theorem 3.4 we can obtain an upper bound for the Jensen's difference,

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i)-f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right).$$

Corollary 3.1: *Under the same assumptions of Theorem 1.1, we have*

$$\begin{aligned}
&\frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(x_i\right)-f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\\
&\leq \frac{1}{b-a}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i\left|x_i-\frac{a+b}{2}\right|+\left|\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i-\frac{a+b}{2}\right|\right)\delta_f.
\end{aligned}$$

3.1.2 Refinements of Integral Converse–Jensen–Mercer Inequalities

The current section based on integral analogous of previous section. For proving techniques, interested readers can see [226]. Throughout this section $\delta_\varphi=\varphi(a)+\varphi(b)-2\varphi\left(\frac{a+b}{2}\right)$.

Theorem 3.5: *Let (Ω, P, μ) be a probability measure space and f is a real valued function that is μ integrable such that $\mu(\Omega)=1$. Let $f:\Omega\longrightarrow[a,b]$ $(-\infty<a<b<\infty)$ be a measurable function, then for any continuous convex function $\varphi:[a,b]\longrightarrow\mathbb{R}$, the following inequality holds*

$$\int_\Omega \varphi(f)d\mu \leq \frac{b-\int_\Omega f d\mu}{b-a}\varphi(a)+\frac{\int_\Omega f d\mu-a}{b-a}\varphi(b)-\left(\int_\Omega \check{f}d\mu\right)\delta_\varphi,$$

where $\check{f}=\frac{1}{2}-\frac{1}{b-a}\left|f-\frac{a+b}{2}\right|$.

Theorem 3.6: *Under the same assumptions of Theorem 3.5 the following inequality holds*

$$\begin{aligned}
&\varphi(a+b-\int_\Omega f d\mu)\leq \int_\Omega \varphi(a+b-f)d\mu\\
&\leq \frac{b-\int_\Omega f d\mu}{b-a}\varphi(b)+\frac{\int_\Omega f d\mu-a}{b-a}\varphi(a)\\
&-\left[\int_\Omega\left(\frac{1}{2}-\frac{1}{b-a}\left|f-\frac{a+b}{2}\right|\right)d\mu\right]\delta_\varphi\\
&\leq \varphi(a)+\varphi(b)-\int_\Omega\varphi(f)d\mu-\left(1-\frac{2}{b-a}\int_\Omega\left|f-\frac{a+b}{2}\right|d\mu\right)\delta_\varphi\\
&\leq \varphi(a)+\varphi(b)-\int_\Omega\varphi(f)d\mu.
\end{aligned}$$

Theorem 3.7: *Under the same assumptions of Theorem 3.5 the following inequality holds:*

$$\begin{aligned}
&\varphi\left(a+b-\int_{\Omega} f d\mu\right) \leq \frac{b-\int_{\Omega} f d\mu}{b-a}\varphi(b)+\frac{\int_{\Omega} f d\mu-a}{b-a}\varphi(a) \\
&-\left(\frac{1}{2}-\frac{1}{b-a}\left|\int_{\Omega} f d\mu-\frac{a+b}{2}\right|\right)\delta_{\varphi} \\
&\leq \varphi(a)+f(b)-\int_{\Omega}\varphi(f)\,d\mu \\
&-\left[1-\frac{1}{b-a}\left(\int_{\Omega}\left|f-\frac{a+b}{2}\right|d\mu+\left|\int_{\Omega} f d\mu-\frac{a+b}{2}\right|\right)\right]\delta_{\varphi} \\
&\leq \varphi(a)+f(b)-\int_{\Omega}\varphi(f)\,d\mu-\left(1-\frac{2}{b-a}\int_{\Omega}\left|f-\frac{a+b}{2}\right|d\mu\right)\delta_{\varphi} \\
&\leq \varphi(a)+\varphi(b)-\int_{\Omega}\varphi(f)\,d\mu.
\end{aligned}$$

Using Theorem 3.7 we can get an upper bound for the difference

$$\int_{\Omega}\varphi(f)\,d\mu-\varphi\left(\int_{\Omega} f d\mu\right).$$

Corollary 3.2: *Under the same assumptions of Theorem 3.5 the following inequality holds*

$$\int_{\Omega}\varphi(f)d\mu-\varphi\left(\int_{\Omega} f d\mu\right) \leq \frac{1}{b-a}\left(\int_{\Omega}\left|f-\frac{a+b}{2}\right|d\mu+\left|\int_{\Omega} f d\mu-\frac{a+b}{2}\right|\right)\delta_{\varphi}.$$

3.2 A Refinement of the Converse–Jessen–Mercer Inequality and A Generalization on Convex Hulls in $\mathbb{R}^k$

Throughout this section we let E be a nonempty set and L be a subspace of the vector space $\mathbb{R}^E$ over $\mathbb{R}$ which contains $\mathbf{1}$, and L having all properties as defined in L1, L2 and L4 as defined in Section *A*.3 along with Positive (Isotonic) Linear Functional as defined in A1, A2 in the same section (see also [173]). Throughout this section, we assume that $\delta_{\varphi}=\varphi(a)+\varphi(b)-2\varphi\left(\frac{a+b}{2}\right)$.

Bakula *et al.* in [29] obtained the following refinement of the converse Jessen's inequality (see Theorem *A*.15)[204, p. 98]).

Theorem 3.8: *Let L satisfy L1, L2, L4 on a nonempty set E, and let A be a positive normalized linear functional. If φ is a convex function on $[a,b]$, then for all $f\in L$ such that $\varphi(f)\in L$ we have $A(f)\in[a,b]$ and*

$$A(\varphi(f)) \leq \frac{b-A(f)}{b-a}\varphi(a)+\frac{A(f)-a}{b-a}\varphi(b)-A\left(\tilde{f}\right)\delta_{\varphi}, \tag{3.1}$$

where $\tilde{f} = \frac{1}{2} - \frac{1}{b-a}\left|f - \frac{a+b}{2}\right|$.

Using inequality (3.1) and the following lemma we shall refine the series of inequalities (1.8) as given in Theorem 1.6.

Lemma 3.1: *Let φ be a convex function on U where U is a convex set in $\mathbb{R}^k$, $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in U^n$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$ is nonnegative n-tuple such that $\sum_{i=1}^{n} w_i = 1$. Then*

$$\min\{w_1, \ldots, w_n\}\left[\sum_{i=1}^{n} \varphi(\boldsymbol{x}_i) - n\varphi\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i\right)\right] \leq \sum_{i=1}^{n} w_i\varphi(\boldsymbol{x}_i) - \varphi\left(\sum_{i=1}^{n} w_i\boldsymbol{x}_i\right)$$
$$\leq \max\{w_1, \ldots, w_n\}\left[\sum_{i=1}^{n} \varphi(\boldsymbol{x}_i) - n\varphi\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i\right)\right].$$

Proof: This is a simple consequence of [183, p. 717, Theorem 1]. □

3.2.1 Refinement of the Converse Jessen–Mercer Inequality

Next two theorems are main results of present subsection. Since proving techniques are similar so we omit the details, interested readers can find complete details in [173].

Theorem 3.9: *Let L satisfy L1, L2 and L4 on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex function on $[a, b]$, then for all $f \in L$ such that $\varphi(f), \varphi(a+b-f) \in L$, we have*

$$\begin{aligned}
&\varphi(a+b-A(f)) \leq A(\varphi(a+b-f)) \\
&\leq \frac{b-A(f)}{b-a}\varphi(b) + \frac{A(f)-a}{b-a}\varphi(a) - A\left(\frac{1}{2} - \frac{1}{b-a}\left|f - \frac{a+b}{2}\right|\right)\delta_{\varphi} \\
&\leq \varphi(a) + \varphi(b) - A(\varphi(f)) - \left[1 - \frac{2}{b-a}A\left(\left|f - \frac{a+b}{2}\right|\right)\right]\delta_{\varphi} \\
&\leq \varphi(a) + \varphi(b) - A(\varphi(f)).
\end{aligned}$$

Theorem 3.10: *Let L satisfy L1, L2 and L4 on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex function on $[a, b]$, then for all $f \in L$ such that $\varphi(f), \varphi(a+b-f) \in L$, we have*

$$
\begin{aligned}
&\varphi\left(a+b-A\left(f\right)\right)\\
&\leq \frac{M-A\left(f\right)}{b-a}\varphi\left(b\right)+\frac{A\left(f\right)-a}{b-a}\varphi\left(a\right)-\left(\frac{1}{2}-\frac{1}{b-a}\left|A\left(f\right)-\frac{a+b}{2}\right|\right)\delta_{\varphi}\\
&\leq \varphi\left(a\right)+\varphi\left(b\right)-A\left(\varphi(f)\right)\\
&-\left[1-\frac{1}{b-a}\left(A\left(\left|f-\frac{a+b}{2}\right|\right)+\left|A\left(f\right)-\frac{a+b}{2}\right|\right)\right]\delta_{\varphi}\\
&\leq \varphi\left(a\right)+\varphi\left(b\right)-A\left(\varphi(f)\right)-\left[1-\frac{2}{b-a}A\left(\left|f-\frac{a+b}{2}\right|\right)\right]\delta_{\varphi}\\
&\leq \varphi\left(a\right)+\varphi\left(b\right)-A\left(\varphi\left(f\right)\right).
\end{aligned}
$$

Using Theorem 3.10 we can get an upper bound for the difference $A\left(\varphi\left(f\right)\right)-\varphi\left(A\left(f\right)\right)$ obtained in [216].

Corollary 3.3: *Let L satisfy L1, L2 and L4 on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex function on $[a,b]$, then for all $f\in L$ such that $\varphi\left(f\right),\varphi\left(a+b-f\right)\in L$, we have*

$$
A\left(\varphi\left(f\right)\right)-\varphi\left(A\left(f\right)\right)\leq \frac{1}{b-a}\left(A\left(\left|f-\frac{a+b}{2}\right|\right)+\left|A\left(f\right)-\frac{a+b}{2}\right|\right)\delta_{\varphi}.
$$

3.2.2 Jensen Type Inequalities on a Convex Hull

We establish two Jensen's type inequalities for convex functions defined on convex subsets of the space $\mathbb{R}^k$ which involve elements of convex hulls. As their direct consequences the comparison theorem for weighted L-conjugate means and a Mercer's result are obtained.

Let U be a convex subset of $\mathbb{R}^k$ and $f:U\to\mathbb{R}$ a function. We define f to be convex on U if

$$
f\left(\lambda \boldsymbol{x}+\left(1-\lambda\right)\boldsymbol{y}\right)\leq \lambda f\left(\boldsymbol{x}\right)+\left(1-\lambda\right)f\left(\boldsymbol{y}\right) \tag{3.2}
$$

for all $\boldsymbol{x},\boldsymbol{y}\in U$ and $\lambda\in(0,1)$. f is concave if the reversed inequality of (3.2) holds.

For a convex function $f:U\to\mathbb{R}$, n $k-$tuples $\boldsymbol{x}_i$ in U and n positive real numbers λ_i with $\Lambda_n=\sum_{i=1}^{n}\lambda_i$, the following Jensen's inequality holds

$$
f\left(\frac{1}{\Lambda_n}\sum_{i=1}^{n}\lambda_i\boldsymbol{x}_i\right)\leq \frac{1}{\Lambda_n}\sum_{i=1}^{n}\lambda_i f\left(\boldsymbol{x}_i\right). \tag{3.3}
$$

If we set the following conditions: $\lambda_1 > 0$, $\lambda_i \leq 0$, for $i \in \{2, \ldots, n\}$ with $\Lambda_n > 0$ and $\frac{1}{\Lambda_n} \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i \in U$, then

$$f\left(\frac{1}{\Lambda_n} \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i\right) \geq \frac{1}{\Lambda_n} \sum_{i=1}^{n} \lambda_i f\left(\boldsymbol{x}_i\right) \tag{3.4}$$

holds. This inequality is known as the reversed Jensen's inequality and it is a simple consequence of the inequality (3.3) (see [204]). We denote by $H\left(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}\right)$ the convex hull of the set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset U$. If $\boldsymbol{y} \in H(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$ then it can be written in a unique way as a convex combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ i.e., $\boldsymbol{y} = \sum_{i=1}^{n} \lambda_i x_i$ where $\lambda_i \geq 0$ for all $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^{n} \lambda_i = 1$ (see [224]).

The next theorem gives a Jensen's type inequality which involves elements of a convex hull.

Theorem 3.11: *Let U be a convex subset of $\mathbb{R}^k$, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in U$ and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m \in H(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$. If $f : U \to \mathbb{R}$ is convex on U then the inequality*

$$f\left(\frac{\sum_{i=1}^{n} w_i \boldsymbol{x}_i - \sum_{j=1}^{m} w_j \boldsymbol{y}_j}{W_n - W_m}\right) \leq \frac{\sum_{i=1}^{n} w_i f\left(\boldsymbol{x}_i\right) - \sum_{j=1}^{m} w_j f\left(\boldsymbol{y}_j\right)}{W_n - W_m} \tag{3.5}$$

holds for all positive real numbers $w_1, \ldots, w_n$ and $w_1, \ldots, w_m$ satisfying the condition

$$w_i \geq W_m \quad \text{for all } i \in \{1, \ldots, n\}, \tag{3.6}$$

where $W_n = \sum_{i=1}^{n} w_i$ and $W_m = \sum_{j=1}^{m} w_j$. If f is concave on U the inequality (3.5) is reversed.

Proof: Let $f : U \to \mathbb{R}$ be a convex function on U and let $w_1, \ldots, w_n$ and $w_1, \ldots, w_m$ be positive real numbers that satisfy the condition (3.6). Since $\boldsymbol{y}_j \in H\left(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}\right)$, there are some $\lambda_i^{(j)} \geq 0$, $i \in \{1, \ldots, n\}$ such that $\sum_{i=1}^{n} \lambda_i^{(j)} = 1$ and

$$\boldsymbol{y}_j = \sum_{i=1}^{n} \lambda_i^{(j)} \boldsymbol{x}_i,$$

for all $j \in \{1, \ldots, m\}$. Since f is convex on U,

$$f\left(\boldsymbol{y}_j\right) = f\left(\sum_{i=1}^{n} \lambda_i^{(j)} \boldsymbol{x}_i\right) \leq \sum_{i=1}^{n} \lambda_i^{(j)} f\left(\boldsymbol{x}_i\right)$$

for all $j \in \{1, \ldots, m\}$.

Now, we can write

$$\frac{\sum\limits_{i=1}^{n} w_i \boldsymbol{x}_i - \sum\limits_{j=1}^{m} w_j \boldsymbol{y}_j}{W_n - W_m} = \frac{\sum\limits_{i=1}^{n} w_i \boldsymbol{x}_i - \sum\limits_{j=1}^{m} w_j \sum\limits_{i=1}^{n} \lambda_i^{(j)} \boldsymbol{x}_i}{W_n - W_m} = \frac{1}{W_n - W_m} \sum_{i=1}^{n} \left(w_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)} \right) \boldsymbol{x}_i.$$

We can easily check that $\dfrac{1}{W_n - W_m} \displaystyle\sum_{i=1}^{n} \left(w_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)} \right) = 1$,

and since for all $i \in \{1, \ldots, n\}$, $w_i \geq W_m \geq \displaystyle\sum_{j=1}^{m} w_j \lambda_i^{(j)}$,

we also have $\dfrac{1}{W_n - W_m} \left(w_i - \displaystyle\sum_{j=1}^{m} w_j \lambda_i^{(j)} \right) \geq 0 \quad (i = 1, \ldots, n)$.

Hence, $\dfrac{1}{W_n - W_m} \{ \displaystyle\sum_{i=1}^{n} w_i \boldsymbol{x}_i - \sum_{j=1}^{m} w_j \boldsymbol{y}_j \}$ is a convex combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in U$ and it belongs to U since U is convex.

Since f is convex on U, we obtain from (3.3) the following:

$$\begin{aligned}
& f\left(\frac{\sum\limits_{i=1}^{n} w_i \boldsymbol{x}_i - \sum\limits_{j=1}^{m} w_j \boldsymbol{y}_j}{W_n - W_m} \right) \\
& = f\left(\frac{1}{W_n - W_m} \sum_{i=1}^{n} \left(w_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)} \right) \boldsymbol{x}_i \right) \\
& \leq \frac{1}{W_n - W_m} \sum_{i=1}^{n} \left(w_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)} \right) f\left(\boldsymbol{x}_i\right) \\
& = \frac{\sum\limits_{i=1}^{n} w_i f\left(\boldsymbol{x}_i\right) - \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} w_j \lambda_i^{(j)} f\left(\boldsymbol{x}_i\right)}{W_n - W_m} \\
& = \frac{\sum\limits_{i=1}^{n} w_i f\left(\boldsymbol{x}_i\right) - \sum\limits_{j=1}^{m} w_j \sum\limits_{i=1}^{n} \lambda_i^{(j)} f\left(\boldsymbol{x}_i\right)}{W_n - W_m} \\
& \leq \frac{\sum\limits_{i=1}^{n} w_i f\left(\boldsymbol{x}_i\right) - \sum\limits_{j=1}^{m} w_j f\left(\boldsymbol{y}_j\right)}{W_n - W_m}.
\end{aligned}$$

It can be easily seen that if f is concave the inequality (3.5) is reversed. □

The following theorem gives a converse inequality of Jensen's type which involves elements of a convex hull.

Theorem 3.12: *Let U be a convex subset of $\mathbb{R}^k$, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in U$, $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m \in H(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$ and let $w_1, \ldots, w_n$ and $w_1, \ldots, w_m$ be positive real numbers such that $W_n - W_m > 0$*

and

$$\frac{\sum\limits_{i=1}^{n} w_i \boldsymbol{x}_i - \sum\limits_{j=1}^{m} w_j \boldsymbol{y}_j}{W_n - W_m} \in U. \tag{3.7}$$

If $f : U \to \mathbb{R}$ is convex on U then

$$f\left(\frac{\sum\limits_{i=1}^{n} w_i \boldsymbol{x}_i - \sum\limits_{j=1}^{m} w_j \boldsymbol{y}_j}{W_n - W_m}\right) \geq \frac{W_n f(\overline{\boldsymbol{x}}) - W_m f(\overline{\boldsymbol{y}})}{W_n - W_m} \geq \frac{W_n f(\overline{\boldsymbol{x}}) - \sum\limits_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{W_n - W_m}, \tag{3.8}$$

where

$$\overline{\boldsymbol{x}} = \frac{1}{W_n} \sum_{i=1}^{n} w_i \boldsymbol{x}_i, \quad \overline{\boldsymbol{y}} = \frac{1}{W_m} \sum_{j=1}^{m} w_j \boldsymbol{y}_j.$$

If f is concave on U the inequalities (3.8) are reversed.

Proof: From (3.4) and then (3.3) we immediately obtain

$$f\left(\frac{W_n\left(\frac{1}{W_n}\sum\limits_{i=1}^{n} w_i \boldsymbol{x}_i\right) - W_m\left(\frac{1}{W_m}\sum\limits_{j=1}^{m} w_j \boldsymbol{y}_j\right)}{W_n - W_m}\right)$$

$$\geq \frac{W_n f(\overline{\boldsymbol{x}}) - W_m f(\overline{\boldsymbol{y}})}{W_n - W_m} \geq \frac{W_n f(\overline{\boldsymbol{x}}) - W_m \frac{1}{W_m}\sum\limits_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{W_n - W_m}.$$

□

Remark 3.1: If positive real numbers $w_1, \ldots, w_n$ and $w_1, \ldots, w_m$ satisfy the condition (3.6) then they obviously satisfy the condition $W_n - W_m > 0$. Also, since U is a convex set and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in U$ they satisfy the condition (3.7). Hence, in this case the inequality (3.5) from Theorem 3.11 can be extended in the following way

$$\frac{W_n f(\overline{\boldsymbol{x}}) - \sum\limits_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{W_n - W_m} \leq \frac{W_n f(\overline{\boldsymbol{x}}) - W_m f(\overline{\boldsymbol{y}})}{W_n - W_m}$$

$$\leq f\left(\frac{\sum\limits_{i=1}^{n} w_i \boldsymbol{x}_i - \sum\limits_{j=1}^{m} w_j \boldsymbol{y}_j}{W_n - W_m}\right) \leq \frac{\sum\limits_{i=1}^{n} w_i f(\boldsymbol{x}_i) - \sum\limits_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{W_n - W_m}.$$

If we consider real valued functions of one variable then the direct consequences of Theorems 3.11 and 3.12 are some results in [29] related to means of n variables.

Corollary 3.4: *Let I be an interval in $\mathbb{R}$ and let $M_1, \ldots, M_m$ be fixed means of n variables $x_1, \ldots, x_n \in I$. If $f : I \to \mathbb{R}$ is convex on I then the inequality*

$$f\left(\frac{\sum_{i=1}^{n} w_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{W_n - W_m}\right) \leq \frac{\sum_{i=1}^{n} w_i f(x_i) - \sum_{j=1}^{m} w_j f(M_j(\boldsymbol{x}))}{W_n - W_m} \tag{3.9}$$

holds for all positive real numbers $w_1, \ldots, w_n$ and $w_1, \ldots, w_m$ satisfying the condition (3.6). If f is concave on I the inequality (3.9) is reversed.

Proof: *Follows from Theorem 3.11, since the convex hull of the set $\{x_1, \ldots, x_n\} \subset I$ is the interval $\left[\min_{i\in\{1,\ldots,n\}} \{x_i\}, \max_{i\in\{1,\ldots,n\}} \{x_i\}\right]$ and for each $j \in \{1, \ldots, m\}$ $M_j(\boldsymbol{x})$ is in that interval.* $\square$

Remark 3.2: The notion of L-conjugate means was introduced in [83]. It was generalized to the notion of the weighted L-conjugate means in [28]. In the same paper the comparison theorem for these means, which immediately follows from Corollary 3.4, was proved.

Corollary 3.5: *Let I be an interval in $\mathbb{R}$, let $M_1, \ldots, M_m$ be fixed means of n variables $x_1, \ldots, x_n \in I$ and let $w_1, \ldots, w_n$ and $w_1, \ldots, w_m$ be positive real numbers such that $W_n - W_m > 0$ and*

$$\frac{\sum_{i=1}^{n} w_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{W_n - W_m} \in I.$$

If $f : I \to \mathbb{R}$ is convex on I then

$$f\left(\frac{\sum_{i=1}^{n} w_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{W_n - W_m}\right) \geq \frac{W_n f(\overline{x}) - W_m f\left(\overline{M}\right)}{W_n - W_m} \geq \frac{W_n f(\overline{x}) - \sum_{j=1}^{m} w_j f(M_j(\boldsymbol{x}))}{W_n - W_m}, \tag{3.10}$$

where

$$\overline{x} = \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i, \quad \overline{M} = \frac{1}{W_m} \sum_{j=1}^{m} w_j M_j(\boldsymbol{x}).$$

If f is concave on I, then the inequalities (3.10) are reversed.

Also, the direct consequence of Theorem 3.11 is the Jensen–Mercer inequality (1.3).

Corollary 3.6: *Let $[a, b]$ be an interval in $\mathbb{R}$, $y_1, \ldots, y_m \in [a, b]$ and $w_1, \ldots, w_m$ positive real* *numbers such that* $W_m = 1$. *If* $f : [a, b] \to \mathbb{R}$ *is convex on* $[a, b]$ *then*

$$f\left(a+b-\sum_{j=1}^{m} w_j y_j\right) \geq 2f\left(\frac{a+b}{2}\right) - f\left(\sum_{j=1}^{m} w_j y_j\right) \geq 2f\left(\frac{a+b}{2}\right) - \sum_{j=1}^{m} w_j f\left(y_j\right).$$

We can also, obtain natural generalizations of Corollary 3.6 to convex functions defined on $\mathbb{R}^k$.

Corollary 3.7: *Let* U *be a simplex with vertices* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^k$ $(n \geq 2)$, $y_1, \ldots, y_m \in U$ *and* $w_1, \ldots, w_m$ *positive real numbers such that* $W_m = 1$. *If* $f : U \to \mathbb{R}$ *is convex on* U *then*

$$\begin{aligned}
&\frac{nf\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\right) - \sum_{j=1}^{m} w_j f\left(\boldsymbol{y}_j\right)}{n-1} \leq \frac{nf\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\right) - f\left(\sum_{j=1}^{m} w_j \boldsymbol{y}_j\right)}{n-1} \\
&\leq f\left(\frac{\sum_{i=1}^{n}\boldsymbol{x}_i - \sum_{j=1}^{m} w_j \boldsymbol{y}_j}{n-1}\right) \leq \frac{\sum_{i=1}^{n} f\left(\boldsymbol{x}_i\right) - \sum_{j=1}^{m} w_j f\left(\boldsymbol{y}_j\right)}{n-1}.
\end{aligned}$$

Proof: Follows from Theorems 3.11 and 3.12 by setting $w_1 = \cdots = w_n = 1$ since in this case $U = H\left(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}\right)$. □

3.2.3 Generalization on Convex Hulls in $\mathbb{R}^k$

We present a generalization of the Jessen–Mercer inequality for convex functions on convex hulls in $\mathbb{R}^k$.

Convex hull of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^k$ is the set

$$\left\{\sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i | \alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\right\}$$

and it is represented by $K = co(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$.

Barycentric coordinates over K are continuous real functions $\lambda_1, \ldots, \lambda_n$ on K with the following properties:

$$\lambda_i(\boldsymbol{x}) \geq 0,\ i \in \{1, \ldots, n\}, \quad \sum_{i=1}^{n} \lambda_i(\boldsymbol{x}) = 1, \quad \boldsymbol{x} = \sum_{i=1}^{n} \lambda_i(\boldsymbol{x}) \boldsymbol{x}_i. \tag{3.11}$$

If $\boldsymbol{x}_2 - \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n - \boldsymbol{x}_1$ are linearly independent vectors, then each $\boldsymbol{x} \in K$ can be written in the unique way as a convex combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ in the form (3.11).

We also consider k-simplex $S = co(\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}\})$ in $\mathbb{R}^k$ which is a convex hull of its vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1} \in \mathbb{R}^k$, where vertices $\boldsymbol{v}_2 - \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1} - \boldsymbol{v}_1 \in \mathbb{R}^k$ are lineary independent. In this case we denote k-simplex by $S = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}]$. Barycentric coordinates $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ over S are nonnegative linear polynomials on S and have a special form (see [38]).

With L^k we denote the linear class of functions $\boldsymbol{g}\colon E \to \mathbb{R}^k$ defined by $\boldsymbol{g}(t) = (g_1(t), \ldots, g_k(t))$, $g_i \in L$, for $i \in \{1, \ldots, k\}$. For a given linear functional A, we also consider linear operator $\widetilde{A} = (A, \ldots, A)\colon\ L^k \to \mathbb{R}^k$ defined by

$$\widetilde{A}(\boldsymbol{g}) = (A(g_1), \ldots, A(g_k)). \tag{3.12}$$

If $A(\mathbf{1}) = 1$ is satisfied, then using (A1) we also have

(A3) $A(f(\boldsymbol{g})) = f(\widetilde{A}(\boldsymbol{g}))$ for every linear function f on $\mathbb{R}^k$.

For $n \in \mathbb{N}$, we denote

$$\Delta_{n-1} = \left\{(\mu_1, \ldots, \mu_n)\colon \mu_i \geq 0, i \in \{1, \ldots, n\}, \sum_{i=1}^{n} \mu_i = 1\right\}.$$

Also, if φ is a function defined on a convex subset $U \subseteq \mathbb{R}^k$ and $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \in U$, we denote

$$S_\varphi^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \sum_{i=1}^{n} \varphi(\boldsymbol{x}_i) - n\varphi\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i\right).$$

Let's notice that δ_φ from Theorem 3.8 is equal to $S_\varphi^2(a, b)$.

Obviously, if φ is convex, $S_\varphi^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \geq 0$.

The next variant of Jensen's inequality was proved by Matković and Pečarić in [169].

Theorem 3.13: *Let U be a convex subset in $\mathbb{R}^k, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in U$ and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m \in co(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$. If φ is a convex function on U, then the inequality*

$$\varphi\left(\frac{\sum_{i=1}^{n} w_i \boldsymbol{x}_i - \sum_{j=1}^{m} w_j \boldsymbol{y}_j}{W_n - W_m}\right) \leq \frac{\sum_{i=1}^{n} w_i \varphi(\boldsymbol{x}_i) - \sum_{j=1}^{m} w_j \varphi(\boldsymbol{y}_j)}{W_n - W_m}$$

holds for all positive real numbers $w_1, \ldots, w_n$ and $w_1, \ldots, w_m$ satisfying the condition $w_i \geq W_m$ for all $i \in \{1, \ldots, n\}$, where $W_n = \sum_{i=1}^{n} w_i$ and $W_m = \sum_{j=1}^{m} w_j$.

Next theorem generalizes and improves Theorem 3.13.

Theorem 3.14: *Let L satisfy properties L1, L2 and L4 on a nonempty set E, A be a positive linear functional on L and $\widetilde{A}$ defined as in (3.12). Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^k$ and $K = co(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$. Let φ be a convex function on K and $\lambda_1, \ldots, \lambda_n$ barycentric coordinates over K. Then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset K$ and $\varphi(\boldsymbol{g}), \lambda_i(\boldsymbol{g}) \in L$, $i \in \{1, \ldots, n\}$ and positive real numbers $w_1, \ldots, w_n$, with $W_n = \sum_{i=1}^n w_i$, satisfying the condition*

$$w_i \geq A(\mathbf{1}) \text{ for all } i \in \{1, \ldots, n\}, \tag{3.13}$$

we have

$$\begin{aligned}
&\varphi\left(\frac{\sum_{i=1}^n w_i \boldsymbol{x}_i - \widetilde{A}(\boldsymbol{g})}{W_n - A(\mathbf{1})}\right) \\
&\leq \frac{\sum_{i=1}^n w_i \varphi(\boldsymbol{x}_i) - \sum_{i=1}^n A(\lambda_i(\boldsymbol{g})) \varphi(\boldsymbol{x}_i) - \min_i \{w_i - A(\lambda_i(\boldsymbol{g}))\}}{W_n - A(\mathbf{1})} \\
&\times S_\varphi^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \\
&\leq \frac{1}{W_n - A(\mathbf{1})} \sum_{i=1}^n w_i \varphi(\boldsymbol{x}_i) - A(\varphi(\boldsymbol{g})) - S_\varphi^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \\
&\times \left[\min_i \{w_i - A(\lambda_i(\boldsymbol{g}))\} + A\left(\min_i \{\lambda_i(\boldsymbol{g})\}\right)\right].
\end{aligned} \tag{3.14}$$

Proof: For each $t \in E$ we have $\boldsymbol{g}(t) \in K$. Using barycentric coordinates we have $\lambda_i(\boldsymbol{g}(t)) \geq 0, i = 1, \ldots, n$, $\sum_{i=1}^n \lambda_i(\boldsymbol{g}(t)) = 1$ and $\boldsymbol{g}(t) = \sum_{i=1}^n \lambda_i(\boldsymbol{g}(t)) \boldsymbol{x}_i$.

Since φ is convex on K, then

$$\varphi(\boldsymbol{g}(t)) \leq \sum_{i=1}^n \lambda_i(\boldsymbol{g}(t)) \varphi(\boldsymbol{x}_i) - \min_i \{\lambda_i(\boldsymbol{g}(t))\} S_\varphi^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n). \tag{3.15}$$

Applying positive linear functional A on (3.15) we get

$$A(\varphi(\boldsymbol{g})) \leq \sum_{i=1}^n A(\lambda_i(\boldsymbol{g})) \varphi(\boldsymbol{x}_i) - A\left(\min_i \{\lambda_i(\boldsymbol{g})\}\right) S_\varphi^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n),$$

where

$$\sum_{i=1}^n A(\lambda_i(\boldsymbol{g})) = A\left(\sum_{i=1}^n \lambda_i(\boldsymbol{g})\right) = A(\mathbf{1})$$

and

$$A(\mathbf{1}) \geq A(\lambda_i(\boldsymbol{g})) \geq 0 \text{ for all } i \in \{1, \ldots, n\}.$$

Also we have

$$\widetilde{A}\left(\boldsymbol{g}\right)=\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)\boldsymbol{x}_i.$$

Now we can write

$$\begin{aligned}\frac{\sum_{i=1}^{n}w_i\boldsymbol{x}_i-\widetilde{A}\left(\boldsymbol{g}\right)}{W_n-A\left(\mathbf{1}\right)}&=\frac{1}{W_n-A\left(\mathbf{1}\right)}\left(\sum_{i=1}^{n}w_i\boldsymbol{x}_i-\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)\boldsymbol{x}_i\right)\\&=\frac{1}{W_n-A\left(\mathbf{1}\right)}\sum_{i=1}^{n}\left(w_i-A\left(\lambda_i(\boldsymbol{g})\right)\right)\boldsymbol{x}_i.\end{aligned}$$

We have

$$\frac{1}{W_n-A\left(\mathbf{1}\right)}\sum_{i=1}^{n}\left(w_i-A\left(\lambda_i(\boldsymbol{g})\right)\right)=1$$

and

$$\frac{1}{W_n-A\left(\mathbf{1}\right)}\left(w_i-A\left(\lambda_i(\boldsymbol{g})\right)\right)\geq 0 \text{ for all } i\in\{1,\dots,n\},$$

since

$$w_i\geq A(\mathbf{1})\geq A\left(\lambda_i(\boldsymbol{g})\right)\ \text{ for all } i\in\{1,\dots,n\}.$$

Therefore, expression $\frac{\sum_{i=1}^{n}w_i\boldsymbol{x}_i-\widetilde{A}\left(\boldsymbol{g}\right)}{W_n-A\left(\mathbf{1}\right)}$ is a convex combination of vectors $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n$ and belongs to K.

Since φ is convex on K, we have

$$\begin{aligned}&\varphi\left(\frac{\sum_{i=1}^{n}w_i\boldsymbol{x}_i-\widetilde{A}\left(\boldsymbol{g}\right)}{W_n-A\left(\mathbf{1}\right)}\right)=\varphi\left(\frac{1}{W_n-A\left(\mathbf{1}\right)}\sum_{i=1}^{n}\left(w_i-A\left(\lambda_i(\boldsymbol{g})\right)\right)\boldsymbol{x}_i\right)\\&\leq\frac{1}{W_n-A\left(\mathbf{1}\right)}\sum_{i=1}^{n}\left(w_i-A\left(\lambda_i(\boldsymbol{g})\right)\right)\varphi\left(\boldsymbol{x}_i\right)-\min_i\left\{\frac{w_i-A(\lambda_i(\boldsymbol{g}))}{W_n-A(\mathbf{1})}\right\}\\&\times S_{\varphi}^{n}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)\\&=\frac{\sum_{i=1}^{n}w_i\varphi\left(\boldsymbol{x}_i\right)-\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)\varphi\left(\boldsymbol{x}_i\right)-\min_i\left\{w_i-A(\lambda_i(\boldsymbol{g}))\right\}}{W_n-A\left(\mathbf{1}\right)}\\&\times S_{\varphi}^{n}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)\\&\leq\frac{\sum_{i=1}^{n}w_i\varphi\left(\boldsymbol{x}_i\right)-A(\varphi(\boldsymbol{g}))-S_{\varphi}^{n}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)}{W_n-A\left(\mathbf{1}\right)}\\&\times\left[\min_i\left\{w_i-A(\lambda_i(\boldsymbol{g}))\right\}+A\left(\min_i\left\{\lambda_i(\boldsymbol{g})\right\}\right)\right].\end{aligned}\tag{3.16}$$

□

Next Corollary shows that Theorem 3.14 is a generalization of Theorem 3.10 for convex hulls.

Corollary 3.8: *Let L satisfy properties L1, L2 and L4 on a nonempty set E and A be a positive* *normalized linear functional on L. Let φ be a convex function on an interval $I=[a,b]\subset\mathbb{R}$. Then for all $g\in L$ such that $g(E)\subset I$ and $\varphi(g)\in L$, we have*

$$\begin{aligned}&\varphi\left(a+b-A(g)\right)\leq\frac{A(g)-a}{b-a}\varphi(a)+\frac{b-A(g)}{b-a}\varphi(b)\\&-\left(\frac{1}{2}-\frac{1}{b-a}\left|A(g)-\frac{a+b}{2}\right|\right)S_{\varphi}^{2}(a,b)\\&\leq\varphi(a)+\varphi(b)-A(\varphi(g))\\&-\left[1-\frac{1}{b-a}\left(\left|A(g)-\frac{a+b}{2}\right|+A\left(\left|g-\frac{a+b}{2}\right|\right)\right)\right]S_{\varphi}^{2}(a,b).\end{aligned}\tag{3.17}$$

Remark 3.3: The inequalities in (3.17) are also improvements of the inequalities obtained in [76].

Theorem 3.15: *Let L satisfy properties $L1$, $L2$ and $L4$ on a nonempty set E, A be a positive linear functional on L and $\widetilde{A}$ defined as in (3.12). Let $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\in\mathbb{R}^k$ and $K=co(\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\})$. Let φ be a convex function on K and $\lambda_1,\ldots,\lambda_n$ barycentric coordinates over K. Then for all $\boldsymbol{g}\in L^k$ such that $\boldsymbol{g}(E)\subset K$ and $\varphi(\boldsymbol{g}),\lambda_i(\boldsymbol{g})\in L$, $i\in\{1,\ldots,n\}$ and positive real numbers $w_1,\ldots,w_n$ satisfying the conditions $W_n-A(\mathbf{1})>0$, where $W_n=\sum_{i=1}^n w_i$, and*

$$\frac{\sum_{i=1}^n w_i\boldsymbol{x}_i-\widetilde{A}\left(\boldsymbol{g}\right)}{W_n-A(\mathbf{1})}\in K,\tag{3.18}$$

we have

$$\begin{aligned}&\varphi\left(\frac{\sum_{i=1}^n w_i\boldsymbol{x}_i-\widetilde{A}\left(\boldsymbol{g}\right)}{W_n-A\left(\mathbf{1}\right)}\right)\geq\frac{W_n\varphi\left(\frac{1}{W_n}\sum_{i=1}^n w_i\boldsymbol{x}_i\right)-A(\mathbf{1})\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}\left(\boldsymbol{g}\right)\right)}{W_n-A\left(\mathbf{1}\right)}\\&\geq\frac{W_n\varphi\left(\frac{1}{W_n}\sum_{i=1}^n w_i\boldsymbol{x}_i\right)-\sum_{i=1}^n A\left(\lambda_i(\boldsymbol{g})\right)\varphi\left(\boldsymbol{x}_i\right)+\min_i\left\{A(\lambda_i(\boldsymbol{g}))\right\}}{W_n-A\left(\mathbf{1}\right)}\\&\times S_{\varphi}^{n}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n).\end{aligned}\tag{3.19}$$

Proof: For each $t\in E$ we have $\boldsymbol{g}(t)\in K$. Using barycentric coordinates we have $\lambda_i(\boldsymbol{g}(t))\geq 0, i=1,\ldots,n$, $\sum_{i=1}^n\lambda_i(\boldsymbol{g}(t))=1$ and

$$\boldsymbol{g}(t)=\sum_{i=1}^n\lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i.$$

Also we have

$$\widetilde{A}\left(\boldsymbol{g}\right)=\sum_{i=1}^n A\left(\lambda_i(\boldsymbol{g})\right)\boldsymbol{x}_i.$$

We can easily see that

$$\frac{1}{A(\mathbf{1})}\widetilde{A}\left(\boldsymbol{g}\right)=\frac{1}{A(\mathbf{1})}\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)\boldsymbol{x}_i\in K,$$

since

$$\frac{1}{A(\mathbf{1})}\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)=1 \text{ and } \frac{1}{A(\mathbf{1})}A\left(\lambda_i(\boldsymbol{g})\right)\geq 0,\ i\in\{1,\dots,n\}.$$

Since φ is convex on K, then using Lemma 3.1

$$\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}\left(\boldsymbol{g}\right)\right)\leq\frac{1}{A(\mathbf{1})}\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)\varphi(\boldsymbol{x}_i)-\min_i\left\{\frac{A(\lambda_i(\boldsymbol{g}))}{A(\mathbf{1})}\right\}S_\varphi^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n). \tag{3.20}$$

Using Theorem 3.8 and (3.20) we have

$$\begin{aligned}
&\varphi\left(\frac{W_n\left(\frac{1}{W_n}\sum_{i=1}^{n}w_i\boldsymbol{x}_i\right)-A(\mathbf{1})\left(\frac{1}{A(\mathbf{1})}\widetilde{A}\left(\boldsymbol{g}\right)\right)}{W_n-A\left(\mathbf{1}\right)}\right)\\
&\geq\frac{W_n\varphi\left(\frac{1}{W_n}\sum_{i=1}^{n}w_i\boldsymbol{x}_i\right)-A(\mathbf{1})\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}\left(\boldsymbol{g}\right)\right)}{W_n-A\left(\mathbf{1}\right)}\\
&\geq\frac{W_n\varphi\left(\frac{1}{W_n}\sum_{i=1}^{n}w_i\boldsymbol{x}_i\right)-A(\mathbf{1})\frac{1}{A(\mathbf{1})}\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)\varphi\left(\boldsymbol{x}_i\right)+\min_i\left\{A(\lambda_i(\boldsymbol{g}))\right\}}{W_n-A\left(\mathbf{1}\right)}\\
&\times S_\varphi^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n).
\end{aligned}$$

□

Remark 3.4: If positive real numbers $w_1,\dots,w_n$ satisfy condition (3.13), then condition (3.18) is also satisfied since K is convex set. Hence (3.14) can be extended as follows

$$\begin{aligned}
&\frac{W_n\varphi\left(\frac{1}{W_n}\sum_{i=1}^{n}w_i\boldsymbol{x}_i\right)-\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)\varphi\left(\boldsymbol{x}_i\right)+\min_i\left\{A(\lambda_i(\boldsymbol{g}))\right\}}{W_n-A\left(\mathbf{1}\right)}S_\varphi^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)\\
&\leq\frac{W_n\varphi\left(\frac{1}{W_n}\sum_{i=1}^{n}w_i\boldsymbol{x}_i\right)-A(\mathbf{1})\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}\left(\boldsymbol{g}\right)\right)}{W_n-A\left(\mathbf{1}\right)}\\
&\leq\varphi\left(\frac{\sum_{i=1}^{n}w_i\boldsymbol{x}_i-\widetilde{A}\left(\boldsymbol{g}\right)}{W_n-A\left(\mathbf{1}\right)}\right)\\
&\leq\frac{\sum_{i=1}^{n}w_i\varphi\left(\boldsymbol{x}_i\right)-\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)\varphi\left(\boldsymbol{x}_i\right)-\min_i\left\{w_i-A(\lambda_i(\boldsymbol{g}))\right\}}{W_n-A\left(\mathbf{1}\right)}S_\varphi^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)\\
&\leq\frac{\sum_{i=1}^{n}w_i\varphi\left(\boldsymbol{x}_i\right)-A(\varphi(\boldsymbol{g}))-S_\varphi^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)}{W_n-A\left(\mathbf{1}\right)}\\
&\times\left[\min_i\left\{w_i-A(\lambda_i(\boldsymbol{g}))\right\}+A\left(\min_i\left\{\lambda_i(\boldsymbol{g})\right\}\right)\right].
\end{aligned}$$

Corollary 3.9: *Let L satisfy properties L1, L2 and L4 on a nonempty set E and A be a positive* *normalized linear functional on* L*. Let* φ *be a convex function on an interval* $I = [a, b] \subset \mathbb{R}$*. Then for all* $g \in L$ *such that* $g(E) \subset I$ *and* $\varphi(g) \in L$*, we have*

$$\begin{aligned}
&\varphi\left(a+b-A(g)\right) \geq 2\varphi\left(\frac{a+b}{2}\right) - \varphi\left(A(g)\right)\\
&\geq 2\varphi\left(\frac{a+b}{2}\right) - \left[\frac{b-A(g)}{b-a}\varphi(a) + \frac{A(g)-a}{b-a}\varphi(b)\right]\\
&+\left(\frac{1}{2} - \frac{1}{b-a}\left|A(g) - \frac{a+b}{2}\right|\right) S_{\varphi}^{2}(a,b).
\end{aligned}$$

Next we give generalizations of Corollary 3.8 and Corollary 3.9 for convex functions defined on k-simplices in $\mathbb{R}^k$.

Let S be a k-simplex in $\mathbb{R}^k$ with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1} \in \mathbb{R}^k$. The barycentric coordinates $\lambda_1, \ldots \lambda_{k+1}$ over S are nonnegative linear polynomials which satisfy Lagrange's property

$$\lambda_i(\boldsymbol{v}_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

It is known (see [38]) that for each $\boldsymbol{x} \in S$ barycentric coordinates $\lambda_1(\boldsymbol{x}), \ldots, \lambda_{k+1}(\boldsymbol{x})$ have the form

$$\begin{aligned}
\lambda_1(\boldsymbol{x}) &= \frac{\mathrm{Vol}_k\left([\boldsymbol{x}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)},\\
\lambda_2(\boldsymbol{x}) &= \frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1, \boldsymbol{x}, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)},\\
&\vdots\\
\lambda_{k+1}(\boldsymbol{x}) &= \frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{x}]\right)}{\mathrm{Vol}_k(S)},
\end{aligned}$$

where $\mathrm{Vol}_k(F)$ denotes the k-dimensional Lebesgue measure of a measurable set $F \subset \mathbb{R}^k$. Here, for example, $[\boldsymbol{v}_1, \boldsymbol{x}, \ldots, \boldsymbol{v}_{k+1}]$ denotes the subsimplex obtained by replacing $\boldsymbol{v}_2$ by $\boldsymbol{x}$, i.e., the subsimplex opposite to $\boldsymbol{v}_2$, when adding $\boldsymbol{x}$ as a new vertex.

Corollary 3.10: *Let* L *satisfy properties L1, L2 and L4 on a nonempty set* E*,* A *be a positive normalized linear functional on* L *and* $\widetilde{A}$ *defined as in* (3.12)*. Let* φ *be a convex function on* k*-simplex* $S = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]$ *in* $\mathbb{R}^k$ *and* $\lambda_1, \ldots, \lambda_{k+1}$ *barycentric coordinates over* S*. Then* *for all* $\boldsymbol{g} \in L^k$ *such that* $\boldsymbol{g}(E) \subset S$ *and* $\varphi(\boldsymbol{g}) \in L$ *we have*

$$\begin{aligned}
&\frac{(k+1)\varphi\left(\frac{1}{k+1}\sum_{i=1}^{k+1}\boldsymbol{v}_i\right)-\sum_{i=1}^{k+1}\lambda_i(\widetilde{A}(\boldsymbol{g}))\varphi(\boldsymbol{v}_i)+\min_i\left\{\lambda_i(\widetilde{A}(\boldsymbol{g}))\right\}}{k}\\
&\times S_{\varphi}^{k+1}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1})\\
&\le \frac{(k+1)\varphi\left(\frac{1}{k+1}\sum_{i=1}^{k+1}\boldsymbol{v}_i\right)-\varphi(\widetilde{A}(\boldsymbol{g}))}{k}\\
&\le \varphi\left(\frac{\sum_{i=1}^{k+1}\boldsymbol{v}_i-\widetilde{A}(\boldsymbol{g})}{k}\right)\\
&\le \frac{\sum_{i=1}^{k+1}\varphi(\boldsymbol{v}_i)-\sum_{i=1}^{k+1}\lambda_i(\widetilde{A}(\boldsymbol{g}))\varphi(\boldsymbol{v}_i)-\min_i\left\{1-\lambda_i(\widetilde{A}(\boldsymbol{g}))\right\}}{k}\\
&S_{\varphi}^{k+1}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1})\\
&\le \frac{\sum_{i=1}^{k+1}\varphi(\boldsymbol{v}_i)-A(\varphi(\boldsymbol{g}))-S_{\varphi}^{k+1}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1})}{k}\\
&\times\left[\min_i\left\{1-\lambda_i(\widetilde{A}(\boldsymbol{g}))\right\}+A\left(\min_i\{\lambda_i(\boldsymbol{g})\}\right)\right]. \qquad (3.21)
\end{aligned}$$

Proof: Since barycentric coordinates $\lambda_1,\ldots,\lambda_{k+1}$ over k-simplex S in $\mathbb{R}^k$ are nonnegative linear polynomials, then $A(\lambda_i(\boldsymbol{g}))=\lambda_i(\widetilde{A}(\boldsymbol{g}))$ for all $i=1,\ldots,k+1$.

Choosing $\boldsymbol{x}_i=\boldsymbol{v}_i$ for all $i\in\{1,\ldots,k+1\}$ and $w_1=w_2=\cdots=w_{k+1}=1$, the inequalities in (3.21) easily follow from (3.14) and (3.19). □

Remark 3.5: As a special case of Corollary 3.10 for $k=1$ and if we take p and q nonnegative real numbers such that $A(g)=\dfrac{pa+qb}{p+q}$ we get right hand side of the inequality (2.3) in [30].

Remark 3.6: Using the same technique and the same special case as in Example 1 and Remark 8 in [217], from (3.21) we get the same results, that is, the k-dimensional version of the Hammer-Bullen inequality, namely

$$\frac{1}{|S|}\int_S f(t)dt-f(\boldsymbol{v}^*)\le\frac{k}{k+1}\sum_{i=1}^{k+1}f(\boldsymbol{v}_i)-\frac{k}{|S|}\int_S f(t)dt,$$

and, as a special case in one dimension, an improvement of classical Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)dt\le\frac{f(a)+f(b)}{2}-\frac{1}{4}S_f^2(a,b).$$

CHAPTER 4

Means of Mercer Type and Exponential Convexity

This chapter examines Mercer-type means and their connections with exponential convexity. We begin with Cauchy means of Mercer type and means related to the Jessen–Mercer inequality, highlighting their role in refining classical bounds. The discussion then extends to mean value theorems and notions of exponential convexity, including n-exponential convexity for Jensen–Mercer and Jessen–Mercer differences.The results presented here provide a deeper understanding of convexity structures underlying Mercer-type inequalities and their applications.

4.1 Cauchy Means of Mercer Type

In this section, we give definition of Cauchy means of Mercer's type. Also, we show that this mean is monotonic. These results are given in [21].

In [175] Mercer gave the definition of generalized power means for sums. In [76] Cheung, Matković, and Pečarić defined these generalized means of Mercer's type for functionals. As a special case of their definition we can define these for integrals as follows:

Definition 4.1: Let Ω be a convex set equipped with probability measure μ, $I = [a, b]$ and $v : \Omega \to I$ is integrable function. Then for $r \in \mathbb{R}$ we define generalized power means of Mercer's type by:

$$M_r(f,\mu) = \begin{cases} \left(a^r + b^r - \int_\Omega f^r(t)d\mu(t)\right)^{\frac{1}{r}}, & r \neq 0; \\ \dfrac{ab}{\exp(\int_\Omega \log(f(t))d\mu(t))}, & r = 0. \end{cases}$$

Following result is extracted from [21].

Theorem 4.1: *Let Ω be a convex set equipped with probability measure μ, and let $f \in C^2(I)$ be a function on an interval $I = [a,b]$ $(-\infty < a < b < \infty)$. Then for a integrable function $g : \Omega \to I$, there exist $\xi \in I$ such that the following equality is valid:*

$$f(a)+f(b)-\int_\Omega f(g(t))d\mu(t)-f\Big(a+b-\int_\Omega g(t)d\mu(t)\Big)=$$
$$\frac{f''(\xi)}{2}\left(a^2+b^2-\Big(\int_\Omega g(t)d\mu(t)\Big)^2-\Big(a+b-\int_\Omega g(t)d\mu(t)\Big)^2\right).$$

Theorem 4.2: *Let $f,h \in C^2(I)$. Then for g and μ as above, we have there exist $\xi \in I$ such that the following equality hold true,*

$$\frac{f''(\xi)}{h''(\xi)}=\frac{f(a)+f(b)-\int_\Omega f(g(t))d\mu(t)-f\Big(a+b-\int_\Omega g(t)d\mu(t)\Big)}{h(a)+h(b)-\int_\Omega h(g(t))d\mu(t)-h\Big(a+b-\int_\Omega g(t)d\mu(t)\Big)}, \tag{4.1}$$

provided that the denominators are non-zero.

Corollary 4.1: *If $\frac{f''}{h''}$ is invertible, then equation (4.1) suggests a new means. That is*

$$\xi=\left(\frac{f''}{h''}\right)^{-1}\left(\frac{f(a)+f(b)-\int_\Omega f(g(t))d\mu(t)-f\Big(a+b-\int_\Omega g(t)d\mu(t)\Big)}{h(a)+h(b)-\int_\Omega h(g(t))d\mu(t)-h\Big(a+b-\int_\Omega g(t)d\mu(t)\Big)}\right)$$

is a new mean, provided that the denominators are non-zero.

Of course we can define generalized quasi-arithmetic means of Mercer's type for function $f(t)$ as follows:

$$\widetilde{M_\alpha}(f,\mu)=\alpha^{-1}\Big(\alpha(a)+\alpha(b)-\int_\Omega \alpha(f(t))d\mu(t)\Big).$$

Theorem 4.3: *Let $I=[a,b]$ be a compact interval and let $\alpha,\beta,\gamma \in C^2(I)$ be strictly monotone functions. Suppose also that Ω is a convex set equipped with a probability measure μ and $f : \Omega \to I$. Then,*

$$\frac{\alpha\widetilde{M_\alpha}(f,\mu)-\alpha\widetilde{M_\gamma}(f,\mu)}{\beta\widetilde{M_\beta}(f,\mu)-\beta\widetilde{M_\gamma}(f,\mu)}=\frac{\alpha''(\eta)\gamma'(\eta)-\alpha'(\eta)\gamma''(\eta)}{\beta''(\eta)\gamma'(\eta)-\beta'(\eta)\gamma''(\eta)}$$

for some η in the image of f, provided that the denominators are non-zero.

Proof: If we choose the functions f and h so that

$$f=\alpha\circ\gamma^{-1},\ h=\beta\circ\gamma^{-1}\ \ m=\gamma(a),\ M=\gamma(b)\ \text{ and }\ g(t)=\gamma(f(t)),$$

and apply Theorem 4.2 to such functions, we find that there exits $\xi \in I$ such that

$$\frac{\alpha(\widetilde{M_\alpha}(f,\mu))-\alpha\widetilde{M_\gamma}(f,\mu)}{\beta\widetilde{M_\beta}(f,\mu)-\beta\widetilde{M_\gamma}(f,\mu)}=\frac{\alpha''(g^{-1}(\xi))\gamma'(g^{-1}(\xi))-\alpha'(g^{-1}(\xi))\gamma''(g^{-1}(\xi))}{\beta''(g^{-1}(\xi))\gamma'(g^{-1}(\xi))-\beta'(g^{-1}(\xi))\gamma''(g^{-1}(\xi))}.$$

Thus, setting $g^{-1}(\xi) = \eta$, find that there exists η in the image of v such that

$$\frac{\alpha\widetilde{M_\alpha}(f,\mu) - \alpha\widetilde{M_\gamma}(f,\mu)}{\beta\widetilde{M_\beta}(f,\mu) - \beta\widetilde{M_\gamma}(f,\mu)} = \frac{\alpha''(\eta)\gamma'(\eta) - \alpha'(\eta)\gamma''(\eta)}{\beta''(\eta)\gamma'(\eta) - \beta'(\eta)\gamma''(\eta)}. \tag{4.2}$$

□

Let

$$\Upsilon(\eta) = \frac{\alpha''(\eta)\gamma'(\eta) - \alpha'(\eta)\gamma''(\eta)}{\beta''(\eta)\gamma'(\eta) - \beta'(\eta)\gamma''(\eta)}.$$

If Υ is invertible then (4.2) suggests new means:

$$\eta = \Upsilon^{-1}\Big(\frac{\alpha\widetilde{M_\alpha}(f,\mu) - \alpha\widetilde{M_\gamma}(f,\mu)}{\beta\widetilde{M_\beta}(f,\mu) - \beta\widetilde{M_\gamma}(f,\mu)}\Big)$$

because η is in the image of $f(t)\ \ t \in \Omega$.

Corollary 4.2: *Let $r, s, l \in \mathbb{R}$ and let Ω be a convex set equipped with a probability measure μ. Then*

$$\frac{M_r^r(f,\mu) - M_s^r(f,\mu)}{M_l^l(f,\mu) - M_s^l(f,\mu)} = \frac{r(r-s)}{l(l-s)}\eta^{r-l} \tag{4.3}$$

for some η in the image of $f(t)$ $(t \in \Omega)$, provided that the denominators are non-zero.

Proof: If we set

$$\alpha(t) = t^r,\ \ \beta(t) = t^l,\ \ \gamma(t) = t^s,\ \ f(t) = f^s(t)$$

in Theorem 4.3 we get the assertion (4.3). □

Since η is in the image of $f(t)$ therefore (4.3) suggests a new mean that is

$$\inf_{t\in\Omega} f(t) \le \Big(\frac{l(l-s)}{r(r-s)}\frac{M_r^r(f,\mu) - M_s^r(f,\mu)}{M_l^l(f,\mu) - M_s^l(f,\mu)}\Big)^{\frac{1}{r-l}} \le \sup_{t\in\Omega} f(t).$$

From (4.3) we can define a new mean $M_{r,l}^s$ as follows:

$$M_{r,l}^s = \ \Big(\tfrac{l(l-s)}{r(r-s)}\tfrac{M_r^r(f,\mu)-M_s^r(f,\mu)}{M_l^l(f,\mu)-M_s^l(f,\mu)}\Big)^{\frac{1}{r-l}},\quad l \ne r \ne s, l, r \ne 0;$$

Now taking $\lim_{l\to 0} M_{r,l}^s(f,\mu)$ we will get

$$\begin{aligned} &M_{r,0}^s(f,\mu) = M_{0,r}^s(f,\mu) = \lim_{l\to 0} M_{r,l}^s(f,\mu) \\ &= \Big(\frac{s[M_r^r(f,\mu) - M_s^r(f,\mu)]}{r(r-s)[\log M_s(f,\mu) - \log M_0(f,\mu)]}\Big)^{\frac{1}{r}}, \qquad r \ne s,\ r, s \ne 0. \end{aligned}$$

Now taking $\lim_{r\to s} M_{r,l}^s(f,\mu)$ for $l \ne s$ and $l, s \ne 0$, we will get

$$\lim_{r\to s} M^s_{r,l}(f,\mu) = M^s_{s,l}(f,\mu) = M^s_{l,s}(f,\mu)$$
$$= \Big(\frac{l(l-s)}{s}\frac{[a^s\log m + b^s\log M - \int_\Omega f(t)^s\log f(t)d\mu(t) - M^s_s(v,\mu)\log M_s(f,\mu)]}{M^l_l(f,\mu)-M^l_s(f,\mu)}\Big)^{\frac{1}{s-l}}.$$

By similar way we can calculate all the cases for $r,s,l\in\mathbb{R}$. finally we get the following definition of $M^s_{r,l}(f,\mu)$:

Definition 4.2: Let Ω be a convex set equipped with probability measure μ and $I=[m,M]$ and $v:\Omega\to I$ is integrable function. Then for $r,s,l\in\mathbb{R}$ we define generalized power means $M^s_{r,l}(f,\mu)$ by:

$$M^s_{r,l}(f,\mu) = \Big(\frac{l(l-s)}{r(r-s)}\frac{M^r_r(f,\mu)-M^r_s(f,\mu)}{M^l_l(f,\mu)-M^l_s(f,\mu)}\Big)^{\frac{1}{r-l}}, \qquad l\neq r\neq s;\ l,r\neq 0$$

other limiting cases of this generalized power mean can be found in [21].

4.1.1 Monotonicity of the Means

Here we state a result related to the monotonicity of $M^s_{r,l}$ defined in Definition 4.2. For idea of the proof we refer the reader to [22]:

Theorem 4.4: *Let $t,r,u,v\in\mathbb{R}$, such that $t\leq v, r\leq u$. Then for $M^s_{r,l}$ as defined in Definition 4.2 we have*

$$M^s_{t,r}\leq M^s_{v,u}.$$

4.2 Means Related to the Jessen–Mercer Inequality

Let $x_1,\ldots,x_n\in[a,b]$ where $a>0$, let $w_1,\ldots,w_n$ *be* nonnegative real numbers with $\sum_{i=1}^n w_i=1$ and let $r\in\mathbb{R}$. The (weighted) *power mean* $M_r(\boldsymbol{x},\boldsymbol{w})$ is defined as

$$M_r(\boldsymbol{x},\boldsymbol{w}) = \begin{cases} \left(\displaystyle\sum_{k=1}^n w_k x_k^r\right)^{\frac{1}{r}}, & r\neq 0,\\ \exp\left(\displaystyle\sum_{k=1}^n w_k\ln x_k\right) & r=0.\end{cases}$$

In [175] Mercer defined the family of functions

$$M_r(a,b,\boldsymbol{x}) = \begin{cases} [a^r+b^r-M_r^r(\boldsymbol{x},\boldsymbol{w})]^{\frac{1}{r}}, & r\neq 0,\\ \dfrac{ab}{M_0(\boldsymbol{x},\boldsymbol{w})}, & r=0\end{cases}$$

and proved the following assertion.

Theorem 4.5: *Let $-\infty<r<s<+\infty$. Then $a<M_r(a,b,\boldsymbol{x})\leq M_s(a,b,\boldsymbol{x})<b$.*

We extend this result on the generalized power means and we give its refinement.

Let E be a nonempty set and let L be a class of functions $f : E \to \mathbb{R}$ having properties L1 and L2. Let A be a positive linear functional on L such that $A(1) = 1$. Assume that $g \in L$ is a function from E to $[a, b]$ $(-\infty < a < b < \infty)$ such that all of the following expressions are well defined. From that especially follows $0 < a < b < \infty$. For any $r, s \in \mathbb{R}$ we define

$$Q(r,g) := \begin{cases} [a^r + b^r - A(g^r)]^{\frac{1}{r}}, & r \neq 0, \\ \dfrac{ab}{\exp\left(A(\ln g)\right)}, & r = 0, \end{cases}$$

$$R(r,s,g) := \begin{cases} \left[A\left([a^r + b^r - g^r]^{\frac{s}{r}}\right)\right]^{\frac{1}{s}}, & r \neq 0, s \neq 0, \\ \exp\left(A\left(\ln\left[a^r + b^r - (g^r)\right]^{\frac{1}{r}}\right)\right), & r \neq 0, s = 0, \\ \left(A\left(\dfrac{ab}{g}\right)^s\right)^{\frac{1}{s}}, & r = 0, s \neq 0, \\ \exp\left(A\left(\ln\dfrac{ab}{g}\right)\right), & r = s = 0 \end{cases}$$

and

$$S(r,s,g) := \begin{cases} \left[\dfrac{b^r - A(g^r)}{b^r - a^r}b^s + \dfrac{A(g^r) - a^r}{b^r - a^r}a^s\right]^{\frac{1}{s}}, & r \neq 0, s \neq 0, \\ \exp\left(\dfrac{b^r - A(g^r)}{b^r - a^r}\ln M + \dfrac{A(g^r) - a^r}{b^r - a^r}\ln a\right), & r \neq 0, s = 0, \\ \left[\dfrac{\ln b - A(\ln g)}{\ln b - \ln a}b^s + \dfrac{A(\ln g) - \ln m}{\ln b - \ln a}a^s\right]^{\frac{1}{s}}, & r = 0, s \neq 0, \\ \exp\left(\dfrac{\ln b - A(\ln g)}{\ln b - \ln a}\ln b + \dfrac{A(\ln g) - \ln a}{\ln b - \ln a}\ln a\right), & r = s = 0. \end{cases}$$

Theorem 4.6: *If $r, s \in \mathbb{R}$ and $r \leq s$, then*

$$Q(r,g) \leq Q(s,g).$$

Furthermore,

$$Q(r,g) \leq R(r,s,g) \leq S(r,s,g) \leq Q(s,g). \tag{4.4}$$

Corollary 4.3: *Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space, and let $g : \Omega \to [a, b]$ $(0 < a < b < \infty)$ be a measurable function. Let A be defined as $A(g) = \int_\Omega g d\mu$. Then for any continuous convex function $\varphi : [a, b] \to R$, and any $r, s \in \mathbb{R}$ with $r \leq s$, (4.4) holds.*

Let ψ, χ be continuous and strictly monotone functions on an interval $I = [a, b]$. Let $g \in L$ be a function such that $\psi(g)$, $\chi(g)$, $\chi\left(\psi^{-1}\left(\psi(a) + \psi(b) - \psi(g)\right)\right) \in L$ (so that $a \leq g(t) \leq b$ for all $t \in E$). Then we can define the *generalized mean of g with respect to the functional A and the function ψ* as (see for example [204, p. 107])

$$M_\psi(g, A) = \psi^{-1}\left(A\left(\psi(g)\right)\right).$$

Observe that if $\psi(a) \leq \psi(g) \leq \psi(b)$ for $t \in E$, then by the isotonic character of A, we have $\psi(a) \leq A(\psi(g)) \leq \psi(b)$, so that M_ψ is well defined. We further define

$$\widetilde{M}_\psi(g, A) = \psi^{-1}(\psi(a) + \psi(b) - A(\psi(g))).$$

From the above observation we know that $\psi(a) \leq \psi(a) + \psi(b) - A(\psi(g)) \leq \psi(b)$ so that $\widetilde{M}_\psi$ is also well defined.

Theorem 4.7: *Under the above hypotheses, we have:*

(i) if either $\chi \circ \psi^{-1}$ is convex and χ is strictly increasing, or $\chi \circ \psi^{-1}$ is concave and χ is strictly decreasing, then

$$\widetilde{M}_\psi(g, A) \leq \widetilde{M}_\chi(g, A) \tag{4.5}$$

holds. In fact, to be more specific we have the following series of inequalities

$$\begin{aligned}\widetilde{M}_\psi(g, A) &\leq \chi^{-1}\left(A\left(\chi\left(\psi^{-1}(\psi(a) + \psi(b) - \psi(g))\right)\right)\right)\\ &\leq \chi^{-1}\left(\frac{\psi(b) - A(\psi(g))}{\psi(b) - \psi(a)}\chi(b) + \frac{A(\psi(g)) - \psi(a)}{\psi(b) - \psi(a)}\chi(a)\right) \leq \widetilde{M}_\chi(g, A);\end{aligned} \tag{4.6}$$

(ii) if either $\chi \circ \psi^{-1}$ is concave and χ is strictly increasing, or $\chi \circ \psi^{-1}$ is convex and χ is strictly decreasing, then the reverse inequalities hold in (4.5) and (4.6).

Proof: Since ψ is strictly monotone and $-\infty < a \leq g(t) \leq b < \infty$, we have $-\infty < \psi(a) \leq \psi(g) \leq \psi(b) < \infty$ or $-\infty < \psi(b) \leq \psi(g) \leq \psi(a) < \infty$.

Suppose that $\chi \circ \psi^{-1}$ is convex. Letting $\varphi = \chi \circ \psi^{-1}$ in Theorem 1.6 we obtain

$$\begin{aligned}&\left(\chi \circ \psi^{-1}\right)(\psi(a) + \psi(b) - A(\psi(g)))\\ &\leq A\left(\left(\chi \circ \psi^{-1}\right)(\psi(a) + \psi(b) - \psi(g))\right)\\ &\leq \frac{\psi(b) - A(\psi(g))}{\psi(b) - \psi(a)}\left(\chi \circ \psi^{-1}\right)(\psi(b)) + \frac{A(\psi(g)) - \psi(a)}{\psi(b) - \psi(a)}\left(\chi \circ \psi^{-1}\right)(\psi(a))\\ &\leq \left(\chi \circ \psi^{-1}\right)(\psi(a)) + \left(\chi \circ \psi^{-1}\right)(\psi(b)) - A\left(\left(\chi \circ \psi^{-1}\right)(\psi(g))\right),\end{aligned}$$

or

$$\begin{aligned}\chi\left(\psi^{-1}(\psi(a) + \psi(b) - A(\psi(g)))\right) &\leq A\left(\chi\left(\psi^{-1}(\psi(a) + \psi(b) - \psi(g))\right)\right)\\ &\leq \frac{\psi(b) - A(\psi(g))}{\psi(b) - \psi(a)}\chi(b) + \frac{A(\psi(g)) - \psi(a)}{\psi(b) - \psi(a)}\chi(a) \leq \chi(a) + \chi(b) - A(\chi(g)).\end{aligned} \tag{4.7}$$

If $\chi \circ \psi^{-1}$ is concave, then we have the reversed inequalities in (4.7).

If χ is strictly increasing, then the inverse function χ^{-1} is also strictly increasing, so that (4.7) implies (4.6). If χ is strictly decreasing, then the inverse function χ^{-1} is also strictly decreasing, so in that case the reverse of (4.7) implies (4.6). Analogously, we get the reverse of (4.6) in the cases when $\chi \circ \psi^{-1}$ is convex and χ is strictly decreasing, or $\chi \circ \psi^{-1}$ is concave and χ is strictly increasing. □

4.3 Mean Value Theorems and Exponential Convexity

Under the assumptions of Theorem 2.1 using (2.2) we define the following functionals:

$$\Psi_1(\mathbf{w},\mathbf{x},f) = D(\mathbf{w},\mathbf{x},f;I) - f\big(a+b-\sum_{i=1}^{n} w_i x_i\big) \geq 0, \tag{4.8}$$

$$\Psi_2(\mathbf{w},\mathbf{x},f) = f(a)+f(b)-\sum_{i=1}^{n} w_i f(x_i) - D(\mathbf{w},\mathbf{x},f;I) \geq 0, \tag{4.9}$$

$$\Psi_3(\mathbf{w},\mathbf{x},f) = f(a)+f(b)-\sum_{i=1}^{n} w_i f(x_i) - \varphi\big(a+b-\sum_{i=1}^{n} w_i x_i\big) \geq 0. \tag{4.10}$$

Similarly, under the assumptions of Theorem 2.23 using (2.30) we define the following functionals:

$$\Psi_4(A,f,\varphi) = \overline{D}(A,f,\varphi;E_1) - f(a+b-A(f)) \geq 0, \tag{4.11}$$

$$\Psi_5(A,f,\varphi) = f(a)+f(b)-A(\varphi(f)) - \overline{D}(A,f,\varphi;E_1) \geq 0, \tag{4.12}$$

$$\Psi_6(A,f,\varphi) = f(a)+f(b)-A(\varphi(f)) - f(a+b-A(f)) \geq 0. \tag{4.13}$$

Now we are in position to give mean value theorems for $\Psi_k(.,.,f)$, $k \in \{1,2,\ldots,6\}$. Since proving techniques are same as stated in Section 4.1 so we omit the details.

Theorem 4.8: *Let $f \in C^2([a,b])$, $\boldsymbol{x} = (x_1,\ldots,x_n) \in [a,b]^n$ and $\boldsymbol{w} = (w_1,\ldots,\, w_n)$ be n-tuple of positive real numbers such that $\sum_{i=1}^{n} w_i = 1$. Then there exists $c_k \in [a,b]$ such that*

$$\Psi_k(\boldsymbol{w},\boldsymbol{x},f) = \frac{f''(c_k)}{2}\Psi_k(\boldsymbol{w},\boldsymbol{x},f_0), \quad \textit{where } f_0(x) = x^2; k \in \{1,2,3\}.$$

Theorem 4.9: *Let $f,\psi \in C^2([a,b])$, $\boldsymbol{x} = (x_1,\ldots,x_n) \in [a,b]^n$ and $\boldsymbol{w} = (w_1,\ldots,w_n)$ be n-tuple of positive real numbers such that $\sum_{i=1}^{n} w_i = 1$. Then there exists $c_k \in [a,b]$ such that*

$$\frac{\Psi_k(\boldsymbol{w},\boldsymbol{x},f)}{\Psi_k(\boldsymbol{w},\boldsymbol{x},\psi)} = \frac{f''(c_j)}{\psi''(c_j)}, \quad k \in \{1,2,3\}.$$

provided that the denominators are non-zero.

Remark 4.1: We can also find mean value theorems for functionals $\Psi_k(.,.,f)$ for $k \in \{4,5,6\}$ (see [151]).

Remark 4.2: If the inverse of $\frac{f''}{\psi''}$ exists, then from the above mean value theorems we can give generalized means

$$c_j = \left(\frac{f''}{\psi''}\right)^{-1}\left(\frac{\Psi_k(.,.,f)}{\Psi_k(.,.,\psi)}\right), \; k \in \{1,2,\ldots,6\}. \tag{4.14}$$

4.3.1 n–Exponential Convexity of Jensen–Mercer and Jessen–Mercer Inequalities

Bernstein [37] and Widder [241] independently introduced an important sub-class of convex functions, which is called class of exponentially convex functions on a given open interval and studied some properties of this newly defined class. Exponentially convex functions have many nice properties, e.g., these functions are analytic on their domain. These functions also provide us positive-semidefinite matrices. Moreover, they play an important role in studying the properties of Stolarsky and Cauchy means, such as monotonicity of these means etc. For further study of the class of exponentially convex functions we refer [13], [122] and [183].

Pečarić and Perić in [215] introduced the notion of $n-$exponentially convex functions which is in fact generalization of the concept of exponentially convex functions. In the present subsection, we discus the same notion of $n-$exponential convexity by describing related definitions and some important results with some remarks from [215]. Before we proceed further first we discuss $\log-$convex functions.

4.3.2 Logarithmically Convex Functions

A number of important inequalities arise from the logarithmic convexity of some functions as one can see in [164]. Logarithmic convexity plays an important role in various fields namely reliability theory and survival analysis, economics, statistics, social sciences, information theory and optimization etc. (see for reference [43]). Its applications can also be found in applied mathematics as well.

Now, we recall some definitions. The following definition is originally given by Jensen in 1906 [123]. Here I is an interval in $\mathbb{R}$.

Definition 4.3: A function $f : I \to \mathbb{R}_+$ is called $\log -$*convex in* $J-$*sense* if the inequality

$$f^2\left(\frac{x_1+x_2}{2}\right) \leq f(x_1)\, f(x_2)$$

holds for each $x_1, x_2 \in I$.

Remark 4.3: A function $f : I \to \mathbb{R}_+$ is $\log-$convex in $J-$sense if and only if the inequality

$$u_1^2 f(x_1) + 2u_1u_2 f\left(\frac{x_1+x_2}{2}\right) + u_1^2 f(x_2) \geq 0$$

holds for each $x_1, x_2 \in I$ and $u_1, u_2 \in \mathbb{R}$.

Definition 4.4: [204, p. 7] A function $f : I \to \mathbb{R}_+$ is called $\log -$*convex* if the inequality

$$f(\lambda x_1 + (1-\lambda)x_2) \leq [f(x_1)]^{\lambda}[f(x_2)]^{(1-\lambda)}$$

holds for each $x_1, x_2 \in I$ and $\lambda \in [0, 1]$.

Remark 4.4: A function $\log-$convex in the $J-$sense is $\log-$convex if it is continuous as well.

4.3.3 n–Exponentially Convex Functions

Definition 4.5: [215] A function $f : J \to \mathbb{R}$ is *n-exponentially convex* in the Jensen sense on the interval J if

$$\sum_{k,l=1}^{n} \alpha_k \alpha_l f\left(\frac{x_k + x_l}{2}\right) \geq 0$$

holds for $\alpha_k \in \mathbb{R}$ and $x_k \in J$, $k \in \{1, 2, \ldots, n\}$.

A function $f : J \to \mathbb{R}$ is *n-exponentially convex* if it is *n-exponentially convex* in the Jensen sense and continuous on J.

Remark 4.5: From the definition it is clear that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponent- ially convex functions in the Jensen sense are m-exponentially convex in the Jensen sense for every $m \in \mathbb{N}, m \leq n$.

Theorem 4.10: *If $f : J \to \mathbb{R}$ is an n-exponentially convex function, then the matrix $\left[f\left(\frac{x_k+x_l}{2}\right)\right]_{k,l=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \leq n$. Particularly,*

$$\det\left[f\left(\frac{x_k + x_l}{2}\right)\right]_{k,l=1}^{m} \geq 0$$

for all $n \in \mathbb{N}$, $m \in \{1, 2, \ldots, n\}$.

Definition 4.6: A function $f : J \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$. A function $f : J \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 4.6: It is easy to show that $f : [a, b] \to \mathbb{R}^+$ is log-convex in the Jensen sense if and only if

$$\alpha^2 f(x) + 2\alpha\beta f\left(\frac{x+y}{2}\right) + \beta^2 f(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in [a, b]$. It follows that a function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

When dealing with functions with different degree of smoothness divided differences are found to be very useful. [138]

Definition 4.7: The second order divided difference of a function $f:[a,b]\to\mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in [a,b]$ is defined recursively by

$$[y_i; f] = f(y_i),\ \ i \in \{0,1,2\}$$

$$[y_i, y_{i+1}; f] = \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i},\ \ i \in \{0,1\}$$

$$[y_0, y_1, y_2; f] = \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}. \tag{4.15}$$

Remark 4.7: The value $[y_0, y_1, y_2; f]$ is independent of the order of the points y_0, y_1 and y_2. By taking limits, this definition may be extended to include the cases in which any two or all three points coincide as follows: $\forall y_0,\ y_1,\ y_2 \in [a,b]$

$$\lim_{y_1\to y_0}[y_0, y_1, y_2; f] = [y_0, y_0, y_2; f] = \frac{f(y_2) - f(y_0) - f^{'}(y_0)(y_2 - y_0)}{(y_2 - y_0)^2},\quad y_2 \neq y_0$$

provided that f' exists, and furthermore, taking the limits $y_i \to y_0, i \in \{1,2\}$ in (4.15), we get

$$[y_0, y_0, y_0; f] \ = \lim_{y_i\to y_0}[y_0, y_1, y_2; f] = \frac{f^{''}(y_0)}{2} \text{ for } \ i \in \{1,2\}$$

provided that $f^{''}$ exist on [a,b].

Concluding as before, we get our results concerning the $n-$exponential convexity and exponential convexity for our functionals $\Psi_k,\ k \in \{1,\ldots,6\}$ as defined in (4.8),. . . ,(4.13). Here we have $[a,b] \subset Dom(f_t)$ where $Dom(f_t)$ being the domain of function f_t. Throughout the section I is an interval in $\mathbb{R}$.

Theorem 4.11: *Let $D_1 = \{f_t \in C[a,b] : t \in I\}$ be a family of functions such that the function $t \mapsto [z_0, z_1, z_2; f_t]$ is $n-$exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0, z_1, z_2 \in [a,b]$. Let $\Psi_k(f_t),\ k \in \{1,\ldots,6\}$ be linear functionals. Then $t \mapsto \Psi_k(f_t)$ is an $n-$exponentially convex function in the $J-$sense on I. If the function $t \mapsto \Psi_k(f_t)$ is continuous on I, then it is $n-$exponentially convex on I.*

As a consequence of the above theorem we can state the following two corollaries.

Corollary 4.4: *Let $D_2 = \{f_t \in C[a,b] : t \in I\}$ be a family of functions such that the function $t \mapsto [z_0, z_1, z_2; f_t]$ is exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0, z_1, z_2 \in [a,b]$. Let $\Psi_k(f_t),\ k \in \{1,\ldots,6\}$ be linear functionals. Then*

(a) The function $t \mapsto \Psi_k(f_t)$ is an exponentially convex function in the $J-$ sense on I. If the function $t \mapsto \Psi_k(f_t)$ is continuous on I, then it is exponentially convex on I.

(*b*) *The matrix* $\left[\Psi_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^m$ *is a positive-semidefinite. Particularly,*

$$\det\left[\Psi_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^m \geq 0$$

for each $m \in \mathbb{N}$ *and* $t_i \in I$ *where* $i \in \{1, \ldots, m\}$.

Corollary 4.5: *Let* $D_3 = \{f_t \in C[a,b] : t \in I\}$ *be a family of functions such that the function* $t \mapsto [z_0, z_1, z_2; f_t]$ *is* $2-$*exponentially convex in the* $J-$*sense on* I *for any three mutually distinct points* $z_0, z_1, z_2 \in [a,b]$. *Let* $\Psi_k(f_t)$, $k \in \{1, \ldots, 6\}$ *be linear functionals. Then the following statements hold:*

(*a*) *If the function* $t \mapsto \Psi_k(f_t)$ *is continuous on* I, *then it is* $2-$*exponentially convex function on* I. *If the function* $t \mapsto \Psi_k(f_t)$ *is additionally positive, then it is also log-convex on* I. *Moreover, the following inequality holds for* $r < s < t$; $r,\ s,\ t \in I$

$$(\Psi_k(f_s))^{t-r} \leq (\Psi_k(f_r))^{t-s} (\Psi_k(f_t))^{s-r}.$$

(*b*) *If the function* $t \mapsto \Psi_k(f_t)$ *is positive and differentiable on* I, *then for every* $s, t, u, v \in I$ *such that* $s \leq u$ *and* $t \leq v$, *we have*

$$\mu_{s,t}(\Psi_k, D) \leq \mu_{u,v}(\Psi_k, D) \tag{4.16}$$

where $\mu_{s,t}$ *is defined as*

$$\mu_{s,t}(\Lambda, \Omega) = \begin{cases} \left(\dfrac{\Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(t)})}\right)^{\frac{1}{s-t}} & , \quad s \neq t, \\ \exp\left(\dfrac{\frac{d}{ds}\Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(s)})}\right) & , \quad s = t. \end{cases} \tag{4.17}$$

Remark 4.8: The proofs of the Theorem 4.11 and Corollaries 4.4 and 4.5 can be found in [151]. We also note that the results from Theorem 4.11, Corollaries 4.4 and 4.5 still hold when any two (three) points $z_0, z_1, z_2 \in [a,b]$ coincide for a family of differentiable (twice differentiable) functions f_t such that the function $t \mapsto [z_0, z_1, z_2; f_t]$ is $n-$exponentially convex, exponentially convex and $2-$expoenetially convex in the $J-$sense, respectively.

Remark 4.9: Some special cases of Corollary 4.5 can be found in [22].

4.3.4 Examples

In present section, we give the same examples as given in the previous chapter and give some means and mean-type results.

Under the assumptions of Theorem 2.23, for the present section, we consider the following conditions to be valid for all $t,\ t_o \in I$:

$$\begin{aligned} \lim_{t\to t_0} A(f_t) &= A\left(\lim_{t\to t_0} f_t\right) = A\left(f_{t_0}\right) \\ \lim_{t\to t_0} \frac{A(f_t) - A(f_{t_0})}{t - t_0} &= A\left(\lim_{t\to t_0} \frac{f_t - f_{t_0}}{t - t_0}\right) = A\left(f'_{t_0}\right) \end{aligned}$$

where I is an interval in $\mathbb{R}$.

From Examples 3.1-3.4 of [136], we note that all the mappings $t \mapsto \Psi_k(f_t)$ for $k \in \{1, \dots, 6\}$ are exponentially convex for $f_t \in \Omega_j$ for $k \in \{1, \dots, 4\}$ in the following examples.

Example 4.1: *Let $\Omega_1 = \{\psi_t : \mathbb{R} \to \mathbb{R}_* : t \in \mathbb{R}\}$ be the family of functions defined by*

$$\psi_t(x) = \begin{cases} \frac{1}{t^2} e^{tx} & , \quad t \neq 0, \\ \frac{1}{2} x^2 & , \quad t = 0. \end{cases}$$

Since, $\psi_t(x)$ is a convex function on $\mathbb{R}$ and $\psi_t''(x)$ is exponentially convex function [122]. Using analogous arguing as in the proof of Theorems 4.11 we have that $t \mapsto [y_0, y_1, y_2; \psi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 4.4 we conclude that $t \mapsto \Psi_k(.,.,\psi_t)$; $k \in \{1, \dots, 6\}$ are exponentially convex in the Jensen sense. It is easy to see that these mappings are continuous, so they are exponentially convex.

Assume that $t \mapsto \Psi_k(.,.,\psi_t) > 0$ $k \in \{1, 2, \dots, 6\}$. By using this family of convex functions in (4.14) for $k \in \{1, 2, \dots, 6\}$, we obtain the following means:

$$\mathfrak{M}_{s,t}(\Psi_k, \Omega_1) = \begin{cases} \frac{1}{s-t} \ln\left(\frac{\Psi_k(\psi_s)}{\Psi_k(\psi_t)}\right) & , \quad s \neq t, \\ \frac{\Psi_k(id.\psi_s)}{\Psi_k(\psi_s)} - \frac{2}{s} & , \quad s = t \neq 0, \\ \frac{\Psi_k(id.\psi_0)}{3\Psi_k(\psi_0)} & , \quad s = t = 0. \end{cases}$$

where id stands for identity function on $\mathbb{R}$.
Since, $\mathfrak{M}_{s,t}(\Psi_k, \Omega_1) = \ln(\mu_{s,t}(\Psi_k, \Omega_1))$ for $k \in \{1, \dots, 6\}$, so by (4.16) these means are monotonic, where $\mu_{s,t}$ is defined in (4.17).

Example 4.2: *Let $\Omega_2 = \{\varphi_t : \mathbb{R}_+ \to \mathbb{R} : t \in \mathbb{R}\}$ be the family of functions defined by*

$$\varphi_t(x) = \begin{cases} \frac{x^t}{t(t-1)} & , \quad t \notin \{0, 1\}, \\ -\ln(x) & , \quad t = 0, \\ x\ln(x) & , \quad t = 1. \end{cases}$$

We assume that $[a, b] \subset \mathbb{R}_+$ and $\Psi_k(\varphi_t) > 0$ for $k \in \{1, \dots, 6\}$. By introducing this family of convex functions in (4.14) for $k \in \{1, \dots, 6\}$ we have the following means:

$$\mu_{s,t}(\Psi_k, \Omega_2) = \begin{cases} \left(\frac{\Psi_k(\varphi_s)}{\Psi_k(\varphi_t)}\right)^{\frac{1}{s-t}} & , \quad s \neq t, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Psi_k(\varphi_0\varphi_s)}{\Psi_k(\varphi_s)}\right) & , \quad s = t \neq 0, 1, \\ \exp\left(1 - \frac{\Psi_k(\varphi_0{}^2)}{2\Psi_k(\varphi_0)}\right) & , \quad s = t = 0, \\ \exp\left(-1 - \frac{\Psi_k(\varphi_0\varphi_1)}{2\Psi_k(\varphi_1)}\right) & , \quad s = t = 1. \end{cases}$$

Monotonicity of these means follows directly from (4.16).

Remark 4.10: If Ψ_k for $k \in \{1, \dots, 6\}$ are positive, then using Theorem 4.9 with $\varphi = \varphi_s \in \Omega_2$ and $\psi = \varphi_t \in \Omega_2$ yield that there exist some $\xi_k \in [a, b]$, $\quad k \in \{1, \dots, 6\}$, such that

$$\xi_k^{s-t} = \frac{\Psi_k(\varphi_s)}{\Psi_k(\varphi_t)}, \quad k \in \{1, \dots, 6\}.$$

Since the function $\xi_k \mapsto \xi_k^{s-t}$ is invertible for $s \neq t$, we then have

$$L_k \leq \left(\frac{\Psi_k(\varphi_s)}{\Psi_k(\varphi_t)}\right)^{\frac{1}{s-t}} \leq U_k, \quad k \in \{1, \ldots, 6\}, \tag{4.18}$$

where $L_k, U_k \in [a, b]$ for $k \in \{1, \ldots, 6\}$, which shows that in this case $\mu_{s,t}(\Psi_k\ , \Omega_2)$ for $k \in \{1, \ldots, 6\}$ are means.

Now, we impose one additional parameter r. For $r \neq 0$ by substituting $s \to \frac{s}{r}$, $t \to \frac{t}{r}$ and $\mathbf{x} \to \mathbf{x}^r$ for $k \in \{7, 8\}$, $g \to g^r$ for $k \in \{9, 10\}$, $f \to f^r$ and $g \to g^r$ for $k \in \{11, 12, 13\}$ and $f \to f^r$ for $k \in \{14, 15, 16\}$ in (4.18), we get

$$\begin{aligned}
L_k &\leq \left(\frac{\Psi_k(\mathbf{x}^r, \mathbf{w}; \varphi_s)}{\Psi_k(\mathbf{x}^r, \mathbf{w}; \varphi_t)}\right)^{\frac{r}{s-t}} \leq U_k \quad \text{for } k \in \{7, 8\}, \\
L_k &\leq \left(\frac{\Psi_k(g^r, \lambda; \varphi_s)}{\Psi_k(g^r, \lambda; \varphi_t)}\right)^{\frac{r}{s-t}} \leq U_k \quad \text{for } k \in \{9, 10\}, \\
L_k &\leq \left(\frac{\Psi_k(f^r, g^r; \varphi_s)}{\Psi_k(f^r, g^r; \varphi_t)}\right)^{\frac{r}{s-t}} \leq U_k \quad \text{for } k \in \{11, 12, 13\}, \\
L_k &\leq \left(\frac{\Psi_k(A, f^r; \varphi_s)}{\Psi_k(A, f^r; \varphi_t)}\right)^{\frac{r}{s-t}} \leq U_k \quad \text{for } k \in \{14, 15, 16\},
\end{aligned}$$

where $\mathbf{x}^r = (x_1^r, \ldots, x_n^r)$.

Here we define new generalized means as follows.

- for $k \in \{7, 8\}$:

$$\mu_{s,t;r}(\Psi_k(\mathbf{x}^r, \mathbf{w}; \varphi_t), \Omega_2) = \begin{cases} \left(\mu_{\frac{s}{r},\frac{t}{r}}(\Psi_k(\mathbf{x}^r, \mathbf{w}; \varphi_t), \Omega_2)\right)^{\frac{1}{r}} & , \quad r \neq 0, \\ \\ \mu_{s,t}(\Psi_k(\ln(\mathbf{x}), \mathbf{w}; \varphi_t), \Omega_1) & , \quad r = 0, \end{cases}$$

- for $k \in \{9, 10\}$:

$$\mu_{s,t;r}(\Psi_k(g^r, \lambda; \varphi_t), \Omega_2) = \begin{cases} \left(\mu_{\frac{s}{r},\frac{t}{r}}(\Psi_k(g^r, \lambda; \varphi_t), \Omega_2)\right)^{\frac{1}{r}} & , \quad r \neq 0, \\ \mu_{s,t}(\Psi_k(\ln(g), \lambda; \varphi_t), \Omega_1) & , \quad r = 0, \end{cases}$$

- for $k \in \{11, 12, 13\}$:

$$\mu_{s,t;r}(\Psi_k(f^r, g^r; \varphi_t), \Omega_2) = \begin{cases} \left(\mu_{\frac{s}{r},\frac{t}{r}}(\Psi_k(f^r, g^r; \varphi_t), \Omega_2)\right)^{\frac{1}{r}} & , \quad r \neq 0, \\ \mu_{s,t}(\Psi_k(\ln(f), \ln(g); \varphi_t), \Omega_1) & , \quad r = 0, \end{cases}$$

- for $k \in \{14, 15, 16\}$:

$$\mu_{s,t;r}(\Psi_k(A, f^r; \varphi_t), \Omega_2) = \begin{cases} \left(\mu_{\frac{s}{r},\frac{t}{r}}(\Psi_k(A, f^r; \varphi_t), \Omega_2)\right)^{\frac{1}{r}} & , \quad r \neq 0, \\ \mu_{s,t}(\Psi_k(A, \ln(f); \varphi_t), \Omega_1) & , \quad r = 0, \end{cases}$$

where $\ln(\mathbf{x}) = (\ln(x_1), \ldots, \ln(x_n))$. These new generalized means are monotonic. If $s, t, u, v \in \mathbb{R}$, $r \neq 0$ such that $s \leq u$, $t \leq v$, then we have

$$\mu_{s,t;r}(\Psi_k, \Omega_2) \leq \mu_{u,v;r}(\Psi_k, \Omega_2), \quad k \in \{1, \ldots, 6\}.$$

The above inequalities are easily followed by using the fact that $\mu_{s,t}(\Psi_k, \Omega_2)$ for $k \in \{1, \dots, 6\}$ are monotonic in both parameters and using the inequalities given below for $k \in \{1, \dots, 6\}$

$$\mu_{\frac{s}{r},\frac{t}{r}}(\Psi_k, \Omega_2) = \left(\frac{\Psi_k(\varphi_{\frac{s}{r}})}{\Psi_k(\varphi_{\frac{t}{r}})}\right)^{\frac{r}{s-t}} \leq \left(\frac{\Psi_k(\varphi_{\frac{u}{r}})}{\Psi_k(\varphi_{\frac{v}{r}})}\right)^{\frac{r}{u-v}} = \mu_{\frac{u}{r},\frac{v}{r}}(\Psi_k, \Omega_2),$$

where $r, s, t, u, v \in \mathbb{R}$, $r \neq 0$ such that $\frac{s}{r} \leq \frac{u}{r}, \frac{t}{r} \leq \frac{v}{r}$. For $r = 0$, we obtain the required result by taking the limit $r \to 0$.

Example 4.3: *Let $\Omega_3 = \{\theta_t : \mathbb{R}_+ \to \mathbb{R}_+ : t \in \mathbb{R}_+\}$ be the family of functions defined by*

$$\theta_t(x) = \frac{e^{-x\sqrt{t}}}{t}.$$

We assume that $[a, b] \subset \mathbb{R}_+$ and $\Psi_k(\theta_t) > 0$, $k \in \{1, \dots, 6\}$. For this family of convex functions $\mu_{s,t}(\Psi_k, \Omega_3)$, $k \in \{1, \dots, 6\}$ from (4.16) become

$$\mu_{s,t}(\Psi_k, \Omega_3) = \begin{cases} \left(\frac{\Psi_k(\theta_s)}{\Psi_k(\theta_t)}\right)^{\frac{1}{s-t}} & , \quad s \neq t, \\ \exp\left(-\frac{\Psi_k(id.\theta_s)}{2\sqrt{s}(\Psi_k(\theta_s))} - \frac{1}{s}\right) & , \quad s = t. \end{cases}$$

Monotonicity of $\mu_{s,t}(\Psi_k, \Omega_3)$ for $k \in \{1, \dots, 6\}$ follow from (4.16). By (4.14) $\mathfrak{M}_{s,t}(\Psi_k, \Omega_3) = -(\sqrt{s} + \sqrt{t}) \ln(\mu_{s,t}(\Psi_k, \Omega_3))$, $\quad k \in \{1, \dots, 6\}$ defines a class of means.

Example 4.4: *Let $\Omega_4 = \{f_t : \mathbb{R}_+ \to \mathbb{R}_+ : t \in \mathbb{R}_+\}$ be the family of functions defined by*

$$f_t(x) = \begin{cases} \frac{t^{-x}}{(\ln(t))^2} & , \quad t \neq 1, \\ \frac{x^2}{2} & , \quad t = 1. \end{cases}$$

We assume that $[a, b] \subset \mathbb{R}_+$ and $\Psi_k(f_t) > 0$, $k \in \{1, \dots, 6\}$. For this family of convex functions $\mu_{s,t}(\Psi_k, \Omega_4)$, $k \in \{1, \dots, 6\}$ from (4.16) become

$$\mu_{s,t}(\Psi_k, \Omega_4) = \begin{cases} \left(\frac{\Psi_k(f_s)}{\Psi_k(f_t)}\right)^{\frac{1}{s-t}} & , \quad s \neq t, \\ \exp\left(-\frac{\Psi_k(id.f_s)}{s\Psi_k(f_s)} - \frac{2}{s\ln(s)}\right) & , \quad s = t \neq 1, \\ \exp\left(\frac{1}{3}\frac{\Psi_k(id.f_1)}{\Psi_k(f_1)}\right) & , \quad s = t = 1. \end{cases}$$

Monotonicity of $\mu_{s,t}(\Psi_k, \Omega_4)$ for $k \in \{1, \dots, 6\}$ follow from (4.16). By (4.14)

$$\mathfrak{M}_{s,t}(\Psi_k, \Omega_4) = -L(s, t) \ln(\mu^j_{s,t}(\Omega_4)), \quad k \in \{1, \dots, 6\}$$

defines a class of means, where $L(s, t)$ is Logarithmic mean defined by

$$L(s, t) = \begin{cases} \frac{s-t}{\ln s - \ln t}, & s \neq t, \\ s, & s = t. \end{cases}$$

Remark 4.11: Motivated by Theorems 3.9 and 3.10, we define two functionals $\Phi_i \colon L_f \to \mathbb{R}, i = 1, 2$, by

$$\Phi_1(\varphi) = \varphi(a) + \varphi(b) - \varphi(a + b - A(f)) - A(\varphi(f)) - \left[1 - \frac{2}{b-a} A\left(\left|f - \frac{a+b}{2}\right|\right)\right] \delta_\varphi$$

and

$$\Phi_2(\varphi) = \varphi(a) + \varphi(b) - \varphi(a+b-A(f)) - A(\varphi(f)) \\ - \left[1 - \frac{1}{b-a}\left(A\left(\left|f - \frac{a+b}{2}\right|\right) + \left|\frac{a+b}{2} - A(f)\right|\right)\right]\delta_\varphi,$$

where A, f and δ_φ are as in Theorem 3.9 and for $[a,b] \subseteq I$,

$$L_f = \{\varphi\colon I \to \mathbb{R}\colon \varphi(f), \varphi(a+b-f) \in L\}.$$

Obviously, Φ_1 and Φ_2 are linear.
If φ is additionally continuous and convex then Theorems 3.9 and 3.10 imply $\Phi_i(f) \geq 0, i \in \{1,2\}$.

We can state all the results as stated in this section for functionals Φ_1 and Φ_2. For further details interested readers see [226].

Remark 4.12: This section also captures some results of [22] as special cases.

CHAPTER 5

Variants of Mercer Type of Some Well-Known Inequalities

In this chapter, we present Mercer-type variants of several well-known inequalities. We begin with a Mercer-type extension of the Jensen–Steffensen inequality. Next, we provide generalizations of the discrete Čebyšev's inequality for two monotone $n-$tuples and also for $k \geq 3$ nonnegative n-tuples monotone in the same direction, followed by their Mercer-type refinements. This is followed by analogous Mercer-type variants of Pečarić's generalizations of discrete Ostrowski inequalities. Finally, we discuss generalizations of Milne's inequality and its converse, together with their Mercer-type variants. These developments illustrate the versatility and broad applicability of Mercer-type refinements across classical inequalities.

5.1 A Variant of the Jensen–Stefensen Inequality

In Chapter 1, as we see from the proof of Theorem 1.1 that the condition $0 < x_1$ is not necessary. We also noted that the condition that the n-tuple $\boldsymbol{x} = (x_1, \ldots, x_n)$ is increasing is also not necessary. Therefore, we state here an extension of Theorem 1.1 and prove it as a simple consequence of Jensen-Steffensen's inequality. Before doing this, note that inequality (1.1) is valid for any function f which is convex on interval containing $[a, b]$ and that $x_1, x_2, \ldots, x_n \in [a, b]$.

Theorem 5.1: *Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in [a, b]^n$, where $a < b$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$ a nonnegative real n-tuple with $\sum_{i=1}^{n} w_i = 1$. Then for any function f which is convex on interval containing $[a, b]$ we have*

$$f\left(a + b - \sum_{i=1}^{n} w_i x_i\right) \leq f(a) + f(b) - \sum_{i=1}^{n} w_i f(x_i).$$

If f is concave, then above inequality is reversed.

Proof: Let $(i_1, i_2, \ldots, i_n)$ be a permutation of $(1, 2, \ldots, n)$ such that $\tilde{\boldsymbol{x}} = (x_{i_1}, \ldots, x_{i_n})$ is increasing, that is, $a \leq x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n} \leq b$, and let $\tilde{\boldsymbol{w}} = (w_{i_1}, \ldots, w_{i_n})$ be corresponding permutation of $\boldsymbol{w}$. Note that

$$\sum_{j=1}^{n} w_{i_j} x_{i_j} = \sum_{i=1}^{n} w_i x_i$$

and for all $k \in \{1, \ldots, n\}$

$$0 \leq \sum_{j=1}^{k} w_{i_j} \leq \sum_{j=1}^{n} w_{i_j} = \sum_{i=1}^{n} w_i = 1.$$

Therefore, if $m = n+2$ and the m-tuples $\boldsymbol{\xi}$ and $\boldsymbol{p}$ are defined by

$$\begin{aligned} \xi_1 &= a, \quad \xi_2 = x_{i_1}, \quad \xi_3 = x_{i_2}, \ldots, \quad \xi_{n+1} = x_{i_n}, \quad \xi_{n+2} = b, \\ p_1 &= 1, \quad p_2 = -w_{i_1}, \quad p_3 = -w_{i_2}, \ldots, \quad p_{n+1} = -w_{i_n}, \quad p_{n+2} = 1, \end{aligned}$$

then it is obvious that the m-tuple $\boldsymbol{\xi}$ is increasing while for $P_k = \sum_{i=1}^{k} p_i$ we have

$$\begin{aligned} P_m &= 1 - \sum_{j=1}^{n} w_{i_j} + 1 = 1 - \sum_{i=1}^{n} w_i + 1 = 1 > 0, \\ 0 &\leq P_k = 1 - \sum_{j=1}^{k-1} w_{i_j} \leq 1 = P_m, \quad k \in \{1, \ldots, n+1\}. \end{aligned}$$

So, we can apply Jensen-Steffensen's inequality to obtain the inequality

$$f\left(\sum_{j=1}^{m} p_j \xi_j\right) \leq \sum_{l=1}^{m} p_j f(\xi_j),$$

which can be rewritten as

$$f\left(a + b - \sum_{i=1}^{n} w_i x_i\right) \leq f(a) + f(b) - \sum_{i=1}^{n} w_i f(x_i).$$

The concave case is analogous. □

Remark 5.1: By careful inspection of the proof above we can easily see that condition "$\boldsymbol{w}$ is a nonnegative real n-tuple with $\sum_{i=1}^{n} w_i = 1$" can be relaxed to condition: $\boldsymbol{w} = (w_1, \ldots, w_n)$ is a real n-tuple satisfying $0 \leq W_k \leq W_n = 1$, for $k \in \{1, 2, \ldots, n-1\}$, where $W_k = \sum_{i=1}^{k} w_i$, but in that case if we want Inequality (5.1) to be valid, we must assume $\boldsymbol{x} = (x_1, \ldots, x_n)$ to be a monotonic n-tuple in $[a, b]^n$.

Now we have the following variant of Jensen–Stefensen inequality:

Theorem 5.2: *Let $f : I \to \mathbb{R}$, where I is any interval in $\mathbb{R}$, and let $[a, b] \subseteq I$, $a < b$. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a monotonic n-tuple in $[a, b]^n$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$ a real n-tuple such that*

$$0 \leq W_k \leq W_n \quad (k \in \{1, 2, \ldots, n\}), \qquad W_n > 0,$$

where $W_k := \sum_{i=1}^{k} w_i$. *If* f *is convex on* I*, then*

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \le f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(x_i\right).$$

If f *is concave on* I*, then the above inequality is reversed.*

Proof: Set

$$w_i = \frac{w_i}{W_n},\ i \in \{1,2,\ldots,n\}$$

and apply Theorem 5.1 and Remark 5.1. □

Now consider two functions $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, where I and J are intervals in $\mathbb{R}$, such that $f(I) \subseteq J$. When each of functions f and g is convex or concave, the following simple technical result can be very usefull.

Lemma 5.1: *Let* $f : I \to \mathbb{R}$ *and* $g : J \to \mathbb{R}$*, where* I *and* J *are intervals in* $\mathbb{R}$*, be two functions such that* $f(I) \subseteq J$*. Then*
(i) *if either* f *is convex on* I *and* g *is nondecreasing and convex on* J *or* f *is concave on* I *and* g *is nonincreasing and convex on* J*, then* $g \circ f$ *is convex on* I*;*
(ii) *if either* f *is convex on* I *and* g *is nonincreasing and concave on* J *or* f *is concave on* I *and* g *is nondecreasing and concave on* J*, then* $g \circ f$ *is concave on* I*.*

Proof: Take $u, v \in I$ and $t \in [0,1]$. If f is convex on I, then

$$f(tu+(1-t)v) \le tf(u)+(1-t)f(v), \tag{5.1}$$

and if f is concave on I, then

$$f(tu+(1-t)v) \ge tf(u)+(1-t)f(v). \tag{5.2}$$

Now, when g *is nondecreasing and convex on* J, from (5.1) we get

$$g\left(f(tu+(1-t)v)\right) \le g\left(tf(u)+(1-t)f(v)\right) \le tg(f(u))+(1-t)g(f(v)).$$

Same inequalities follow from (5.2) in case when g *is nonincreasing and convex on* J, It follows in both cases that $g \circ f$ *is convex on* I which proves the first claim. The second claime is proved by analogous argument. □

So, in each of the two cases described in Lemma 5.1 we can apply Jensen's inequalities (*A*.7) and (1.2) to $g \circ f$. For example, if f is convex on I and g is nondecreasing and convex on J, then

$$(g \circ f)\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i \xi_i\right) \le \frac{1}{P_m}\sum_{i=1}^{m} p_i \left(g \circ f\right)\left(\xi_i\right) \tag{5.3}$$

holds for any m-tuple $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ in I^m and any nonnegative m-tuple $\boldsymbol{p} = (p_1, \ldots, p_m)$ such that $P_m = \sum_{i=1}^m p_i > 0$. Actually, under this assumptions on f and g the above inequality can be refined to the following

$$(g \circ f)\left(\frac{1}{P_m}\sum_{i=1}^m p_i\xi_i\right) \leq g\left(\frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i)\right) \leq \frac{1}{P_m}\sum_{i=1}^m p_i (g \circ f)(\xi_i).$$

(Some more general results of this type can be found in [10].) Clearly, if we assume $\boldsymbol{\xi}$ to be a monotonic m-tuple in I^m and $\boldsymbol{p}$ to be a real m-tuple satisfying conditions ($A.9$), then (5.3) still holds but the above refinement need not be true unless f is also monotonic (nondecreasing or nonincreasing). Similar observations can be made in all other cases considered by Lemma 5.1. More precisely, we have the following result:

Theorem 5.3: *Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, where I and J are intervals in $\mathbb{R}$, be two functions such that $f(I) \subseteq J$. Assume f to be monotonic on I. Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ be any monotonic m-tuple in I^m and $\boldsymbol{p} = (p_1, \ldots, p_m)$ any real m-tuple such that*

$$0 \leq P_k \leq P_m, \quad k \in \{1, 2, \ldots, m\}, \qquad P_m > 0,$$

where $P_k = \sum_{i=1}^k p_i$.
(i) *If either f is convex on I and g is nondecreasing and convex on J or f is concave on I and g is nonincreasing and convex on J, then*

$$(g \circ f)\left(\frac{1}{P_m}\sum_{i=1}^m p_i\xi_i\right) \leq g\left(\frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i)\right) \leq \frac{1}{P_m}\sum_{i=1}^m p_i (g \circ f)(\xi_i). \tag{5.4}$$

(ii) *If either f is convex on I and g is nonincreasing and concave on J or f is concave on I and g is nondecreasing and concave on J, then*

$$(g \circ f)\left(\frac{1}{P_m}\sum_{i=1}^m p_i\xi_i\right) \geq g\left(\frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i)\right) \geq \frac{1}{P_m}\sum_{i=1}^m p_i (g \circ f)(\xi_i). \tag{5.5}$$

Proof: Let us denote

$$f(\boldsymbol{\xi}) = (f(\xi_1), \ldots, f(\xi_m)).$$

Since $\boldsymbol{\xi}$ is a monotonic m-tuple in I^m, $f(I) \subseteq J$ and f is assumed to be monotonic on I, $f(\boldsymbol{\xi})$ is monotonic m-tuple in J^m. Further, in case f is convex we can apply Jensen–Stefensen inequality to obtain

$$f\left(\frac{1}{P_m}\sum_{i=1}^m p_i\xi_i\right) \leq \frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i), \tag{5.6}$$

while in case f is concave we get

$$f\left(\frac{1}{P_m}\sum_{i=1}^m p_i\xi_i\right) \geq \frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i). \tag{5.7}$$

Now, the second inequality in (5.4) is a consequence of Jensen–Stefensen inequality for convex function g and the fact that $f(\boldsymbol{\xi})$ is monotonic m-tuple in J^m. The first inequality in (5.4) follows from (5.6) in case g is nondecreasing, and from (5.7) in case g is nonincreasing. Similarly, the second inequality in (5.5) is a consequence of Jensen–Stefensen inequality for concave function g, while the first inequality in (5.5) follows from (5.7) in case g is nondecreasing, and from (5.6) in case g is nonincreasing. □

In our next result we obtain variants of the inequalities in Theorem 5.3 analogous to those obtained in Theorem 5.2.

Theorem 5.4: *Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, where I and J are intervals in $\mathbb{R}$, be two functions such that $f(I) \subseteq J$. Assume also that f is monotonic on I. Let $[a, b] \subseteq I$, $a < b$. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be monotonic n-tuple in $[a, b]^n$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be a real n-tuple such that $0 \leq Q_k \leq Q_n \quad (k \in \{1, 2, \ldots, n\})$, $\quad Q_n > 0$, where $Q_k = \sum_{i=1}^{k} q_i$.*
(i) *If either f is convex on I and g is nondecreasing and convex on J or f is concave on I and g is nonincreasing and convex on J, then*

$$(g \circ f)\left(a + b - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right) \leq g\left(f(a) + f(b) - \frac{1}{Q_n}\sum_{i=1}^{n} q_i f(x_i)\right)$$
$$\leq (g \circ f)(a) + (g \circ f)(b) - \frac{1}{Q_n}\sum_{i=1}^{n} q_i (g \circ f)(x_i).$$

(ii) *If either f is convex on I and g is nonincreasing and concave on J or f is concave on I and g is nondecreasing and concave on J, then*

$$(g \circ f)\left(a + b - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right)$$
$$\geq g\left(f(a) + f(b) - \frac{1}{Q_n}\sum_{i=1}^{n} q_i f(x_i)\right)$$
$$\geq (g \circ f)(a) + (g \circ f)(b) - \frac{1}{Q_n}\sum_{i=1}^{n} q_i (g \circ f)(x_i).$$

Proof: Set $m = n + 2$ and define the m-tuple $\boldsymbol{p}$ as

$$p_1 = 1, \quad p_2 = -\frac{q_1}{Q_n}, \quad p_3 = -\frac{q_2}{Q_n}, \ldots, \quad p_{n+1} = -\frac{q_n}{Q_n}, \quad p_{n+2} = 1.$$

Furthermore, if $\boldsymbol{x}$ is nondecreasing i.e. $a \leq x_1 \leq \cdots \leq x_n \leq b$, then define $\boldsymbol{\xi}$ as

$$\xi_1 = a, \quad \xi_2 = x_1, \quad \xi_3 = x_2, \ldots, \quad \xi_{n+1} = x_n, \quad \xi_{n+2} = b,$$

while in case that $\boldsymbol{x}$ is nonincreasing i.e., $a \leq x_n \leq \cdots \leq x_1 \leq b$, define $\boldsymbol{\xi}$ as

$$\xi_1 = b, \quad \xi_2 = x_1, \quad \xi_3 = x_2, \ldots, \quad \xi_{n+1} = x_n, \quad \xi_{n+2} = a.$$

Now it is easy to see that in both cases we can apply Theorem 5.3 to obtain the proposed inequalities. □

Remark 5.2: In case $\boldsymbol{q}$ is also nonnegative, the n-tuple $\boldsymbol{x}$ need not be monotonic and the inequalities proposed in Theorem 5.4 are still valid.

It is interesting that these results concerning two functions, such that each of which is either convex or concave, can be generalized by induction to any set of functions satisfying certain conditions.

Consider a set of $r+1$ $(r \geq 1)$ functions

$$f : I \to \mathbb{R},\ g_1 : I_1 \to \mathbb{R},\ \ldots\ ,\ g_r : I_r \to \mathbb{R},$$

where I, I_1, $\ldots$, I_r are intervals in $\mathbb{R}$ such that

$$f(I) \subseteq I_1,\ g_k(I_k) \subseteq I_{k+1}, \quad k \in \{1, 2, \ldots, r-1\}. \tag{5.8}$$

Then the following two sets of auxiliary functions $F_k,\ G_k,\ k \in \{1, 2, \ldots, r\}$, such that $F_k : I \to \mathbb{R}$, $G_k : I_k \to \mathbb{R}, \quad k \in \{1, 2, \ldots, r\}$, are well defined as

$$F_k : I \to \mathbb{R},\ F_k = g_k \circ g_{k-1} \circ \cdots \circ g_1 \circ f, \tag{5.9}$$

$$G_k : I_k \to \mathbb{R},\ G_k = g_r \circ g_{r-1} \circ \cdots \circ g_k. \tag{5.10}$$

Furthermore, for any given monotonic $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ in I^m and real $\boldsymbol{p} = (p_1, \ldots, p_m)$ satisfying condition $(A.9)$ we define the value $\overline{\xi} \in I$ as

$$\overline{\xi} = \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i. \tag{5.11}$$

Also, we define $f(\boldsymbol{\xi})$ and $F_k(\boldsymbol{\xi})$ $(k \in \{1, 2, \ldots, r\})$, and the values $\overline{f(\boldsymbol{\xi})}$ and $\overline{F_k(\boldsymbol{\xi})}$ by

$$\begin{gathered} f(\boldsymbol{\xi}) = \left(f\left(\xi_1\right), \ldots, f\left(\xi_m\right)\right), \quad F_k(\boldsymbol{\xi}) = \left(F_k\left(\xi_1\right), \ldots, F_k\left(\xi_m\right)\right), \\ \overline{f(\boldsymbol{\xi})} = \frac{1}{P_m} \sum_{i=1}^{m} p_i f\left(\xi_i\right), \quad \overline{F_k(\boldsymbol{\xi})} = \frac{1}{P_m} \sum_{i=1}^{m} p_i F_k\left(\xi_i\right). \end{gathered} \tag{5.12}$$

Now, we assume that each of the considered functions of the set $f, g_1, \ldots, g_r$ is either convex or concave and that following monotonicity condition is fulfilled.

Monotonicity Condition. *Denote $g_0 = f$. We say that a set of functions $g_0, g_1. \cdots, g_r$ satisfies the Monotonicity Condition (MC),* if *all $k \in \{0, 1, \ldots, r-1\}$, and all pairs (g_k, g_{k+1}) satisfiy the following:*

(i) *when both functions g_k and g_{k+1} are either convex or concave, then g_{k+1} is nondecreasing:*

(ii) *when either g_k is convex and g_{k+1} is concave or g_k is concave and g_{k+1} is convex, then g_{k+1} is nonincreasing.*

Note that when the functions f, $g_1, \dots, g_r$ satisfy the above stated MC, then all of them except $g_0 = f$ are monotonic. Also, we have:

Theorem 5.5: *Let the functions f, $g_1, \dots, g_r$ be as above and satisfy MC. Let F_k, $k \in \{1, 2, \dots, r\}$, be defined by (5.9). Then for all $k \in \{1, \dots, r\}$ we have*
(i) *if g_k is convex on I, then F_k is convex on I;*
(ii).*if g_k is concave on I, then F_k is concave on I.*

Proof: For $k = 1$ we set $g = g_1$ and apply Lemma 5.1, while for $k > 1$ the proposed conclusions easily follow by induction. The details are left to the reader. □

It is now obvious that we can extend the result stated in Theorem 5.4 to the following general result:

Theorem 5.6: *Let $f : I \to \mathbb{R}$ and $g_k : I_k \to \mathbb{R}$, $k \in \{1, 2, \dots, r\}$, be either convex or concave, where I and I_k $k \in \{1, 2, \dots, r\}$, are intervals in $\mathbb{R}$ satisfying (5.8) Define auxiliary functions F_k and G_k by (5.9) and (5.10), respectively. Assume that f and g_k, $k \in \{1, 2, \dots, r\}$, satisfy MC and additionally assume f to be monotonic. Then for any monotonic m-tuple $\boldsymbol{\xi} \in I^m$ and real m-tuple $\boldsymbol{p}$ satisfying (A.9) we have:*
(i) *if g_r is convex on I, then*

$$F_r(\bar{\xi}) \le G_1\left(\overline{f(\boldsymbol{\xi})}\right) \le G_2\left(\overline{F_1(\boldsymbol{\xi})}\right) \le G_3\left(\overline{F_2(\boldsymbol{\xi})}\right) \le \cdots \le G_r\left(\overline{F_{r-1}(\boldsymbol{\xi})}\right) \le \overline{F_r(\boldsymbol{\xi})},$$

(ii) *if g_r is concave on I, then*

$$F_r(\bar{\xi}) \ge G_1\left(\overline{f(\boldsymbol{\xi})}\right) \ge G_2\left(\overline{F_1(\boldsymbol{\xi})}\right) \ge G_3\left(\overline{F_2(\boldsymbol{\xi})}\right) \ge \cdots \ge G_r\left(\overline{F_{r-1}(\boldsymbol{\xi})}\right) \ge \overline{F_r(\boldsymbol{\xi})},$$

where the value $\bar{\xi}$ is defined by (5.11) and values $\overline{f(\boldsymbol{\xi})}$ and $\overline{F_k(\boldsymbol{\xi})}$ are defined by (5.12).

Proof: Under the given assumptions, the monotonicity of f ensures that all $f(\boldsymbol{\xi}) \in I_1^m$ and $F_k(\boldsymbol{\xi}) \in I_{k+1}^m$ $(k \in \{1, 2, \dots, r-1\})$ are monotonic. Now, for $r = 1$ we set $g = g_1$ and it is easily seen that the proposed statement is in fact the statement of Theorem 5.4. The general case then easily follows by induction. The details are left to the reader. □

Corollary 5.1: *Let all the assumptions of Theorem 5.6 be satisfied. Let $[a, b] \subseteq I$, $a < b$. Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be monotonic in $[a, b]^n$ and $\boldsymbol{q} = (q_1, \dots, q_n)$ be real n-tuple such that $0 \le Q_k \le Q_n$ $(k \in \{1, 2, \dots, n\})$, $Q_n > 0$, where $Q_k = \sum_{i=1}^{k} q_i$. Then*
(i) *if g_r is convex on I, then*

$$F_r(\bar{\xi}) \le G_1\left(\overline{f(\boldsymbol{\xi})}\right) \le G_2\left(\overline{F_1(\boldsymbol{\xi})}\right) \le G_3\left(\overline{F_2(\boldsymbol{\xi})}\right) \le \cdots \le G_r\left(\overline{F_{r-1}(\boldsymbol{\xi})}\right) \le \overline{F_r(\boldsymbol{\xi})},$$

(ii) *if g_r is concave on I, then*

$$F_r(\bar{\xi}) \geq G_1\left(\overline{f(\boldsymbol{\xi})}\right) \geq G_2\left(\overline{F_1(\boldsymbol{\xi})}\right) \geq G_3\left(\overline{F_2(\boldsymbol{\xi})}\right) \geq \cdots \geq G_r\left(\overline{F_{r-1}(\boldsymbol{\xi})}\right) \geq \overline{F_r(\boldsymbol{\xi})},$$

where

$$\bar{\xi} = a + b - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i, \ \overline{F_k(\boldsymbol{\xi})} = F_k(a) + F_k(b) - \frac{1}{Q_n}\sum_{i=1}^{n} q_i F_k(x_i).$$

Proof: Define $\boldsymbol{\xi}$ and $\boldsymbol{p}$ exactly as in the proof of Theorem 5.4 and then apply Theorem 5.6. □

5.1.1 Jensen–Stefensen Inequality and Quasi-Arithmetic Means

Let $f : I \to \mathbb{R}$ be strictly monotonic and continuous function, where I is an interval in $\mathbb{R}$. Then for a given m-tuple $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ in I^m and nonnegative m-tuple $\boldsymbol{p} = (p_1, \ldots, p_m)$ with $P_m = \sum_{i=1}^{m} p_i > 0$, the value

$$M_f(\boldsymbol{\xi}, \boldsymbol{p}) = f^{-1}\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i f(\xi_i)\right)$$

is well defined and is called quasi-arithmetic f-mean of $\boldsymbol{\xi}$ with weights $\boldsymbol{p}$.

If $\boldsymbol{\xi}$ is assumed to be monotonic m-tuple in I^m and $\boldsymbol{p}$ any real m-tuple satisfying condition ($A.9$), then $M_f(\boldsymbol{\xi}, \boldsymbol{p})$ is still well defined. Moreover, the following result is true:

Theorem 5.7: *Let f and g be two continuous strictly monotone functions on an interval I. The inequality*

$$f^{-1}\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i f(\xi_i)\right) \leq g^{-1}\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i g(\xi_i)\right) \tag{5.13}$$

holds for all monotonic m-tuples $\boldsymbol{\xi}$ in I^m and real m-tuples $\boldsymbol{p}$ satisfying conditions ($A.9$) if and only if the function $g \circ f^{-1}$ is convex and g is strictly increasing or the function $g \circ f^{-1}$ is concave and g is strictly decreasing. The reverse inequality

$$f^{-1}\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i f(\xi_i)\right) \geq g^{-1}\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i g(\xi_i)\right) \tag{5.14}$$

holds for all monotonic m-tuples $\boldsymbol{\xi}$ in I^m and real m-tuples $\boldsymbol{p}$ satisfying conditions ($A.9$) if and only if either $g \circ f^{-1}$ is concave and g is strictly increasing or $g \circ f^{-1}$ is convex and g is strictly decreasing. In case the function $g \circ f^{-1}$ is strictly convex (concave) equalities in (5.13) and (5.14) hold if and only if $\xi_i = c$ for all $i \in \{j : p_j \neq 0\} \subseteq \{1, \ldots, m\}$.

Proof: Since f and g are continuous and strictly monotone we know that $f(I) = J_1$ and $g(I) = J_2$ are intervals in $\mathbb{R}$. Now take any monotonic m-tuple $\boldsymbol{\xi}$ in I^m and any real m-tuple $\boldsymbol{p}$

satisfying condition $(A.9)$. Since f and g are strictly monotonic, then

$$f(\boldsymbol{\xi}) = (f(\xi_1), \ldots, f(\xi_m)) \in J_1^m, \quad g(\boldsymbol{\xi}) = (g(\xi_1), \ldots, g(\xi_m)) \in J_2^m$$

are both monotonic and consequently we have

$$\frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i) \in J_1, \quad \frac{1}{P_m}\sum_{i=1}^m p_i g(\xi_i) \in J_2.$$

So that both sides of (5.13) ((5.14)) are well defined.

If the function $g \circ f^{-1}$ is convex on J_1, then for all m-tuples $\boldsymbol{\xi}$ and $\boldsymbol{p}$ satisfying the above conditions, Jensen-Steffensen's inequality $(A.7)$ gives

$$\left(g \circ f^{-1}\right)\left(\frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i)\right) \leq \frac{1}{P_m}\sum_{i=1}^m p_i \left(g \circ f^{-1}\right)(f(\xi_i))$$

which can be rewritten as

$$g\left(f^{-1}\left(\frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i)\right)\right) \leq \frac{1}{P_m}\sum_{i=1}^m p_i g(\xi_i). \tag{5.15}$$

In case $g \circ f^{-1}$ is concave, we obtain the reverse of the above inequality

$$g\left(f^{-1}\left(\frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i)\right)\right) \geq \frac{1}{P_m}\sum_{i=1}^m p_i g(\xi_i). \tag{5.16}$$

If the function g is strictly increasing, then inverse function g^{-1} is also strictly increasing, so that (5.15) implies (5.13). If g is strictly decreasing, then g^{-1} is strictly decreasing too, so that in this case inequality (5.16) implies (5.13). On the other hand, analogous arguments give us the reverse of (5.13), that is, we get inequality (5.14) in cases when $g \circ f^{-1}$ is convex and g is strictly decreasing or $g \circ f^{-1}$ is concave and g is strictly increasing.

If $\xi_i = c$ for all $i \in \{j : p_j \neq 0\} \subseteq \{1, \ldots, m\}$, then

$$\begin{aligned} f^{-1}\left(\frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i)\right) &= f^{-1}(f(c)) = c \\ &= g^{-1}(g(c)) = g^{-1}\left(\frac{1}{P_m}\sum_{i=1}^m p_i g(\xi_i)\right). \end{aligned}$$

Conversly, assume that

$$f^{-1}\left(\frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i)\right) = g^{-1}\left(\frac{1}{P_m}\sum_{i=1}^m p_i g(\xi_i)\right).$$

This can be rewritten as

$$\left(g\circ f^{-1}\right)\left(\frac{1}{P_m}\sum_{i=1}^{m}p_i f\left(\xi_i\right)\right)=\frac{1}{P_m}\sum_{i=1}^{m}p_i\left(g\circ f^{-1}\right)\left(f\left(\xi_i\right)\right).$$

For strictly convex (strictly concave) function $g\circ f^{-1}$ the above assumption implies

$$f\left(\xi_i\right)=c',\quad \text{for all}\quad i\in\{j:p_j\neq 0\},$$

and since f is strictly monotonic, this means

$$\xi_i=f^{-1}(c')=c,\quad \text{for all}\quad i\in\{j:p_j\neq 0\}.$$

This completes the proof. $\square$

Theorem 5.7 can be extended in the following way.

Theorem 5.8: *Let $F_k : I\to\mathbb{R}$, $k\in\{0,1,\ldots,r\}$, be continuous and strictly monotonic functions on an interval I. in $\mathbb{R}$. Let functions $g_1,g_2,\ldots,g_r$ be defined as*

$$g_k=F_k\circ F_{k-1}^{-1}:I_k\to I_{k+1},\quad k\in\{1,\ldots,r\},$$

where $I_j=F_{j-1}^{-1}(I)$, $j\in\{1,\ldots,r+1\}$ are intervals in $\mathbb{R}$. Assume that each function g_k, $k\in\{1,\ldots,r\}$, is either convex or concave on I_k and that the set of functions $g_1,g_2,\cdots,g_r$ is satisfying MC. Then for all monotonic m-tuples $\boldsymbol{\xi}$ in I^m and all real m-tuples $\boldsymbol{p}$ satisfying conditions (A.9) we have:

(i) *if either $F_1\circ F_0^{-1}$ is convex and F_1 is strictly increasing or $F_1\circ F_0^{-1}$ is concave and F_1 is strictly decreasing, then*

$$M_{F_0}\left(\boldsymbol{\xi},\boldsymbol{p}\right)\leq M_{F_1}\left(\boldsymbol{\xi},\boldsymbol{p}\right)\leq M_{F_2}\left(\boldsymbol{\xi},\boldsymbol{p}\right)\leq\cdots\leq M_{F_r}\left(\boldsymbol{\xi},\boldsymbol{p}\right);\tag{5.17}$$

(ii) *if either $F_1\circ F_0^{-1}$ is concave and F_1 is strictly increasing or $F_1\circ F_0^{-1}$ is convex and F_1 is strictly decreasing, then*

$$M_{F_0}\left(\boldsymbol{\xi},\boldsymbol{p}\right)\geq M_{F_1}\left(\boldsymbol{\xi},\boldsymbol{p}\right)\geq M_{F_2}\left(\boldsymbol{\xi},\boldsymbol{p}\right)\geq\cdots\geq M_{F_r}\left(\boldsymbol{\xi},\boldsymbol{p}\right);\tag{5.18}$$

where

$$M_{F_k}\left(\boldsymbol{\xi},\boldsymbol{p}\right)=F_k^{-1}\left(\frac{1}{P_m}\sum_{i=1}^{m}p_iF_k\left(\xi_i\right)\right),\quad k\in\{0,1,\ldots,r\}.$$

Proof: The first inequality in (5.17) as well as the first inequality in (5.18) follows directly by Theorem 5.7. The other inequalities in (5.17) and in (5.18) are consequences of Theorem 5.7 and the fact that set $g_1,g_2,\ldots,g_r$ is satisfying Monotonicity Condition MC. For example, assume that $F_1\circ F_0^{-1}$ is convex and F_1 is strictly increasing. Then for $F_2\circ F_1^{-1}$ we have two possibilities: either it is convex or it is concave. If $F_2\circ F_1^{-1}$ is convex, then it must be strictly increasing by MC, and $F_2=\left(F_2\circ F_1^{-1}\right)\circ F_1$ is strictly increasing too, while in case $F_2\circ F_1^{-1}$ is concave, it must be strictly decreasing by MC and $F_2 = \left(F_2 \circ F_1^{-1}\right) \circ F_1$ is strictly decreasing too. Hence, in both cases applying Theorem 5.7 with $f = F_1$ and $g = F_2$ we obtain the second inequality in (5.17). In all other cases we argue similarly. $\square$

Corollary 5.2: *Let all the assumptions of Theorem 5.8 be satisfied. Let $[a,b] \subseteq I$, $a < b$. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a monotonic n-tuple in $[a,b]^n$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ real n-tuple such that $0 \leq Q_k \leq Q_n,\ k \in \{1, 2, \ldots, n\},\ Q_n > 0$, where $Q_k = \sum_{i=1}^{k} q_i$. Then we have:*
(i) *if either $F_1 \circ F_0^{-1}$ is convex and F_1 is strictly increasing or $F_1 \circ F_0^{-1}$ is concave and F_1 is strictly decreasing, then*

$$\tilde{M}_{F_0}\left(a, b, \boldsymbol{x}, \boldsymbol{q}\right) \leq \tilde{M}_{F_1}\left(a, b, \boldsymbol{x}, \boldsymbol{q}\right) \leq \tilde{M}_{F_2}\left(a, b, \boldsymbol{x}, \boldsymbol{q}\right) \leq \cdots \leq \tilde{M}_{F_r}\left(a, b, \boldsymbol{x}, \boldsymbol{q}\right);$$

(ii) *if either $F_1 \circ F_0^{-1}$ is concave and F_1 is strictly increasing or $F_1 \circ F_0^{-1}$ is convex and F_1 is strictly decreasing, then*

$$\tilde{M}_{F_0}\left(a, b, \boldsymbol{x}, \boldsymbol{q}\right) \geq \tilde{M}_{F_1}\left(a, b, \boldsymbol{x}, \boldsymbol{q}\right) \geq \tilde{M}_{F_2}\left(a, b, \boldsymbol{x}, \boldsymbol{q}\right) \geq \cdots \geq \tilde{M}_{F_r}\left(a, b, \boldsymbol{x}, \boldsymbol{q}\right);$$

where

$$\tilde{M}_{F_k}\left(a, b, \boldsymbol{x}, \boldsymbol{q}\right) = F_k^{-1}\left(F_k\left(a\right) + F_k\left(b\right) - \frac{1}{Q_n}\sum_{i=1}^{n} q_i F_k\left(x_i\right)\right),\ k \in \{0, 1, \ldots, r\}.$$

Proof: Define $\boldsymbol{\xi}$ and $\boldsymbol{p}$ exactly as in the proof of Theorem 5.4 and then apply Theorem 5.8. $\square$

5.1.2 Examples

In 2002 [175], Mercer considered some monotonicity properties of power means. He proved the following theorem:

Theorem 5.9: *Suppose that $0 < a < b$ and $a \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq b$ with at least one of the x_k satisfying $a < x_k < b$. If $\boldsymbol{w} = (w_1, \ldots, w_n)$ is a positive n-tuple with $\sum_{i=1}^{n} w_i = 1$ and $-\infty < r < s < +\infty$ then*

$$a < M_r\left(a, b, \boldsymbol{x}\right) < M_s\left(a, b, \boldsymbol{x}\right) < b,$$

where

$$M_k\left(a, b, \boldsymbol{x}\right) \equiv \left(a^k + b^k - \sum_{i=1}^{n} w_i x_i^k\right)^{\frac{1}{k}}$$

for all real $k \neq 0$, and

$$M_0\left(a, b, x\right) \equiv \frac{ab}{G}, \quad G = \prod_{i=1}^{n} x_i^{w_i}.$$

Now we show how Theorem 5.9 can be obtained as a special case of Theorem 5.7.

Example 5.1: *For $r, s \in \mathbb{R} \setminus \{0\}$ let us define two functions f and g on interval $I = (0, \infty)$ as*

$f(x)=x^r$, $g(x)=x^s$. Then

$$\left(g\circ f^{-1}\right)(x)=x^{\frac{s}{r}},\qquad \left(g\circ f^{-1}\right)''(x)=s(s-r)\frac{x^{\frac{s}{r}-2}}{r^2}.$$

We have three possible cases:

Case 5.10: $r<s<0$. In this case we have $\left(g\circ f^{-1}\right)''(x)<0$ for all $x\in I_1$ which means that $g\circ f^{-1}$ is concave and g strictly decreasing.

Case 5.11: $r<0<s$. In this case we have $\left(g\circ f^{-1}\right)''(x)>0$ for all $x\in I_1$ which means that $g\circ f^{-1}$ is convex and g strictly increasing.

Case 5.12: $0<r<s$. In the last case we have $\left(g\circ f^{-1}\right)''(x)>0$ for all $x\in I_1$ which means that $g\circ f^{-1}$ is convex and g strictly increasing.

Since conditions of Theorem 5.7 are satisfied for such f and g, with adequate m-tuples $\boldsymbol{\xi}$ and $\boldsymbol{p}$ we can obtain inequality (5.13) in all cases.
Now, let us have n-tuples $\boldsymbol{x}$ and $\boldsymbol{w}$ as in Theorem 5.9. If we choose m-tuples $\boldsymbol{\xi}$ and $\boldsymbol{p}$ $(m=n+2)$ such that

$$\begin{aligned}\xi_1 &= a,\xi_2=x_1,\xi_3=x_2,\ldots,\xi_{n+1}=x_n,\xi_{n+2}=b,\\ p_1 &= 1,p_2=-w_1,p_3=-w_2,\ldots,p_{n+1}=-w_n,p_{n+2}=1,\end{aligned}$$

we can easily see that all conditions on them in Theorem 5.7 are satisfied, so for $f,g,\boldsymbol{\xi}$ and $\boldsymbol{p}$ as the above, Theorem 5.7 gives us

$$\left(\frac{1}{P_m}\sum_{i=1}^{m}p_i\xi_i^r\right)^{\frac{1}{r}}\leq\left(\frac{1}{P_m}\sum_{i=1}^{m}p_i\xi_i^s\right)^{\frac{1}{s}}$$

which can be written as

$$\left(a^r+b^r-\sum_{i=1}^{n}w_ix_i^r\right)^{\frac{1}{r}}<\left(a^s+b^s-\sum_{i=1}^{n}w_ix_i^s\right)^{\frac{1}{s}},$$

and this is the middle inequality in Mercer's result. The left and the right bound are obtained from

$$\lim_{r\to-\infty}\left(\frac{1}{P_m}\sum_{i=1}^{m}p_i\xi_i^r\right)^{\frac{1}{r}}=a,\qquad \lim_{r\to+\infty}\left(\frac{1}{P_m}\sum_{i=1}^{m}p_i\xi_i^r\right)^{\frac{1}{r}}=b.$$

Example 5.2: *If we consider the other interesting choices of functions f and g in Theorem 5.7, we can get more inequalities from paper [175]. For instance, if on $I=(0,\infty)$ is chosen $f(x)=x^{-r}$, for $r>0$ and $g(x)=\ln x$, we obtain*

$$\left(g\circ f^{-1}\right)(x)=-\frac{\ln x}{r},\qquad \left(g\circ f^{-1}\right)''(x)=\frac{1}{rx^2}.$$

It is obvious that g is strictly increasing and $g\circ f^{-1}$ convex on I, so for adequate n-tuples $\boldsymbol{x}$ and

$\boldsymbol{w}$. *Theorem 5.7 gives us* $\left(a^{-r}+b^{-r}-\sum_{i=1}^{n} w_i x_i^{-r}\right)^{-\frac{1}{r}} < \frac{ab}{G}$. *On the other hand, for* $f(x)=\ln x$, *and* $g(x)=x^r$, *for* $r>0$, *we obtain*

$$\left(g\circ f^{-1}\right)(x)=e^{rx}, \qquad \left(g\circ f^{-1}\right)''(x)=r^2e^{rx},$$

from which we can see that g *is strictly increasing and* $g\circ f^{-1}$ *convex on* I_1, *so again, Theorem 5.7 gives us*

$$\frac{ab}{G}<\left(a^r+b^r-\sum_{i=1}^{n} w_i x_i^r\right)^{\frac{1}{r}}.$$

Example 5.3: *Let us choose in (5.18)* $F_1(x)=x^r$, $F_2(x)=r\log x$ *for* $0<r\le 1$ *and* $[a,b]\subset[0,\infty]$. *Since functions* F_1 *and* $F_2\circ F_1^{-1}$ *are concave and increasing, it is easy to verify the following inequality for* n*-tuples* $\boldsymbol{x}$ *and* $\boldsymbol{w}$ *as in Corollary 5.2:*

$$\prod_{i=1}^{n}\left(\frac{ab}{x_i}\right)^{w_i}\le\left(\sum_{i=1}^{n} w_i\left(a^r+b^r-x_i^r\right)\right)^{\frac{1}{r}}\le\sum_{i=1}^{n} w_i\left(a+b-x_i\right). \tag{5.19}$$

Example 5.4: *Let* $p_1>1,\ p_2<0,\ 0\le w_i\le 1$, *where* $\sum_{i=1}^{n} w_i=1$, *and let* $x_i\in[a,b]\subset[0,\infty]$ $(i\in\{1,2,\dots,n\})$. *Then we have*

$$\sum_{i=1}^{n} w_i\left(a^{p_1p_2}+b^{p_1p_2}-x_i^{p_1p_2}\right)\ge\prod_{i=1}^{n}\left(\frac{ab}{x_i}\right)^{w_ip_1p_2}$$

$$\ge\left(\sum_{i=1}^{n} w_i\left(a+b-x_i\right)\right)^{p_1p_2}\ge\sum_{i=1}^{n} w_i\left(a^{p_1}+b^{p_1}-x_i^{p_1}\right)^{p_2},$$

and if also $p_2<-1$ *we have*

$$\sum_{i=1}^{n} w_i\left(a^{p_1p_2}+b^{p_1p_2}-x_i^{p_1p_2}\right)\ge\left(\sum_{i=1}^{n} w_i\left(a^{p_2}+b^{p_2}-x_i^{p_2}\right)\right)^{p_1}\ge\prod_{i=1}^{n}\left(\frac{ab}{x_i}\right)^{w_ip_1p_2}$$

$$\ge\left(\sum_{i=1}^{n} w_i\left(a+b-x_i\right)\right)^{p_1p_2}\ge\left(\sum_{i=1}^{n} w_i\left(a^{-p_2}+b^{-p_2}-x_i^{-p_2}\right)\right)^{-p_1}.$$

Proof: *If in (5.18) we choose* $F_1(x)=x^{p_1p_2}$, $F_2(x)=p_1p_2\log x$, *and if we take into consideration that* $p_1p_2<0$, *we obtain*

$$\sum_{i=1}^{n} w_i\left(a^{p_1p_2}+b^{p_1p_2}-x_i^{p_1p_2}\right)\ge\prod_{i=1}^{n}\left(\frac{a^{p_1p_2}b^{p_1p_2}}{x_i^{p_1p_2}}\right)^{w_i}=\left(\prod_{i=1}^{n}\left(\frac{ab}{x_i}\right)^{w_i}\right)^{p_1p_2}.$$

Since we know that (5.19) holds, we can deduce that

$$\left(\prod_{i=1}^{n}\left(\frac{ab}{x_i}\right)^{w_i}\right)^{p_1p_2}\ge\left(\sum_{i=1}^{n} w_i\left(a+b-x_i\right)\right)^{p_1p_2}$$

$$=\left(\sum_{i=1}^{n} w_i\left(\left(a^{p_1}\right)^{\frac{1}{p_1}}+\left(b^{p_1}\right)^{\frac{1}{p_1}}-\left(x_1^{p_1}\right)^{\frac{1}{p_1}}\right)\right)^{p_1p_2}.$$

On the other hand, we know that $0 < \frac{1}{p_1} < 1$, *so* (5.19) *gives us again*

$$\left(\sum_{i=1}^{n} w_i \left((a^{p_1})^{\frac{1}{p_1}} + (b^{p_1})^{\frac{1}{p_1}} - (x_1^{p_1})^{\frac{1}{p_1}}\right)\right)^{p_1 p_2} \geq \left(\sum_{i=1}^{n} w_i \left(a^{p_1} + b^{p_1} - x_i^{p_1}\right)\right)^{p_2}.$$

If $p_2 < -1$ *then* $0 < -\frac{1}{p_2} < 1$, *so remaining inequalities are obtained easily in the same way.*
□

5.2 Two Variants of Čebyšev's Inequality

A discrete analog of the classical Čebyšev's inequality (1882, 1883) is stated as it follows.

Theorem 5.13: *Let* $\boldsymbol{w}$ *be a nonnegative* n*-tuple. If real* n*-tuples* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *and* $\boldsymbol{y} = (y_1, \ldots, y_n)$ *are monotone in the same direction then*

$$\sum_{i=1}^{n} w_i x_i \sum_{i=1}^{n} w_i y_i \leq \sum_{i=1}^{n} w_i \sum_{i=1}^{n} w_i x_i y_i. \tag{5.20}$$

If $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are monotone in opposite directions the inequality in* (5.20) *is reversed.*

Although the proof of the following generalization of the inequality (5.20) has been already known (see [205]) for the sake of the clarity we shall briefly present it here.

Theorem 5.14: *Let* $\boldsymbol{w} = (w_1, \ldots, w_n)$ *be a real* n*-tuple such that* (A.9) *is satisfied. Then for any real* n*-tuples* $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n)$ *monotone in the same direction the inequality* (5.20)*. holds. If* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are monotone in opposite directions* (5.20) *is reversed.*

Proof: Using the well-known Abel's identity it can be proved that the following identity holds.

$$\begin{aligned}\sum_{i=1}^{n} w_i \sum_{i=1}^{n} w_i x_i y_i - \sum_{i=1}^{n} w_i x_i \sum_{i=1}^{n} w_i y_i = \sum_{k=1}^{n-1} \Bigg[& \sum_{l=1}^{k-1} \overline{W}_{k+1} W_l \left(x_{l+1} - x_l\right)\left(y_{k+1} - y_k\right) \\ & + \sum_{l=k+1}^{n} \overline{W}_l W_k \left(x_l - x_{l-1}\right)\left(y_{k+1} - y_k\right) \Bigg],\end{aligned} \tag{5.21}$$

where $\overline{W}_k = \sum_{i=k}^{n} w_i$. Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in the same direction. Then $(x_{i+1} - x_i)(y_{j+1} - y_j) \geq 0$ for all $i, j \in \{1, \ldots, n-1\}$. Furthermore, the conditions (A.9) on n-tuple $\boldsymbol{w}$ imply that also $\overline{W}_k \geq 0,\ k \in \{1, \ldots, n\}$, so from identity (5.21) we may conclude

$$\sum_{i=1}^{n} w_i \sum_{i=1}^{n} w_i x_i y_i - \sum_{i=1}^{n} w_i x_i \sum_{i=1}^{n} w_i y_i \geq 0.$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in opposite directions we have $(x_{i+1} - x_i)(y_{j+1} - y_j) \leq 0$ for all $i, j \in \{1, \ldots, n-1\}$, so the reverse of (5.20) immediately follows. □

In the next theorem we give a Mercer's type variant of the inequality (5.20).

Theorem 5.15: *Let $n \geq 2$ and let $\boldsymbol{w}$ be a real n-tuple such that (A.9) is satisfied. Let $[a,b]$ and $[c,d]$ be intervals in $\mathbb{R}$, where $a < b,\ c < d$. Then for any real n-tuples $\boldsymbol{x} \in [a,b]^n$ and $\boldsymbol{y} \in [c,d]^n$ monotone in the same direction we have*

$$\left(a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\left(c + d - \frac{1}{W_n}\sum_{i=1}^{n} w_i y_i\right) \leq ac + bd - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i y_i. \tag{5.22}$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in opposite directions the inequality in (5.22) is reversed.

Proof: Without any loss of generality we may suppose that n-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ are both decreasing (in other cases proof is similar). We define $(n+2)$-tuples $\boldsymbol{w}' = \left(w_1', \ldots, w_{n+2}'\right)$, $\boldsymbol{x}' = \left(x_1', \ldots, x_{n+2}'\right)$ and $\boldsymbol{y}' = \left(y_1', \ldots, y_{n+2}'\right)$ as

$$\begin{array}{lllll} w_1' = 1, & w_2' = -\frac{w_1}{W_n}, & \ldots, & w_{n+1}' = -\frac{w_n}{W_n}, & w_{n+2}' = 1, \\ x_1' = b, & x_2' = x_1, & \ldots, & x_{n+1}' = x_n, & x_{n+2}' = a, \\ y_1' = d, & y_2' = y_1, & \ldots, & y_{n+1}' = y_n, & y_{n+2}' = c, \end{array}$$

Obviously, $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are both decreasing and we have

$$0 \leq W_k' \leq 1,\ k \in \{1, \ldots, n+1\},\ W_{n+2}' = 1,$$

so we may apply Theorem 5.14 on $(n+2)$-tuples $\boldsymbol{w}'$, $\boldsymbol{x}'$ and $\boldsymbol{y}'$ to obtain

$$\sum_{i=1}^{n+2} w_i' x_i' \sum_{i=1}^{n+2} w_i' y_i' \leq \sum_{i=1}^{n+2} w_i' \sum_{i=1}^{n+2} w_i' x_i' y_i'$$

from which we can easily get (5.22). $\square$

Čebyšev's inequality can be generalized for $k \geq 3$ nonnegative n-tuples monotone in the same direction with nonnegative weights $\boldsymbol{w}$ (see for example [204, p. 198]). Here we give an analogous generalization of Čebyšev's inequality for $k \geq 3$ nonnegative n-tuples in which weights $\boldsymbol{w}$ satisfy the conditions (A.9). In order to simplify our results, we shall consider only weights $\boldsymbol{w}$ with $\sum_{i=1}^{n} w_i = 1$.

Theorem 5.16: *Let $n \geq 2$ and let $\boldsymbol{w}$ be a real n-tuple such that*

$$0 \leq W_k \leq 1\ \left(k \in \{1, \ldots, n-1\}\right),\ W_n = 1. \tag{5.23}$$

Then for $k \geq 2$ and for any $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)} \in [0, +\infty\rangle^k$ such that

$$\boldsymbol{x}^{(1)} \leq \cdots \leq \boldsymbol{x}^{(n)} \quad \text{or} \quad \boldsymbol{x}^{(1)} \geq \cdots \geq \boldsymbol{x}^{(n)}.$$

Then the following inequality holds

$$\prod_{i=1}^{k}\sum_{j=1}^{n} w_j x_i^{(j)} \leq \sum_{j=1}^{n} w_j \prod_{i=1}^{k} x_i^{(j)}. \tag{5.24}$$

Proof: The proof of (5.24) is by induction on k. The case $k=2$ follows from Theorem 5.14. Suppose that (5.24) is valid for all $l,\ 2 \leq l \leq k$. We have

$$\sum_{j=1}^{n} w_j \prod_{i=1}^{k+1} x_i^{(j)} = \sum_{j=1}^{n} w_j \prod_{i=1}^{k} x_i^{(j)} x_{k+1}^{(j)} \tag{5.25}$$

and we know that

$$\prod_{i=1}^{k}\sum_{j=1}^{n} w_j x_i^{(j)} \geq 0,\ \sum_{j=1}^{n} w_j \prod_{i=1}^{k} x_i^{(j)} \geq 0,\ \sum_{j=1}^{n} w_j x_{k+1}^{(j)} \geq 0$$

(see Remark 1.1). We define nonnegative n-tuple $\boldsymbol{y}$ as

$$y_j = \prod_{i=1}^{k} x_i^{(j)},\ j \in \{1, \ldots, n\}.$$

It can be easily seen that $\boldsymbol{y}$ is monotone in the same sense as $\left(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}\right)$, i.e., $\boldsymbol{y}$ is monotone in the same sense as $\left(x_{k+1}^{(1)}, \ldots, x_{k+1}^{(n)}\right)$, so we may apply (5.20) and our induction hypothesis in (5.25) to obtain

$$\begin{aligned}
\sum_{j=1}^{n} w_j \prod_{i=1}^{k+1} x_i^{(j)} &= \sum_{j=1}^{n} w_j \prod_{i=1}^{k} x_i^{(j)} x_{k+1}^{(j)} \\
&\geq \left(\sum_{j=1}^{n} w_j \prod_{i=1}^{k} x_i^{(j)}\right)\left(\sum_{j=1}^{n} w_j x_{k+1}^{(j)}\right) \\
&\geq \left(\prod_{i=1}^{k}\sum_{j=1}^{n} w_j x_i^{(j)}\right)\left(\sum_{j=1}^{n} w_j x_{k+1}^{(j)}\right) \\
&= \prod_{i=1}^{k+1}\sum_{j=1}^{n} w_j x_i^{(j)},
\end{aligned}$$

so by induction the result holds. □

In the next theorem we give a Mercer's type variant of (5.24).

Theorem 5.17: *Let $n \geq 2$ and let $\mathbf{w}$ be a real n-tuple such that (5.23). is satisfied. Let $k \geq 2$ and let $K = [a_1, b_1] \times \cdots \times [a_k, b_k] \subset [0, +\infty\rangle^k$. Then for any $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)} \in K$ such that*

$$\boldsymbol{x}^{(1)} \leq \cdots \leq \boldsymbol{x}^{(n)} \quad \textit{or} \quad \boldsymbol{x}^{(1)} \geq \cdots \geq \boldsymbol{x}^{(n)},$$

we have

$$\prod_{i=1}^{k}\left(a_i+b_i-\sum_{j=1}^{n}w_jx_i^{(j)}\right)\leq\prod_{i=1}^{k}a_i+\prod_{i=1}^{k}b_i-\sum_{j=1}^{n}w_j\prod_{i=1}^{k}x_i^{(j)}. \tag{5.26}$$

Proof: Suppose that $\boldsymbol{x}^{(1)}\leq\cdots\leq\boldsymbol{x}^{(n)}$. We define k-tuples $\boldsymbol{\xi}^{(j)}\in[0,+\infty\rangle^k$, $j\in\{1,\ldots,n+2\}$ and weights $\boldsymbol{w}'$ as

$$\boldsymbol{\xi}_i^{(1)}=a_i,\ \boldsymbol{\xi}_i^{(n+2)}=b_i,\ i\in\{1,\ldots,k\},$$

$$\boldsymbol{\xi}^{(j)}=\boldsymbol{x}^{(j-1)},\ j\in\{2,\ldots,n+1\},$$

$$w_1'=1,\ w_2'=-w_1,\ldots,w_{n+1}'=w_n,\ w_{n+2}'=1.$$

Obviously $\boldsymbol{\xi}^{(1)}\leq\cdots\leq\boldsymbol{\xi}^{(n+2)}$ and

$$0\leq W_k'\leq 1,\ k\in\{1,\ldots,n+1\},\ W_{n+2}'=1,$$

so we can apply Theorem 5.16 on $\boldsymbol{\xi}^{(j)},j\in\{1,\ldots,n+2\}$ and $\boldsymbol{w}'$, to obtain

$$\prod_{i=1}^{k}\sum_{j=1}^{n+2}w_j'\xi_i^{(j)}\leq\sum_{j=1}^{n+2}w_j'\prod_{i=1}^{k}\xi_i^{(j)},$$

from which (5.26) immediately follows.

For the case $\boldsymbol{x}^{(1)}\geq\cdots\geq\boldsymbol{x}^{(n)}$ proof is similar, so we omit the details. □

5.3 Variants of Pečarić's Inequalities

In 1984, J. Pečarić [210] proved several generalizations of the discrete Ostrowski's inequalities. Here we give two of them which are interesting to us because they are refinements of Theorem 5.14.

Theorem 5.18: *Let $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\boldsymbol{y}=(y_1,\ldots,y_n)$ be real n-tuples monotone in the same direction and let $\boldsymbol{w}=(w_1,\ldots,w_n)$ be a real n-tuple such that*

$$0\leq W_k\leq W_n,\ k\in\{1,\ldots,n-1\}. \tag{5.27}$$

If m and r are nonnegative real numbers such that

$$|x_{k+1}-x_k|\geq m,\ |y_{k+1}-y_k|\geq r,\ k\in\{1,\ldots,n-1\},$$

then

$$T(\boldsymbol{x},\boldsymbol{y};\boldsymbol{w})\geq mrT(\boldsymbol{e},\boldsymbol{e};\boldsymbol{w})\geq 0,$$

where

$$T(\boldsymbol{x},\boldsymbol{y};\boldsymbol{w})=\sum_{i=1}^{n}w_i\sum_{i=1}^{n}w_ix_iy_i-\sum_{i=1}^{n}w_ix_i\sum_{i=1}^{n}w_iy_i,$$

$$\boldsymbol{e} = (0, 1, \ldots, n-1).$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in opposite directions, then

$$T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{w}) \leq -mrT(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{w}) \leq 0.$$

Theorem 5.19: *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be real n-tuples such that*

$$|x_{k+1} - x_k| \leq M, \ |y_{k+1} - y_k| \leq R, \ k \in \{1, \ldots, n-1\},$$

hold for some nonnegative real numbers M and R, and let $\boldsymbol{w}$ be a real n-tuple such that (5.27) is valid. Then

$$|T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{w})| \leq MRT(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{w}).$$

In the next two theorems we give Mercer's type variants of Theorem 5.18 and Theorem 5.19 which are refinements of Theorem 5.15

Theorem 5.20: *Let $n \geq 2$ and let $\boldsymbol{w}$ be a real n-tuple such that (5.23) is valid. Let $[a, b]$, $[c, d]$ be intervals in $\mathbb{R}$, where $a < b$, $c < d$. Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in [a, b]^n$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in [c, d]^n$ be monotone n-tuples, and let m and r be nonnegative real numbers such that*

$$\begin{array}{ll} \min_{1 \leq i \leq n} x_i - a \geq m, & b - \max_{1 \leq i \leq n} x_i \geq m, \\ |x_{k+1} - x_k| \geq m, & k \in \{1, \ldots, n-1\}, \\ \min_{1 \leq i \leq n} y_i - c \geq r, & d - \max_{1 \leq i \leq n} y_i \geq r, \\ |y_{k+1} - y_k| \geq r, & k \in \{1, \ldots, n-1\}. \end{array}$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in the same direction then

$$\widetilde{T}(\boldsymbol{x}, \boldsymbol{y}, a, b, c, d; \boldsymbol{w}) \geq mr\left[\widetilde{T}(\boldsymbol{f}, \boldsymbol{f}, 1, n, 1, n; \boldsymbol{w}) + 2n\right] \geq 0,$$

where

$$\begin{aligned} &\widetilde{T}(\boldsymbol{x}, \boldsymbol{y}, a, b, c, d; \boldsymbol{w}) \\ &= ac + bd - \sum_{i=1}^{n} w_i x_i y_i - \left(a + b - \sum_{i=1}^{n} w_i x_i\right)\left(c + d - \sum_{i=1}^{n} w_i y_i\right), \end{aligned}$$

$$\boldsymbol{f} = (1, \ldots, n).$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in opposite directions then

$$\widetilde{T}(\boldsymbol{x}, \boldsymbol{y}, a, b, c, d; \boldsymbol{w}) \leq -mr\left[\widetilde{T}(\boldsymbol{f}, \boldsymbol{f}, 1, n, 1, n; \boldsymbol{w}) + 2n\right] \leq 0.$$

Proof: Suppose that n-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ are both decreasing (if $\boldsymbol{x}$ and $\boldsymbol{y}$ are increasing proof is similar). We define $(n+2)$-tuples $\boldsymbol{w}' = (w'_1, \ldots, w'_{n+2})$, $\boldsymbol{x}\prime = (x'_1, \ldots, x'_{n+2})$ and $\boldsymbol{y}\prime = (y'_1, \ldots, y'_{n+2})$ as

$$\begin{array}{llll} w_1' = 1, & w_2' = -w_1, & \ldots, & w_{n+1}' = -w_n, \quad w_{n+2}' = 1, \\ x_1' = b, & x_2' = x_1, & \ldots, & x_{n+1}' = x_n, \quad x_{n+2}' = a, \\ y_1' = d, & y_2' = y_1, & \ldots, & y_{n+1}' = y_n, \quad y_{n+2}' = c, \end{array}$$

Obviously, $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are both decreasing and we have

$$0 \leq W_k' \leq 1, \ k \in \{1, \ldots, n+1\}, \ W_{n+2}' = 1,$$

$$\left|x_{k+1}' - x_k'\right| \geq m, \ \left|y_{k+1}' - y_k'\right| \geq r, \ k \in \{1, \ldots, n+1\}.$$

From Theorem 5.18 we have

$$T\left(\boldsymbol{x}', \boldsymbol{y}'; \boldsymbol{w}'\right) \geq mrT\left(\boldsymbol{e}', \boldsymbol{e}'; \boldsymbol{w}'\right) \geq 0,$$

where $\boldsymbol{e}' = (0, 1, \ldots, n+1)$. From that we immediately obtain

$$\begin{aligned} & ac + bd - \sum_{i=1}^{n} w_i x_i y_i - \left(a + b - \sum_{i=1}^{n} w_i x_i\right)\left(c + d - \sum_{i=1}^{n} w_i y_i\right) \\ & \geq mr\left[\sum_{i=1}^{n+2} w_i'(i-1)^2 - \left(\sum_{i=1}^{n+2} w_i'(i-1)\right)^2\right] \\ & = mr\left[(n+1)^2 - \sum_{i=1}^{n} w_i i^2 - \left(n + 1 - \sum_{i=1}^{n} w_i i\right)^2\right] \geq 0, \end{aligned}$$

i.e.,

$$\widetilde{T}(\boldsymbol{x}, \boldsymbol{y}, a, b, c, d; \boldsymbol{w}) \geq mr\left[\widetilde{T}(\boldsymbol{f}, \boldsymbol{f}, 1, n, 1, n; \boldsymbol{w}) + 2n\right] \geq 0.$$

If n-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in opposite directions proof is similar. □

Theorem 5.21: *Let $n \geq 2$ and let $\boldsymbol{w}$ be a real n-tuple such that (5.23) is valid. Let $[a, b]$, $[c, d]$ be intervals in $\mathbb{R}$, where $a < b$, $c < d$. Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in [a, b]^n$, $\boldsymbol{y} = (y_1, \ldots, y_n) \in [c, d]^n$ and let M and R be nonnegative real numbers such that*

$$\begin{array}{ll} x_1 - a \leq M, & b - x_n \leq M, \\ |x_{k+1} - x_k| \leq M, & k \in \{1, \ldots, n-1\}, \\ y_1 - c \leq R, & d - y_n \leq R, \\ |y_{k+1} - y_k| \leq R, & k \in \{1, \ldots, n-1\}. \end{array}$$

Then

$$\left|\widetilde{T}(\boldsymbol{x}, \boldsymbol{y}, a, b, c, d; \boldsymbol{w})\right| \leq MR\left[\widetilde{T}(\boldsymbol{f}, \boldsymbol{f}, 1, n, 1, n; \boldsymbol{w}) + 2n\right].$$

Proof: Similarly as in Theorem 5.20. □

Corollary 5.3: *Let $n \geq 2$ and let $[a, b]$ be an interval in $\mathbb{R}$ where $a < b$. Then for all $\boldsymbol{x} =$*

$(x_1,\ldots,x_n)\in[a,b]^n$ we have

$$\left[na^2+nb^2-\sum_{i=1}^{n}x_i^2-\frac{1}{n}\left(na+nb-\sum_{i=1}^{n}x_i\right)^2\right]\frac{12}{n\left(n+1\right)\left(5n+7\right)}\geq m^2,$$

where $m=\min_{0\leq i<j\leq n+1}\left|x_i-x_j\right|,\ x_0=a,\quad x_{n+1}=b.$

Proof: Directly from Theorem 5.20 □

Corollary 5.4: *Let $\boldsymbol{x}=(x_1,\ldots,x_n)$, $\boldsymbol{y}=(y_1,\ldots,y_n)$, M and R be defined as in Theorem 5.21. Then*

$$\left|nac+nbd-\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\left(na+nb-\sum_{i=1}^{n}x_i\right)\left(nc+nd-\sum_{i=1}^{n}y_i\right)\right|$$
$$\leq\frac{n\left(n+1\right)\left(5n+7\right)}{12}MR.$$

Proof: Directly from Theorem 5.21. □

Remark 5.3: The above results are variants of some Lupaş' results [162].

5.4 Inequality of Milne and its Converse

In 1925, Milne [181] obtained the following interesting integral inequality for positive functions f and g which are integrable on $[a,b]$:

$$\int_a^b\frac{f\left(x\right)g\left(x\right)}{f\left(x\right)+g\left(x\right)}\mathrm{d}x\int_a^b\left[f\left(x\right)+g\left(x\right)\right]\mathrm{d}x\leq\int_a^b f\left(x\right)\mathrm{d}x\int_a^b g\left(x\right)\mathrm{d}x.$$

In 2000, Rao [221] combined Milne's inequality and the well known inequality between arithmetic and geometric mean to obtain the following double inequality for sums.

Theorem 5.22: *Let $n\geq 2$ and let $w_i>0$, $i\in\{1,2,\ldots,n\}$ be real numbers with $\sum_{i=1}^n w_i=1$. Then for all real numbers $p_i\in\langle-1,1\rangle$, $i\in\{1,\ldots,n\}$ the following inequality holds*

$$\sum_{i=1}^{n}\frac{w_i}{1-p_i^2}\leq\left(\sum_{i=1}^{n}\frac{w_i}{1-p_i}\right)\left(\sum_{i=1}^{n}\frac{w_i}{1+p_i}\right)\leq\left(\sum_{i=1}^{n}\frac{w_i}{1-p_i^2}\right)^2. \tag{5.28}$$

Two years later Alzer and Kovačec obtained the following refinement of (5.28).

Theorem 5.23 ***([18, Theorem 1]):*** *Let $n\geq 2$ and let $w_i>0$, $i\in\{1,2,\ldots,n\}$ be real numbers with $\sum_{i=1}^n w_i=1$. Then for all real numbers $p_i\in[0,1\rangle$, $i\in\{1,\ldots,n\}$*

$$\left(\sum_{i=1}^{n}\frac{w_i}{1-p_i^2}\right)^{\alpha}\leq\left(\sum_{i=1}^{n}\frac{w_i}{1-p_i}\right)\left(\sum_{i=1}^{n}\frac{w_i}{1+p_i}\right)\leq\left(\sum_{i=1}^{n}\frac{w_i}{1-p_i^2}\right)^{\beta} \tag{5.29}$$

holds, with the best possible exponents $\alpha=1$ and $\beta=2-\min\limits_{1\leq i\leq n}w_i$.

We note here that the crucial step in the proof of Theorem 5.23 was performed by using a discrete variant of the Čebyšev's inequality which itself was generalized in Subsection 5.2. That enables us to give the following generalization of Theorem 5.23.

Theorem 5.24: *Let $n \geq 2$ and let $\boldsymbol{w} = (w_1, \dots, w_n)$ be a real n-tuple such that (5.23) is satisfied. Then for all $\alpha \in \langle -\infty, 1]$, $\beta \in [2 - \min_{1\leq i\leq n} W_i, +\infty\rangle$ and for all monotone n-tuples $\boldsymbol{p} = (p_1, \dots, p_n) \in [0,1\rangle^n$*

$$\left(\sum_{i=1}^{n} \frac{w_i}{1-p_i^2}\right)^{\alpha} \leq \left(\sum_{i=1}^{n} \frac{w_i}{1-p_i}\right)\left(\sum_{i=1}^{n} \frac{w_i}{1+p_i}\right) \leq \left(\sum_{i=1}^{n} \frac{w_i}{1-p_i^2}\right)^{\beta} \tag{5.30}$$

holds, with the best possible exponents $\alpha = 1$ and $\beta = 2 - \min\limits_{1\leq i\leq n} W_i$.

Proof: We follow the idea of the proof given in [18]. □

In the next theorem we give a Mercer's type variant of (5.30).

Theorem 5.25: *Let $n \geq 2$ and let $\boldsymbol{w} = (w_1, \dots, w_n)$ be a real n-tuple such that (5.23)is satisfied. Then for all $\alpha \in \langle -\infty, 1]$, $\beta \in [2, +\infty\rangle$ and for all monotone n-tuples $\boldsymbol{p} = (p_1, \dots, p_n) \in [p,q]^n$, where $[p,q] \subseteq [0,1\rangle$ and $p < q$,*

$$\begin{aligned}
&\left(\frac{1}{1-p^2} + \frac{1}{1-q^2} - \sum_{i=1}^{n} \frac{w_i}{1-p_i^2}\right)^{\alpha} \\
&\leq \left(\frac{1}{1-p} + \frac{1}{1-q} - \sum_{i=1}^{n} \frac{w_i}{1-p_i}\right)\left(\frac{1}{1+p} + \frac{1}{1+q} - \sum_{i=1}^{n} \frac{w_i}{1+p_i}\right) \\
&\leq \left(\frac{1}{1-p^2} + \frac{1}{1-q^2} - \sum_{i=1}^{n} \frac{w_i}{1-p_i^2}\right)^{\beta},
\end{aligned} \tag{5.31}$$

holds, with the best possible exponents $\alpha = 1$ and $\beta = 2$.

Proof: Suppose that $q \geq p_1 \geq p_2 \geq \cdots \geq p_n \geq p$. We define $(n+2)$ −tuples $\boldsymbol{w}' = (w_1', \dots, w_{n+2}')$ and $\boldsymbol{p}' = (p_1', \dots, p_{n+2}') \in [0,1\rangle^n$ as

$$\begin{aligned}
&w_1' = 1, \quad w_2' = -w_1, \quad \dots, \quad w_{n+1}' = -w_n, \quad w_{n+2}' = 1, \\
&p_1' = q, \quad p_2' = p_1, \quad \dots, \quad p_{n+1}' = p_n, \quad p_{n+2}' = p.
\end{aligned}$$

Then $0 \leq W_k' \leq 1,\ k \in \{1, \dots, n+1\},\ W_{n+2}' = 1,\ \min\limits_{1\leq i\leq n} W_i' = 0$. From Remark 1.1 we know that $\sum\limits_{i=1}^{n+2} \frac{w_i'}{1-p_i'^2} \geq 1$, so the left side and the right side of (5.31) are well defined. If we apply Theorem 5.24 on $(n+2)$ −tuples $\boldsymbol{w}'$ and $\boldsymbol{p}'$ we obtain

$$\left(\sum_{i=1}^{n+2} \frac{w_i'}{1-p_i'^2}\right)^{\alpha} \leq \left(\sum_{i=1}^{n+2} \frac{w_i'}{1-p_i'}\right)\left(\sum_{i=1}^{n+2} \frac{w_i'}{1+p_i'}\right) \leq \left(\sum_{i=1}^{n+2} \frac{w_i'}{1-p_i'^2}\right)^{\beta},$$

from which (5.31) immediately follows. If $p \leq p_1 \leq \cdots \leq p_n \leq q$ proof is similar. □

CHAPTER 6

On Normalized Jensen–Mercer Functional

This chapter focuses on the normalized Jensen–Mercer functional and its analytical properties. We begin by introducing the normalized form of the Jensen–Mercer functional and deriving bounds that sharpen classical estimates. The discussion then explores fundamental properties of the functional, highlighting structural features that underlie Mercer-type inequalities. These results provide a foundation for further refinements and applications in subsequent chapters.

6.1 Normalized Jensen–Mercer Functional

Let $\mathcal{P}_n$ denotes the set of all nonnegative real n-tuples $(p_1, \ldots, p_n)$ with the property $\sum_{i=1}^{n} p_i = 1$. For any convex function $f : [a, b] \to \mathbb{R}$ and for any choice of n-tuples $\boldsymbol{x} = (x_1, \ldots, x_n) \in [a, b]^n$ and $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathcal{P}_n$ we define

$$\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) := f(a) + f(b) - \sum_{i=1}^{n} p_i f(x_i) - f\left(a + b - \sum_{i=1}^{n} p_i x_i\right) \tag{6.1}$$

and we call it the normalized Jensen–Mercer functional. For a fixed function f and n-tuple $\boldsymbol{x}$, $\mathcal{M}_n(f, \boldsymbol{x}, \cdot)$ can be observed as a function on $\mathcal{P}_n$. Note that $\mathcal{P}_n$ is obviously a convex subset in $\mathbb{R}^n$ and because of the Jensen–Mercer inequality, $\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) \geq 0$ for all $\boldsymbol{p} \in \mathcal{P}_n$.

We establish the inequalities of type

$$M\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{q}) \geq \mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) \geq m\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{q}),$$

where m and M are real constants satisfying certain conditions. We prove them for the case when $\boldsymbol{p}$ and $\boldsymbol{q}$ are nonnegative n-tuples and when $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfy the conditions for the Jensen-Steffensen inequality. We also give the integral versions of all results [32].

6.2 Bounds for the Normalized Jensen–Mercer Functional

Theorem 6.1: *Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be two n-tuples from $\mathcal{P}_n$. Let m and M be any real constants such that*

$$m \geq 0, \quad p_i - mq_i \geq 0, \quad Mq_i - p_i \geq 0, \quad i \in \{1, \ldots, n\}. \tag{6.2}$$

If $f : [a, b] \to \mathbb{R}$ is a convex function and $\boldsymbol{x} = (x_1, \ldots, x_n)$ is any n-tuple from $[a, b]^n$, then

$$M\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{q}) \geq \mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) \geq m\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{q}). \tag{6.3}$$

Proof: Suppose that $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}_n$ and $m, M \in \mathbb{R}$ satisfy (6.2). From $p_i - mq_i \geq 0 \ \ (i \in \{1, \ldots, n\})$ follows that $1 - m = \sum_{i=1}^n (p_i - mq_i) \geq 0$ i.e., $m \leq 1$, and from $Mq_i - p_i \geq 0 \ \ (i \in \{1, \ldots, n\})$ follows that $M - 1 = \sum_{i=1}^n (Mq_i - p_i) \geq 0$ i.e., $M \geq 1$. If $m = 1$ or $M = 1$, then $\boldsymbol{p} = \boldsymbol{q}$ and (6.3) obviously holds. Hence, it remains to consider the case when $m < 1$ and $M > 1$.

Applying Remark 1.2 with $w_i := p_i - mq_i$ and using the convexity of the function f we obtain the right inequality in (6.3).

Similarly, applying Remark 1.2 with $w_i := Mq_i - p_i$ and using the convexity of the function f we obtain the left inequality in (6.3). □

Corollary 6.1: *Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be two n-tuples from $\mathcal{P}_n$ such that $q_i > 0 \ \ (i \in \{1, \ldots, n\})$. Let*

$$m = m(\boldsymbol{p}, \boldsymbol{q}) := \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\}, \quad M = M(\boldsymbol{p}, \boldsymbol{q}) := \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\}. \tag{6.4}$$

If $f : [a, b] \to \mathbb{R}$ is a convex function and $\boldsymbol{x} = (x_1, \ldots, x_n)$ is any n-tuple from $[a, b]^n$, then (6.3) holds.

Proof: *Obviously $m \geq 0$ and*

$$\frac{p_i}{q_i} - m \geq 0, \quad M - \frac{p_i}{q_i} \geq 0, \quad i \in \{1, \ldots, n\},$$

which implies

$$p_i - mq_i \geq 0, \quad Mq_i - p_i \geq 0, \quad i \in \{1, \ldots, n\}.$$

Hence, m and M satisfy the conditions of Theorem 6.1. □

Let $\tilde{\mathcal{P}}_n$ denotes the set of all real n-tuples $\boldsymbol{p} = (p_1, \ldots, p_n)$ satisfying the following Jensen-Steffensen conditions

$$0 \leq P_k \leq 1, \quad k \in \{1, \ldots, n-1\}, \quad P_n = 1, \tag{6.5}$$

where $P_k := \sum_{i=1}^k p_i$ for $k \in \{1, \ldots, n\}$. Since any n-tuple $\boldsymbol{p}$ from $\mathcal{P}_n$ obviously satisfies (6.5), $\mathcal{P}_n \subseteq \tilde{\mathcal{P}}_n$. Notice that $\tilde{\mathcal{P}}_n$ is also a convex subset of $\mathbb{R}^n$.

Let $f : [a, b] \to \mathbb{R}$ be a convex function, $\boldsymbol{x} = (x_1, \ldots, x_n) \in [a, b]^n$ any monotonic n-tuple and $\boldsymbol{p}=(p_1,\ldots,p_n)\in\tilde{\mathcal{P}}_n$. Then $\sum_{i=1}^n p_i x_i\in[a,b]$ (see for example [8]) and $\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{p})$ is well defined. Also, because of Theorem 2 of [8] (or see Theorem 1.4), $\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{p})\geq 0$ for all $\boldsymbol{p}\in\tilde{\mathcal{P}}_n$.

Theorem 6.2: *Let $\boldsymbol{p}=(p_1,\ldots,p_n)$ and $\boldsymbol{q}=(q_1,\ldots,q_n)$ be two n-tuples from $\tilde{\mathcal{P}}_n$. Let m and M be any real constants such that*

$$m\geq 0,\quad P_k-mQ_k\geq 0,\quad (1-P_k)-m\,(1-Q_k)\geq 0,\quad k\in\{1,\ldots,n-1\} \tag{6.6}$$

and

$$MQ_k-P_k\geq 0,\quad M\,(1-Q_k)-(1-P_k)\geq 0,\quad k\in\{1,\ldots,n-1\}, \tag{6.7}$$

where $P_k=\sum_{i=1}^k p_i$, $Q_k=\sum_{i=1}^k q_i$. If $f:[a,b]\to\mathbb{R}$ is a convex function and $\boldsymbol{x}=(x_1,\ldots,x_n)\in[a,b]^n$ is any monotonic n-tuple, then

$$M\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{q})\geq\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{p})\geq m\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{q}). \tag{6.8}$$

Proof: Assume that $\boldsymbol{p},\boldsymbol{q}\in\tilde{\mathcal{P}}_n$ and $m,M\in\mathbb{R}$ satisfy (6.6) and (6.7). From (6.6) follows that it has to be $m\leq 1$, and from (6.7) follows that it has to be $M\geq 1$. If $m=1$ or $M=1$, then $\boldsymbol{p}=\boldsymbol{q}$ and (6.8) obviously holds. Hence, it remains to consider the case when $m<1$ and $M>1$.

To prove the right inequality in (6.8) we consider the n-tuple $\boldsymbol{w}=(w_1,\ldots,\, w_n)$ defined by $w_i:=p_i-mq_i$, $i\in\{1,\ldots,n\}$. From (6.6) follows that $\boldsymbol{w}$ satisfies conditions (A.9) (or (6.5)). Now, we follow our proof of Theorem 6.1, but instead of using the Remark 1.2 we use Theorem 2 of [8] (or see Theorem 1.4).

To prove the left inequality in (6.8) we consider the n-tuple $\boldsymbol{w}=(w_1,\ldots,w_n)$ defined by $w_i:=Mq_i-p_i$, $i\in\{1,\ldots,n\}$. From (6.7) follows that $\boldsymbol{w}$ satisfies conditions (A.9) (or (6.5)). Again, we follow our proof of Theorem 6.1, but using Theorem 2 of [8] (or see Theorem 1.4) instead of Remark 1.2. □

Corollary 6.2: *Let $\boldsymbol{p}=(p_1,\ldots,p_n)$ and $\boldsymbol{q}=(q_1,\ldots,q_n)$ be two n-tuples from $\tilde{\mathcal{P}}_n$. For $k\in\{1,\ldots,n\}$ denote $P_k:=\sum_{i=1}^k p_i$, $Q_k:=\sum_{i=1}^k q_i$. Assume that $0<Q_k<1$ for all $k\in\{1,\ldots,n-1\}$ and define*

$$\tilde{m}=\tilde{m}(\boldsymbol{p},\boldsymbol{q}):=\min\left\{\frac{P_k}{Q_k},\frac{1-P_k}{1-Q_k}:k\in\{1,\ldots,n-1\}\right\}, \tag{6.9}$$

$$\tilde{M}=\tilde{M}(\boldsymbol{p},\boldsymbol{q}):=\max\left\{\frac{P_k}{Q_k},\frac{1-P_k}{1-Q_k}:k\in\{1,\ldots,n-1\}\right\}. \tag{6.10}$$

If $f:[a,b]\to\mathbb{R}$ is a convex function and if $\boldsymbol{x}=(x_1,\ldots,x_n)\in[a,b]^n$ is any monotonic n-tuple, then

$$\tilde{M}\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{q})\geq\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{p})\geq\tilde{m}\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{q}). \tag{6.11}$$

Proof: Since $0<Q_k<1$ for all $k\in\{1,\ldots,n-1\}$, $\tilde{m}$ and $\tilde{M}$ are well defined and obviously (6.6) and (6.7) are satisfied for $m=\tilde{m}$ and $M=\tilde{M}$. Therefore we can apply Theorem 6.2 to obtain (6.11). □

We can consider the uniform distribution $\boldsymbol{u} = (\frac{1}{n}, \dots, \frac{1}{n})$ and the corresponding nonweighted Jensen–Mercer functional

$$\mathcal{M}_n(f, \boldsymbol{x}) := \mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{u}) = f(a) + f(b) - \frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(a + b - \frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Then we can state the following special case of Corollary 6.2.

Corollary 6.3: *Let $\boldsymbol{p} = (p_1, \dots, p_n)$ be n-tuple from $\tilde{\mathcal{P}}_n$. For $k \in \{1, \dots, n\}$ denote $P_k := \sum_{i=1}^{k} p_i$ and define*

$$\tilde{m}_0 := n \cdot \min\left\{\frac{P_k}{k}, \frac{1 - P_k}{n - k} : k \in \{1, \dots, n-1\}\right\},$$
$$\tilde{M}_0 := n \cdot \max\left\{\frac{P_k}{k}, \frac{1 - P_k}{n - k} : k \in \{1, \dots, n-1\}\right\}.$$

If $f : [a, b] \to \mathbb{R}$ is a convex function and if $\boldsymbol{x} = (x_1, \dots, x_n) \in [a, b]^n$ is any monotonic n-tuple, then

$$\tilde{M}_0 \mathcal{M}_n(f, \boldsymbol{x}) \geq \mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) \geq \tilde{m}_0 \mathcal{M}_n(f, \boldsymbol{x}).$$

Next, we show that Theorem 6.2 in some way provides an improvement of Corollary 6.1. Denote by Π_n the set off all permutations of $(1, 2, \dots, n)$. Suppose that $\pi = (\pi(1), \pi(2), \dots, \pi(n)) \in \Pi_n$ and denote

$$\boldsymbol{a}_\pi := (a_{\pi(1)}, a_{\pi(2)}, \dots, a_{\pi(n)})$$

for any n-tuple $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$. First we prove one simple auxiliary result.

Lemma 6.1: *Let $\boldsymbol{p} = (p_1, \dots, p_n)$ and $\boldsymbol{q} = (q_1, \dots, q_n)$ be two nonnegative n-tuples from $\mathcal{P}_n$. If $q_i > 0$ for all $i \in \{1, \dots, n\}$, then $\tilde{m}(\boldsymbol{p}, \boldsymbol{q})$ and $\tilde{M}(\boldsymbol{p}, \boldsymbol{q})$ are well defined by (6.9) and (6.10) and*

$$\max_{1 \leq i \leq n}\left\{\frac{p_i}{q_i}\right\} \geq \tilde{M}(\boldsymbol{p}, \boldsymbol{q}), \quad \tilde{m}(\boldsymbol{p}, \boldsymbol{q}) \geq \min_{1 \leq i \leq n}\left\{\frac{p_i}{q_i}\right\}. \tag{6.12}$$

Proof: Since $q_i > 0$ for all $i \in \{1, \dots, n\}$, it is obvious that $0 < Q_k < 1$ for all $k \in \{1, \dots, n-1\}$, so that $\tilde{m}(\boldsymbol{p}, \boldsymbol{q})$ and $\tilde{M}(\boldsymbol{p}, \boldsymbol{q})$ are well defined by (6.9) and (6.10). Also, for any $k \in \{1, \dots, n-1\}$ we can write

$$P_k := \sum_{i=1}^{k} p_i = \sum_{i=1}^{k} \frac{p_i}{q_i} q_i, \quad 1 - P_k = \sum_{i=k+1}^{n} p_i = \sum_{i=k+1}^{n} \frac{p_i}{q_i} q_i.$$

Now,

$$\max_{1 \leq i \leq n}\left\{\frac{p_i}{q_i}\right\} Q_k = \max_{1 \leq i \leq n}\left\{\frac{p_i}{q_i}\right\} \sum_{i=1}^{k} q_i \geq P_k \geq \min_{1 \leq i \leq n}\left\{\frac{p_i}{q_i}\right\} \sum_{i=1}^{k} q_i = \min_{1 \leq i \leq n}\left\{\frac{p_i}{q_i}\right\} Q_k,$$

i.e.,

$$\max_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\ge\frac{P_k}{Q_k}\ge\min_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}.$$

Similarly,

$$\max_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\ge\frac{1-P_k}{1-Q_k}\ge\min_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\quad\text{for all } k\in\{1,\dots,n-1\},$$

and (6.12) immediately follows. □

Remark 6.1: It is clear that inequalities stated in Lemma 6.1 can be strict. For example, if $n=5$, $\boldsymbol{p}=(\frac{1}{6},\frac{1}{3},\frac{1}{9},\frac{1}{6},\frac{2}{9})$ and $\boldsymbol{q}=(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5})$, then

$$\max_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}=\frac{5}{3}>\tilde{M}(\boldsymbol{p},\boldsymbol{q})=\frac{5}{4},\quad \tilde{m}(\boldsymbol{p},\boldsymbol{q})=\frac{5}{6}>\min_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}=\frac{5}{9}.$$

It is not hard to see that generally

$$\max_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}=\max_{\pi\in\Pi_n}\tilde{M}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi),\quad \min_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}=\min_{\pi\in\Pi_n}\tilde{m}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi).$$

Theorem 6.3: *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $\boldsymbol{x}=(x_1,\dots,x_n)\in[a,b]^n$ be any n-tuple. Let $\pi=(\pi(1),\pi(2),\dots,\pi(n))$ be a permutation of $(1,2,\dots,n)$ such that $\boldsymbol{x}_\pi$ is monotonic (nondecreasing or nonincreasing). If $\boldsymbol{p}=(p_1,\dots,p_n)$ and $\boldsymbol{q}=(q_1,\dots,q_n)$ are two n-tuples from $\mathcal{P}_n$ such that $q_i>0$ for all $i\in\{1,\dots,n\}$, then*

$$\begin{aligned}\max_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{q})&\ge\tilde{M}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{q})\ge\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{p})\\&\ge\tilde{m}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{q})\ge\min_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{q}),\end{aligned}\tag{6.13}$$

where $\tilde{m}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)$ and $\tilde{M}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)$ are defined as in (6.9) and (6.10). The first and the last inequality can be strict.

Proof: Since π is chosen so that $\boldsymbol{x}_\pi$ is monotonic we can apply Corollary 6.2 and Lemma 6.1 to the n-tuples $\boldsymbol{p}_\pi$ and $\boldsymbol{q}_\pi$ to get

$$\begin{aligned}\max_{1\le i\le n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}\mathcal{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi)&\ge\tilde{M}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathcal{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi)\ge\mathcal{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{p}_\pi)\\&\ge\tilde{m}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathcal{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi)\ge\min_{1\le i\le n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}\mathcal{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi).\end{aligned}$$

Since $\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{p})$ doesn't change if we simultaneously permute the components of $\boldsymbol{x}$ and $\boldsymbol{p}$, we have $\mathcal{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{p}_\pi)=\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{p})$ and $\mathcal{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi)=\mathcal{M}_n(f,\boldsymbol{x},\boldsymbol{q})$. Also it is obvious that

$$\max_{1\le i\le n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}=\max_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\quad\text{and}\quad\min_{1\le i\le n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}=\min_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}.$$

Therefore, sequence of inequalities (6.13) holds. By Remark 6.1 the first and the last inequality in the sequence can be strict. □

From Corollary 1.1 we have the inequality for $\varphi = f$ and $f = x$:

$$f\left(a+b-\int_{\Omega} x\,\mathrm{d}\mu\right) \leq f(a)+f(b)-\int_{\Omega} f(x)\,\mathrm{d}\mu. \tag{6.14}$$

It can analogously be proved that for a measure space $(\Omega, \mathcal{A}, \mu)$ with $0 < \mu(\Omega) < \infty$ the integral version of the Jensen–Mercer inequality

$$f\left(a+b-\frac{1}{\mu(\Omega)}\int_{\Omega} x\,\mathrm{d}\mu\right) \leq f(a)+f(b)-\frac{1}{\mu(\Omega)}\int_{\Omega} f(x)\,\mathrm{d}\mu \tag{6.15}$$

holds. In a special case when $\Omega = [\alpha, \beta]$, where $-\infty < \alpha < \beta < \infty$ and $\lambda : [\alpha, \beta] \to \mathbb{R}$ is any nondecreasing function such that $\lambda(\beta) \neq \lambda(\alpha)$ inequality (6.15) becomes

$$f\left(a+b-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} x(t)\mathrm{d}\lambda(t)\right) \leq f(a)+f(b)-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} f(x(t))\mathrm{d}\lambda(t). \tag{6.16}$$

Also, we can prove that (6.16) remains valid even in the case when the condition "λ is nondecreasing function" is somewhat relaxed. We use the following integral variant of the Jensen-Steffensen inequality given by R. P. Boas [39] (see also [204, p. 59]):

Theorem 6.4: *Let $x : [\alpha, \beta] \to (a, b)$ be a continuous and monotonic function (either nondecreasing or nonincreasing), where $-\infty < \alpha < \beta < \infty$ and $-\infty \leq a < b \leq \infty$, and let function $\lambda : [\alpha, \beta] \to \mathbb{R}$ be either continuous or of bounded variation satisfying*

$$\lambda(\alpha) \leq \lambda(t) \leq \lambda(\beta) \quad \textit{for all } t \in [\alpha, \beta], \quad \lambda(\beta)-\lambda(\alpha) > 0. \tag{6.17}$$

If $f : (a, b) \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} x(t)\mathrm{d}\lambda(t)\right) \leq \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} f(x(t))\mathrm{d}\lambda(t). \tag{6.18}$$

holds.

Theorem 6.5: *Let $x : [\alpha, \beta] \to [a, b]$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < +\infty$ and $-\infty < a < b < +\infty$. Let function $\lambda : [\alpha, \beta] \to \mathbb{R}$ be either continuous or of bounded variation satisfying (6.17). Then for any continuous convex function $f : [a, b] \to \mathbb{R}$ inequality (6.16) holds.*

Proof: For proof see [32]. □

There is no loss in generality if we assume $\lambda(\beta) - \lambda(\alpha) = 1$ and consider the normalized Jensen–Mercer functional

$$\mathcal{M}(f, x, \lambda) := f(a)+f(b)-\int_{\alpha}^{\beta} f(x(t))\mathrm{d}\lambda(t) - f\left(a+b-\int_{\alpha}^{\beta} x(t)\mathrm{d}\lambda(t)\right). \tag{6.19}$$

Under the appropriate assumptions on f, x and λ, we always have $\mathcal{M}(f,x,\lambda) \geq 0$.

For $-\infty < \alpha < \beta < \infty$ let $\Lambda_{[\alpha,\beta]}$ denotes the class of all functions $\lambda : [\alpha,\beta] \to \mathbb{R}$ which are either continuous or of bounded variation and satisfy the condition

$$\lambda(\alpha) \leq \lambda(t) \leq \lambda(\beta) \quad \text{for all } t \in [\alpha,\beta], \quad \lambda(\beta) - \lambda(\alpha) = 1. \tag{6.20}$$

Notice that any nondecreasing function $\lambda : [\alpha,\beta] \to \mathbb{R}$ with $\lambda(\beta) - \lambda(\alpha) = 1$ belongs to $\Lambda_{[\alpha,\beta]}$.

Now we can state the integral analogues of the results from the previous section. First, we give the integral version of Theorem 6.1.

Theorem 6.6: *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$. Let $x : [\alpha,\beta] \to [a,b]$, $-\infty < a < b < \infty$ be a continuous function and let $f : [a,b] \to \mathbb{R}$ be a convex function.*
(i) If μ is nondecreasing and if $m \geq 0$ is a constant such that the function $\rho : [\alpha,\beta] \to \mathbb{R}$ defined by

$$\rho(t) := \lambda(t) - m\mu(t), \quad t \in [\alpha,\beta] \tag{6.21}$$

is also nondecreasing, then

$$\mathcal{M}(f,x,\lambda) \geq m\mathcal{M}(f,x,\mu). \tag{6.22}$$

(ii) If λ is nondecreasing and if $M > 0$ is a constant such that the function $\sigma : [\alpha,\beta] \to \mathbb{R}$ defined by

$$\sigma(t) := M\mu(t) - \lambda(t), \quad t \in [\alpha,\beta] \tag{6.23}$$

is also nondecreasing, then

$$M\mathcal{M}(f,x,\mu) \geq \mathcal{M}(f,x,\lambda). \tag{6.24}$$

Proof: (i) Since μ and $\rho = \lambda - m\mu$ are assumed to be nondecreasing and $m \geq 0$, the function $\lambda = \rho + m\mu$ is nondecreasing too. Hence, $\mathcal{M}(f,x,\lambda)$ and $\mathcal{M}(f,x,\mu)$ are well defined. Since ρ is nondecreasing it has to be $m \leq 1$. If $m = 1$ then (6.22) holds with equality sign. Hence, it remains to consider the case when $m < 1$. In this case $\rho(\beta) - \rho(\alpha) = 1 - m > 0$ and ρ is nondecreasing by our assumption so that (6.16) can be applied to ρ. Now, using (6.16) and then the convexity of f we obtain (6.22).

(ii) Since λ and $\sigma = M\mu - \lambda$ are assumed to be nondecreasing and $M > 0$, the function $\mu = \frac{1}{M}(\sigma + \lambda)$ is nondecreasing too. Hence, $\mathcal{M}(f,x,\lambda)$ and $\mathcal{M}(f,x,\mu)$ are well defined. Since ρ is nondecreasing it has to be $M \geq 1$. If $M = 1$ then (6.24) holds with equality sign. Hence, it remains to consider the case when $M > 1$. In this case $\sigma(\beta) - \sigma(\alpha) = M - 1 > 0$ and σ is nondecreasing by our assumption so that (6.16) can be applied to σ. Now, using (6.16) and then the convexity of f we obtain (6.24). This completes the proof. □

Corollary 6.4: *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$, $x : [\alpha,\beta] \to [a,b]$, $-\infty < a < b < \infty$ be a continuous function and $f : [a,b] \to \mathbb{R}$ be a convex function.*

(i) Let μ is strictly increasing and define

$$\tilde{m} = \tilde{m}(\lambda, \mu) := \inf_{\alpha<t<\beta} \left\{ \inf \left\{ \frac{\lambda(t) - \lambda(s)}{\mu(t) - \mu(s)} : \alpha \le s \le \beta,\ s \ne t \right\} \right\}.$$

If $\widetilde{m} \ge 0$, then

$$\mathcal{M}(f, x, \lambda) \ge \tilde{m}\mathcal{M}(f, x, \mu). \tag{6.25}$$

(ii) Let λ is nondecreasing, μ is strictly increasing and define

$$\tilde{M} = \tilde{M}(\lambda, \mu) := \sup_{\alpha<t<\beta} \left\{ \sup \left\{ \frac{\lambda(t) - \lambda(s)}{\mu(t) - \mu(s)} : \alpha \le s \le \beta,\ s \ne t \right\} \right\}.$$

If $\widetilde{M} < \infty$, then

$$\tilde{M}\mathcal{M}(f, x, \mu) \ge \mathcal{M}(f, x, \lambda). \tag{6.26}$$

Proof: (i) Since μ is strictly increasing it is injective and $\tilde{m}$ is well defined real number in $[-\infty, \infty)$. If $\tilde{m} \ge 0$, then $\rho = \lambda - \tilde{m}\mu$ is well defined nondecreasing function on $[\alpha, \beta]$ and we can apply Theorem 6.6 a) with $m = \tilde{m}$ to obtain (6.25).

(ii) Since λ is nondecreasing and μ is strictly increasing $\tilde{M}$ is well defined real number in $(0, \infty]$. If $\tilde{M} < \infty$ then $\sigma = \tilde{M}\mu - \lambda$ is well defined nondecreasing function on $[\alpha, \beta]$ and we can apply Theorem 6.6 b) with $M = \tilde{M}$ to obtain (6.26). □

Remark 6.2: If $\tilde{m} = 0$, then the inequality (6.25) is trivially fulfilled. Similarly, if $\tilde{M} = \infty$, then the inequality (6.26) is trivially fulfilled. Two simple examples which illustrate such cases are as follows.
For $[\alpha, \beta] = [0, 1]$ let $\lambda(t) = t^3$, $\mu(t) = t$, $t \in [0, 1]$. Then $\tilde{m} = 0$, $\tilde{M} = 3$.
For $[\alpha, \beta] = [0, 1]$ let $\lambda(t) = t$, $\mu(t) = t^3$, $t \in [0, 1]$. Then $\tilde{m} = \frac{1}{3}$, $\tilde{M} = \infty$.

As in the discrete case we can consider the uniform distribution i.e., the function $\upsilon \in \Lambda_{[\alpha,\beta]}$ defined by $\upsilon(t) := \frac{1}{\beta-\alpha}t$, $t \in [\alpha, \beta]$ and the corresponding nonweighted integral Jensen–Mercer functional

$$\mathcal{M}(f, x) := \mathcal{M}(f, x, \upsilon) = f\,(a) + f\,(b) - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x(t))\mathrm{d}t - f\left(a + b - \frac{1}{\beta-\alpha}\int_\alpha^\beta x(t)\mathrm{d}t\right).$$

Then we can state the following special case of Corollary 6.4.

Corollary 6.5: *Let λ be a function from $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$, $x : [\alpha, \beta] \to [a, b]$, $-\infty < a < b < \infty$ be a continuous function and $f : [a, b] \to \mathbb{R}$ be a convex function.*
(i) If $\tilde{m}_0 := (\beta - \alpha) \cdot \inf_{\alpha<t<\beta} \left\{ \inf \left\{ \frac{\lambda(t)-\lambda(s)}{t-s} : \alpha \le s \le \beta,\ s \ne t \right\} \right\} \ge 0$, then $\mathcal{M}(f, x, \lambda) \ge \tilde{m}_0\mathcal{M}(f, x)$.
(ii) If λ is nondecreasing and

$$\tilde{M}_0 := (\beta - \alpha) \cdot \sup_{\alpha<t<\beta} \left\{ \sup \left\{ \frac{\lambda(t) - \lambda(s)}{t - s} : \alpha \le s \le \beta,\ s \ne t \right\} \right\} < \infty,$$

then $\tilde{M}_0\mathcal{M}(f,x) \geq \mathcal{M}(f,x,\lambda)$.

Next, we give the integral version of Theorem 6.2.

Theorem 6.7: *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$, either both continuous or both of bounded variation. Let $x : [\alpha,\beta] \to [a,b]$, $-\infty < a < b < \infty$ be a monotonic function (either nondecreasing or nonincreasing) and $f : [a,b] \to \mathbb{R}$ be a convex function.*
(i) If $m \geq 0$ is a constant such that for all $\alpha < t < \beta$

$$\lambda(t) - \lambda(\alpha) - m(\mu(t) - \mu(\alpha)) \geq 0, \quad \lambda(\beta) - \lambda(t) - m(\mu(\beta) - \mu(t)) \geq 0, \tag{6.27}$$

then

$$\mathcal{M}(f,x,\lambda) \geq m\mathcal{M}(f,x,\mu). \tag{6.28}$$

(ii) If $M > 0$ is a constant such that for all $\alpha < t < \beta$

$$M(\mu(t) - \mu(\alpha)) - (\lambda(t) - \lambda(\alpha)) \geq 0, \quad M(\mu(\beta) - \mu(t)) - (\lambda(\beta) - \lambda(t)) \geq 0, \tag{6.29}$$

then

$$M\mathcal{M}(f,x,\mu) \geq \mathcal{M}(f,x,\lambda). \tag{6.30}$$

Proof: First note that under the given assumptions on λ, μ, x and f, $\mathcal{M}(f,x,\lambda)$ and $\mathcal{M}(f,x,\mu)$ are well defined and nonnegative.

(i) From (6.27) follows that it has to be $m \leq 1$. If $m = 1$ then (6.28) obviously holds with equality sign. Hence, it remains to consider the case $0 \leq m < 1$. We define the function $\rho : [\alpha,\beta] \to \mathbb{R}$ by $\rho(t) := \lambda(t) - m\mu(t)$, $t \in [\alpha,\beta]$. If λ and μ are both continuous (both of bounded variation), then ρ is continuous (of bounded variation) too. Further, from (6.27) follows that for all $\alpha < t < \beta$

$$\rho(t) - \rho(\alpha) = \lambda(t) - \lambda(\alpha) - m(\mu(t) - \mu(\alpha)) \geq 0,$$
$$\rho(\beta) - \rho(t) = \lambda(\beta) - \lambda(t) - m(\mu(\beta) - \mu(t)) \geq 0.$$

Also, since $\lambda,\mu \in \Lambda_{[\alpha,\beta]}$

$$\rho(\beta) - \rho(\alpha) = \lambda(\beta) - \lambda(\alpha) - m(\mu(\beta) - \mu(\alpha)) = 1 - m > 0.$$

We conclude that the normalized function $\frac{1}{1-m}\rho$ belongs to the class $\Lambda_{[\alpha,\beta]}$. Now, we follow our proof of Theorem 6.6 (i), but instead of using (6.16) we apply Theorem 6.5. Thus we obtain (6.28).

(ii) From (6.29) follows that it has to be $M \geq 1$. If $M = 1$ then (6.30) holds with equality sign. Hence, it remains to consider the case $M > 1$. We define the function $\sigma : [\alpha,\beta] \to \mathbb{R}$ by

$\sigma(t) := M\mu(t) - \lambda(t)$, $t \in [\alpha, \beta]$. If λ and μ are both continuous (both of bounded variation), then σ is continuous (of bounded variation) too. Further, from (6.29) follows that for all $\alpha < t < \beta$

$$\sigma(t) - \sigma(\alpha) = M(\mu(t) - \mu(\alpha)) - (\lambda(t) - \lambda(\alpha)) \geq 0,$$
$$\sigma(\beta) - \sigma(t) = M(\mu(\beta) - \mu(t)) - (\lambda(\beta) - \lambda(t)) \geq 0.$$

Also, since $\lambda,\mu \in \Lambda_{[\alpha,\beta]}$

$$\sigma(\beta) - \sigma(\alpha) = M(\mu(\beta) - \mu(\alpha)) - (\lambda(\beta) - \lambda(\alpha)) = M - 1 > 0.$$

We conclude that the normalized function $\frac{1}{M-1}\sigma$ belongs to the class $\Lambda_{[\alpha,\beta]}$. Now we follow our proof of Theorem 6.6 (ii) but instead of using (6.16) we apply Theorem 6.5. Thus we obtain (6.30). □

Corollary 6.6: *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$, either both continuous or both of bounded variation. Let $x : [\alpha, \beta] \to [a, b]$, $-\infty < a < b < \infty$ be a monotonic function (either nondecreasing or nonincreasing) and $f : [a, b] \to \mathbb{R}$ be a convex function. Assume that $\mu(\alpha) < \mu(t) < \mu(\beta)$ for all $\alpha < t < \beta$, and define*

$$\tilde{m} = \tilde{m}(\lambda, \mu) := \inf\left\{\frac{\lambda(t) - \lambda(\alpha)}{\mu(t) - \mu(\alpha)}, \frac{\lambda(\beta) - \lambda(t)}{\mu(\beta) - \mu(t)} : \alpha < t < \beta\right\},$$
$$\tilde{M} = \tilde{M}(\lambda, \mu) := \sup\left\{\frac{\lambda(t) - \lambda(\alpha)}{\mu(t) - \mu(\alpha)}, \frac{\lambda(\beta) - \lambda(t)}{\mu(\beta) - \mu(t)} : \alpha < t < \beta\right\}.$$

Then

$$\tilde{M}\mathcal{M}(f, x, \mu) \geq \mathcal{M}(f, x, \lambda) \geq \tilde{m}\mathcal{M}(f, x, \mu). \tag{6.31}$$

Proof: Since $\mu(\alpha) < \mu(t) < \mu(\beta)$ *for all* $\alpha < t < \beta$, $\tilde{m}$ and $\tilde{M}$ are well defined, and obviously $\tilde{m} \in [0, \infty)$ and $\tilde{M} \in (0, \infty]$. Therefore, the right inequality in (6.31) follows from Theorem 6.7 (i) with $m = \tilde{m}$. If $\tilde{M} = \infty$, then the left inequality in (6.31) holds trivially, while for $\tilde{M} < \infty$ it follows from Theorem 6.7 (ii) with $M = \tilde{M}$. □

As in the previous case we can consider the uniform distribution υ and the corresponding nonweighted integral Jensen–Mercer functional $\mathcal{M}(f, x)$ and state the following special case of Corollary 6.6:

Corollary 6.7: *Let λ be a function from $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$. Let $x : [\alpha, \beta] \to [a, b]$, $-\infty < a < b < \infty$ be a monotonic function (either nondecreasing or nonincreasing) and $f : [a, b] \to \mathbb{R}$ be a convex function. If $\tilde{m}_0$ and $\tilde{M}_0$ are defined by*

$$\tilde{m}_0 := (\beta - \alpha) \cdot \inf\left\{\frac{\lambda(t) - \lambda(\alpha)}{t - \alpha}, \frac{\lambda(\beta) - \lambda(t)}{\beta - t} : \alpha < t < \beta\right\},$$
$$\tilde{M}_0 := (\beta - \alpha) \cdot \sup\left\{\frac{\lambda(t) - \lambda(\alpha)}{t - \alpha}, \frac{\lambda(\beta) - \lambda(t)}{\beta - t} : \alpha < t < \beta\right\},$$

then $\tilde{M}_0\mathcal{M}(f,x) \geq \mathcal{M}(f,x,\lambda) \geq \tilde{m}_0\mathcal{M}(f,x)$.

6.3 Some Properties of Jensen–Mercer Functional

It is worth mentioning that the results of this section are extracted from [130].

6.3.1 Properties of Discrete Jensen–Mercer's Functional

In the following theorem we are concerned with proving the superadditivity property of the functional (6.1).

Theorem 6.8: *Let* $\boldsymbol{p} = (p_1, \ldots, p_n)$ *and* $\boldsymbol{q} = (q_1, \ldots, q_n)$ *be two* n*-tuples from* $\mathcal{P}_n$*: If* $f : [a,b] \to \mathbb{R}$*,* $[a,b] \subset \mathbb{R}$*, is a convex function and if* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *is an* n*-tuple in* $[a,b]^n$*, then* $\mathcal{M}_n(f, \boldsymbol{x}, \cdot)$ *defined by* (6.1) *is superadditive on* $\mathcal{P}_n$*; i.e.*

$$\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p}+\boldsymbol{q}) \geq \mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) + \mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{q}) \geq 0.$$

The functional (6.1) satisfies the monotonicity property, as is shown in the sequel.

Theorem 6.9: *Let* $\boldsymbol{p} = (p_1, \ldots, p_n)$ *and* $\boldsymbol{q} = (q_1, \ldots, q_n)$ *be two* n*-tuples from* $\mathcal{P}_n$ *such that* $\boldsymbol{p} \geq \boldsymbol{q}$ *(i.e.,* $p_i \geq q_i$*,* $i = 1, \ldots, n$*). If* $f : [a,b] \to \mathbb{R}$*,* $[a,b] \subset \mathbb{R}$*, is a convex function and if* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *is an* n*-tuple in* $[a,b]^n$*, then for functional* $\mathcal{M}_n(f, \boldsymbol{x}, \cdot)$ *defined by* (6.1)*, the following inequality holds on* $\mathcal{P}_n$*.*

$$\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) \geq \mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{q})$$

Remark 6.3: We can easily obtain the result of Theorem A from [32] by means of Theorem 6.9. Let $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}_n$ and let m and M be real constants such that $\boldsymbol{p} - m\boldsymbol{q}$ and $M\boldsymbol{q} - \boldsymbol{p}$ are in $\mathcal{P}_n$. If $f : [a,b] \to \mathbb{R}$, $[a,b] \subset \mathbb{R}$, is a convex function and if $\boldsymbol{x} = (x_1, \ldots, x_n)$ is an n-tuple in $[a,b]$, then by Theorem 6.9, $\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) \geq \mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p} - m\boldsymbol{q}) + \mathcal{M}_n(f, \boldsymbol{x}, m\boldsymbol{q}) \geq m\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{q})$. Similarly, we get $\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) \leq M\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{q})$; that is, $M\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{q}) \geq \mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) \geq m\mathcal{M}_n(f, \boldsymbol{x}, \boldsymbol{q})$.

Applying Theorem 6.9, we are able to give the result on bounding the functional (6.1) by a non-weighted functional.

Corollary 6.8: *Let* $\boldsymbol{p}, \boldsymbol{x}, f$*, and functional* M *be as in Theorem* 6.9*. Then*

$$\max_{1\leq i\leq n} f(p_i) \cdot M_{\mathcal{N}}(f, \boldsymbol{x}) \geq M(f, \boldsymbol{x}, \boldsymbol{p}) \geq \min_{1\leq i\leq n} f(p_i) \cdot M_{\mathcal{N}}(f, \boldsymbol{x}),$$

where

$$M_{\mathcal{N}}(f, \boldsymbol{x}) = [f(a) + f(b)] - \sum_{i=1}^{n} f(x_i) - nf(a+b) - \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Remark 6.4: The similar results as stated in this section can also be stated for discrete Jensen–Mercer's functional under Jensen-Steffensen's conditions. Further, similar results and remarks as stated in this section can also be stated for integral version of Jensen–Mercer functional. For details we refer the reader to [130].

CHAPTER 7

Jensen–Mercer and Jessen–Mercer Inequalities for Superquadratic Functions

This chapter explores Jensen–Mercer and Jessen–Mercer inequalities in the context of superquadratic functions. We first present variants of the Jensen–Steffensen inequality for positive superquadratic functions, highlighting refinements and structural improvements. The discussion then extends to Mercer-type variants of Jessen's inequality for superquadratic functions, illustrating how classical inequalities can be sharpened within this nonlinear framework. These results provide a foundation for further generalizations and applications in higher-order convexity settings.

As Jensen's inequality for convex functions is a generalization of Hölder's inequality for $f(x) = x^p,\ p \geq 1$, so the inequalities satisfied by superquadratic functions are generalizations of the inequalities satisfied by the superquadratic functions $f(x) = x^p,\ p \geq 2$ (see [5], [6]). The results of this section are extracted from [7]

First we quote some definitions and state a list of basic properties of superquadratic functions.

Definition 7.1: A function $f : [0, \infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C(x) \in \mathbb{R}$ such that

$$f(y) - f(x) - f(|y - x|) \geq C(x)(y - x) \tag{7.1}$$

for all $y \geq 0$.

Definition 7.2: A function $f : [0, \infty) \to \mathbb{R}$ is said to be strictly superquadratic if (7.1) is strict for all $x \neq y$ where $xy \neq 0$.

In [4] the following refinement of Jensen's Steffensen's type inequality for nonnegative superquadratic functions was proved:

Theorem 7.1: *[4, Theorem 1] Let $f : [0,\infty) \to [0,\infty)$ be differentiable and super- quadratic function, let ξ be a nonnegative monotonic m-tuple in $\mathbb{R}^m$ and $\boldsymbol{w}$ a real m-tuple, $m \geq 3$, satisfying*

$$0 \leq W_j \leq W_m \ , \ j \in \{1,2,\ldots,m\} \ , \qquad W_m > 0 \ ,$$

Let $\overline{\xi}$ be defined as $\overline{\xi} = \frac{1}{W_m}\sum_{i=1}^{m} w_i\xi_i$. Then

$$\begin{aligned}
&\sum_{i=1}^{m} w_i f(\xi_i) - W_m f(\overline{\xi}) \\
&\geq \sum_{i=1}^{k-1} W_i f(\xi_{i+1}-\xi_i) + W_k f(\overline{\xi}-\xi_k) \\
&+ \overline{W}_{k+1} f(\xi_{k+1}-\overline{\xi}) + \sum_{i=k+2}^{m} \overline{W}_i f(\xi_i - \xi_{i-1}) \\
&\geq \left[\sum_{i=1}^{k} W_i + \sum_{i=k+1}^{m} \overline{W}_i\right] f\left(\frac{\sum_{i=1}^{m} w_i\left|\overline{\xi}-\xi_i\right|}{\sum_{i=1}^{k} W_i + \sum_{i=k+1}^{m}\overline{W}_i}\right) \\
&\geq (m-1) W_m f\left(\frac{\sum_{i=1}^{m} w_i\left|\overline{\xi}-\xi_i\right|}{(m-1) W_m}\right) \quad ,
\end{aligned}$$

where $\overline{W}_i = \sum_{j=i}^{m} w_j$ and $k \in \{1,\ldots,m-1\}$ satisfies $\xi_k \leq \overline{\xi} \leq \xi_{k+1}$.

In case f is also strictly superquadratic, inequality

$$\sum_{i=1}^{m} w_i f(\xi_i) - W_m f(\overline{\xi}) > (m-1) W_m f\left(\frac{\sum_{i=1}^{m} w_i\left(\left|\xi_i-\overline{\xi}\right|\right)}{(m-1) W_m}\right)$$

holds for $\xi > \mathbf{0}$ unless one of the following two cases occurs:

(1) *either $\overline{\xi} = \xi_1$ or $\overline{\xi} = \xi_m$,*

(2) *there exists $k \in \{3,\ldots,m-2\}$ such that $\overline{\xi} = \xi_k$ and*

$$\begin{cases} W_j(\xi_j - \xi_{j+1}) = 0, & j \in \{1,\ldots,k-1\} \\ \overline{W}_j(\xi_j - \xi_{j-1}) = 0, & j \in \{k+1,\ldots,m\}. \end{cases}$$

In these two cases

$$\sum_{i=1}^{m} w_i f(\xi_i) - W_m f(\overline{\xi}) = 0.$$

7.1 Variants of Jensen-Steffensen's Inequality for Positive Superquadratic Functions

In this section we refine in two ways the Jensen–Mercer inequality (1.4) for functions which are superquadratic and positive. The refinement in Theorem 7.2 follows by showing that it is a special case of Theorem 7.1 for specific $\boldsymbol{w}$. The refinement in Theorem 7.3 follows the steps in the proof of Theorem 1.1 given by Witkowski in [242]. Therefore the second refinement is confined only to the specific $\boldsymbol{w}$ given in Theorem 1.1, which means that what we get is a variant of Jensen's inequality and not of the more general Jensen-Steffensen's inequality.

Theorem 7.2: *Let $f:[0,\infty)\to[0,\infty)$ and let $[a,b]\subseteq[0,\infty)$. Let $\boldsymbol{x}=(x_1,\dots,x_n)$ be a monotonic n-tuple in $[a,b]^n$ and $\boldsymbol{v}=(v_1,\dots,v_n)$ a real n-tuple such that $v_i\neq 0$, $i\in\{1,2,\dots,n\}$, $0\le V_j\le V_n$, $j\in\{1,2,\dots,n\}$, and $V_n>0$, where $V_j=\sum_{i=1}^{j}v_i$. If f is differentiable and superquadratic, then*

$$
\begin{aligned}
&f(a)+f(b)-\frac{1}{V_n}\sum_{i=1}^{n}v_if(x_i)-f\left(a+b-\frac{1}{V_n}\sum_{i=1}^{n}v_ix_i\right) \\
&\ge (n+1)\,f\left(\frac{b-a-\frac{1}{V_n}\sum_{i=1}^{n}v_i\left|a+b-x_i-\frac{1}{V_n}\sum_{j=1}^{n}v_jx_j\right|}{n+1}\right).
\end{aligned}
\tag{7.2}
$$

In case f is also strictly superquadratic and $a>0$, inequality (7.2) is strict unless one of the following two cases occurs:

(1) *either $\overline{x}=a$ or $\overline{x}=b$,*

(2) *there exists $l\in\{2,\dots,n-1\}$ such that $\overline{x}=x_1+x_n-x_l$ and*

$$
\begin{cases}
x_1=a,\ x_n=b \quad\text{or}\quad x_1=b,\ x_n=a \\
\overline{V}_j\,(x_{j-1}-x_j)=0\ ,\ j\in\{2,\dots,l\}, \\
V_j\,(x_j-x_{j+1})=0\ ,\ j\in\{l,\dots,n-1\},
\end{cases}
$$

where $\overline{V}_j=\sum_{i=j}^{n}v_i$, $j\in\{1,2,\dots,n\}$, and $\overline{x}=\frac{1}{V_n}\sum_{i=1}^{n}v_ix_i$.

In these two cases we have $f(a)+f(b)-\frac{1}{V_n}\sum_{i=1}^{n}v_if(x_i)-f\left(a+b-\frac{1}{V_n}\sum_{i=1}^{n}v_ix_i\right)=0.$

In the special case where $\boldsymbol{v}>0$ and f is also strictly superquadratic, the equality holds in (7.2) iff $x_i=a$, or $x_i=b$, for $i\in\{1,2,\dots,n\}$.

In the following theorem we will prove a refinement of Theorem 1.1. Without loss of generality we assume that $\sum_{i=1}^{n}v_i=1$.

Theorem 7.3: *Let $f:[0,\infty)\to[0,\infty)$ and let $[a,b]\subseteq[0,\infty)$, $a<b$. Let $\boldsymbol{x}=(x_1,\dots,x_n)$ be an n-tuple in $[a,b]^n$ and $\boldsymbol{v}=(v_1,\dots,v_n)$ a real n-tuple such that $\boldsymbol{v}\ge 0$ and $\sum_{i=1}^{n}v_i=1$. If f is superquadratic we have*

$$
\begin{aligned}
&f(a)+f(b)-\sum_{i=1}^{n}v_if(x_i)-f\left(a+b-\sum_{i=1}^{n}v_ix_i\right) \\
&\ge\sum_{i=1}^{n}v_if\left(\left|\sum_{j=1}^{n}v_jx_j-x_i\right|\right)+2\sum_{i=1}^{n}v_i\left[\frac{x_i-a}{b-a}f(b-x_i)+\frac{b-x_i}{b-a}f(x_i-a)\right] \\
&\ge\sum_{i=1}^{n}v_if\left(\left|\sum_{j=1}^{n}v_jx_j-x_i\right|\right)+2\sum_{i=1}^{n}v_if\left(\frac{2(x_i-a)(b-x_i)}{b-a}\right).
\end{aligned}
\tag{7.3}
$$

If f is strictly superquadratic and $\boldsymbol{v}>0$ equality holds in (7.3) iff $x_i=a$, or $x_i=b$, for $i\in\{1,2,\dots,n\}$.

Corollary 7.1: *Let $\boldsymbol{v} = (v_1, \ldots, v_n)$ be a real n-tuple such that $\boldsymbol{v} \geq 0$, $\sum_{i=1}^{n} v_i = 1$ and let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an n-tuple in $[a, b]^n$, $0 < a < b$. Then for any real numbers r and s such that $\frac{s}{r} \geq 2$ we have*

$$\begin{aligned}
\left(\frac{M_s(a,b;\boldsymbol{x})}{M_r(a,b;\boldsymbol{x})}\right)^s - 1 &\geq \frac{1}{M_r(a,b;\boldsymbol{x})^s}\left[\sum_{i=1}^{n} v_i \left|\sum_{j=1}^{n} v_j x_j^r - x_i^r\right|^{\frac{s}{r}}\right. \\
&\left. +2\sum_{i=1}^{n} v_i \left(\frac{x_i^r - a^r}{b^r - a^r}\left(b^r - x_i^r\right)^{\frac{s}{r}} + \frac{b^r - x_i^r}{b^r - a^r}\left(x_i^r - a^r\right)^{\frac{s}{r}}\right)\right] \qquad (7.4) \\
&\geq \frac{1}{M_r(a,b;\boldsymbol{x})^s}\left[\sum_{i=1}^{n} v_i \left|\sum_{j=1}^{n} v_j x_j^r - x_i^r\right|^{\frac{s}{r}} + 2\sum_{i=1}^{n} v_i \left(\frac{2\left(x_i^r - a^r\right)\left(b^r - x_i^r\right)}{b^r - a^r}\right)^{\frac{s}{r}}\right],
\end{aligned}$$

where $M_p(a,b;\boldsymbol{x}) = \left(a^p + b^p - \sum_{i=1}^{n} v_i x_i\right)^{\frac{1}{p}}$, $p \in \mathbb{R} \setminus \{0\}$.
If $\frac{s}{r} > 2$ and $\boldsymbol{v} > 0$ the equalities hold in (7.4) iff $x_i = a$, or $x_i = b$, $i \in \{1, \ldots, n\}$.

Remark 7.1: It is a immediate result of Corollary 1 that if $\frac{s}{r} > 2$ *and there is at least one* $j \in \{1, \ldots, n\}$ *such that* $v_j\left(x_j^r - a^r\right)\left(b^r - x_j^r\right) > 0$ *then for this j we have*

$$\left(\frac{M_s(a,b;\boldsymbol{x})}{M_r(a,b;\boldsymbol{x})}\right)^s - 1 > \frac{2v_j}{M_r(a,b;\boldsymbol{x})^s}\left(\frac{2\left(x_j^r - a^r\right)\left(b^r - x_j^r\right)}{b^r - a^r}\right)^{\frac{s}{r}} > 0.$$

7.2 Variants of Jessen–Mercer Inequality for Superquadratic Functions

The results of this section are selected from [9].

Here we give the main result of this section which is an analogue of Theorem 1.6 for superquadratic functions.

Theorem 7.4: *Let L satisfy properties $L1$ and $L2$, on a nonempty set E, $\varphi : [0,1) \to \mathbb{R}$ be a continuous superquadratic function, and $0 \leq m < M < 1$. Assume that A is an isotonic linear functional on L with $A(\mathbf{1}) = 1$. If $g \in L$ is such that $m \leq g(t) \leq M$, for all $t \in E$, and such that $\varphi(g)$, $\varphi(m + M - g)$, $(M - g)\varphi(g - m)$, $(g - m)\varphi(M - g) \in L$, then we have*

$$\begin{aligned}
\varphi(m + M - A(g)) &\leq \frac{A(g) - m}{M - m}\varphi(m) + \frac{M - A(g)}{M - m}\varphi(M) \\
&\quad - \frac{1}{M - m}\Big[(A(g) - m)\varphi(M - A(g)) + (M - A(g))\varphi(A(g) - m)\Big] \\
&\leq \varphi(m) + \varphi(M) - A(\varphi(g)) \qquad (7.5) \\
&\quad - \frac{1}{M - m}A\Big((g - m)\varphi(M - g) + (M - g)\varphi(g - m)\Big) \\
&\quad - \frac{1}{M - m}\Big[(A(g) - m)\varphi(M - A(g)) + (M - A(g))\varphi(A(g) - m)\Big].
\end{aligned}$$

If the function φ is subquadratic, then all the inequalities above are reversed.

Proof: See [9]. □

Remark 7.2: If a function φ is superquadratic and nonnegative, then it is convex [5, Lemma 2.2]. Hence, in this case inequality (7.5) is a refinement of inequality (1.6).

On the other hand, we can get one more inequality in (7.5) if we use a result of S. Banić and S. Varošanec [31] on Jessen's inequality for superquadratic functions:

Theorem 7.5: *([31, Theorem 8, Remark 1]). Let L satisfy properties $L1$, $L2$, on a nonempty set E, and let $\varphi : [0,1) \to \mathbb{R}$ be a continuous superquadratic function. Assume that A is an isotonic linear functional on L with $A(\mathbf{1}) = 1$. If $f \in L$ is nonnegative and such that $\varphi(f)$, $\varphi(|f - A(f)|) \in L$, then we have*

$$\varphi(A(f)) \leq A(\varphi(f)) - A(\varphi(|f - A(f)|)).$$

If the function φ is subquadratic, then the inequality above is reversed.

Using Theorem 7.5 and some basic properties of superquadratic functions we can prove the next theorem.

Theorem 7.6: *Let L be a linear class of functions satisfy properties $L1$, $L2$ on a non-empty set E, and let $\varphi : [0,1) \to \mathbb{R}$ be a continuous superquadratic function and let $0 \leq a < b < \infty$. Assume that A is positive linear functional on L with $A(\mathbf{1}) = 1$. If $f \in L$ such that $a \leq f(t) \leq b$ for all $t \in E$ and such that $\varphi(f)$, $\varphi(a+b-f)$, $(b-f)\varphi(f-a)$, $(f-a)\varphi(b-f)$, $\varphi(|f - A(f)|) \in L$, then we have*

$$\begin{aligned}
&\varphi(a+b-A(f)) \\
&\leq A(\varphi(a+b-f)) - \varphi(|f-A(f)|) \\
&\leq \frac{b-A(f)}{b-a}\varphi(b) + \frac{A(f)-a}{b-a}\varphi(a) \\
&- \frac{1}{b-a}A((b-f)\varphi(f-a) + (f-a)\varphi(b-f)) - \varphi(|f-A(f)|) \\
&\leq \varphi(a) + \varphi(b) - A(\varphi(f)) \\
&- \frac{2}{b-a}A((b-f)\varphi(f-a) + (f-a)\varphi(b-f)) - \varphi(|f-A(f)|).
\end{aligned}$$

If the function φ is subquadratic, then all inequality stated above are reversed.

Proof: See [9]. □

Remark 7.3: Some applications and related remarks can be found in [9].

Remark 7.4: For latest work on Jessen–Mercer inequality for superquadratic functions and related applications we refer the reader to [11].

CHAPTER 8

Jensen–Mercer Inequality for Multivariate Functions

This chapter examines Jensen–Mercer inequalities in the multivariate setting. We begin with inequalities for P-convex functions and functions with nondecreasing increments, followed by generalizations of Čebyšev's and Hölder's inequalities. The chapter then considers inequalities for generalized quasi-arithmetic means and presents refinements using index set functions. Finally, refinements of both Čebyšev's and Hölder's inequalities are discussed, highlighting sharper bounds and structural improvements in the multivariate context.

8.1 Jensen–Mercer Inequalities for Functions with Nondecreasing Increments and P–Convex Functions

In this section we give Jensen–Mercer inequalities for P-convex functions and functions with nondecreasing increments and consider some of their applications. We use them to obtain variants of Čebyšev's inequality, Hölder's inequality for monotone sequences in which weights satisfy conditions as in Jensen-Steffensen's inequality, Beck's inequalities for generalized quasi-arithmetic means, and another generalization of Hölder's inequality and other inequalities involving quasi-arithmetic means. Most of the content of the present chapter can be found in [169, 25, 26].

For the sake of clarity, for the use of some substitutions, we rewrite Theorem 1.4 for weights $\boldsymbol{p}$.

Theorem 8.1: *Let I be an interval in $\mathbb{R}$, and let $[a,b] \subseteq I$, $a < b$. Let $\boldsymbol{x}$ be a monotone n-tuple in $[a,b]^n$ and let $\boldsymbol{p}$ be a real n-tuple such that*

$$0 \leq P_k \leq P_n,\ k \in \{1,\ldots,n-1\},\ P_n > 0, \tag{8.1}$$

where $P_k = \sum_{i=1}^{k} p_i$*. If function* $f : I \to R$ *is convex, then*

$$f\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le f(a)+f(b)-\frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i\right)$$

holds.

The Jensen–Mercer inequality for P-convex functions is given in the following theorem.

Theorem 8.2: *Let* $\boldsymbol{x} \in [a,b]^n$ *and* $\boldsymbol{y} \in [c,d]^n$ *be two real* n*-tuples monotone in the same direction, and let* $\boldsymbol{p}$ *be a real* n*-tuple such that conditions (8.1) are satisfied. Then for any* P*-convex function* $\varphi : [a,b] \times [c,d] \to \mathbb{R}$ *the following inequality holds:*

$$\varphi\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, c+d-\frac{1}{P_n}\sum_{i=1}^{n} p_i y_i\right) \le \varphi\left(a,c\right)+\varphi\left(b,d\right)-\frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi\left(x_i, y_i\right). \tag{8.2}$$

If function φ *is* P*-concave the inequality in (8.2) is reversed.*

The Jensen–Mercer inequality for functions with nondecreasing increments is given in the next theorem.

Theorem 8.3: *Let* $f : K \to \mathbb{R}$*, where* $K = [a_1, b_1] \times \cdots \times [a_k, b_k] \subset \mathbb{R}^k$*, be a continuous function with nondecreasing increments, let* $\boldsymbol{p}$ *be a real* n*-tuple such that (8.1) is satisfied. Then for any* $\boldsymbol{x}^{(i)} \in K$ $(i \in \{1, 2, \ldots, n\})$ *such that*

$$\boldsymbol{x}^{(1)} \le \cdots \le \boldsymbol{x}^{(n)} \quad or \quad \boldsymbol{x}^{(1)} \ge \cdots \ge \boldsymbol{x}^{(n)},$$

inequality

$$f\left(\boldsymbol{a}+\boldsymbol{b}-\frac{1}{P_n}\sum_{i=1}^{n} p_i \boldsymbol{x}^{(i)}\right) \le f\left(\boldsymbol{a}\right)+f\left(\boldsymbol{b}\right)-\frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(\boldsymbol{x}^{(i)}\right),$$

holds, where $\boldsymbol{a} = (a_1, \ldots, a_k)$*,* $\boldsymbol{b} = (b_1, \ldots, b_k)$*.*

Remark 8.1: Interested readers can find new refinements of Jensen–Mercer type inequalities for functions with nondecreasing increments in [92].

8.1.1 Generalizations of Čebyšev's and Hölder's Inequality

First, we give an alternative proof of Theorem 5.14 using the Jensen-Steffensen's inequality for P-convex functions.

Theorem 8.4: *Let* $\boldsymbol{w}$ *be a real* m*-tuple such that (A.15) is satisfied. Then for any real* m*-tuples* $\boldsymbol{x}$*,* $\boldsymbol{y}$ *monotone in the same direction,*

$$\sum_{i=1}^{m} w_i x_i \sum_{i=1}^{m} w_i y_i \le W_m \sum_{i=1}^{m} w_i x_i y_i \tag{8.3}$$

holds. If $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are monotone in opposite directions the inequality in (8.3) is reversed.*

Proof: The function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x,y) = xy$ is P-convex on $\mathbb{R}$. If m-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in the same direction, we can apply Theorem A.20 on φ and on the m-tuples $\boldsymbol{\xi} = \boldsymbol{x}$, $\boldsymbol{\eta} = \boldsymbol{y}$ and $\boldsymbol{w}$, after which we immediately obtain (8.3).

If m-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in opposite directions, we can apply Theorem A.20 on the P-concave function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined as $\varphi(x,y) = -xy$ and on the m-tuples $\boldsymbol{\xi} = \boldsymbol{x}$, $\boldsymbol{\eta} = -\boldsymbol{y}$ (now monotone in the same direction) and $\boldsymbol{w}$ to obtain reversed (8.3). □

Theorem 8.5: *Let $\boldsymbol{p}$ be a real n-tuple such that (8.1) is satisfied. Then for any real n-tuples $\boldsymbol{x} \in [a,b]^n$ and $\boldsymbol{y} \in [c,d]^n$ monotone in the same direction,*

$$\left(a + b - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)\left(c + d - \frac{1}{P_n}\sum_{i=1}^{n} p_i y_i\right) \leq ac + bd - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i y_i. \tag{8.4}$$

holds. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in opposite directions the inequality in (8.4) is reversed.

Proof: Let n-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ be monotone in the same direction. In order to obtain (8.4) we simply apply Theorem 8.2 on the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined as in Theorem 8.4 and on the n-tuples $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{p}$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in opposite directions we apply Theorem 8.2 on the P-concave function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined as $\varphi(x,y) = -xy$ and on the n-tuples $\boldsymbol{x}$, $-\boldsymbol{y} \in [-d,-c]^n$ and $\boldsymbol{p}$, in which case we obtain the reversed (8.4). □

Theorem 8.6: *Let $\boldsymbol{w}$ be a real m-tuple such that (A.15) is satisfied and let $\boldsymbol{p} \in \langle -\infty, 0\rangle^k \cup \langle 0, 1]^k$. Then for any $\boldsymbol{x}^{(i)} \in K \subset \langle 0, +\infty\rangle^k$, $i \in \{1,2,\ldots,m\}$ such that*

$$\boldsymbol{x}^{(1)} \leq \cdots \leq \boldsymbol{x}^{(m)} \quad \text{or} \quad \boldsymbol{x}^{(1)} \geq \cdots \geq \boldsymbol{x}^{(m)},$$

inequality

$$\prod_{i=1}^{k}\left(\frac{1}{W_m}\sum_{j=1}^{m} w_j \left(x_i^{(j)}\right)^{p_i}\right)^{\frac{1}{p_i}} \leq \frac{1}{W_m}\sum_{j=1}^{m} w_j \prod_{i=1}^{k} x_i^{(j)} \tag{8.5}$$

holds.

Proof: We define function $\varphi : \langle 0, +\infty\rangle^k \to \mathbb{R}$ as

$$\varphi(\boldsymbol{x}) = \prod_{i=1}^{k} x_i^{\frac{1}{p_i}}, \quad p_i \in \mathbb{R}\setminus\{0\}, \; i \in \{1,2,\ldots,k\}. \tag{8.6}$$

We have

$$\varphi_{x_i x_i}(\boldsymbol{x}) = \frac{1}{p_i}\left(\frac{1}{p_i} - 1\right) x_i^{-2}\prod_{j=1}^{k} x_j^{\frac{1}{p_j}}, \quad i \in \{1,2,\ldots,k\},$$

$$\varphi_{x_i x_j}(\boldsymbol{x}) = \frac{1}{p_i}\frac{1}{p_j} x_i^{-1} x_j^{-1}\prod_{j=1}^{k} x_j^{\frac{1}{p_j}}; \quad i,j \in \{1,2,\ldots,k\}, \; i \neq j,$$

function φ has nondecreasing increments iff p_i $(i \in \{1,2,\ldots,k\})$ are all negative or if p_i for $i \in \{1,2,\ldots,k\}$ are all positive and not greater then 1. Suppose that $\boldsymbol{x}^{(1)} \leq \cdots \leq \boldsymbol{x}^{(m)}$ and $\boldsymbol{p} \in \langle -\infty, 0\rangle^k \cup \langle 0,1]^k$. We define k-tuples $\boldsymbol{\xi}^{(j)} \in \langle 0,+\infty\rangle^k$ $(j \in \{1,2,\ldots,m\})$ as

$$\boldsymbol{\xi}_i^{(j)} = \left(\boldsymbol{x}_i^{(j)}\right)^{p_i}, \quad i \in \{1,2,\ldots,k\}.$$

Obviously, if $\boldsymbol{p} \in \langle -\infty, 0\rangle^k$ then $\boldsymbol{\xi}^{(1)} \geq \cdots \geq \boldsymbol{\xi}^{(m)}$, and if $\boldsymbol{p} \in \langle 0,1]^k$ then $\boldsymbol{\xi}^{(1)} \leq \cdots \leq \boldsymbol{\xi}^{(m)}$. So, in both cases we can apply Theorem A.21 on $\boldsymbol{w}$, $\boldsymbol{\xi}^{(1)},\ldots,\boldsymbol{\xi}^{(m)}$ and on the function φ to obtain (8.5).

If $\boldsymbol{x}^{(1)} \geq \cdots \geq \boldsymbol{x}^{(m)}$ proof is similar. $\square$

Remark 8.2: If in Theorem 8.6 we let $k=2$, *then for* $(p,q) \in \langle -\infty,0\rangle^2 \cup \langle 0,1]^2$, a real m*-tuple* $\boldsymbol{w}$ *such that* (*A*.15) *is satisfied* and any positive m-tuples $\boldsymbol{x}$, $\boldsymbol{y}$ monotone in the same direction we obtain

$$\left(\frac{1}{W_m}\sum_{i=1}^m w_i x_i^p\right)^{\frac{1}{p}} \left(\frac{1}{W_m}\sum_{i=1}^m w_i y_i^q\right)^{\frac{1}{q}} \leq \frac{1}{W_m}\sum_{i=1}^m w_i x_i y_i.$$

Theorem 8.7: *Let $K = [a_1,b_1] \times \cdots \times [a_k,b_k] \subset \langle 0,+\infty\rangle^k$, let $\boldsymbol{w}$ be a real n-tuple such that* (*A*.15) *is satisfied and let $\boldsymbol{p} \in \langle -\infty,0\rangle^k \cup \langle 0,1]^k$. Then for any $\boldsymbol{x}^{(i)} \in K$ $(i \in \{1,2,\ldots,n\})$ such that*

$$\boldsymbol{x}^{(1)} \leq \cdots \leq \boldsymbol{x}^{(n)} \quad \text{or} \quad \boldsymbol{x}^{(1)} \geq \cdots \geq \boldsymbol{x}^{(n)},$$

inequality

$$\prod_{i=1}^k \left(a_i^{p_i} + b_i^{p_i} - \frac{1}{W_n}\sum_{j=1}^n w_j \left(x_i^{(j)}\right)^{p_i}\right)^{\frac{1}{p_i}} \leq \prod_{i=1}^k a_i + \prod_{i=1}^k b_i - \frac{1}{W_n}\sum_{j=1}^n w_j \prod_{i=1}^k x_i^{(j)}$$

holds.

Proof: Suppose that $\boldsymbol{x}^{(1)} \leq \cdots \leq \boldsymbol{x}^{(n)}$ and $\boldsymbol{p} \in \langle -\infty,0\rangle^k \cup \langle 0,1]^k$. We define k-tuples $\boldsymbol{\xi}^{(j)} \in \langle 0,+\infty\rangle^k$, $j \in \{1,\ldots,n\}$ as

$$\boldsymbol{\xi}_i^{(j)} = \left(\boldsymbol{x}_i^{(j)}\right)^{p_i}, \quad i \in \{1,2,\ldots,k\}.$$

Obviously, if $\boldsymbol{p} \in \langle -\infty,0\rangle^k$ then $\boldsymbol{\xi}^{(1)} \geq \cdots \geq \boldsymbol{\xi}^{(n)}$ and $\boldsymbol{\xi}^{(i)} \in [b_1^{p_1}, a_1^{p_1}] \times \cdots \times \left[b_k^{p_k}, a_k^{p_k}\right]$ $(i \in \{1,2,\ldots,n\})$, and if $\boldsymbol{p} \in \langle 0,1]^k$ then $\boldsymbol{\xi}^{(1)} \leq \cdots \leq \boldsymbol{\xi}^{(n)}$ and $\boldsymbol{\xi}^{(i)} \in [a_1^{p_1}, b_1^{p_1}] \times \cdots \times \left[a_k^{p_k}, b_k^{p_k}\right]$ $(i \in \{1,2,\ldots,n\})$. In both cases we can apply Theorem 8.3 on the function φ defined as in (8.6) to obtain desired inequality.

If $\boldsymbol{x}^{(1)} \geq \cdots \geq \boldsymbol{x}^{(n)}$ proof is similar. $\square$

8.1.2 Inequalities for Generalized Quasi-Arithmetic Means

Let I_M be an interval in $\mathbb{R}$ and suppose that $M : I_M \to \mathbb{R}$ is a continuous and strictly monotone function. Let $\boldsymbol{x} \in I_M^m$ be a monotonic m-tuple and let $\boldsymbol{w}$ *be a* real m-tuple *such that* ($A.15$) *is satisfied.* Then the generalized quasy-arithmetic mean of $\boldsymbol{x}$ with weights $\boldsymbol{w}$ is defined as

$$M_m(\boldsymbol{x};\boldsymbol{w}) = M^{-1}\left(\frac{1}{W_m}\sum_{i=1}^{m} w_i M(x_i)\right).$$

Clearly,

$$M_m(\boldsymbol{x};\boldsymbol{w}) = M^{-1}\left(A_m\left(M(\boldsymbol{x});\boldsymbol{w}\right)\right),$$

where A_m denotes the arithmetic mean and $M(\boldsymbol{x}) = (M(x_1),\ldots,M(x_m))$. Since M is continuous and strictly monotone we know that $M(I_M)$ is an interval in $\mathbb{R}$. From Remark 1.1 we also know that if (for example) $\boldsymbol{x}$ and M are increasing, then $\frac{1}{W_m}\sum_{i=1}^{m} w_i M(x_i) \in [M(x_1), M(x_m)]$, i. e., $M^{-1}(\frac{1}{W_m} \times \sum_{i=1}^{m} w_i M(x_i))$ is from $[x_1, x_m]$, which means that $M_m(\boldsymbol{x};\boldsymbol{w})$ is well defined.

Comparison theorems for such generalized quasy-arithmetic means are given in [8].

Remark 8.3: Using the above notation the inequality ($A.16$) may be rewritten as

$$\varphi\left(A_m(\boldsymbol{\xi};\boldsymbol{w}), A_m(\boldsymbol{\eta};\boldsymbol{w})\right) \leq A_m\left(\varphi(\boldsymbol{\xi};\boldsymbol{\eta});\boldsymbol{w}\right),$$

where $\varphi(\boldsymbol{\xi};\boldsymbol{\eta}) = (\varphi(\xi_1,\eta_1),\ldots,\varphi(\xi_m,\eta_m))$.

Theorem 8.8: *Let I_M, I_L, I_K be intervals in $\mathbb{R}$, let $M : I_M \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $K : I_K \to \mathbb{R}$ be strictly monotone and continuous functions, where K and L are monotone in the same direction, and let $f : I_K \times I_L \to I_M$ be a continuous function. Let $\boldsymbol{w}$ be a real m-tuple such that* ($A.15$) *is satisfied, and let function $H : K(I_K) \times L(I_L) \to \mathbb{R}$ be defined as*

$$H(x,y) = M\left(f\left(K^{-1}(x), L^{-1}(y)\right)\right).$$

If H is P-convex (P-concave) and M is increasing (decreasing), then for any m-tuples $\boldsymbol{x} \in I_K^m$ and $\boldsymbol{y} \in I_L^m$ monotone in the same direction

$$f\left(K_m(\boldsymbol{x};\boldsymbol{w}), L_m(\boldsymbol{y};\boldsymbol{w})\right) \leq M_m\left(f(\boldsymbol{x},\boldsymbol{y});\boldsymbol{w}\right). \tag{8.7}$$

If H is P-convex (P-concave) and M is decreasing (increasing) the inequality in (8.7) *is reversed.*

Proof: Since K and L are monotone in the same direction, the m-tuples $K(\boldsymbol{x})$ and $L(\boldsymbol{y})$ remain monotone in the same direction. If we apply Theorem A.20 on the P-convex function H and on the m-tuples $\boldsymbol{\xi} = K(\boldsymbol{x})$, $\boldsymbol{\eta} = L(\boldsymbol{y})$ and $\boldsymbol{w}$, we obtain

$$\begin{aligned} H\left(A_m(K(\boldsymbol{x});\boldsymbol{w}), A_m(L(\boldsymbol{y});\boldsymbol{w})\right) &\leq A_m\left(H(K(\boldsymbol{x}), L(\boldsymbol{y}));\boldsymbol{w}\right) \\ &= A_m\left(M(f(\boldsymbol{x},\boldsymbol{y}));\boldsymbol{w}\right). \end{aligned}$$

From this we have $M\left(f\left(K_m\left(\boldsymbol{x};\boldsymbol{w}\right),L_m\left(\boldsymbol{y};\boldsymbol{w}\right)\right)\right)\leq A_m\left(M\left(f\left(\boldsymbol{x},\boldsymbol{y}\right)\right);\boldsymbol{w}\right).$

If M is increasing (8.7) immediately follows, and if M is decreasing (8.7) is reversed.

If H is P-concave proof is similar. □

The inequality (8.7) is interesting because making different choices of function H we may obtain variants of other well known inequalities in which weights $\boldsymbol{w}$ satisfy conditions (A.15).

Denote by $A=\dfrac{K''}{K'},\quad B=\dfrac{L''}{L'},\quad C=\dfrac{M''}{M'}.$

Theorem 8.9: *Let K, L, M and $\boldsymbol{w}$ be as in Theorem 8.8 and suppose that derivatives K', K'', L', L'', M', M'' exist. If M is strictly increasing (decreasing) convex (concave) function and if*

$$\begin{aligned}A\left(x\right)&\leq C\left(x+y\right),\\ B\left(y\right)&\leq C\left(x+y\right),\end{aligned}\tag{8.8}$$

for all $x\in I_K$, $y\in I_L$, then for any m-tuples $\boldsymbol{x}\in I_K^m$ and $\boldsymbol{y}\in I_L^m$ monotone in the same direction

$$K_m\left(\boldsymbol{x};\boldsymbol{w}\right)+L_m\left(\boldsymbol{y};\boldsymbol{w}\right)\leq M_m\left(\boldsymbol{x}+\boldsymbol{y};\boldsymbol{w}\right)\tag{8.9}$$

holds. If M is strictly decreasing (increasing) convex (concave) function and if reversed inequalities (8.8) hold for all $x\in I_K$, $y\in I_L$, the inequality in (8.9) is reversed.

Proof: Let $f:I_K\times I_L\to I_M$ be defined as $f\left(x,y\right)=x+y$. Then the function H from Theorem 8.8 is defined as $H\left(x,y\right)=M\left(K^{-1}\left(x\right)+L^{-1}\left(y\right)\right)$. The condition that function H is P-convex is for such K, L and M equivalent to the conditions

$$\begin{aligned}&M''\left(x+y\right)\geq M'\left(x+y\right)\frac{K''\left(x\right)}{K'\left(x\right)},\\ &M''\left(x+y\right)\geq 0,\\ &M''\left(x+y\right)\geq M'\left(x+y\right)\frac{L''\left(y\right)}{L'\left(y\right)},\end{aligned}$$

for all $x\in I_K,\ y\in I_L$. If M is strictly increasing (decreasing) function the conditions (8.8) (respectively reversed (8.8)) immediately follow, and M obviously must be convex. In either case the inequality (8.9) (its reverse) follows from Theorem 8.8.

If H is P-concave proof is similar. □

Corollary 8.1: *Let $p\geq 1$ and let $\boldsymbol{w}$ be a real m-tuple such that (A.15) is satisfied. Then for any positive m-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ monotone in the same direction*

$$G_m\left(\boldsymbol{x};\boldsymbol{w}\right)+G_m\left(\boldsymbol{y};\boldsymbol{w}\right)\leq M_m^{[p]}\left(\boldsymbol{x}+\boldsymbol{y};\boldsymbol{w}\right),\tag{8.10}$$

holds, where G_m denotes geometric mean and $M_m^{[p]}$ power mean of order p.

Proof: Let functions K, L and M from Theorem 8.9 be defined as

$$K(x) = L(x) = \ln x,\ M(x) = x^p,$$

for all $x \in \langle 0, +\infty \rangle$, where p is a real number. Then

$$A(x) = B(x) = -\frac{1}{x},\quad C(x) = \frac{p-1}{x},$$

and the conditions (8.8) become

$$-\frac{1}{x} \le \frac{p-1}{x+y}, \text{and} \quad -\frac{1}{y} \le \frac{p-1}{x+y}, \tag{8.11}$$

for all $x, y \in \langle 0, +\infty \rangle$. We must consider three cases:

1) If $p \ge 1$ (M is convex and strictly increasing) then the conditions (8.11) are satisfied, so from Theorem 8.9 we know that for any positive m-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ monotone in the same direction inequality

$$\begin{aligned}
&\exp\left(\frac{1}{W_m}\sum_{i=1}^{m} w_i \ln x_i\right) + \exp\left(\frac{1}{W_m}\sum_{i=1}^{m} w_i \ln y_i\right) \\
&= \left(\prod_{i=1}^{m} x_i^{w_i}\right)^{\frac{1}{W_m}} + \left(\prod_{i=1}^{m} y_i^{w_i}\right)^{\frac{1}{W_m}} \le \left(\frac{1}{W_m}\sum_{i=1}^{m} w_i (x_i + y_i)^p\right)^{\frac{1}{p}}
\end{aligned}$$

holds, so (8.10) immediately follows.

2) If $0 < p < 1$ (M is concave and strictly increasing) then reversed conditions (8.11) can not be satisfied.

3) If $p < 0$ (M is convex and strictly decreasing) then reversed conditions (8.11) can not be satisfied again. □

Corollary 8.2: *Let $p \ge 1$ and let $\boldsymbol{w}$ be a real m-tuple such that (A.15) is satisfied. Then for any positive m-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ monotone in the same direction,*

$$H_m(\boldsymbol{x}; \boldsymbol{w}) + H_m(\boldsymbol{y}; \boldsymbol{w}) \le M_m^{[p]}(\boldsymbol{x} + \boldsymbol{y}; \boldsymbol{w}),$$

holds, where H_m denotes harmonic mean.

Proof: Let functions K, L and M from Theorem 8.9 be defined as

$$K(x) = L(x) = \frac{1}{x},\ M(x) = x^p,$$

for all $x \in \langle 0, +\infty \rangle$, where p is a real number. Then

$$A(x) = B(x) = -\frac{2}{x},\quad C(x) = \frac{p-1}{x},$$

and the conditions (8.8) become

$$-\frac{2}{x} \leq \frac{p-1}{x+y}, \qquad -\frac{2}{y} \leq \frac{p-1}{x+y},$$

for all $x, y \in \langle 0, +\infty \rangle$. Very similar analysis as in Corollary 8.1 leads us to the conclusion that inequality

$$\frac{W_m}{\sum\limits_{i=1}^{m} \frac{w_i}{x_i}} + \frac{W_m}{\sum\limits_{i=1}^{m} \frac{w_i}{y_i}} \leq \left(\frac{1}{W_m} \sum_{i=1}^{m} w_i (x_i + y_i)^p \right)^{\frac{1}{p}}$$

holds for all positive m-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ monotone in the same direction only if $p \geq 1$. □

Theorem 8.10: *Let K, L, M and $\boldsymbol{w}$ be as in Theorem 8.8 and suppose that derivatives K', K'', L', L'', M', M'' exist. If M is strictly increasing (decreasing) and if inequalities*

$$\begin{aligned} C(xy)\, xy &\geq -1, \\ y\left[C(xy)\, y - A(x)\right] &\geq 0, \\ x\left[C(xy)\, x - B(y)\right] &\geq 0, \end{aligned} \tag{8.12}$$

(or respectively the reversed (8.12)) hold for all $x \in I_K$, $y \in I_L$, then for any m-tuples $\boldsymbol{x} \in I_K^m$ and $\boldsymbol{y} \in I_L^m$ monotone in the same direction

$$K_m(\boldsymbol{x}; \boldsymbol{w})\, L_m(\boldsymbol{y}; \boldsymbol{w}) \leq M_m(\boldsymbol{xy}; \boldsymbol{w}), \tag{8.13}$$

where $\boldsymbol{xy} = (x_1 y_1, \ldots, x_m y_m)$.
If M is strictly decreasing (increasing) and if inequalities (8.12) (or respectively the reversed (8.12)) hold for all $x \in I_K$, $y \in I_L$, the inequality in (8.13) is reversed.

Proof: Let $f : I_K \times I_L \to I_M$ be defined as $f(x, y) = xy$. Then H from Theorem 8.8 is defined as $H(x, y) = M\left(K^{-1}(x)\, L^{-1}(y)\right)$. The condition that the function H is P-convex is for such K, L and M equivalent to the conditions

$$\begin{aligned} M''(xy)\, xy + M'(xy) &\geq 0, \\ y\left[M''(xy)\, y - M'(xy)\frac{K''(x)}{K'(x)}\right] &\geq 0, \\ x\left[M''(xy)\, x - M'(xy)\frac{L''(y)}{L'(y)}\right] &\geq 0, \end{aligned}$$

for all $x \in I_K$, $y \in I_L$.

The rest of the proof is similar as in Theorem 8.9 □

Corollary 8.3: *Let p, q and r be real numbers such that $pq > 0$, and let $\boldsymbol{w}$ be a real m-tuple such that (A.15) is satisfied. If $r > 0$ and $r \geq \max\{p, q\}$, then for any positive m-tuples $\boldsymbol{x}$ and*

$\boldsymbol{y}$ monotone in the same direction inequality

$$\left(\frac{1}{W_m}\sum_{i=1}^{m} w_i x_i^p\right)^{\frac{1}{p}} \left(\frac{1}{W_m}\sum_{i=1}^{m} w_i y_i^q\right)^{\frac{1}{q}} \leq \left(\frac{1}{W_m}\sum_{i=1}^{m} w_i x_i^r y_i^r\right)^{\frac{1}{r}} \tag{8.14}$$

holds. If $r<0$ and $r \leq \min\{p,q\}$ the inequality in (8.14) is reversed.

Proof: We define functions $K,\ L,\ M : \langle 0,+\infty\rangle \to \mathbb{R}$ as

$$K(x) = x^p,\ L(x) = x^q,\ M(x) = x^r.$$

In this case the function M is strictly increasing if $r>0$ and strictly decreasing if $r<0$.

If $r>0$ the conditions (8.12) from Theorem 8.10 are equivalent to the conditions $r \geq p$ and $r \geq q$, and if $r<0$ to the conditions $r \leq p$ and $r \leq q$. In both cases the inequality (8.14) (its reversal) follows from Theorem 8.10. □

Remark 8.4: Inequality (8.14) can be rewritten as

$$M_m^{[p]}(\boldsymbol{x};\boldsymbol{w})\, M_m^{[q]}(\boldsymbol{y};\boldsymbol{w}) \leq M_m^{[r]}(\boldsymbol{xy};\boldsymbol{w}).$$

In paper [8] another variant of generalized quasi-arithmetic means was investigated. Let $I_M \subset \mathbb{R}$ and suppose that function $M : I_M \to \mathbb{R}$ is continuous and strictly monotonic. For $[m_1,m_2] \subset I_M$ let $\boldsymbol{x} \in [m_1,m_2]^n$ be a monotonic n-tuple and let $\boldsymbol{p}$ be a real n-tuple such that (8.1) is satisfied. We define the generalized quasy-arithmetic mean of Mercer's typo of $\boldsymbol{x}$ with weights $\boldsymbol{p}$ as

$$\widetilde{M}_n(m_1;m_2;\boldsymbol{x};\boldsymbol{p}) = M^{-1}\left(M(m_1)+M(m_2)-\frac{1}{P_n}\sum_{i=1}^{n} p_i M(x_i)\right).$$

Theorem 8.11: *Let $M : I_M \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $K : I_K \to \mathbb{R}$, $I_M, I_L, I_K \subset \mathbb{R}$, be strictly monotone and continuous functions, where K and L are monotone in the same direction, and let $f : I_K \times I_L \to I_M$ be a continuous function. Let $\boldsymbol{p}$ be a real n-tuple such that (8.1) is satisfied and let function $H : K(I_K) \times L(I_L) \to \mathbb{R}$ be defined as*

$$H(x,y) = M\left(f\left(K^{-1}(x), L^{-1}(y)\right)\right).$$

If H is P-convex (P-concave) and M is increasing (decreasing), then for any n-tuples $\boldsymbol{x} \in [k_1,k_2]^n \subset I_K^n$ and $\boldsymbol{y} \in [l_1,l_2]^n \subset I_L^n$ monotone in the same direction inequality

$$f\left(\widetilde{K}_n(k_1;k_2;\boldsymbol{x};\boldsymbol{p}), \widetilde{L}_n(l_1;l_2;\boldsymbol{y};\boldsymbol{p})\right) \leq \widetilde{M}_n\left(f(k_1,l_1); f(k_2,l_2); f(\boldsymbol{x},\boldsymbol{y}); \boldsymbol{p}\right). \tag{8.15}$$

holds. If H is P-convex (P-concave) and M is decreasing (increasing) the inequality in (8.15) is reversed.

Proof: Similarly as in the proof of Theorem 8.8, but using Theorem 8.2 instead of Theorem A.20 Suppose that H is P-convex. We immediately have

$$\begin{aligned}
&H\left(K\left(k_1\right)+K\left(k_2\right)-\frac{1}{P_n}\sum_{i=1}^{n}p_iK\left(x_i\right),L\left(l_1\right)+L\left(l_2\right)-\frac{1}{P_n}\sum_{i=1}^{n}p_iL\left(y_i\right)\right)\\
&\le H\left(K\left(k_1\right),L\left(l_1\right)\right)+H\left(K\left(k_2\right),L\left(l_2\right)\right)-\frac{1}{P_n}\sum_{i=1}^{n}p_iH\left(K\left(x_i\right),L\left(y_i\right)\right)\\
&=M\left(f\left(k_1,l_1\right)\right)+M\left(f\left(k_2,l_2\right)\right)-\frac{1}{P_n}\sum_{i=1}^{n}p_iM\left(f\left(x_i,y_i\right)\right),
\end{aligned}$$

from which we easily obtain

$$f\left(\widetilde{K}_n\left(k_1;k_2;\boldsymbol{x};\boldsymbol{p}\right),\widetilde{L}_n\left(l_1;l_2;\boldsymbol{y};\boldsymbol{p}\right)\right)\le\widetilde{M}_n\left(f\left(k_1,l_1\right);f\left(k_2,l_2\right);f\left(\boldsymbol{x},\boldsymbol{y}\right);\boldsymbol{p}\right)$$

if M is increasing, or

$$f\left(\widetilde{K}_n\left(k_1;k_2;\boldsymbol{x};\boldsymbol{p}\right),\widetilde{L}_n\left(l_1;l_2;\boldsymbol{y};\boldsymbol{p}\right)\right)\ge\widetilde{M}_n\left(f\left(k_1,l_1\right);f\left(k_2,l_2\right);f\left(\boldsymbol{x},\boldsymbol{y}\right);\boldsymbol{p}\right)$$

if M is decreasing.

If H is P-concave proof is similar. □

Theorem 8.12: *Let K, L, M and $\boldsymbol{p}$ be as in Theorem 8.11 and suppose that the derivatives K', K'', L', L'', M', M'' exist. If M is strictly increasing (decreasing) convex (concave) function and if the conditions (8.8) are satisfied, then for any n-tuples $\boldsymbol{x}\in[k_1,k_2]^n\subset I_K^n$ and $\boldsymbol{y}\in[l_1,l_2]^n\subset I_L^n$ monotone in the same direction, then*

$$\widetilde{K}_n\left(k_1;k_2;\boldsymbol{x};\boldsymbol{p}\right)+\widetilde{L}_n\left(l_1;l_2;\boldsymbol{y};\boldsymbol{p}\right)\le\widetilde{M}_n\left(k_1+l_1;k_2+l_2;\boldsymbol{x}+\boldsymbol{y};\boldsymbol{p}\right)\tag{8.16}$$

holds. If M is strictly decreasing (increasing) convex (concave) function and if the reversals of (8.8) hold, than the inequality in (8.16) is reversed.

Proof: Similarly as in Theorem 8.9 □

Corollary 8.4: *Let $p\ge 1$ and let $\boldsymbol{p}$ be a real n-tuple such that (A.15) is satisfied. Then for any positive n-tuples $\boldsymbol{x}\in[k_1,k_2]^n\subset\langle 0,+\infty\rangle^n$ and $\boldsymbol{y}\in[l_1,l_2]^n\subset\langle 0,+\infty\rangle^n$ monotone in the same direction,*

$$\widetilde{G}_n\left(k_1;k_2;\boldsymbol{x};\boldsymbol{p}\right)+\widetilde{G}_n\left(l_1;l_2;\boldsymbol{y};\boldsymbol{p}\right)\le\widetilde{M}_n^{[p]}\left(k_1+l_1;k_2+l_2;\boldsymbol{x}+\boldsymbol{y};\boldsymbol{p}\right)$$

and

$$\widetilde{H}_n\left(k_1;k_2;\boldsymbol{x};\boldsymbol{p}\right)+\widetilde{H}_n\left(l_1;l_2;\boldsymbol{y};\boldsymbol{p}\right)\le\widetilde{M}_n^{[p]}\left(k_1+l_1;k_2+l_2;\boldsymbol{x}+\boldsymbol{y};\boldsymbol{p}\right)$$

hold.

Proof: Similarly as in Corollary 8.1 and Corollary 8.2. □

Theorem 8.13: *Let K, L, M and $\boldsymbol{p}$ be as in Theorem 8.11 and suppose that the derivatives K', K'', L', L'', M', M'' exist. If M is strictly increasing (decreasing) and if the inequalities*

(8.12) *(or respectively reversed inequalities* (8.12)*) hold for all* $x \in I_K$, $y \in I_L$, *then for any* n*-tuples* $\boldsymbol{x} \in [k_1, k_2]^n \subset I_K^n$ *and* $\boldsymbol{y} \in [l_1, l_2]^n \subset I_L^n$ *monotone in the same direction*

$$\widetilde{K}_n\left(k_1; k_2; \boldsymbol{x}; \boldsymbol{p}\right) \widetilde{L}_n\left(l_1; l_2; \boldsymbol{y}; \boldsymbol{p}\right) \leq \widetilde{M}_n\left(k_1 l_1; k_2 l_2; \boldsymbol{xy}; \boldsymbol{p}\right) \tag{8.17}$$

holds, where $\boldsymbol{xy} = (x_1 y_1, \ldots, x_m y_m)$.
If M *is strictly decreasing (increasing) and if the inequalities* (8.12) *(or respectively the reversed* (8.12)*) hold for all* $x \in I_K$, $y \in I_L$, *the inequality in* (8.17) *is reversed.*

Proof: Similarly as in Theorem 8.10 □

Corollary 8.5: *Let* p, q *and* r *be real numbers such that* $pq > 0$, *and let* $\boldsymbol{w}$ *be a real* n*-tuple such that* (A.15) *is satisfied. If* $r > 0$ *and* $r \geq \max\{p, q\}$, *then for any* n*-tuples* $\boldsymbol{x} \in [a, b]^n \subset \langle 0, +\infty\rangle^n$ *and* $\boldsymbol{y} \in [c, d]^n \subset \langle 0, +\infty\rangle^n$ *monotone in the same direction,*

$$\left(a^p + b^p - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{1}{p}} \left(c^q + d^q - \frac{1}{W_n}\sum_{i=1}^{n} w_i y_i^q\right)^{\frac{1}{q}} \leq \left(a^r c^r + b^r d^r - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^r y_i^r\right)^{\frac{1}{r}}$$

hold. If $r < 0$ *and* $r \leq \min\{p, q\}$ *the inequality in* (8.39) *is reversed.*

Proof: Similarly as in Corollary 8.3 □

8.2 Refinements

In this section we give another variant of Jensen–Mercer inequality for functions with nondecreasing increments and also its refinement. Furthermore, we show that the monotonicity condition on k-tuples $\boldsymbol{\xi}^{(i)}$ ($i \in \{1, 2, \ldots, n\}$) can be relaxed on the monotonicity in mean condition. Then we show how these results can be used to obtain analogous refinements of Čebyšev's inequality and Hölder's inequality for monotone sequences.

8.2.1 Index Set Function and Refinements

Denote by K a k-dimensional rectangle $[a_1, b_1] \times \cdots \times [a_k, b_k] \subset \mathbb{R}^k$, $a_i < b_i$ ($i \in \{1, 2, \ldots, k\}$), and by $\boldsymbol{a}$ and $\boldsymbol{b}$ two specially chosen k-tuples from the rectangle K:

$$\boldsymbol{a} = (a_1, \ldots, a_k), \quad \boldsymbol{b} = (b_1, \ldots, b_k).$$

In the next theorem we give a variant of the Jensen–Mercer inequality for functions with nondecreasing increments.

Theorem 8.14: *Let* $f : K \to \mathbb{R}$ *be a continuous function with nondecreasing increments and let* $\boldsymbol{x}^{(i)} \in K$ ($i \in \{1, 2, \ldots, n\}$) *satisfy the condition*

$$\boldsymbol{x}^{(1)} \leq \cdots \leq \boldsymbol{x}^{(n)} \quad or \quad \boldsymbol{x}^{(1)} \geq \cdots \geq \boldsymbol{x}^{(n)}.$$

If $\boldsymbol{p}$ is a nonnegative n-tuple such that $P_n > 0$, then

$$f\left(\boldsymbol{a}+\boldsymbol{b}-\frac{1}{P_n}\sum_{i=1}^{n} p_i \boldsymbol{x}^{(i)}\right) \leq f(\boldsymbol{a})+f(\boldsymbol{b})-\frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(\boldsymbol{x}^{(i)}\right). \tag{8.18}$$

If $\boldsymbol{p}$ is a real n-tuple such that $p_1 > 0, \quad p_i \leq 0, \quad i \in \{2,\ldots,n\}, \quad P_n > 0$, and if

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \boldsymbol{x}^{(i)} \in K, \tag{8.19}$$

then (8.18) remains valid.

In order to present our next result we need to introduce some more notation.

Let $f : K \to \mathbb{R}$ and let I be a nonempty finite set of positive integers. Let $\boldsymbol{X} = \left\{\boldsymbol{x}^{(i)} \mid i \in I\right\}$ be a sequence in K such that $\boldsymbol{a} \leq \boldsymbol{x}^{(i)} \leq \boldsymbol{b}$ $(i \in I)$. Let $\boldsymbol{p} = \{p_i \mid i \in I\}$ be a real sequence such that $\boldsymbol{a}+\boldsymbol{b}-\frac{1}{P_I}\sum_{i\in I} p_i \boldsymbol{x}^{(i)} \in K$ and $P_I = \sum_{i\in I} p_i \neq 0$. We define the index set function F by

$$F(I) = P_I\left(f(\boldsymbol{a})+f(\boldsymbol{b})\right)-\sum_{i\in I} p_i f\left(\boldsymbol{x}^{(i)}\right)-P_I f\left(\boldsymbol{a}+\boldsymbol{b}-\frac{1}{P_I}\sum_{i\in I} p_i \boldsymbol{x}^{(i)}\right).$$

We also define the arithmetic mean $\boldsymbol{A}_I$ over the index set I as

$$\boldsymbol{A}_I(\boldsymbol{X};\boldsymbol{p}) = \frac{1}{P_I}\sum_{i\in I} p_i \boldsymbol{x}^{(i)}.$$

To simplify the notation, we sometimes write $\boldsymbol{A}_I$ instead of $\boldsymbol{A}_I(\boldsymbol{X};\boldsymbol{p})$.

Theorem 8.15: *Let $f : K \to \mathbb{R}$ be a continuous function with nondecreasing increments and let I and J be finite nonempty sets of positive integers such that $I \cap J = \varnothing$. Let $\boldsymbol{p} = \{p_i \mid i \in I \cup J\}$ be a real sequence such that $P_{I\cup J} > 0$, and let $\boldsymbol{X} = \left\{\boldsymbol{x}^{(i)} \mid i \in I \cup J\right\}$ be a sequence in K such that $\boldsymbol{A}_S(\boldsymbol{X};\boldsymbol{p}) \in K$ $(S = I, J, I\cup J)$. If $P_I > 0$, $P_J > 0$ and if*

$$\boldsymbol{A}_I(\boldsymbol{X};\boldsymbol{p}) \leq \boldsymbol{A}_J(\boldsymbol{X};\boldsymbol{p}) \quad \text{or} \quad \boldsymbol{A}_J(\boldsymbol{X};\boldsymbol{p}) \leq \boldsymbol{A}_I(\boldsymbol{X};\boldsymbol{p}), \tag{8.20}$$

then

$$F(I\cup J) \geq F(I)+F(J). \tag{8.21}$$

If $P_I P_J < 0$ and if (8.20) holds, then (8.21) is reversed.

Remark 8.5: It can be easily shown that the conditions $\boldsymbol{A}_I \leq \boldsymbol{A}_J$ or $\boldsymbol{A}_J \leq \boldsymbol{A}_I$ are equivalent to the conditions $\boldsymbol{A}_I \leq \boldsymbol{A}_{I\cup J}$ or $\boldsymbol{A}_{J\cup I} \leq \boldsymbol{A}_I$.

Corollary 8.6: *Let $f : K \to \mathbb{R}$ be a continuous function with nondecreasing increments, let $\boldsymbol{x}^{(i)} \in K$ $(i \in \{1,2,\ldots,n\})$ satisfy the condition*

$$\boldsymbol{x}^{(1)} \leq \cdots \leq \boldsymbol{x}^{(n)} \quad \text{or} \quad \boldsymbol{x}^{(1)} \geq \cdots \geq \boldsymbol{x}^{(n)},$$

and let $\boldsymbol{p}$ be a real n-tuple such that $p_1 > 0$. If

$$p_i \geq 0, \quad i \in \{2, \ldots, n\}, \tag{8.22}$$

then

$$0 \leq F(I_1) \leq \ldots \leq F(I_{n-1}) \leq F(I_n), \tag{8.23}$$

where $I_j = \{1, \ldots, j\}$. If

$$p_i \leq 0 \quad (i \in \{2, \ldots, n\}), \quad P_n > 0 \tag{8.24}$$

and

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \boldsymbol{x}^{(i)} \in K,$$

then

$$0 \leq F(I_n) \leq F(I_{n-1}) \leq \ldots \leq F(I_1). \tag{8.25}$$

In [211] J. Pečarić proved that the monotonicity conditions on k-tuples $\boldsymbol{\xi}^{(i)}$ for $i \in \{1, 2, \ldots, n\}$ in Theorem A.22 can be relaxed on the monotonicity in mean conditions. In the next theorem we show that it can be also done in Corollary 8.6

Theorem 8.16: *Let $f : K \to \mathbb{R}$ be a continuous function with nondecreasing increments and let $\boldsymbol{p}$ be a real n-tuple such that $p_1 > 0$. Let $\boldsymbol{x}^{(i)} \in K$ $(i \in \{1, 2, \ldots, n\})$ satisfy the condition*

$$\boldsymbol{x}^{(1)} \leq \boldsymbol{A}_{I_2} \leq \cdots \leq \boldsymbol{A}_{I_n} \quad \text{or} \quad \boldsymbol{x}^{(1)} \geq \boldsymbol{A}_{I_2} \geq \cdots \geq \boldsymbol{A}_{I_n}.$$

(i) If $\boldsymbol{p}$ satisfies the condition (8.22) then (8.23) holds.
(ii) If $\boldsymbol{p}$ satisfies the conditions (8.24) and (8.19), then (8.25) holds.

Proof: Suppose that $\boldsymbol{p}$ satisfies the condition (8.22). If $\boldsymbol{A}_{I_{n-1}} \leq \boldsymbol{A}_{I_n}$, then

$$\sum_{i \in I_n} \frac{P_n - P_{n-1}}{P_n P_{n-1}} p_i \boldsymbol{x}^{(i)} \leq \frac{p_n}{P_n} \boldsymbol{x}^{(n)},$$

which can be rewritten as $\boldsymbol{A}_{I_{n-1}} \leq \boldsymbol{x}^{(n)}$. Using substitutions $I = I_{n-1}$ and $J = \{n\}$, from Theorem 8.15,

$$F(I_n) \geq F(I_{n-1})$$

follows, from where we get (8.23) by iteration. The other case $\boldsymbol{A}_{I_{n-1}} \geq \boldsymbol{A}_{I_n}$ is analogous.

We can prove the second part of the theorem similarly. □

Remark 8.6: Similar assertions can be formulated for P-convex functions.

8.2.2 Refinements of Čebyšev's Inequality

In this subsection we show how Corollary 8.6 can be used to obtain a refinement of the Mercer's type variant of Čebyšev's inequality for two monotonic sequences.

If we define the index set function C, for a real n-tuple $\boldsymbol{p}$ and two monotonic real n-tuples $\boldsymbol{x} \in [a,b]^n$ and $\boldsymbol{y} \in [c,d]^n$, by

$$C(I) = P_I(ac+bd) - \sum_{i \in I} p_i x_i y_i - P_I \left(a+b-\frac{1}{P_I}\sum_{i\in I} p_i x_i \right) \left(c+d-\frac{1}{P_I}\sum_{i\in I} p_i y_i \right).$$

then the following theorem is valid.

Theorem 8.17: *Let a, b, c and d be real numbers such that $a<b$, $c<d$, and let $\boldsymbol{x} \in [a,b]^n$ and $\boldsymbol{y} \in [c,d]^n$. Let $\boldsymbol{p}$ be a real n-tuple such that $p_1>0$.*
If $\boldsymbol{p}$ satisfies the condition (8.22) and if $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in the same direction, then

$$0 \le C(I_1) \le \cdots \le C(I_{n-1}) \le C(I_n). \tag{8.26}$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in the opposite directions, then all inequalities in (8.26) are reversed.
If $\boldsymbol{p}$ satisfies the conditions (8.24), if

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \in [a,b], \qquad \frac{1}{P_n}\sum_{i=1}^{n} p_i y_i \in [c,d],$$

and if $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in the same direction, then

$$0 \le C(I_n) \le C(I_{n-1}) \le \cdots \le C(I_1). \tag{8.27}$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in the opposite directions, then all inequalities in (8.27) are reversed.

Proof: Suppose that $\boldsymbol{p}$ satisfies the condition (8.22) and that the n-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in the same direction. In order to obtain (8.26) we simply apply Corollary 8.6 to the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x,y) = xy$ and n-tuples $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{p}$. If the n-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ are monotone in the opposite directions we apply Corollary 8.6 to $\boldsymbol{x}$, $-\boldsymbol{y} \in [-d,-c]^n$ and $\boldsymbol{p}$, to obtain reversed (8.26).

The second case can be proved in a similar way. □

Remark 8.7: Similarly as in Theorem 8.16, the "same monotonicity" condition on $\boldsymbol{x}$ and $\boldsymbol{y}$ can be replaced with the conditions

$$x_1 \le \frac{1}{P_2}\sum_{i=1}^{2} p_i x_i \le \cdots \le \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \tag{8.28}$$

$$y_1 \le \frac{1}{P_2}\sum_{i=1}^{2} p_i y_i \le \cdots \le \frac{1}{P_n}\sum_{i=1}^{n} p_i y_i, \tag{8.29}$$

or with (8.28) and (8.29) both reversed, and the "opposite monotonicity" condition with

$$x_1 \leq \frac{1}{P_2}\sum_{i=1}^{2} p_i x_i \leq \cdots \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \tag{8.30}$$

$$y_1 \geq \frac{1}{P_2}\sum_{i=1}^{2} p_i y_i \geq \cdots \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i y_i, \tag{8.31}$$

or with (8.30) and (8.31) both reversed.

8.2.3 Refinements of Hölder's Inequality

In this subsection we present two refinements of Hölder's inequality. The first refinement is for $k \geq 2$ monotonic sequences in which exponents p and q need not to be conjugate. The second refinement is for two monotonic sequences with exponents p, q and r.

Denote by K a positive k-dimensional rectangle $[a_1, b_1] \times \cdots \times [a_k, b_k]$, where $0 < a_i < b_i$, $i \in \{1, 2, \ldots, k\}$. For k-tuples $\boldsymbol{x}^{(i)} \in K$, $i \in \{1, 2, \ldots, n\}$ such that $\boldsymbol{x}^{(1)} \leq \cdots \leq \boldsymbol{x}^{(n)}$ or $\boldsymbol{x}^{(1)} \geq \cdots \geq \boldsymbol{x}^{(n)}$ we define the index set function H by

$$\begin{aligned} H(I) = {} & W_I \left(\prod_{j=1}^{k} a_j + \prod_{j=1}^{k} b_j \right) - \sum_{i \in I} w_i \prod_{j=1}^{k} x_j^{(i)} \\ & - W_I \prod_{j=1}^{k} \left(a_j^{p_j} + b_j^{p_j} - \frac{1}{W_I} \sum_{i \in I} w_i \left(x_j^{(i)} \right)^{p_j} \right)^{\frac{1}{p_j}}. \end{aligned}$$

Theorem 8.18: *Let $\boldsymbol{p} \in \langle -\infty, 0\rangle^k \cup \langle 0, 1]^k$, let $\boldsymbol{x}^{(i)} \in K$ $(i \in \{1, 2, \ldots, n\})$ be such that*

$$\boldsymbol{x}^{(1)} \leq \cdots \leq \boldsymbol{x}^{(n)} \quad \text{or} \quad \boldsymbol{x}^{(1)} \geq \cdots \geq \boldsymbol{x}^{(n)},$$

and let $\boldsymbol{w}$ be a real n-tuple such that $w_1 > 0$ If

$$w_i \geq 0, \quad i \in \{2, \ldots, n\}, \tag{8.32}$$

then

$$0 \leq H(I_1) \leq \cdots \leq H(I_{n-1}) \leq H(I_n). \tag{8.33}$$

If

$$w_i \leq 0, \quad i \in \{2, \ldots, n\}, \quad W_n > 0 \tag{8.34}$$

and if

$$\frac{1}{W_n} \sum_{i=1}^{n} w_i \boldsymbol{x}^{(i)} \in K,$$

then

$$0 \leq H(I_n) \leq H(I_{n-1}) \leq \cdots \leq H(I_1).$$

Proof: For some fixed $p_j \in \mathbb{R} \setminus \{0\}$, $j \in \{1, \ldots, k\}$ we define the function $\varphi : \langle 0, +\infty\rangle^k \to \mathbb{R}$

as

$$\varphi(\boldsymbol{x}) = \prod_{j=1}^{k} x_j^{\frac{1}{p_j}}. \tag{8.35}$$

Then we have

$$\varphi_{x_j x_j}(\boldsymbol{x}) = \frac{1}{p_j}\left(\frac{1}{p_j} - 1\right) x_j^{-2} \prod_{i=1,}^{k} x_i^{\frac{1}{p_i}}, \quad j \in \{1, 2, \ldots, k\},$$

$$\varphi_{x_i x_j}(\boldsymbol{x}) = \frac{1}{p_i}\frac{1}{p_j} x_i^{-1} x_j^{-1} \prod_{l=1,}^{k} x_l^{\frac{1}{p_l}}, \quad i, j \in \{1, 2, \ldots, k\}, \ i \neq j,$$

so it can be easily seen that φ is continuous and that it has nondecreasing increments if p_j for $j \in \{1, 2, \ldots, k\}$ are all negative or if p_j for $j \in \{1, 2, \ldots, k\}$ are all positive and not greater then 1.

Suppose that $\boldsymbol{x}^{(1)} \leq \cdots \leq \boldsymbol{x}^{(n)}$ and $\boldsymbol{p} \in \langle -\infty, 0\rangle^k \cup \langle 0, 1]^k$
(if $\boldsymbol{x}^{(1)} \geq \cdots \geq \boldsymbol{x}^{(n)}$ proof is similar). We define k-tuples $\boldsymbol{\xi}^{(i)}$, $i \in \{1, 2, \ldots, n\}$ as

$$\boldsymbol{\xi}_j^{(i)} = \left(\boldsymbol{x}_j^{(i)}\right)^{p_j}, \quad j \in \{1, 2, \ldots, k\}.$$

Obviously, if $\boldsymbol{p} \in \langle -\infty, 0\rangle^k$ then $\boldsymbol{\xi}^{(1)} \geq \cdots \geq \boldsymbol{\xi}^{(n)}$ and $\boldsymbol{\xi}^{(i)} \in [b_1^{p_1}, a_1^{p_1}] \times \cdots \times \left[b_k^{p_k}, a_k^{p_k}\right]$ ($i \in \{1, 2, \ldots, n\}$), and if $\boldsymbol{p} \in \langle 0, 1]^k$ then $\boldsymbol{\xi}^{(1)} \leq \cdots \leq \boldsymbol{\xi}^{(n)}$ and $\boldsymbol{\xi}^{(i)} \in [a_1^{p_1}, b_1^{p_1}] \times \cdots \times \left[a_k^{p_k}, b_k^{p_k}\right]$ ($i \in \{1, 2, \ldots, n\}$). In both cases we can apply Corollary 8.6 to $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$, $\boldsymbol{w}$ satisfying (8.32) and φ, and hence obtain (8.33).

The other case can be proved analogously. □

Corollary 8.7: *Let $\boldsymbol{w}$ be a real n-tuple such that $w_1 > 0$ and let $(p, q) \in \langle -\infty, 0\rangle^2 \cup \langle 0, 1]^2$, and let $\boldsymbol{x} \in [a, b]^n$, $0 < a < b$ and $\boldsymbol{y} \in [c, d]^n$, $0 < c < d$ be two n-tuples monotonic in the same direction.*
If $\boldsymbol{w}$ satisfies the condition (8.32), then

$$\left(a^p + b^p - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{1}{p}} \left(c^q + d^q - \frac{1}{W_n}\sum_{i=1}^{n} w_i y_i^q\right)^{\frac{1}{q}} \leq ac + bd - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i y_i. \tag{8.36}$$

If $\boldsymbol{w}$ satisfies conditions (8.34) and if

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \in [a, b], \qquad \frac{1}{W_n}\sum_{i=1}^{n} w_i y_i \in [c, d], \tag{8.37}$$

then (8.36) remains valid.

Proof: This variant of Hölder's inequality is a special case of Theorem 8.18 for $k = 2$. □

Now, we define the index set function R as

$$R(I) = W_I\left(a^r c^r + b^r d^r\right) - \sum_{i\in I} w_i x_i^r y_i^r$$
$$- W_I\left(a^p + b^p - \frac{1}{W_I}\sum_{i\in I} w_i x_i^p\right)^{\frac{r}{p}}\left(c^q + d^q - \frac{1}{W_I}\sum_{i\in I} w_i y_i^q\right)^{\frac{r}{q}},$$

where $\boldsymbol{w}$ is a real n-tuple and $\boldsymbol{x} \in [a,b]^n$, $0 < a < b$ and $\boldsymbol{y} \in [c,d]^n$, $0 < c < d$ are two n-tuples monotone in the same direction.

Theorem 8.19: *Let p, q be real numbers such that $pq > 0$ and let r be a real number such that $r > 0$ and $r \geq \max\{p,q\}$, or $r < 0$ and $r \leq \min\{p,q\}$. Let $\boldsymbol{x} \in [a,b]^n$, $0 < a < b$ and $\boldsymbol{y} \in [c,d]^n$, $0 < c < d$ be two n-tuples monotone in the same direction and let $\boldsymbol{w}$ be a real n-tuple such that $w_1 > 0$. If $\boldsymbol{w}$ satisfies the condition (8.32), then*

$$0 \leq R(I_1) \leq \cdots \leq R(I_{n-1}) \leq R(I_n). \tag{8.38}$$

If $\boldsymbol{w}$ satisfies the conditions (8.34) and (8.37), then

$$0 \leq R(I_n) \leq R(I_{n-1}) \leq \cdots \leq R(I_1).$$

Proof: For some fixed $p, q, r \in \mathbb{R} \setminus \{0\}$ we define the function $\varphi : \langle 0, +\infty\rangle^2 \to \mathbb{R}$ as

$$\varphi(x,y) = x^{\frac{r}{p}} y^{\frac{r}{q}}.$$

Obviously, φ is continuous and we have

$$\varphi_{xx}(x,y) = \frac{r(r-p)}{p^2} x^{\frac{r}{p}-2} y^{\frac{r}{q}},$$
$$\varphi_{xx}(x,y) = \frac{r^2}{pq} x^{\frac{r}{p}-1} y^{\frac{r}{q}-1},$$
$$\varphi_{yy}(x,y) = \frac{r(r-q)}{q^2} x^{\frac{r}{p}} y^{\frac{r}{q}-2},$$

so it can be easily seen that the function φ has nondecreasing increments if $pq > 0$ and $r > 0$, $r \geq \max\{p,q\}$ or $r < 0$, $r \leq \min\{p,q\}$.

Suppose that $\boldsymbol{w}$ satisfies (8.32). Without any loss of generality we may assume that both $\boldsymbol{x}$ and $\boldsymbol{y}$ are increasing. We define

$$\xi_i = x_i^p, \quad (i \in \{1,2,\ldots,n\}), \qquad \eta_i = y_i^q, \quad (i \in \{1,2,\ldots,n\}).$$

If p and q are both positive we have

$$a^p \leq \xi_1 \leq \cdots \leq \xi_n \leq b^p, \qquad c^q \leq \eta_1 \leq \cdots \leq \eta_n \leq d^q,$$

and if they are both negative we have

$$b^p \leq \xi_n \leq \cdots \leq \xi_1 \leq a^p, \qquad d^q \leq \eta_n \leq \cdots \leq \eta_1 \leq c^q.$$

Assume that $r > 0$ and $r \geq \max\{p, q\}$ or $r < 0$ and $r \leq \min\{p, q\}$. If we apply Corollary 8.6 to $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{w}$ and φ we obtain (8.38).

The other case can be proved analogously. □

Remark 8.8: Similarly as in the previous subsection, the monotonicity conditions in Theorem 8.18 and Theorem 8.19 can be relaxed on the monotonicity in mean conditions.

Corollary 8.8: *Let $\boldsymbol{x} \in [a, b]^n$, $0 < a < b$ and $\boldsymbol{y} \in [c, d]^n$, $0 < c < d$ be two n-tuples monotonic in the same direction. Let $\boldsymbol{w}$ be a real n-tuple with $w_1 > 0$ and satisfying the condition (8.32) or the conditions (8.34) and (8.37). Let p, q be real numbers such that $pq > 0$ and let r be a real number such that $r > 0$ and $r \geq \max\{p, q\}$. Then*

$$\left(a^p + b^p - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{1}{p}} \left(c^q + d^q - \frac{1}{W_n}\sum_{i=1}^{n} w_i y_i^q\right)^{\frac{1}{q}} \leq \left(a^r c^r + b^r d^r - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^r y_i^r\right)^{\frac{1}{r}}. \tag{8.39}$$

If $r < 0$ and $r \leq \min\{p, q\}$ inequality (8.39) is reversed.

Proof: Suppose that $pq > 0$, $r > 0$ and $r \geq \max\{p, q\}$. From Theorem 8.19,

$$W_n \left(a^p + b^p - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{r}{p}} \left(c^q + d^q - \frac{1}{W_n}\sum_{i=1}^{n} w_i y_i^q\right)^{\frac{r}{q}} \leq W_n \left(a^r c^r + b^r d^r\right) - \sum_{i=1}^{n} w_i x_i^r y_i^r \tag{8.40}$$

follows. Since $W_n > 0$ and $r > 0$, we immediately obtain (8.39).

If $pq > 0$, $r < 0$ and $r \leq \min\{p, q\}$ then inequality (8.40) follows from Theorem 8.19 again, but since r is negative in the end we obtain reversed inequality (8.39). □

CHAPTER 9

On Jensen–Mercer Operator Inequality

This chapter develops Jensen–Mercer inequalities in the operator setting on Hilbert spaces. We begin with the classical Jensen–Mercer inequality and its applications to operator means of Mercer's type, including power and quasi-arithmetic operator means. The discussion then addresses refinements using index set functions, extensions to superquadratic functions, and continuous fields of operators. Further results include new refinements with applications, Hermite–Hadamard–Mercer type inequalities, and inequalities for h-convex operators. Finally, we examine operator inequalities acting on a Hilbert space and their consequences, providing a comprehensive framework for Mercer-type inequalities in operator theory.

It is worth mentioning that most of the results of this chapter are extracted from [171, 172, 33, 118, 166, 170, 168].

9.1 Jensen–Mercer Inequality on a Hilbert Space

In this section we give an extension of the Jensen–Mercer inequality to selfadjoint operators on a Hilbert space. Using the obtained inequality we prove that related operator power means of Mercer's type have a monotonicity property. In the same way, we give some inequalities among related quasi-arithmetic means.

9.1.1 The Inequality

For a selfadjoint operator $A \in \mathcal{B}(H)$, we write $mI \leq A \leq MI$, where I is the identity operator on H, if $m(x,x) \leq (Ax,x) \leq M(x,x)$ for all $x \in H$.

Theorem 9.1: *Let $A_1, \ldots, A_k \in \mathcal{B}(H)$ be selfadjoint operators with $mI \leq A_j \leq MI$ for some scalars $m < M$ and let $x_1, \ldots x_k \in H$ satisfy $\sum_{j=1}^{k} (x_j, x_j) = 1$. If $f \in C([m, M])$ is convex on $[m, M]$, then*

$$f\left(m + M - \sum_{j=1}^{k} (A_j x_j, x_j)\right) \leq f(m) + f(M) - \sum_{j=1}^{k} (f(A_j) x_j, x_j) \tag{9.1}$$

holds. In fact, to be more specific, we have the following series of inequalities

$$\begin{aligned} f\left(m + M - \sum_{j=1}^{k} (A_j x_j, x_j)\right) &\leq \sum_{j=1}^{k} (f((m+M)I - A_j) x_j, x_j) \\ &\leq \frac{M - \sum_{j=1}^{k} (A_j x_j, x_j)}{M - m} f(M) + \frac{\sum_{j=1}^{k} (A_j x_j, x_j) - m}{M - m} f(m) \\ &\leq f(m) + f(M) - \sum_{j=1}^{k} (f(A_j) x_j, x_j). \end{aligned} \tag{9.2}$$

If a function f is concave the inequalities in (9.1) and (9.2) are reversed.

Proof: From the conditions $m(x_j, x_j) \leq (A_j x_j, x_j) \leq M(x_j, x_j)$, for $j = 1, \ldots, k$ and $\sum_{j=1}^{k} (x_j, x_j) = 1$, by summing it follows that $m \leq \sum_{j=1}^{k} (A_j x_j, x_j) \leq M$, and hence, $m \leq m + M - \sum_{j=1}^{k} (A_j x_j, x_j) \leq M$.

Since f is continuous and convex, the same is also true for the function $g : [m, M] \to \mathbb{R}$, $g(t) = f(m + M - t)$, $t \in [m, M]$. By Theorem A.16, $g\left(\sum_{j=1}^{k} (A_j x_j, x_j)\right) \leq \sum_{j=1}^{k} (g(A_j) x_j, x_j)$ holds, i.e., $f\left(m + M - \sum_{j=1}^{k} (A_j x_j, x_j)\right) \leq \sum_{j=1}^{k} (f((m+M)I - A_j) x_j, x_j)$.

Applying Theorem A.17 to g and then to f, we obtain

$$\begin{aligned} &\sum_{j=1}^{k} (f((m+M)I - A_j) x_j, x_j) \\ &\leq \frac{M - \sum_{j=1}^{k} (A_j x_j, x_j)}{M - m} g(m) + \frac{\sum_{j=1}^{k} (A_j x_j, x_j) - m}{M - m} g(M) \\ &= \frac{M - \sum_{j=1}^{k} (A_j x_j, x_j)}{M - m} f(M) + \frac{\sum_{j=1}^{k} (A_j x_j, x_j) - m}{M - m} f(m) \\ &= f(m) + f(M) \\ &- \left[\frac{M - \sum_{j=1}^{k} (A_j x_j, x_j)}{M - m} f(m) + \frac{\sum_{j=1}^{k} (A_j x_j, x_j) - m}{M - m} f(M)\right] \\ &\leq f(m) + f(M) - \sum_{j=1}^{k} (f(A_j) x_j, x_j). \end{aligned}$$

The last statement follows immediately from the fact that if f is concave then $-f$ is convex.

□

9.1.2 Applications to the Operator Means of Mercer's Type

Throughout this subsection we suppose that:

(i) $\boldsymbol{A} = (A_1, \ldots, A_k)$, where $A_j \in \mathcal{B}(H)$ are selfadjoint operators such that $mI \leq A_j \leq MI$ $(j \in \{1, 2, \ldots, k\})$ for some scalars $0 < m < M$,

(ii) $\boldsymbol{x} = (x_1, \ldots, x_k)$, where $x_j \in H$ $(j \in \{1, 2, \ldots, k\})$ satisfy $\sum_{j=1}^{k} (x_j, x_j) = 1$.

We define, for any $r \in \mathbb{R}$

$$\widetilde{M}_r(\boldsymbol{A}, \boldsymbol{x}) := \begin{cases} \left[m^r + M^r - \sum_{j=1}^{k} \left(A_j^r x_j, x_j\right)\right]^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\ln mM - \sum_{j=1}^{k} \left((\ln A_j)\, x_j, x_j\right)\right), & r = 0. \end{cases}$$

Observe that, since $0 < m\,(x_j, x_j) \leq (A_j x_j, x_j) \leq M\,(x_j, x_j)$, it follows that:

- $0 < m^r\,(x_j, x_j) \leq \left(A_j^r x_j, x_j\right) \leq M^r\,(x_j, x_j)$, for all $r > 0$,
- $0 < M^r\,(x_j, x_j) \leq \left(A_j^r x_j, x_j\right) \leq m^r\,(x_j, x_j)$, for all $r < 0$,
- $(\ln m)\,(x_j, x_j) \leq ((\ln A_j)\, x_j, x_j) \leq (\ln M)\,(x_j, x_j)$ $(j \in \{1, 2, \ldots, k\})$.

By summing, it follows that:

- $0 < m^r \leq \sum_{j=1}^{k} \left(A_j^r x_j, x_j\right) \leq M^r$, for all $r > 0$,
- $0 < M^r \leq \sum_{j=1}^{k} \left(A_j^r x_j, x_j\right) \leq m^r$, for all $r < 0$,
- $\ln m \leq \sum_{j=1}^{k} ((\ln A_j)\, x_j, x_j) \leq \ln M$,

since $\sum_{j=1}^{k} (x_j, x_j) = 1$. Hence, $\widetilde{M}_r\,(\boldsymbol{A}, \boldsymbol{x})$ is well defined.

Furthermore, we define, for any $r, s \in \mathbb{R}$

$$R(r, s, \boldsymbol{A}, \boldsymbol{x}) := \begin{cases} \left[\sum_{j=1}^{k} \left(\left((m^r + M^r)\, I - A_j^r\right)^{\frac{s}{r}} x_j, x_j\right)\right]^{\frac{1}{s}}, & r \neq 0, s \neq 0, \\ \exp\left(\sum_{j=1}^{k} \left(\ln\left((m^r + M^r)\, I - A_j^r\right)^{\frac{1}{r}} x_j, x_j\right)\right), & r \neq 0, s = 0, \\ \left[\sum_{j=1}^{k} \left(\exp s\left((\ln mM)\, I - \ln A_j\right) x_j, x_j\right)\right]^{\frac{1}{s}}, & r = 0, s \neq 0, \end{cases}$$

$$S(r, s, \boldsymbol{A}, \boldsymbol{x}) := \begin{cases} \left[\frac{M^r - S_r}{M^r - m^r} M^s + \frac{S_r - m^r}{M^r - m^r} m^s\right]^{\frac{1}{s}}, & r \neq 0, s \neq 0, \\ \exp\left(\frac{M^r - S_r}{M^r - m^r} \ln M + \frac{S_r - m^r}{M^r - m^r} \ln m\right), & r \neq 0, s = 0, \\ \left[\frac{(\ln M) - S_0}{\ln M - \ln m} M^s + \frac{S_0 - (\ln m)}{\ln M - \ln m} m^s\right]^{\frac{1}{s}}, & r = 0, s \neq 0, \end{cases}$$

where $S_r = \sum_{j=1}^{k} \left(A_j^r x_j, x_j \right)$ and $S_0 = \sum_{j=1}^{k} \left((\ln A_j) x_j, x_j \right)$. It is easy to see that $R(r, s, \boldsymbol{A}, \boldsymbol{x})$ and $S(r, s, \boldsymbol{A}, \boldsymbol{x})$ are also well defined.

Theorem 9.2: *If $r, s \in \mathbb{R}$ and $r < s$, then*

$$\widetilde{M}_r (\boldsymbol{A}, \boldsymbol{x}) \leq \widetilde{M}_s (\boldsymbol{A}, \boldsymbol{x}) .$$

Furthermore,

$$\widetilde{M}_r (\boldsymbol{A}, \boldsymbol{x}) \leq R(r, s, \boldsymbol{A}, \boldsymbol{x}) \leq S(r, s, \boldsymbol{A}, \boldsymbol{x}) \leq \widetilde{M}_s (\boldsymbol{A}, \boldsymbol{x}) .$$

Let $\varphi, \psi \in C([m, M])$ be strictly monotonic functions on an interval $[m, M]$. We define

$$\widetilde{M}_\varphi (\boldsymbol{A}, \boldsymbol{x}) = \varphi^{-1} \left(\varphi(m) + \varphi(M) - \sum_{j=1}^{k} (\varphi(A_j) x_j, x_j) \right) .$$

Theorem 9.3: *Under the above hypotheses, the following is valid.*

(*i*) *If either $\psi \circ \varphi^{-1}$ is convex and ψ is strictly increasing, or $\psi \circ \varphi^{-1}$ is concave and ψ is strictly decreasing, then*

$$\widetilde{M}_\varphi (\boldsymbol{A}, \boldsymbol{x}) \leq \widetilde{M}_\psi (\boldsymbol{A}, \boldsymbol{x}) . \tag{9.3}$$

In fact, to be more specific, we have the following series of inequalities

$$\begin{aligned} \widetilde{M}_\varphi (\boldsymbol{A}, \boldsymbol{x}) &\leq \psi^{-1} \left(\sum_{j=1}^{k} \left(\left(\psi \circ \varphi^{-1}\right) \left((\varphi(m) + \varphi(M)) I - \varphi(A_j) \right) x_j, x_j \right) \right) \\ &\leq \psi^{-1} \left(\frac{\varphi(M) - \sum_{j=1}^{k} (\varphi(A_j) x_j, x_j)}{\varphi(M) - \varphi(m)} \psi(M) \right. \\ &\qquad \left. + \frac{\sum_{j=1}^{k} (\varphi(A_j) x_j, x_j) - \varphi(m)}{\varphi(M) - \varphi(m)} \psi(m) \right) \leq \widetilde{M}_\psi (\boldsymbol{A}, \boldsymbol{x}) . \end{aligned} \tag{9.4}$$

(*ii*) *If either $\psi \circ \varphi^{-1}$ is concave and ψ is strictly increasing, or $\psi \circ \varphi^{-1}$ is convex and ψ is strictly decreasing, then the inequalities in (9.3) and (9.4) are reversed.*

Proof: Suppose that $\psi \circ \varphi^{-1}$ is convex. If in Theorem 9.1 we let $f = \psi \circ \varphi^{-1}$ and replace A_j, m and M with $\varphi(A_j)$, $\varphi(m)$ and $\varphi(M)$ respectively, then we obtain

$$\begin{aligned} &\left(\psi \circ \varphi^{-1}\right) \left(\varphi(m) + \varphi(M) - \sum_{j=1}^{k} (\varphi(A_j) x_j, x_j) \right) \\ &\leq \sum_{j=1}^{k} \left(\left(\psi \circ \varphi^{-1}\right) \left((\varphi(m) + \varphi(M)) I - \varphi(A_j) \right) x_j, x_j \right) \\ &\leq \frac{\varphi(M) - \sum_{j=1}^{k} (\varphi(A_j) x_j, x_j)}{\varphi(M) - \varphi(m)} \psi(M) + \frac{\sum_{j=1}^{k} (\varphi(A_j) x_j, x_j) - \varphi(m)}{\varphi(M) - \varphi(m)} \psi(m) \\ &\leq \psi(m) + \psi(M) - \sum_{j=1}^{k} (\psi(A_j) x_j, x_j) . \end{aligned} \tag{9.5}$$

If $\psi \circ \varphi^{-1}$ is concave then we obtain the reverse of inequalities (9.5).

If ψ is strictly increasing, then the inverse function ψ^{-1} is also strictly increasing, so that (9.5) implies (9.4). If ψ is strictly decreasing, then the inverse function ψ^{-1} is also strictly decreasing, so that in this case the reverse of (9.5) implies (9.4). Analogously, we get the reverse of (9.4) in the cases when $\psi \circ \varphi^{-1}$ is convex and ψ is strictly decreasing, or $\psi \circ \varphi^{-1}$ is concave and ψ is strictly increasing. □

9.2 Jensen–Mercer Operator Inequality

In this section we give an extension of the Jensen–Mercer inequality (1.3) to selfadjoint operators and positive linear maps. This variant of Jensen's operator inequality holds for arbitrary convex functions, while the well known Davis-Choi-Jensen inequality (A.13) and its generalization (A.14) hold only for operator convex functions. Using the obtained inequality we prove that related operator power means of Mercer's type have a monotonicity property. In the same way, we give some inequalities among related quasi−arithmetic means. The results of this section are carefully selected from [168, 172].

Let us consider that $\mathcal{B}(H)$ and $\mathcal{B}(K)$ are the C^*-algebras of all bounded operators on the appropriate Hilbert spaces and $\mathbf{P}[\mathcal{B}(H), \mathcal{B}(K)]$ is the set of all positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$.

Theorem 9.4: *Let $A_1, \ldots, A_k \in \mathcal{B}(H)$ be selfadjoint operators with spectra in $[m, M]$ for some scalars $m < M$ and $\Phi_1, \ldots, \Phi_k \in \mathbf{P}[\mathcal{B}(H), \mathcal{B}(K)]$ positive linear maps with $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$. If $f \in C([m, M])$ is convex on $[m, M]$, then the Jensen–Mercer operator inequality*

$$f\left((m+M)1_K - \sum_{j=1}^{k} \Phi_j(A_j)\right) \le (f(m) + f(M))1_K - \sum_{j=1}^{k} \Phi_j(f(A_j)) \tag{9.6}$$

holds. In fact, the following series of inequalities

$$\begin{aligned} & f\left((m+M)1_K - \sum_{j=1}^{k} \Phi_j(A_j)\right) \\ & \le \frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M-m} f(M) + \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M-m} f(m) \\ & \le (f(m) + f(M))1_K - \sum_{j=1}^{k} \Phi_j(f(A_j)) \end{aligned} \tag{9.7}$$

holds. If a function f is concave, then the inequalities in (9.6) and (9.7) are reversed.

Proof: Since f is continuous and convex, the same is also true for the function $g : [m, M] \to \mathbb{R}$,

$g(t) = f(m+M-t)$, $t \in [m, M]$. Hence, the following inequalities hold for every $t \in [m, M]$ (see ($A.2$) in Section A.1):

$$f(t) \leq \frac{t-m}{M-m} f(M) + \frac{M-t}{M-m} f(m), \qquad g(t) \leq \frac{t-m}{M-m} g(M) + \frac{M-t}{M-m} g(m).$$

Since $m1_H \leq A_j \leq M1_H$ ($j \in \{1, 2, \ldots, k\}$) and $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$, it follows that $m1_K \leq \sum_{j=1}^{k} \Phi_j(A_j) \leq M1_K$. Now, using the functional calculus we have

$$g\left(\sum_{j=1}^{k} \Phi_j(A_j)\right) \leq \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M-m} g(M) + \frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M-m} g(m),$$

i.e.,

$$\begin{aligned} f\left((m+M)1_K - \sum_{j=1}^{k} \Phi_j(A_j)\right) &\leq \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M-m} f(m) + \frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M-m} f(M) \\ = (f(m) + f(M))1_K &- \left[\frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M-m} f(m) + \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M-m} f(M)\right]. \end{aligned} \tag{9.8}$$

On the other hand, using the functional calculus we also have

$$f(A_j) \leq \frac{A_j - m1_H}{M-m} f(M) + \frac{M1_H - A_j}{M-m} f(m).$$

Applying positive linear maps Φ_j and summing, it follows that

$$\sum_{j=1}^{k} \Phi_j(f(A_j)) \leq \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M-m} f(M) + \frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M-m} f(m). \tag{9.9}$$

Using inequalities (9.8) and (9.9), we obtain desired inequalities (9.6) and (9.7).

The last statement follows immediately from the fact that if φ is concave then $-\varphi$ is convex.

For an operator convex function f the Jensen–Mercer operator inequality can be refined in the following way. $\square$

Theorem 9.5: *Let $A_1, \ldots, A_k \in \mathcal{B}(H)$ be selfadjoint operators with spectra in $[m, M]$ for some scalars $m < M$ and $\Phi_1, \ldots, \Phi_k \in \mathbf{P}[\mathcal{B}(H), \mathcal{B}(K)]$ positive linear maps with $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$. If $f \in C([m, M])$ is operator convex on $[m, M]$, then the following series of inequalities*

$$\begin{aligned} f\left((m+M)1_K - \sum_{j=1}^{k} \Phi_j(A_j)\right) &\leq \sum_{j=1}^{k} \Phi_j(f((m+M)1_H - A_j)) \\ &\leq \frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M-m} f(M) + \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M-m} f(m) \\ &\leq (f(m) + f(M))1_K - \sum_{j=1}^{k} \Phi_j(f(A_j)). \end{aligned} \tag{9.10}$$

holds. If fa unction f is operator concave then the inequalities in (9.10) *are reversed.*

Proof: Similar to proof of Theorem 9.1. □

9.2.1 Power Operator Means of Mercer Type

We suppose that:

(i) $\boldsymbol{A} = (A_1, \ldots, A_k)$, where $A_j \in \mathcal{B}(H)$ are positive invertible operators with $Sp(A_j) \subseteq [m, M]$ for some scalars $0 < m < M$,

(ii) $\boldsymbol{\Phi} = (\Phi_1, \ldots, \Phi_k)$, where $\Phi_j \in \mathbf{P}[\mathcal{B}(H), \mathcal{B}(K)]$ are positive linear maps with $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$,

(iii) $\Delta(m, M, p) = K\left(m^p, M^p, \frac{1}{p}\right) = \frac{p(m^p M - M^p m)}{(1-p)(M^p - m^p)}\left(\frac{(1-p)(M-m)}{m^p M - M^p m}\right)^{\frac{1}{p}}$, for $0 < m < M$ and $p \in \mathbb{R}$, $p \neq 0$. Set:

$$\Delta(m, M, 0) = \lim_{p \to 0} \Delta(m, M, p) = S\left(\frac{M}{m}, 1\right) = \frac{M - m}{\ln M - \ln m} \exp\left(\frac{m(1 + \ln M) - M(1 + \ln m)}{M - m}\right).$$

We define, for any $r \in \mathbb{R}$

$$\widetilde{M}_r(\boldsymbol{A}, \boldsymbol{\Phi}) := \begin{cases} \left[(m^r + M^r) 1_K - \sum_{j=1}^{k} \Phi_j\left(A_j^r\right)\right]^{\frac{1}{r}}, & r \neq 0, \\ \exp\left((\ln mM) 1_K - \sum_{j=1}^{k} \Phi_j(\ln A_j)\right), & r = 0. \end{cases}$$

Furthermore, we define, for any $r, s \in \mathbb{R}$

$$S(r, s, \boldsymbol{A}, \boldsymbol{\Phi}) := \begin{cases} \left[\frac{M^r 1_K - S_r}{M^r - m^r} M^s + \frac{S_r - m^r 1_K}{M^r - m^r} m^s\right]^{\frac{1}{s}}, & r \neq 0, s \neq 0, \\ \exp\left(\frac{M^r 1_K - S_r}{M^r - m^r} \ln M + \frac{S_r - m^r 1_K}{M^r - m^r} \ln m\right), & r \neq 0, s = 0, \\ \left[\frac{(\ln M) 1_K - S_0}{\ln M - \ln m} M^s + \frac{S_0 - (\ln m) 1_K}{\ln M - \ln m} m^s\right]^{\frac{1}{s}}, & r = 0, s \neq 0, \end{cases}$$

where $S_r = \sum_{j=1}^{k} \Phi_j\left(A_j^r\right)$ and $S_0 = \sum_{j=1}^{k} \Phi_j(\ln A_j)$. It is easy to see that $S(r, s, \mathbf{A}, \mathbf{\Phi})$ is also well defined.

Theorem 9.6: *Let $r, s \in \mathbb{R}$, $r < s$.*
(i) If either $r \leq -1$ or $s \geq 1$, then

$$\widetilde{M}_r(\boldsymbol{A}, \boldsymbol{\Phi}) \leq \widetilde{M}_s(\boldsymbol{A}, \boldsymbol{\Phi}).$$

(ii) If $-1 < r$ and $s < 1$, then

$$\widetilde{M}_r(\boldsymbol{A}, \boldsymbol{\Phi}) \leq \Delta(m, M, s) \widetilde{M}_s(\boldsymbol{A}, \boldsymbol{\Phi}).$$

If we use inequalities (9.7) instead of the (9.6), then we have the following results.

Theorem 9.7: *Let $r, s \in \mathbb{R}$, $r < s$.*

(*i*) *If* $s \geq 1$, *then*

$$\widetilde{M}_r(\boldsymbol{A}, \boldsymbol{\Phi}) \leq S(r, s, \boldsymbol{A}, \boldsymbol{\Phi}) \leq \widetilde{M}_s(\boldsymbol{A}, \boldsymbol{\Phi}).$$

If $r \leq -1$, *then*

$$\widetilde{M}_r(\boldsymbol{A}, \boldsymbol{\Phi}) \leq S(s, r, \boldsymbol{A}, \boldsymbol{\Phi}) \leq \widetilde{M}_s(\boldsymbol{A}, \boldsymbol{\Phi}).$$

(*ii*) *If* $-1 < r$ *and* $s < 1$, *then*

$$\frac{1}{\Delta(m, M, s)}\widetilde{M}_r(\boldsymbol{A}, \boldsymbol{\Phi}) \leq S(r, s, \boldsymbol{A}, \boldsymbol{\Phi}) \leq \Delta(m, M, s)\widetilde{M}_s(\boldsymbol{A}, \boldsymbol{\Phi}).$$

Remark 9.1: Some considerations in Theorem 9.6 and Theorem 9.7 can be shortened using the obvious properties $\widetilde{M}_{-s}\left(\boldsymbol{A}^{-1}, \boldsymbol{\Phi}\right) = \widetilde{M}_s(\boldsymbol{A}, \boldsymbol{\Phi})^{-1}$ and $S(-s, -r, \boldsymbol{A}^{-1}, \boldsymbol{\Phi}) = S(s, r, \boldsymbol{A}, \boldsymbol{\Phi})^{-1}$, where $\boldsymbol{A}^{-1} = \left(A_1^{-1}, \ldots, A_k^{-1}\right)$.

Remark 9.2: Since obviously $S(r, r, \boldsymbol{A}, \boldsymbol{\Phi}) = \widetilde{M}_r(\boldsymbol{A}, \boldsymbol{\Phi})$, inequalities in Theorem 9.7 (*i*) give us

$$S(r, r, \boldsymbol{A}, \boldsymbol{\Phi}) \leq S(r, s, \boldsymbol{A}, \boldsymbol{\Phi}) \leq S(s, s, \boldsymbol{A}, \boldsymbol{\Phi}), \ r < s, \ s \geq 1$$

and

$$S(r, r, \boldsymbol{A}, \boldsymbol{\Phi}) \leq S(s, r, \boldsymbol{A}, \boldsymbol{\Phi}) \leq S(s, s, \boldsymbol{A}, \boldsymbol{\Phi}), \ r < s, \ r \leq -1.$$

An open problem is to give list of inequalities comparing "mixed means" $S(r, s, \mathbf{A}, \boldsymbol{\Phi})$ in remaining cases.

It is easy to see that for any $r, s \in \mathbb{R}$,

$$R(r, s, \boldsymbol{A}, \boldsymbol{\Phi}) := \begin{cases} \left[\sum_{j=1}^k \Phi_j \left(\left[(m^r + M^r)\, 1_K - A_j^r\right]^{\frac{s}{r}}\right)\right]^{\frac{1}{s}}, & r \neq 0, s \neq 0, \\ \exp\left(\sum_{j=1}^k \Phi_j \left(\ln\left[(m^r + M^r)\, 1_K - A_j^r\right]^{\frac{1}{r}}\right)\right), & r \neq 0, s = 0, \\ \left[\sum_{j=1}^k \Phi_j \left(\exp s\left[(\ln mM)\, 1_K - \ln A_j\right]\right)\right]^{\frac{1}{s}}, & r = 0, s \neq 0 \end{cases}$$

is well defined and that $R(r, r, \boldsymbol{A}, \boldsymbol{\Phi}) = S(r, r, \boldsymbol{A}, \boldsymbol{\Phi}) = \widetilde{M}_r(\boldsymbol{A}, \boldsymbol{\Phi})$ (including the case when $r = 0$). To simplify notations, in what follows we will write $\widetilde{M}_r, R(r, s)$,$S(r, s)$ instead of $\widetilde{M}_r(\mathbf{A}, \boldsymbol{\Phi}), R(r, s, \mathbf{A}, \boldsymbol{\Phi})$,$S(r, s, \mathbf{A}, \boldsymbol{\Phi})$, respectively.

The following figure illustrates regions $(i) - (vii)$ which determine the seven cases occurring in Theorem 9.8 (compare with the figure on p. 117 in [100]).

Theorem 9.8: *Let* $r, s \in \mathbb{R}$, $r < s$.
(*i*) *If* $1 \leq r$, *then*

$$\widetilde{M}_r \leq S(s, r) \leq R(s, r) \leq \widetilde{M}_s.$$

(*ii*)*If* $s \leq -1$, *then*

$$\widetilde{M}_r \leq R(r, s) \leq S(r, s) \leq \widetilde{M}_s.$$

(iii) If $r \leq -1$ and $s \geq 1$, then

$$\begin{aligned}\widetilde{M}_r &\leq R(r,-1) \leq S(r,-1) \leq \widetilde{M}_{-1} \leq S(1,-1)\\ &\leq R(1,-1) \leq \widetilde{M}_1 \leq S(s,1) \leq R(s,1) \leq \widetilde{M}_s.\end{aligned}$$

(iv) If $\frac{1}{2} < r < 1 < s$, then

$$\widetilde{M}_r \leq R(r,1) \leq S(r,1) \leq \widetilde{M}_1 \leq S(s,1) \leq R(s,1) \leq \widetilde{M}_s.$$

(v) If $r < -1 < s < -\frac{1}{2}$, then

$$\widetilde{M}_r \leq R(r,-1) \leq S(r,-1) \leq \widetilde{M}_{-1} \leq S(s,-1) \leq R(s,-1) \leq \widetilde{M}_s.$$

(vi) If $-1 < r \leq \frac{1}{2}$ and $s \geq 1$, or $-s \leq r < s \leq 1$, then

$$\widetilde{M}_r \leq \Delta\left(m,M,r\right) S(s,r) \leq \Delta\left(m,M,r\right)^2 R(s,r) \leq \Delta\left(m,M,r\right)^3 \widetilde{M}_s.$$

(vii) If $r \leq -1$ and $-\frac{1}{2} \leq s < 1$, or $-1 \leq r < s \leq -r$, then

$$\widetilde{M}_r \leq \Delta\left(m,M,s\right) R(r,s) \leq \Delta\left(m,M,s\right)^2 S(r,s) \leq \Delta\left(m,M,s\right)^3 \widetilde{M}_s.$$

Remark 9.3: Besides these results in Theorem 9.8, one can prove in the same way that for $r < s < 2r$ and $s > 1$,

$$\widetilde{M}_r \leq R(r,s) \leq S(r,s) \leq \widetilde{M}_s$$

holds, and that for $r < s < \frac{1}{2}r$ and $r < -1$,

$$\widetilde{M}_r \leq S(s,r) \leq R(s,r) \leq \widetilde{M}_s$$

also holds, but to include these cases in the figure we should compare sequences of inequalities in common regions (see Remark 9.4).

Remark 9.4: If we define by $M_r(\boldsymbol{A},\boldsymbol{\Phi}) := \left(\sum_{j=1}^{k}\Phi_j(A_j^r)\right)^{\frac{1}{r}}$ (the weighted power mean), by $\widetilde{M}_r(B) := (m^r 1 + M^r 1 - B^r)^{\frac{1}{r}}$ (the Mercer mean for positive invertible operator B with $Sp(B) \subset [m,M]$, $0 < m < M$) and by $\widetilde{M}_r(\boldsymbol{A}) := (\widetilde{M}_r(A_1),\ldots,\widetilde{M}_r(A_k))$ (for an $k-$tuple $\boldsymbol{A}$ of positive invertible operators), we can write:

$$\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) = \widetilde{M}_r(M_r(\boldsymbol{A},\boldsymbol{\Phi}))$$

$$R(r,s,\boldsymbol{A},\boldsymbol{\Phi}) = M_s(\widetilde{M}_r(\boldsymbol{A}),\boldsymbol{\Phi}),$$

so we can describe inequalities in Theorem 9.8 as mixed mean inequalities. We can also state the following open problem: What is the complete set of inequalities among mixed means $M_r(\widetilde{M}_s(\boldsymbol{A}),\boldsymbol{\Phi})$, $\widetilde{M}_s(M_r(\boldsymbol{A},\boldsymbol{\Phi}))$, $\widetilde{M}_r(M_s(\boldsymbol{A},\boldsymbol{\Phi}))$ and $M_s(\widetilde{M}_r(\boldsymbol{A}),\boldsymbol{\Phi})$ Some special cases are given in Theorem 9.8 and Remark 9.3. Also, it is easy to see that

$$\widetilde{M}_r(M_r(\boldsymbol{A},\boldsymbol{\Phi})) \leq \widetilde{M}_s(M_r(\boldsymbol{A},\boldsymbol{\Phi}))$$

reduces to monotonicity property of Mercer's means, and that in some cases,

$$\widetilde{M}_s(M_s(\boldsymbol{A},\boldsymbol{\Phi})) \leq \widetilde{M}_s(M_r(\boldsymbol{A},\boldsymbol{\Phi}))$$

reduces to inequalities between $\left(\sum_{j=1}^{k}\Phi_j(A_j^r)\right)^{s/r}$ and $\sum_{j=1}^{k}\Phi(A_j^s)$.

9.2.2 Chaotic Order Among Power Operator Means of Mercer's Type

The result of this subsection can be found in [170].

Definition 9.1: *For positive invertible operators $A, B \in \mathcal{B}(H)$, we denote by $A \gg B$ if $\ln A \geq \ln B$ and we call it the **chaotic order**.*

Remark 9.5: The chaotic order is based on the fact that $\ln t$ is operator monotone on $\langle 0, +\infty\rangle$, by which it is weaker than the usual operator order $\geq$.

Let $\boldsymbol{A}$, $\boldsymbol{\Phi}$, $\widetilde{M}_r$, $R(r,s)$ and $S(r,s)$ be as in the previous subsection.

Theorem 9.9: *If $r, s \in \mathbb{R}$, $r < s$, then*

$$\widetilde{M}_r \ll \widetilde{M}_s.$$

Proof: See [170]. $\square$

Theorem 9.10: *Let $r, s \in \mathbb{R}$, $r < s$.*
(i) If $r \geq 0$, then

$$\widetilde{M}_r \ll S(s,r) \ll R(s,r) \ll \widetilde{M}_s.$$

(ii) If $s \leq 0$, then

$$\widetilde{M}_r \ll R(r,s) \ll S(r,s) \ll \widetilde{M}_s.$$

(iii) If $r < 0 < s$, then

$$\widetilde{M}_r \ll R(r,0) \ll S(r,0) \ll \widetilde{M}_0 \ll S\,(s,0) \ll R\,(s,0) \ll \widetilde{M}_s.$$

Proof: See [170]. $\square$

Remark 9.6: If we define by $\widetilde{M}_r(B) = (m^r 1 + M^r 1 - B^r)^{\frac{1}{r}}$ and $\widetilde{M}_r(\boldsymbol{A})$ as in Remark 9.4, we can write:

$$\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) = \widetilde{M}_r(M_r(\boldsymbol{A},\boldsymbol{\Phi}))$$

$$R(r,s,\boldsymbol{A},\boldsymbol{\Phi}) = M_s(\widetilde{M}_r(\boldsymbol{A}),\boldsymbol{\Phi}),$$

so we can describe inequalities in Theorem 9.10 as mixed mean inequalities. One can also ask the question: What is the complete set of inequalities among mixed means $M_r(\widetilde{M}_s(\boldsymbol{A}),\boldsymbol{\Phi})$, $\widetilde{M}_s(M_r(\boldsymbol{A},\boldsymbol{\Phi}))$, $\widetilde{M}_r(M_s(\boldsymbol{A},\boldsymbol{\Phi}))$ and $M_s(\widetilde{M}_r(\boldsymbol{A}),\boldsymbol{\Phi})$ under the chaotic order One part of the answer is in Theorems 9.9 and 9.10.

9.2.3 Quasi-Arithmetic Operator Means of Mercer's Type

Let $\mathbf{A}$ and $\mathbf{\Phi}$ be as in the previous subsection. Let $\varphi, \psi \in C([m, M])$ be strictly monotonic functions on an interval $[m, M]$. We define

$$\widetilde{M}_{\varphi}(\boldsymbol{A}, \boldsymbol{\Phi}) = \varphi^{-1}\left((\varphi(m) + \varphi(M)) 1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j))\right).$$

Theorem 9.11: *Under the above hypotheses, the following is valid.*

(i) If either $\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator increasing, or $\psi \circ \varphi^{-1}$ is concave and ψ^{-1} is operator decreasing, then

$$\widetilde{M}_{\varphi}(\boldsymbol{A}, \boldsymbol{\Phi}) \leq \widetilde{M}_{\psi}(\boldsymbol{A}, \boldsymbol{\Phi}). \tag{9.11}$$

In fact, to be more specific, we have the following series of inequalities

$$\begin{aligned}\widetilde{M}_{\varphi}(\boldsymbol{A}, \boldsymbol{\Phi}) \leq \psi^{-1}\Bigg(&\frac{\varphi(M) 1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j))}{\varphi(M) - \varphi(m)} \psi(M) \\ &+ \frac{\sum_{j=1}^{k} \Phi_j(\varphi(A_j)) - \varphi(m) 1_K}{\varphi(M) - \varphi(m)} \psi(m)\Bigg) \leq \widetilde{M}_{\psi}(\boldsymbol{A}, \boldsymbol{\Phi})\end{aligned} \tag{9.12}$$

(ii) If either $\psi \circ \varphi^{-1}$ is concave and ψ^{-1} is operator increasing, or $\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator decreasing, then inequalities (9.11) and (9.12) are reversed.

Proof: Suppose that $\psi \circ \varphi^{-1}$ is convex. If in Theorem 9.4 we let $f = \psi \circ \varphi^{-1}$ and replace A_j, m and M with $\varphi(A_j)$, $\varphi(m)$ and $\varphi(M)$ respectively, then we obtain

$$\begin{aligned}&\left(\psi \circ \varphi^{-1}\right)\left((\varphi(m) + \varphi(M)) 1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j))\right) \\ &\leq \frac{\varphi(M) 1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j))}{\varphi(M) - \varphi(m)} \left(\psi \circ \varphi^{-1}\right)(\varphi(M)) \\ &\quad + \frac{\sum_{j=1}^{k} \Phi_j(\varphi(A_j)) - \varphi(m) 1_K}{\varphi(M) - \varphi(m)} \left(\psi \circ \varphi^{-1}\right)(\varphi(m)) \\ &\leq \left(\left(\psi \circ \varphi^{-1}\right)(\varphi(m)) + \left(\psi \circ \varphi^{-1}\right)(\varphi(M))\right) 1_K - \sum_{j=1}^{k} \Phi_j\left(\left(\psi \circ \varphi^{-1}\right)(\varphi(A_j))\right),\end{aligned}$$

i.e.,

$$\begin{aligned}&\psi\left(\varphi^{-1}\left((\varphi(m) + \varphi(M)) 1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j))\right)\right) \\ &\leq \frac{\varphi(M) 1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j))}{\varphi(M) - \varphi(m)} \psi(M) + \frac{\sum_{j=1}^{k} \Phi_j(\varphi(A_j)) - \varphi(m) 1_K}{\varphi(M) - \varphi(m)} \psi(m) \\ &\leq (\psi(m) + \psi(M)) 1_K - \sum_{j=1}^{k} \Phi_j(\psi(A_j)).\end{aligned} \tag{9.13}$$

If $\psi \circ \varphi^{-1}$ is concave then we obtain the reverse of inequalities (9.13).

If ψ^{-1} is operator increasing, then (9.13) implies (9.12). If ψ^{-1} is operator decreasing, then the reverse of (9.13) implies (9.12). Analogously, we get the reverse of (9.12) in the cases when $\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator decreasing, or $\psi \circ \varphi^{-1}$ is concave and ψ^{-1} is operator increasing. □

Theorem 9.12: *Under the assumptions of Theorem 9.11, the following is valid.*

(*i*) *If either $\psi \circ \varphi^{-1}$ is operator convex and ψ^{-1} is operator increasing, or $\psi \circ \varphi^{-1}$ is operator concave and ψ^{-1} is operator decreasing, then*

$$\begin{aligned}
&\widetilde{M}_{\varphi}(\boldsymbol{A}, \boldsymbol{\Phi}) \\
&\leq \psi^{-1}\left(\sum_{j=1}^{k} \Phi_j\left(\left(\psi \circ \varphi^{-1}\right)\left(\left(\varphi(m)+\varphi(M)\right) 1_H-\varphi\left(A_j\right)\right)\right)\right) \\
&\leq \psi^{-1}\left(\frac{\varphi(M) 1_K-\sum_{j=1}^{k} \Phi_j\left(\varphi\left(A_j\right)\right)}{\varphi(M)-\varphi(m)} \psi(M)\right. \\
&\quad\left.+\frac{\sum_{j=1}^{k} \Phi_j\left(\varphi\left(A_j\right)\right)-\varphi(m) 1_K}{\varphi(M)-\varphi(m)} \psi(m)\right) \\
&\leq \widetilde{M}_{\psi}(\boldsymbol{A}, \boldsymbol{\Phi}) .
\end{aligned} \tag{9.14}$$

(*ii*) *If either $\psi \circ \varphi^{-1}$ is operator concave and ψ^{-1} is operator increasing, or $\psi \circ \varphi^{-1}$ is operator convex and ψ^{-1} is operator decreasing, then the reversed inequalities in (9.14) hold.*

Proof: Suppose that $\psi \circ \varphi^{-1}$ is operator convex. If in Theorem 9.5 we let $f=\psi \circ \varphi^{-1}$ and replace A_j, m and M with $\varphi(A_j)$, $\varphi(m)$ and $\varphi(M)$ respectively, then we have

$$\begin{aligned}
&\left(\psi \circ \varphi^{-1}\right)\left(\left(\varphi(m)+\varphi(M)\right) 1_K-\sum_{j=1}^{k} \Phi_j\left(\varphi\left(A_j\right)\right)\right) \\
&\leq \sum_{j=1}^{k} \Phi_j\left(\left(\psi \circ \varphi^{-1}\right)\left(\left(\varphi(m)+\varphi(M)\right) 1_H-\varphi\left(A_j\right)\right)\right) \\
&\leq \frac{\varphi(M) 1_K-\sum_{j=1}^{k} \Phi_j\left(\varphi\left(A_j\right)\right)}{\varphi(M)-\varphi(m)}\left(\psi \circ \varphi^{-1}\right)(\varphi(M)) \\
&\quad+\frac{\sum_{j=1}^{k} \Phi_j\left(\varphi\left(A_j\right)\right)-\varphi(m) 1_K}{\varphi(M)-\varphi(m)}\left(\psi \circ \varphi^{-1}\right)(\varphi(m)) \\
&\leq\left(\left(\psi \circ \varphi^{-1}\right)(\varphi(m))+\left(\psi \circ \varphi^{-1}\right)(\varphi(M))\right) 1_K-\sum_{j=1}^{k} \Phi_j\left(\left(\psi \circ \varphi^{-1}\right)\left(\varphi\left(A_j\right)\right)\right),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\psi\left(\varphi^{-1}\left(\left(\varphi(m)+\varphi(M)\right)1_K-\sum_{j=1}^{k}\Phi_j\left(\varphi\left(A_j\right)\right)\right)\right)\\
&\le\sum_{j=1}^{k}\Phi_j\left(\psi\left(\varphi^{-1}\left(\left(\varphi\left(m\right)+\varphi\left(M\right)\right)1_H-\varphi\left(A_j\right)\right)\right)\right)\\
&\le\frac{\varphi\left(M\right)1_K-\sum_{j=1}^{k}\Phi_j\left(\varphi\left(A_j\right)\right)}{\varphi\left(M\right)-\varphi\left(m\right)}\psi\left(M\right)+\frac{\sum_{j=1}^{k}\Phi_j\left(\varphi\left(A_j\right)\right)-\varphi\left(m\right)1_K}{\varphi\left(M\right)-\varphi\left(m\right)}\psi\left(m\right)\\
&\le\left(\psi\left(m\right)+\psi\left(M\right)\right)1_K-\sum_{j=1}^{k}\Phi_j\left(\psi\left(A_j\right)\right).
\end{aligned}\tag{9.15}$$

If $\psi\circ\varphi^{-1}$ is operator concave then we have reversed inequalities (9.15).

If ψ^{-1} is operator increasing, then (9.15) implies (9.14). If ψ^{-1} is operator decreasing, then the reverse of (9.15) implies (9.14). Analogously, we get the reverse of (9.14) in the cases when $\psi\circ\varphi^{-1}$ is operator convex and ψ^{-1} is operator decreasing, or $\psi\circ\varphi^{-1}$ is operator concave and ψ^{-1} is operator increasing. □

Theorem 9.13: *Under the assumptions of Theorem 9.11, the following is valid.*

(i) If either φ is operator concave and φ^{-1} is operator increasing or φ is operator convex and φ^{-1} is operator decreasing, and either ψ is operator convex and ψ^{-1} is operator increasing or ψ is operator concave and ψ^{-1} is operator decreasing, then

$$\begin{aligned}
\widetilde{M}_{\varphi}\left(\boldsymbol{A},\boldsymbol{\Phi}\right)&\le\varphi^{-1}\left(\frac{M1_K-\sum_{j=1}^{k}\Phi_j\left(A_j\right)}{M-m}\varphi(M)+\frac{\sum_{j=1}^{k}\Phi_j\left(A_j\right)-m1_K}{M-m}\varphi(m)\right)\\
&\le\varphi^{-1}\left(\sum_{j=1}^{k}\Phi_j\left(\varphi\left(\left(m+M\right)1_H-A_j\right)\right)\right)\le\widetilde{M}_1\left(\boldsymbol{A},\boldsymbol{\Phi}\right)\\
&\le\psi^{-1}\left(\sum_{j=1}^{k}\Phi_j\left(\psi\left(\left(m+M\right)1_H-A_j\right)\right)\right)\\
&\le\psi^{-1}\left(\frac{M1_K-\sum_{j=1}^{k}\Phi_j\left(A_j\right)}{M-m}\psi(M)+\frac{\sum_{j=1}^{k}\Phi_j\left(A_j\right)-m1_K}{M-m}\psi(m)\right)\le\widetilde{M}_{\psi}\left(\boldsymbol{A},\boldsymbol{\Phi}\right).
\end{aligned}\tag{9.16}$$

(ii) If either φ is operator convex and φ^{-1} is operator increasing or φ is operator concave and φ^{-1} is operator decreasing, and either ψ is operator concave and ψ^{-1} is operator increasing or ψ is operator convex and ψ^{-1} is operator decreasing, then the reversed inequalities in (9.16) hold.

Proof: Suppose that φ is operator concave and φ^{-1} is operator increasing, and ψ is operator convex and ψ^{-1} is operator increasing. By Theorem 9.5, we have

$$\begin{aligned}
&\varphi\left((m+M)1_K-\sum_{j=1}^{k}\Phi_j(A_j)\right)\\
&\geq \sum_{j=1}^{k}\Phi_j\left(\varphi\left((m+M)1_H-A_j\right)\right)\\
&\geq \frac{M1_K-\sum_{j=1}^{k}\Phi_j(A_j)}{M-m}\varphi(M)+\frac{\sum_{j=1}^{k}\Phi_j(A_j)-m1_K}{M-m}\varphi(m)\\
&\geq \left(\varphi(m)+\varphi(M)\right)1_K-\sum_{j=1}^{k}\Phi_j\left(\varphi(A_j)\right).
\end{aligned}$$

Since φ^{-1} is operator increasing, it follows that

$$\begin{aligned}
&\widetilde{M}_\varphi(\boldsymbol{A},\boldsymbol{\Phi})\leq \varphi^{-1}\left(\frac{M1_K-\sum_{j=1}^{k}\Phi_j(A_j)}{M-m}\cdot\varphi(M)+\frac{\sum_{j=1}^{k}\Phi_j(A_j)-m1_K}{M-m}\cdot\varphi(m)\right)\\
&\leq \varphi^{-1}\left(\sum_{j=1}^{k}\Phi_j\left(\varphi\left(m1_H+M1_H-A_j\right)\right)\right)\leq \widetilde{M}_1(\boldsymbol{A},\boldsymbol{\Phi}).
\end{aligned}$$

Also, by Theorem 9.5, we have

$$\begin{aligned}
&\psi\left((m+M)1_K-\sum_{j=1}^{k}\Phi_j(A_j)\right)\\
&\leq \sum_{j=1}^{k}\Phi_j\left(\psi\left((m+M)1_H-A_j\right)\right)\\
&\leq \frac{M1_K-\sum_{j=1}^{k}\Phi_j(A_j)}{M-m}\psi(M)+\frac{\sum_{j=1}^{k}\Phi_j(A_j)-m1_K}{M-m}\psi(m)\\
&\leq \left(\psi(m)+\psi(M)\right)1_K-\sum_{j=1}^{k}\Phi_j\left(\psi(A_j)\right).
\end{aligned}$$

Since ψ^{-1} is operator increasing, it follows that

$$\begin{aligned}
&\widetilde{M}_1(\boldsymbol{A},\boldsymbol{\Phi})\leq \psi^{-1}\left(\sum_{j=1}^{k}\Phi_j\left(\psi\left((m+M)1_H-A_j\right)\right)\right)\\
&\leq \psi^{-1}\left(\frac{M1_K-\sum_{j=1}^{k}\Phi_j(A_j)}{M-m}\psi(M)+\frac{\sum_{j=1}^{k}\Phi_j(A_j)-m1_K}{M-m}\psi(m)\right)\leq \widetilde{M}_\psi(\boldsymbol{A},\boldsymbol{\Phi}).
\end{aligned}$$

Hence, we have inequalities (9.16). In remaining cases the proof is analogous. □

9.2.4 Index Set Function and Refinements of the Jensen–Mercer Operator Inequality

We give refinements of the inequality

$$f\left((m+M)1_H - \frac{1}{W_k}\sum_{j=1}^{k} w_j A_j\right) \le (f(m)+f(M))1_H - \frac{1}{W_k}\sum_{j=1}^{k} w_j f(A_j), \tag{9.17}$$

which is a special case of the Jensen–Mercer operator inequality (9.6), and then we use them to refine some inequalities among power and quasi-arithmetic operator means of Mercer's type.

The following assertion is a special case of Theorem 9.5, when for $A \in \mathcal{B}(H)$ and $j \in \{1,2,\ldots,k\}$ positive linear maps $\Phi_j \in \boldsymbol{P}[\mathcal{B}(H),\mathcal{B}(H)]$ are given by $\Phi_j(A) = w_j A$, where w_j ($j \in \{1,2,\ldots,k\}$) are positive real numbers.

Results of present subsection are taken from [171].

Theorem 9.14: *If $f \in C([m,M])$ is an operator convex function on $[m,M]$, then*

$$\begin{aligned} & f\left((m+M)1_H - \frac{1}{W_k}\sum_{j=1}^{k} w_j A_j\right) \\ & \le \frac{1}{W_k}\sum_{j=1}^{k} w_j f((m+M)1_H - A_j) \\ & \le (f(m)+f(M))1_H - \frac{1}{W_k}\sum_{j=1}^{k} w_j f(A_j). \end{aligned} \tag{9.18}$$

If a function f is operator concave, then the inequalities in (9.18) are reversed.

Let I be a finite nonempty set of positive integers. Let $A_i\,(i \in I)$ be selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in $[m,M]$ for some scalars $m < M$, and let $w_i\,(i \in I)$ be positive real numbers. Observe that spectra of $\frac{1}{W_I}\sum_{i\in I} w_i A_i$ is also contained in $[m,M]$.

If we define the index set function F as

$$F(I) = W_I\left[(f(m)+f(M))1_H - \frac{1}{W_I}\sum_{i\in I} w_i f(A_i) - f\left((m+M)1_H - \frac{1}{W_I}\sum_{i\in I} w_i A_i\right)\right],$$

then the following theorem is valid.

Theorem 9.15: *Let I and J be finite nonempty sets of positive integers such that $I \cap J = \varnothing$. Let $w_i\,(i \in I \cup J)$ be positive real numbers and let $A_i\,(i \in I \cup J)$ be selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in $[m,M]$. If $f \in C([m,M])$ is an operator convex function, then*

$$F(I \cup J) \ge F(I) + F(J). \tag{9.19}$$

If a function f is operator concave, then the inequality in (9.19) is reversed.

Corollary 9.1: *Let $I_1,\ldots,I_k$ be finite nonempty sets of positive integers such that $I_i\cap I_j=\varnothing$, for all $i\neq j\in\{1,\ldots,k\}$. Let $w_i\left(i\in\cup_{j=1}^k I_j\right)$ be positive real numbers and $A_i\left(i\in\cup_{j=1}^k I_j\right)$ selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in $[m,M]$. If $f\in C([m,M])$ is an operator convex function, then*

$$F\left(\bigcup_{j=1}^k I_j\right)\geq\sum_{j=1}^k F(I_j). \tag{9.20}$$

If a function f is operator concave, then the inequality in (9.20) is reversed.

The following corollaries give refinements of inequality (9.17).

Corollary 9.2: *Let $I_k=\{1,\ldots,k\}$ $(k=1,\ldots,n)$. Let $w_i\,(i\in I_n)$ be positive real numbers and $A_i\,(i\in I_n)$ selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in $[m,M]$. If $f\in C([m,M])$ is an operator convex function, then*

$$F(I_n)\geq F(I_{n-1})\geq\cdots\geq F(I_2)\geq F(I_1)\geq 0. \tag{9.21}$$

If a function f is operator concave, then the inequalities in (9.21) are reversed.

Corollary 9.3: *Let $I_k=\{1,\ldots,k\}$ $(k=1,\ldots,n)$. Let $w_i\,(i\in I_n)$ be positive real numbers and $A_i\,(i\in I_n)$ selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in $[m,M]$. If $f\in C([m,M])$ is an operator convex function, then*

$$\begin{aligned}F(I_n)\geq(w_i+w_j)\Bigg[&(f(m)+f(M))\,1_H-\frac{w_if(A_i)+w_jf(A_j)}{w_i+w_j}\\&-f\left((m+M)\,1_H-\frac{w_iA_i+w_jA_j}{w_i+w_j}\right)\Bigg],\end{aligned} \tag{9.22}$$

for all $1\leq i<j\leq n$ and

$$F(I_n)\geq w_i\left[(f(m)+f(M))\,1_H-f(A_i)\,f((m+M)\,1_H-A_i)\right], \tag{9.23}$$

for all $1\leq i\leq n$. If a function f is operator concave, then the inequalities in (9.22) and (9.23) are reversed.

If we define the (weighted) power operator mean of Mercer's type for selfadjoint operators $A_i\in\mathcal{B}(H)$ with spectra contained in $[m,M]$ $(0<m<M)$, and positive real numbers w_i $(i=1,\ldots,n)$, as

$$\widetilde{M}_n^{[r]}:=\begin{cases}\left((m^r+M^r)\,1_H-\frac{1}{W_n}\sum_{i=1}^n w_iA_i^r\right)^{\frac{1}{r}}, & r\neq 0,\\ \exp\left((\ln mM)\,1_H-\frac{1}{W_n}\sum_{i=1}^k w_i\ln(A_i)\right), & r=0,\end{cases}$$

then we have the following results.

Theorem 9.16:

$$W_n\left(\ln \widetilde{M}_n^{[0]}-\ln \widetilde{M}_n^{[1]}\right) \leq W_{n-1}\left(\ln \widetilde{M}_{n-1}^{[0]}-\ln \widetilde{M}_{n-1}^{[1]}\right) \leq \cdots \leq W_1\left(\ln \widetilde{M}_1^{[0]}-\ln \widetilde{M}_1^{[1]}\right) \leq 0.$$

Corollary 9.4:

$$W_n\left(\ln \widetilde{M}_n^{[-1]}-\ln \widetilde{M}_n^{[0]}\right) \leq W_{n-1}\left(\ln \widetilde{M}_{n-1}^{[-1]}-\ln \widetilde{M}_{n-1}^{[0]}\right) \leq \cdots \leq W_1\left(\ln \widetilde{M}_1^{[-1]}-\ln \widetilde{M}_1^{[0]}\right) \leq 0.$$

Theorem 9.17: *If $r \leq -1$ or $\frac{1}{2} \leq r \leq 1$, then*

$$W_n\left(\widetilde{M}_n^{[1]}-\widetilde{M}_n^{[r]}\right) \geq W_{n-1}\left(\widetilde{M}_{n-1}^{[1]}-\widetilde{M}_{n-1}^{[r]}\right) \geq \cdots \geq W_1\left(\widetilde{M}_1^{[1]}-\widetilde{M}_1^{[r]}\right) \geq 0. \tag{9.24}$$

If $r \geq 1$, then the inequalities in (9.24) are reversed.

Theorem 9.18: *Let $r, s \in \mathbb{R}$, $r \leq s$.*

(*i*) *If $0<r$ and $s \leq 2r$, or $0<s \leq -r$, then*

$$W_n\left(\left(\widetilde{M}_n^{[s]}\right)^s-\left(\widetilde{M}_n^{[r]}\right)^s\right) \geq W_{n-1}\left(\left(\widetilde{M}_{n-1}^{[s]}\right)^s-\left(\widetilde{M}_{n-1}^{[r]}\right)^s\right) \geq \cdots \geq W_1\left(\left(\widetilde{M}_1^{[s]}\right)^s-\left(\widetilde{M}_1^{[r]}\right)^s\right) \geq 0. \tag{9.25}$$

(*ii*) *If $s<0$, then the inequalities in (9.25) are reversed.*

(*iii*) *If $s<0$ and $2s \leq r$, or $0<-r \leq s$, then*

$$W_n\left(\left(\widetilde{M}_n^{[r]}\right)^r-\left(\widetilde{M}_n^{[s]}\right)^r\right) \geq W_{n-1}\left(\left(\widetilde{M}_{n-1}^{[r]}\right)^r-\left(\widetilde{M}_{n-1}^{[s]}\right)^r\right) \geq \cdots \geq W_1\left(\left(\widetilde{M}_1^{[r]}\right)^r-\left(\widetilde{M}_1^{[s]}\right)^r\right) \geq 0. \tag{9.26}$$

(*iv*) *If $0<r$, then the inequalities in (9.26) are reversed.*

Let $\varphi \in C([m, M])$ be strictly monotonic function on an interval $[m, M]$. If we define the quasi-arithmetic operator mean of Mercer's type for selfadjoint operators $A_i \in \mathcal{B}(H)$ with spectra contained in $[m, M]$ $(0<m<M)$, with positive weights w_i $(i=1, \ldots, n)$, as

$$\widetilde{M}_{\varphi}^{[n]}:=\varphi^{-1}\left((\varphi(m)+\varphi(M)) 1_H-\frac{1}{W_n} \sum_{i=1}^n w_i \varphi\left(A_i\right)\right),$$

then we have the following results.

Theorem 9.19: *Let $\varphi, \psi \in C([m, M])$ be strictly monotonic functions. If $\psi \circ \varphi^{-1}$ is operator* *convex, then*

$$W_n\left(\psi\left(\widetilde{M}_{\psi}^{[n]}\right)-\psi\left(\widetilde{M}_{\varphi}^{[n]}\right)\right)\geq W_{n-1}\left(\psi\left(\widetilde{M}_{\psi}^{[n-1]}\right)-\psi\left(\widetilde{M}_{\varphi}^{[n-1]}\right)\right) \geq\cdots\geq W_1\left(\psi\left(\widetilde{M}_{\psi}^{[1]}\right)-\psi\left(\widetilde{M}_{\varphi}^{[1]}\right)\right)\geq 0. \tag{9.27}$$

If $\psi\circ\varphi^{-1}$ is operator concave, then the inequalities in (9.27) are reversed.

Remark 9.7: Theorems 9.16, 9.17 and 9.18 follow from Theorem 9.19 by ch-oosing adequate functions φ and ψ, and appropriate substitutions.

Theorem 9.20: *Let $\varphi,\psi\in C([m,M])$ be strictly monotonic functions. If $\psi\circ\varphi^{-1}$ is operator convex, then*

$$W_n\left(\psi\left(\widetilde{M}_{\psi}^{[n]}\right)-\psi\left(\widetilde{M}_{\varphi}^{[n]}\right)\right)\geq (w_i+w_j)\left[(\psi(m)+\psi(M))1_H-\frac{w_i\psi(A_i)+w_j\psi(A_j)}{w_i+w_j} -\left(\psi\circ\varphi^{-1}\right)\left((\varphi(m)+\varphi(M))1_H-\frac{w_i\varphi(A_i)+w_j\varphi(A_j)}{w_i+w_j}\right)\right], \tag{9.28}$$

for all $1\leq i<j\leq n$ and

$$W_n\left(\psi\left(\widetilde{M}_{\psi}^{[n]}\right)-\psi\left(\widetilde{M}_{\varphi}^{[n]}\right)\right)\geq w_i\left[(\psi(m)+\psi(M))1_H-\psi(A_i) -\left(\psi\circ\varphi^{-1}\right)((\varphi(a)+\varphi(M))1_H-\varphi(A_i))\right], \tag{9.29}$$

for all $1\leq i\leq n$. If $\psi\circ\varphi^{-1}$ is operator concave, then the inequalities in (9.28) and (9.29) are reversed.

Remark 9.8: From Theorem 9.20, analogous assertions for the power means $\widetilde{M}_n^{[r]}$ follow by choosing adequate functions φ and ψ, and appropriate substitutions.

9.2.5 Jensen–Mercer Operator Inequality for Superquadratic Functions

We give an analogue of Theorem 9.4 for superquadratic functions, and we use that result to derive some refinements of the inequalities among power operator means of Mercer's type and quasi-arithmetic operator means of Mercer's type. The results of this section are selected from [33].

Theorem 9.21: *Let $A\in\mathcal{B}(H)$ be a selfadjoint operator with spectra in $[m,M]$ for some scalars $0<m<M$ and $\Phi\in\mathbf{P}[\mathcal{B}(H),\mathcal{B}(K)]$ a unital positive linear map. If $f\in C([0,\infty))$ is superquadratic, then*

$$\begin{aligned}&f((m+M)1_K-\Phi(A))\\&+\frac{1}{M-m}((\Phi(A)-m1_K)f(M1_K-\Phi(A))\\&+(M1_K-\Phi(A))f(\Phi(A)-m1_K))\\&\leq(f(m)+f(M))1_K-\Phi(f(A))\\&-\frac{1}{M-m}(\Phi((A-m1_H)f(M1_H-A))+\Phi((M1_H-A)f(A-m1_H)))\end{aligned} \tag{9.30}$$

holds. If f is subquadratic, then the inequality in (9.30) is reversed.

Corollary 9.5: *Let $A_1, \dots, A_k \in \mathcal{B}(H)$ be selfadjoint operators with spectra in $[m, M]$ for some scalars $0 < m < M$ and $\Phi_1, \dots, \Phi_k \in \mathbf{P}[\mathcal{B}(H), \mathcal{B}(K)]$ positive linear maps with $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$. If $f \in C([0, \infty))$ is superquadratic, then*

$$
\begin{aligned}
& f\left((m+M)1_K - \sum_{j=1}^{k} \Phi_j(A_j)\right) \\
& + \frac{1}{M-m}\left(\sum_{j=1}^{k} \Phi_j(A_j) - m1_K\right) f\left(M1_K - \sum_{j=1}^{k} \Phi_j(A_j)\right) \\
& + \frac{1}{M-m}\left(M1_K - \sum_{j=1}^{k} \Phi_j(A_j)\right) f\left(\sum_{j=1}^{k} \Phi_j(A_j) - m1_K\right) \\
& \le (f(m) + f(M))1_K - \sum_{j=1}^{k} \Phi_j(f(A_j)) \qquad (9.31) \\
& - \frac{1}{M-m} \sum_{j=1}^{k} \Phi_j\left((A_j - m1_H) f(M1_H - A_j)\right) \\
& - \frac{1}{M-m} \sum_{j=1}^{k} \Phi_j\left((M1_H - A_j) f(A_j - m1_H)\right)
\end{aligned}
$$

holds. If f is subquadratic, then the inequality in (9.31) is reversed.

Remark 9.9: If a function f is superquadratic and nonnegative, then it is convex [5, Lemma 2.2]. Hence, in this case inequality (9.31) is a refinement of inequality (9.6) since these inequalities are invariant under a transposition $f \mapsto f + \alpha$.

For further remarks and observations we refer the reader to [33].

Here we state another important result:

Theorem 9.22: *Let $r, s \in \mathbb{R}$, $2r \le s$.*
(i) If $r > 0$ and $s \ge 1$, then

$$\widetilde{M}_r(\boldsymbol{A}, \boldsymbol{\Phi}) \le \left(\left[\widetilde{M}_s(\boldsymbol{A}, \boldsymbol{\Phi})\right]^s - \square(m, M, r, s, \boldsymbol{A}, \boldsymbol{\Phi})\right)^{\frac{1}{s}}.$$

(ii) If $s \le -1$, then

$$\widetilde{M}_r(\boldsymbol{A}, \boldsymbol{\Phi}) \le \left(\left[\widetilde{M}_s(\boldsymbol{A}, \boldsymbol{\Phi})\right]^s - \square(M, m, r, s, \boldsymbol{A}, \boldsymbol{\Phi})\right)^{\frac{1}{s}}.$$

(iii) If $r > 0$ and $s < 1$, then

$$\widetilde{M}_r(\boldsymbol{A}, \boldsymbol{\Phi}) \le \Delta(m, M, r)\left(\left[\widetilde{M}_s(\boldsymbol{A}, \boldsymbol{\Phi})\right]^s - \square(m, M, r, s, \boldsymbol{A}, \boldsymbol{\Phi})\right)^{\frac{1}{s}}.$$

(iv) If $-1 < s < 0$*, then*

$$\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \leq \Delta(m,M,r)\left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^s - \square\left(M,m,r,s,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{s}}.$$

Theorem 9.23: *Let* $r,s \in \mathbb{R}$*,* $2r \geq s$*.*
(i) If $s \geq 1$*, then reversed inequality* (9.22) *holds.*
(ii) If $r < 0$ *and* $s \leq -1$*, then reversed inequality* (9.22) *holds.*
(iii) If $0 < s < 1$*, then*

$$\Delta(m,M,r)\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \geq \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^s - \square\left(m,M,r,s,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{s}}.$$

(iv) If $r < 0$ *and* $s > -1$*, then*

$$\Delta(m,M,r)\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \geq \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^s - \square\left(M,m,r,s,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{s}}.$$

Theorem 9.24: *Let* $r,s \in \mathbb{R}$*,* $2s \leq r$*.*
(i) If $s > 0$ *and* $r \geq 1$*, then*

$$\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \geq \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^r + \square\left(m,M,s,r,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{r}}.$$

(ii) If $r \leq -1$*, then*

$$\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \geq \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^r + \square\left(M,m,s,r,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{r}}.$$

(iii) If $s > 0$ *and* $r < 1$*, then*

$$\Delta\left(m,M,r\right)\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \geq \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^r + \square\left(m,M,s,r,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{r}}.$$

(iv) If $-1 < r < 0$*, then*

$$\Delta\left(m,M,r\right)\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \geq \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^r + \square\left(M,m,s,r,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{r}}.$$

Theorem 9.25: *Let* $r,s \in \mathbb{R}$*,* $2s \geq r$*.*
(i) If $r \geq 1$*, then reversed inequality* (9.24) *holds.*
(ii) If $s < 0$ *and* $r \leq -1$*, then reversed inequality* (9.24) *holds.*
(iii) If $0 < r < 1$*, then*

$$\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \leq \Delta\left(m,M,r\right)\left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^r + \square\left(m,M,s,r,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{r}}.$$

(iv) If $s < 0$ *and* $r > -1$*, then*

$$\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \leq \Delta\left(m,M,r\right)\left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^r + \square\left(M,m,s,r,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{r}}.$$

Remark 9.10: Notice that some areas in the last four theorems have common parts. In some of them we can establish double inequalities. For example, for $r,s \in \mathbb{R}$ such that $2r \leq s \leq -1$

and $2s \leq r \leq -1$ we have

$$\left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^r + \square\left(M,m,s,r,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{r}} \leq \widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \leq \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^s - \square\left(M,m,r,s,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{s}}.$$

Theorem 9.26: *Let $r,s \in \mathbb{R}$, $2r \leq s$.*
(i) If $r > 0$, then

$$\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \ll \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^s - \square\left(m,M,r,s,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{s}}. \tag{9.32}$$

(ii) If $s < 0$, then

$$\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \ll \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^s - \square\left(M,m,r,s,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{s}}. \tag{9.33}$$

Theorem 9.27: *Let $r,s \in \mathbb{R}$, $2r \geq s$.*
(i) If $s > 0$, then reversed inequality (9.32) holds.
(ii) If $r < 0$, then reversed inequality (9.33) holds.

Theorem 9.28: *Let $r,s \in \mathbb{R}$, $2s \leq r$.*
(i) If $s > 0$, then

$$\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \gg \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^r + \square\left(m,M,s,r,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{r}}. \tag{9.34}$$

(ii) If $r < 0$, then

$$\widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \gg \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^r + \square\left(M,m,s,r,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{r}}. \tag{9.35}$$

Theorem 9.29: *Let $r,s \in \mathbb{R}$, $2s \geq r$.*
(i) If $r > 0$, then reversed inequality (9.34) holds.
(ii) If $s < 0$, then reversed inequality (9.35) holds.

Remark 9.11: Combining the last four theorems we can establish two double inequalities. If $0 < \frac{r}{2} \leq s \leq 2r$, then

$$\left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^r + \square\left(m,M,s,r,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{r}} \gg \widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \gg \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^s - \square\left(m,M,r,s,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{s}}.$$

If $2r \leq s \leq \frac{r}{2} < 0$, then

$$\left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^r + \square\left(M,m,s,r,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{r}} \ll \widetilde{M}_r(\boldsymbol{A},\boldsymbol{\Phi}) \ll \left(\left[\widetilde{M}_s(\boldsymbol{A},\boldsymbol{\Phi})\right]^s - \square\left(M,m,r,s,\boldsymbol{A},\boldsymbol{\Phi}\right)\right)^{\frac{1}{s}}.$$

Theorem 9.30: *Let $\varphi \in C([m,M])$ be strictly increasing and nonnegative and let $\psi \in C([m,M])$ be strictly monotonic.*
(i) If either $\psi \circ \varphi^{-1}$ is superquadratic and ψ^{-1} is operator increasing, or $\psi \circ \varphi^{-1}$ is subquadratic

and ψ^{-1} is operator decreasing, then

$$\widetilde{M}_{\varphi}(\mathbf{A}, \mathbf{\Phi}) \le \psi^{-1}\left(\psi\left(\widetilde{M}_{\psi}(\mathbf{A}, \mathbf{\Phi})\right) - \square(m, M, \varphi, \psi, \boldsymbol{A}, \mathbf{\Phi})\right). \tag{9.36}$$

(ii) If either $\psi \circ \varphi^{-1}$ is subquadratic and ψ^{-1} is operator increasing, or $\psi \circ \varphi^{-1}$ is superquadratic and ψ^{-1} is operator decreasing, then the inequality in (9.36) is reversed.

Remark 9.12: If the function φ in Theorem 9.30 is strictly decreasing then the inequality (9.36) and its reversal also hold under the same assumptions, but with m and M interchanged.

Remark 9.13: Obviously, Theorems 9.22 – 9.29 follow from Theorem 9.30 and Remark 9.12 by choosing $\varphi(t) = t^r$ and $\psi(t) = t^s$, or vice versa.

9.2.6 Jensen–Mercer Operator Inequality and Continuous Fields of Operators

Let T be a locally compact Hausdorff space, and let $\mathcal{A}$ be a C^*-algebra of operators on a Hilbert space H. We say that a field $(x_t)_{t\in T}$ of operators in $\mathcal{A}$ is continuous if the function $t \to x_t$ is norm continuous on T. If in addition μ is a Radon measure on T and the function $t \to \|x_t\|$ is integrable, then we can form the Bochner integral $\int_T x_t \,\mathrm{d}\mu(t)$, which is the unique element in $\mathcal{A}$ such that $\varphi\left(\int_T x_t \,\mathrm{d}\mu(t)\right) = \int_T \varphi(x_t)\,\mathrm{d}\mu(t)$ for every linear functional φ in the norm dual $\mathcal{A}^*$, cf. [109].

Let $(\phi_t)_{t\in T}$ is a field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$ of operators on a Hilbert space K. We say that such a field is continuous if the function $t \to \phi_t(x)$ is continuous for every $x \in \mathcal{A}$. If in addition the field $t \to \phi_t(\mathbf{1})$ is integrable with integral $\mathbf{1}$, we say that $(\phi_t)_{t\in T}$ is *unital*. Hansen, Pečarić and Perić [107] proved the following general form of Jensen's operator inequality.

Theorem 9.31: *Let $f : I \to \mathbb{R}$ be an operator convex function defined on an interval I, and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras with $\mathcal{B}$ acting on a Hilbert space K. If $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then the inequality*

$$f\left(\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right) \le \int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t) \tag{9.37}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in I.

Note here that inequality (9.37) holds for the class of operator convex functions which is a proper subclass of the clas of convex function. We will prove that our general form of the Jensen–Mercer operator inequality holds for the larger class of all convex functions.

Theorem 9.32: *Let $f \in C([m, M])$ be a convex function, and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-*

algebras with $\mathcal{B}$ acting on a Hilbert space K. If $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : A \to B$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then the inequality

$$f\left((m+M)\mathbf{1} - \int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right) \le (f(m)+f(M))\mathbf{1} - \int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t) \tag{9.38}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in $[m,M]$.

Note that we also proved the following series of inequalities:

$$\begin{aligned}
& f\left((m+M)\mathbf{1} - \int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right) \\
& \le \frac{\int_T \phi_t(x_t)\,\mathrm{d}\mu(t) - m\mathbf{1}}{M-m} f(m) + \frac{M\mathbf{1} - \int_T \phi_t(x_t)\,\mathrm{d}\mu(t)}{M-m} f(M) \\
& \le (f(m)+f(M))\mathbf{1} - \int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t).
\end{aligned}$$

Next, we obtain refinement of inequality (9.38) for operator convex functions.

Theorem 9.33: *Let $f \in C([m,M])$ be an operator convex function, and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras with $\mathcal{B}$ acting on a Hilbert space K. If $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : A \to B$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then*

$$\begin{aligned}
& f\left((m+M)\mathbf{1} - \int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right) \\
& \le \int_T \phi_t(f((m+M)\mathbf{1} - x_t))\,\mathrm{d}\mu(t) \\
& \le (f(m)+f(M))\mathbf{1} - \int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t)
\end{aligned} \tag{9.39}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in $[m,M]$.

9.3 New Refinement of Jensen–Mercer Operator Inequality with Applications

Let H be a complex Hilbert space and $S(I)$ be the class of all self-adjoint bounded operators on H whose spectra are contained in an interval $I \subset \mathbb{R}$. The spectrum of a bounded operator A on H is denoted by $\mathrm{sp}(A)$. Let $I \subset \mathbb{R}$ be an interval then the function $f : S(I) \to \mathbb{R}$ is said to be operator-convex if f is continuous on $S(I)$ and

$$f(sA+tB) \le sf(A)+tf(B)$$

for all A, $B \in S(I)$ and for all positive numbers s and t such that $s+t=1$. The function f is called operator-concave on $S(I)$ if $-f$ is operator-convex on $S(I)$.

Theorem 9.34: *Jensen's operator inequality ([184]): Let $I \subset \mathbb{R}$ be an interval and $f : S(I) \to \mathbb{R}$ be an operator-convex function on I. If $A_i \in S(I)$ and $w_i > 0$; $i \in \{1, 2, \ldots, n\}$ such that $W_n = \sum\limits_{i=1}^{n} w_i$, then*

$$f\Big(\frac{1}{W_n}\sum_{i=1}^{n} w_i A_i\Big) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(A_i). \tag{9.40}$$

If f is an operator-concave function, then (9.40) is reversed.

A self-adjoint bounded operator A on H is called strictly positive if it is positive and invertible, or equivalently, $\mathrm{Sp}(A) \subset [m, M]$ for some $0 < m < M$. The power mean for strictly positive operators $\mathbf{A} := (A_1, \ldots, A_n)$ with positive weights $\mathbf{w} := (w_1, \ldots, w_n)$ is defined in [184] as

$$M_r(\mathbf{w}, \mathbf{A}) := \left(\frac{1}{W_n}\sum_{i=1}^{n} w_i A_i^r\right)^{\frac{1}{r}},$$

where $r \in \mathbb{R} \setminus \{0\}$ and $W_n = \sum\limits_{i=1}^{n} w_i$.

The next result is borrowed from [184] (see also [100]).

Theorem 9.35: *Let $I \subset \mathbb{R}$ be an interval and $\boldsymbol{A}$ be an n-tuple of strictly positive operators with positive weights $\boldsymbol{w} := (w_1, \ldots, w_n)$ such that $W_n = \sum\limits_{i=1}^{n} w_i$. Then*

$$M_s(\boldsymbol{w}, \boldsymbol{A}) \leq M_r(\boldsymbol{w}, \boldsymbol{A}); \quad r, s \in \mathbb{R} \setminus \{0\}$$

if either

(i) $s \leq r; r, s \notin (-1, 1)$ *or*
(ii) $\frac{1}{2} \leq s \leq 1 \leq r$ *or*
(iii) $s \leq -1 \leq r \leq -\frac{1}{2}$ *holds.*

The inequalities for convex function are widely studied. Jensen–Mercer operator inequality and refinements of the operator Jensen–Mercer inequality are studied in [118] and [154] respectively, where as a variant of the Jensen–Mercer operator inequality for superquadratic functions is discussed in [33]. We give a new refinement of Jensen–Mercer operator inequality for operator-convex in the next subsection. The contents of this section can be found in [143].

9.3.1 Refinement of Jensen–Mercer Operator Inequality

Let $\phi : I \to \mathbb{R}$ be an operator convex function on $I \subset \mathbb{R}$. If $A_i \in S(I)$ and $w_i > 0$ with $W_n = \sum\limits_{i=1}^{n} w_i = 1$ then for any nonempty proper subset J of $\{1, 2, \ldots, n\}$ we put $\bar{J} := \{1, 2, \ldots, n\} \setminus J$

and define $W_J = \sum_{i \in J} w_i$ and $W_{\bar{J}} = 1 - \sum_{i \in J} w_i$. Now for an operator convex function ϕ, the n-tuples $\mathbf{A} := (A_1, \ldots, A_n)$ and $\mathbf{w} := (w_1, \ldots, w_n)$ as above, we can define the following functional

$$D(\phi, \mathbf{w}, \mathbf{A}; J) := W_J \phi(\frac{1}{W_J} \sum_{i \in J} w_i A_i) + W_{\bar{J}} \phi(\frac{1}{W_{\bar{J}}} \sum_{i \in \bar{J}} w_i A_i).$$

In case if $J = \{k\}$, $k \in \{1, 2, \ldots, n\}$ then we have functional

$$D_k(\phi, \mathbf{w}, \mathbf{A}) := D(\phi, \mathbf{w}, \mathbf{A}; \{k\}) = w_k \phi(A_k) + (1 - w_k) \phi \left(\frac{\sum_{i=1}^{n} w_i A_i - w_k A_k}{1 - w_k} \right),$$

which has been investigated for convex function in [86] and earlier in [87].

Theorem 9.36: *Let $\phi : S(I) \to \mathbb{R}$ be an operator convex function on $I \subset \mathbb{R}$. If $A_i \in S(I)$ and $w_i > 0$; $i \in \{1, 2, \ldots, n\}$ with $W_n = \sum_{i=1}^{n} w_i = 1$ then for any nonempty proper subset J of $\{1, 2, \ldots, n\}$,*

$$\sum_{i=1}^{n} w_i \phi(A_i) \geq D(\phi, \boldsymbol{w}, \boldsymbol{A}; J) \geq \phi(\sum_{i=1}^{n} w_i A_i). \tag{9.41}$$

If ϕ is concave, then (9.41) is reversed.

Now for fixed values of k we assume the (non-empty) subsets $J_i^k, \ldots, J_l^k$ of $\{1, 2, \ldots, n\}$, $1 \leq l \leq k \leq n$ such that $J_i^k \cup \ldots \cup J_l^k = \{1, 2, \ldots, n\}$ and $J_i^k \cap J_j^k = \phi$ for $i \neq j$, where $i, j = 1, 2, \ldots, n$. Consider the functional

$$D_k^{J_1^k, \ldots, J_l^k}(\phi, \mathbf{w}, \mathbf{A}) := \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} \phi((a + b)I - \frac{1}{W_{J_j^k}} \sum_{i \in J_l^k} w_i A_i),$$

for the function $\phi : S([a, b]) \to \mathbb{R}$ where $\mathbf{A}$ is an n-tuple of selfadjoint operators defined on $[a, b] \subset \mathbb{R}$ and $w_i > 0$; $i \in \{1, 2, \ldots, n\}$ also $W_J = \sum_{i \in J} w_i$; $J \subset \{1, 2, \ldots, n\}$.

Theorem 9.37: *Let $\boldsymbol{A}$ be an n-tuple of selfadjoint operators defined on $[a, b] \subset \mathbb{R}$ with $w_i > 0$; $i \in \{1, 2, \ldots, n\}$ such that $W_n = \sum_{i=1}^{n} w_i$. If $\phi : S([a, b]) \to \mathbb{R}$ is an operator-convex function, then*

$$\phi(aI) + \phi(bI) - \frac{1}{W_n} \sum_{i=1}^{n} w_i \phi(A_i) \geq M_k \geq M_{k-1}$$
$$\geq \ldots \geq M_2 \geq \phi((a + b)I - \frac{1}{W_n} \sum_{i=1}^{n} w_i A_i), \tag{9.42}$$

where

$$M_k := \max_{J_1^k, \ldots, J_l^k} [D_k^{J_1^k, \ldots, J_l^k}(\phi, \boldsymbol{w}, \boldsymbol{A})]$$

If f is an operator-concave function, then (9.42) is reversed.

Theorem 9.38: *Let $\boldsymbol{A}$ be an n-tuple of selfadjoint operators defined on $[a,b] \subset \mathbb{R}$ with $w_i > 0$; $i \in \{1,2,\ldots,n\}$ such that $W_n = \sum_{i=1}^{n} w_i$. If $\phi : [a,b] \to \mathbb{R}$ is an operator-convex function, then we have*

$$\begin{aligned} &\phi(aI) + \phi(bI) - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i \phi(A_i) \geq H_k \geq H_{k-1} \geq \ldots \geq H_3 \geq H_2 \geq \\ &\geq \phi((a+b)I - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i A_i) \end{aligned} \tag{9.43}$$

where

$$H_k := \min_{J_1^k,\ldots,J_l^k} [D_k^{J_1^k,\ldots,J_l^k}(\phi, \boldsymbol{w}, \boldsymbol{A})]$$

If f is an operator-concave function then (9.43) is reversed.

Proof: Its proof is similar to Theorem 9.37 but we use the minimum of the functional for l index sets instead functionals for $l-1$ index set. □

9.3.1.1. Applications to Operator Power Means

Mixed-symmetric means according to (9.41), (9.42) and (9.43) are defined as

$$M(r,s;\mathbf{w},\mathbf{A}) := \left(W_J \left(\frac{1}{W_J} \sum_{i \in J} w_i A_i^r \right)^{\frac{s}{r}} + W_{\bar{J}} \left(\frac{1}{W_{\bar{J}}} \sum_{i \in \bar{J}} w_i A_i^r \right)^{\frac{s}{r}} \right)^{\frac{1}{s}},$$

$$M_n(s,r;\mathbf{w},\mathbf{A};k) := \left(\max_{J_1^k,\ldots,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} \left((a+b)I - \frac{1}{W_{J_j^k}} \sum_{i \in J_j^k} w_i A_i{}^r \right)^{\frac{s}{r}} \right)^{\frac{1}{s}}$$

and

$$H_n(s,r;\mathbf{w},\mathbf{A};k) := \left(\min_{J_1^k,\ldots,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} \left((a+b)I - \frac{1}{W_{J_j^k}} \sum_{i \in J_j^k} w_i A_i{}^r \right)^{\frac{s}{r}} \right)^{\frac{1}{s}},$$

where $r,s \in \mathbb{R} \setminus \{0\}$ and $1 \leq l \leq k \leq n$.

The idea of above means for convex functions is given in [184].

Corollary 9.6: *Let $\boldsymbol{A}$ be n-tuple of strictly positive operators with $w_i > 0$; $i \in \{1,2,\ldots,n\}$ such that $W_n = \sum_{i=1}^{n} w_i = 1$. Then for any nonempty proper subset J of $\{1,2,\ldots,n\}$, we have*

$$M_s(\boldsymbol{w},\boldsymbol{A}) \leq M(r,s;\boldsymbol{w},\boldsymbol{A}) \leq M_r(\boldsymbol{w},\boldsymbol{A}), \tag{9.44}$$

if either

(i) $1 \leq s \leq r$ *or*
(ii) $-r \leq s \leq -1$ *or*
(iii) $r \geq s \leq 2r,\ s \leq -1$

holds. While (9.44) is reversed if either

(*iv*) $r \leq s \leq -1$ *or*
(*v*) $1 \leq s \leq -r$ *or*
(*vi*) $r \leq s \leq 2r,\ s \geq 1$ *holds.*

Corollary 9.7: *Let* $[a,b] \subset \mathbb{R}$ *and* $\boldsymbol{A}$ *be* n*-tuple of strictly positive operators with* $w_i > 0$*;* $i \in \{1,2,\ldots,n\}$ *such that* $W_n = \sum_{i=1}^{n} w_i$*. Then we have*

$$((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_i {A_i}^r)^{\frac{1}{r}} \geq M_n(s,r;\mathbf{w},\mathbf{A};2) \geq \ldots \geq M_n(s,r;\mathbf{w},\mathbf{A};k) \geq$$

$$\geq ((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n}\sum_{i=1}^{n} w_i {A_i}^s)^{\frac{1}{s}} \quad (9.45)$$

if either

(*i*) $1 \leq s \leq r$ *or*
(*ii*) $-r \leq s \leq -1$ *or*
(*iii*) $r \geq s \leq 2r,\ s \leq -1$

holds. While (9.45) *is reversed if either*

(*iv*) $r \leq s \leq -1$ *or*
(*v*) $1 \leq s \leq -r$ *or*
(*vi*) $r \leq s \leq 2r,\ s \geq 1$ *holds.*

Corollary 9.8: *Let* $[a,b] \subset \mathbb{R}$ *and* $\boldsymbol{A}$ *be an* n*-tuple of strictly positive operators with* $w_i > 0$*;* $i \in \{1,2,\ldots,n\}$ *such that* $W_n = \sum_{i=1}^{n} w_i$*. Then we have*

$$((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_i {A_i}^r)^{\frac{1}{r}} \geq H_n(s,r;\mathbf{w},\mathbf{A};2) \geq \ldots \geq H_n(s,r;\mathbf{w},\mathbf{A};k)$$

$$\geq ((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n}\sum_{i=1}^{n} w_i {A_i}^s)^{\frac{1}{s}}, \quad (9.46)$$

if either

(*i*) $1 \leq s \leq r$ *or*
(*ii*) $-r \leq s \leq -1$ *or*
(*iii*) $r \geq s \leq 2r,\ s \leq -1$

holds. While (9.46) *is reversed if either*

(*iv*) $r \leq s \leq -1$ *or*
(*v*) $1 \leq s \leq -r$ *or*
(*vi*) $r \leq s \leq 2r,\ s \geq 1$ *holds.*

Now we introduce the following expressions for operator-convex functions in the same way as these are defined in [184] for convex functions:

$$g(\phi;p;\mathbf{A}) := \left\{ W_J\phi[\frac{1}{W_J}\sum_{i\in J} w_i A_i^p]^{\frac{1}{p}} + W_{\bar{J}}\phi[\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^p]^{\frac{1}{p}} \right\}^{\frac{1}{p}},$$

$$g_{k,n}(\phi;p;\mathbf{A}) := \left\{ \max_{J_1^k,\ldots,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} \phi[((a+b)I - \frac{1}{W_{J_j^k}} \sum_{i\in J_l^k} w_i A_i^p)^{\frac{1}{p}}]^p \right\}^{\frac{1}{p}}$$

and

$$h_{k,n}(\phi;p;\mathbf{A}) := \left\{ \min_{J_1^k,\ldots,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} \phi[((a+b)I - \frac{1}{W_{J_j^k}} \sum_{i\in J_l^k} w_i A_i^p)^{\frac{1}{p}}]^p \right\}^{\frac{1}{p}}.$$

Corollary 9.9: *Let $\boldsymbol{A}$ be an n-tuple of strictly positive operators with $w_i > 0$, $i \in \{1, 2, \ldots, n\}$ such that $W_n = \sum_{i=1}^{n} w_i = 1$, ϕ is a positive operator monotone function on $S(0, \infty)$ and J is any nonempty proper subset of $\{1, 2, \ldots, n\}$. Now if $p \geq 1$, then we have*

$$\left(\sum_{i=1}^{n} w_i \phi(A_i^p)\right)^{\frac{1}{p}} \leq g(\phi;p;\boldsymbol{A}) \leq \phi\left(\left(\sum_{i=1}^{n} w_i A_i^p\right)^{\frac{1}{p}}\right). \tag{9.47}$$

If $p \leq -1$, then (9.47) is reversed.

Corollary 9.10: *Let $I \subset \mathbb{R}$ be an interval and $\boldsymbol{A}$ be n-tuple of strictly positive operators with $w_i > 0$; $i \in \{1, 2, \ldots, n\}$ such that $W_n = \sum_{i=1}^{n} w_i$ and ϕ is a positive operator monotone function on $S(0, \infty)$. Now if $p \geq 1$, then we have*

$$\begin{aligned}(\phi((aI)^{\frac{1}{p}})^p + \phi((bI)^{\frac{1}{p}})^p - \frac{1}{W_n}\sum_{i=1}^{n} w_i \phi(A_i)^p)^{\frac{1}{p}} &\leq g_{k,n}(\phi;p;\boldsymbol{A}) \leq \\ \ldots \leq g_{2,n}(\phi;p;\boldsymbol{A}) &\leq \phi(((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_i {A_i}^p)^{\frac{1}{p}}),\end{aligned} \tag{9.48}$$

where $1 \leq l \leq k \leq n$. If $p \leq -1$, then (9.48) is reversed.

Corollary 9.11: *Let the conditions of Corollary 9.10 be satisfied. Then we have the following inequality*

$$\begin{aligned}(\phi((aI)^{\frac{1}{p}})^p + \phi((bI)^{\frac{1}{p}})^p - \frac{1}{W_n}\sum_{i=1}^{n} w_i \phi(A_i)^p)^{\frac{1}{p}} &\leq h_{k,n}(\phi;p;\boldsymbol{A}) \\ \leq \ldots \leq h_{2,n}(\phi;p;\boldsymbol{A}) &\leq \phi(((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_i {A_i}^p)^{\frac{1}{p}}),\end{aligned} \tag{9.49}$$

where $1 \leq l \leq k \leq n$. If $p \leq -1$, then (9.49) is reversed.

9.3.2 Refinements of Jensen–Mercer Type Operator Inequality: Hermite–Hadamard–Mercer Type Inequalities

In this section we present a Hermite–Hadamard type inequality using the Mercer inequality (1.3) and then give its operator extension.

Theorem 9.39: *Let f be a convex function on $[m, M]$. Then*

$$f\left(M+m-\frac{x+y}{2}\right) \leq f(M)+f(m)-\int_0^1 f(tx+(1-t)y)dt \leq f(M)+f(m)-f\left(\frac{x+y}{2}\right),$$

and

$$f\left(M+m-\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(M+m-t)dt \leq f(M)+f(m)-\frac{f(x)+f(y)}{2} \tag{9.50}$$

for all $x, y \in [m, M]$.

The following operator version of inequality (9.50) holds true.

Theorem 9.40: *If f is convex on $[m, M]$, then*

$$\int_0^1 f(M+m-(t\Phi(A)+(1-t)\Phi(B)))dt \leq f(M)+f(m)-\frac{\Phi(f(A))+\Phi(f(B))}{2}, \tag{9.51}$$

$$f\left(m+M-\frac{\Phi(A)+\Phi(B)}{2}\right) \leq f(M)+f(m)-\int_0^1 f(t\Phi(A)+(1-t)\Phi(B))dt \tag{9.52}$$

for all self-adjoint operators A, B with spectra in $[m, M]$ and a unital positive linear map Φ. Furthermore, if f is operator convex, then

$$\begin{aligned} f\left(M+m-\frac{\Phi(A)+\Phi(B)}{2}\right) &\leq \int_0^1 f(M+m-(t\Phi(A)+(1-t)\Phi(B)))dt \\ &\leq f(M)+f(m)-\frac{\Phi(f(A))+\Phi(f(B))}{2}. \end{aligned}$$

Example 9.1: *If $f : J \to \mathbb{R}$ is a convex function, then the inequality*

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tA+(1-t)B)dt \leq \frac{f(A)+f(B)}{2}$$

may not hold in general [190]. To see this consider the convex function $f(t) = t^4$ which appears in some counter-examples, starting with a work of M.-D. Choi [77], and Hermitian matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Nevertheless, inequalities (9.51) and (9.52) are valid for all $m, M \in J$ provided that the spectra of A and B are contained in $[m, M]$.

9.4 A Variant of the Jensen–Mercer Inequality for Operators

We use an idea from [237] to obtain one of the main results of present section.

Theorem 9.41: *Let A_i for $i \in \{1, 2, \ldots, n\}$ be positive operators acting on a finite dimensional*

Hilbert space $\mathscr{H}$ *with* $\sum_{i=1}^{n} A_i = I$*. If* f *is convex on an interval* $[m, M]$ *containing* 0*, then*

$$f\left(M + m - \sum_{i=1}^{n} x_i A_i\right) \leq f(M) + f(m) - \sum_{i=1}^{n} f(x_i) A_i \tag{9.53}$$

for all $x_1, \ldots, x_n \in [m, M]$.

Corollary 9.12: *Let* f *and* x_i *for* $i \in \{1, 2, \ldots, n\}$ *be as in Theorem* 9.41*. If* $\sum_{i=1}^{n} A_i \leq I$*, then inequality* (9.53) *remains true.*

The following particular case of (9.53) is of special interest.

Corollary 9.13: *Let* A_i *for* $i \in \{1, 2, \ldots, n\}$ *be positive operators acting on a finite dimensional Hilbert space* $\mathscr{H}$ *with* $\sum_{i=1}^{n} A_i = I$*. If* f *is convex on the interval* $[m, M]$ *containing* -1 *and* 1*, then*

$$f\left(M + m + \sum_{i=1}^{k} A_i - \sum_{i=k+1}^{n} A_i\right) \leq f(M) + f(m) - f(-1) \sum_{i=1}^{k} A_i - f(1) \sum_{i=k+1}^{n} A_i.$$

Remark 9.14: It should be mentioned that inequality (9.53) implies a weaker version of the Jensen–Mercer operator inequality. Assume that X_i for $i \in \{1, 2, \ldots, n\}$ are self-adjoint operators on a finite dimensional Hilbert space $\mathscr{H}$ with spectra in $[m, M]$ and $w_i \in [0, 1]$ with $\sum_{i=1}^{n} w_i = 1$. Let $X_i = \sum_{j=1}^{k} \lambda_{ij} P_{ij}$ $(i \in \{1, 2, \ldots, n\})$ be the spectral decomposition of X_i so that $\lambda_{ij} \in [m, M]$. It follows from $\sum_{i=1}^{n} \sum_{j=1}^{k} w_i P_{ij} = I$ that

$$\begin{aligned} f\left(M + m - \sum_{i=1}^{n} w_i X_i\right) &= f\left(M + m - \sum_{i=1}^{n} \sum_{j=1}^{k} w_i \lambda_{ij} P_{ij}\right) \\ &\leq f(M) + f(m) - \sum_{i=1}^{n} \sum_{j=1}^{k} f(\lambda_{ij}) w_i P_{ij} \quad (\text{by } (9.53)) \\ &= f(M) + f(m) - \sum_{i=1}^{n} w_i f\left(\sum_{j=1}^{k} \lambda_{ij} P_{ij}\right) \\ &= f(M) + f(m) - \sum_{i=1}^{n} w_i f(X_i). \end{aligned}$$

Theorem 9.42: *Let* $m < M$ *and let* Φ *be a positive linear map on* $\mathbb{B}(\mathscr{H})$ *with* $0 < \Phi(I) \leq I$*. Let*

$$m' = \min\{m\langle\Phi(I)x, x\rangle;\ \ \|x\| = 1\} \quad \textit{and} \quad M' = \max\{M\langle\Phi(I)x, x\rangle;\ \ \|x\| = 1\}.$$

If $f : J \to \mathbb{R}$ *is an operator convex function with* $f(0) \leq 0$ *and* $[m, M] \cup [m', M'] \subseteq J$*, Then*

$$f((m + M)\Phi(I) - \Phi(A)) \leq f(m) + f(M) - \Phi(f(A))$$

for any self-adjoint operator A *with spectrum contained in* $[m, M]$.

Corollary 9.14: *Let* f *and* $\mathscr{H}$ *be as in Theorem* 9.42*. Let* Φ_i $(i \in \{1, 2, \ldots, n\})$ *be positive* *linear maps on* $\mathbb{B}(\mathscr{H})$ *with* $0 < \Phi(I) = \sum_{i=1}^{n} \Phi_i(I) \leq I$. *Then*

$$f\left((m+M)\Phi(I) - \sum_{i=1}^{n} \Phi_i(A_i)\right) \leq f(m) + f(M) - \sum_{i=1}^{n} \Phi_i(f(A_i))$$

for all self-adjoint operators A_i *with spectra in* $[m, M]$.

The next theorem yields a Jensen–Mercer operator type inequality for separately operator convex functions.

Definition 9.2: We say that a function $f : [m, M] \times [m', M'] \longrightarrow \mathbb{R}$ is separately operator convex if the functions g_t and h_s defined by $g_t(s) = f(t, s) = h_s(t)$ are operator convex, respectively, on $[m', M']$ and $[m, M]$ for each $t \in [m, M]$ and $s \in [m', M']$.

Theorem 9.43: *Let* $f : [m, M] \times [m', M'] \longrightarrow \mathbb{R}$ *be a separately operator convex function. Let* Φ_i, Ψ_j, $(1 \leq i \leq r, 1 \leq j \leq k)$ *be positive linear maps on* $\mathbb{B}(\mathscr{H})$ *with* $\sum_{i=1}^{r} \Phi_i(I) = I = \sum_{j=1}^{k} \Psi_j(I)$. *Then*

$$\begin{aligned}
&f\left(M + m - \sum_{i=1}^{r} \Phi_i(A_i), M' + m' - \sum_{j=1}^{k} \Psi_j(B_j)\right) \\
&\leq f(m, m') + f(m, M') + f(M, m') + f(M, M') - 2f\left(\sum_{i=1}^{r} \Phi_i(A_i), \frac{M' + m'}{2}\right) \\
&- 2f\left(\frac{m+M}{2}, \sum_{j=1}^{k} \Psi_j(B_j)\right) + f\left(\sum_{i=1}^{r} \Phi_i(A_i), \sum_{j=1}^{k} \Psi_j(B_j)\right)
\end{aligned}$$

for all self-adjoint operators A_i *with spectra in* $[m, M]$ *and* B_j *with spectra in* $[m', M']$.

9.5 Operator h–Convex Functions and Related Results

Dragomir in [30] introduced an even more general definition of operator convex functions.

Definition 9.3: Let J be an interval included in $\mathbb{R}$ with $(0, 1) \subset J$. Let $h : J \to \mathbb{R}$ be a non-negative and identically nonzero function. We shall say that a continuous function $f : I \to \mathbb{R}$, defined on an interval $I \subset \mathbb{R}$, is an operator h-convex for operators in K if

$$f(tA + (1-t)B) \leq h(t)f(A) + h(1-t)f(B)$$

for all $t \in (0, 1)$ and $A, B \in K \subseteq B(H)^+$ such that $\mathrm{Sp}(A) \subset I$ and $\mathrm{Sp}(B) \subset I$.

Using this definition Dragomir proved that

Theorem 9.44: *Let* f *be an operator* h*-convex function. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{A + 2B}{2}\right) \leq \int_0^1 f(tB + (1-t)A)\, dt \leq (f(A) + f(B)) \int_0^1 h(t)\, dt$$

From [79], we recall some of the results as under:

Theorem 9.45: *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a positive and h-convex function, and let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers in I, and w_k $(1 \leq k \leq n)$ positive numbers such that $\sum_{k=1}^{n} w_k = 1$. Then we have*

$$f\left(x_1 + x_n - \sum_{k=1}^{n} w_k x_k\right) \leq M\left(f(x_1) + f(x_n)\right) - \sum_{k=1}^{n} h(w_k) f(x_k)$$

where $M = \sup\{h(t) : t \in (0,1)\}$.

Remark 9.15: If $h(t) = t$ we have $M = \sup\{h(t) : t \in (0,1)\} = 1$, so the inequality in Theorem 3.2 of [79] takes the form

$$f\left(x_1 + x_n - \sum_{k=1}^{n} w_k x_k\right) \leq f(x_1) + f(x_n) - \sum_{k=1}^{n} w_k f(x_k)$$

and it coincides with the Jensen–Mercer inequality (1.3).

Remark 9.16: If $h(t) = ts; (0 < s \leq 1)$ we have $M = \sup\{fts : t \in (0,1)\} = 1$, and so, for s-convex functions we have

$$f\left(x_1 + x_n - \sum_{k=1}^{n} w_k x_k\right) \leq f(x_1) + f(x_n) - \sum_{k=1}^{n} w_s^k f(x_k).$$

Corollary 9.15: *Let $A_1, A_2, \ldots, A_n \in B(H)$ be self-adjoint operators with spectra in $[a,b]$ for some scalars $a < b$ and w_k $(1 \leq k \leq n)$ positive numbers such that $\sum_{k=1}^{n} w_k = 1$. Then*

$$f\left(A_1 + A_n - \sum_{k=1}^{n} w_k A_k\right) \leq M\left(f(A_1) + f(A_n)\right) - \sum_{k=1}^{n} h(w_k) f(A_k)$$

where $M = \sup\{h(t) : t \in (0,1)\}$.

Theorem 9.46: *Let h be a super-additive function and $A_1, A_2, \ldots,\ A_n \in B(H)$ be self-adjoint operators with spectra in $[a,b]$ for some scalars $a < b$ and $\Phi_1, \Phi_2,\ \ldots, \Phi_n \in P[B(H); B(K)]$ positive linear maps with $\sum_{i=1}^{n} \Phi_i(I_H) = I_K$. If $f \in C([a,b])$ is an operator h-convex on $[a,b]$, then*

$$f\left(aI_K + bI_K - \sum_{i=1}^{n} \Phi_i(A_i)\right) \leq f(a)h(I_K) + f(b)h(I_K) - \sum_{i=1}^{n} \Phi_i(f(A_i)).$$

Let h be an integrable function and f a $h-$convex function on $[m, M]$. Then

$$\begin{aligned} f\left(M + m + \frac{x+y}{2}\right) &\leq f(M) + f(m) - \int_0^1 f(tx + (1-t)y)dt \\ &\leq f(M) + f(m) - 2h(1/2) f\left(\frac{x+y}{2}\right) \end{aligned}$$

and

$$f\left(M + m + \frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(M+m-t)dt \leq f(M) + f(m) - (f(x) + f(y))\int_0^1 h(t)dt$$

for all $x, y \in [m, M]$.

Theorem 9.47: *Let h be an integrable function and f a h-convex function on $[m, M]$; then*

$$\int_0^1 f\left(M + m - (t\Phi(A) + (1-t)\Phi(B))\right) dt \\ \leq f(M) + f(m) - (\Phi(f(A)) + \Phi(f(B))) \int_0^1 h(t)\, dt$$

and

$$f\left(M + m - \Phi(A) + \Phi(B)\right) \leq \frac{1}{2} \left\| 2h\left(\frac{1}{2}\right) \int_0^1 f\left(M + m - t\Phi(A) - (1-t)\Phi(B)\right) dt \right\|$$

for all self-adjoint operators A, B with spectra in $[m, M]$ and a unitary positive linear map Φ. Furthermore, if f is operator h-convex, then

$$f\left(M + m - \Phi(A) + \Phi(B)\right) \leq \int_0^1 f\left(M + m - (t\Phi(A) + (1-t)\Phi(B))\right) dt$$

$$\leq f(M) + f(m) - (\Phi(f(A)) + \Phi(f(B))) \int_0^1 h(t)\, dt$$

Remark 9.17: From last two results can get various inequalities by putting different values of $h(t)$ see for reference [79].

One can find further results on Mercer type inequalities for operator $h-$convex functions in [1].

9.6 An Operator Inequality and its Consequences

In their study of Hadamard's inequalities for co-ordinated convex functions, Hwang, Tseng and Yang proved that if $f : [a, b] \to \mathbb{R}$ is a convex function and $a \leq y_1 \leq x_1 \leq x_2 \leq y_2 \leq b$ and $x_1 + x_2 = y_1 + y_2$, then $f(x_1) + f(x_2) \leq f(y_1) + f(y_2)$.

In this section, we extend this inequality to operators acting on a Hilbert space and apply it to obtain a series of operator inequalities including Jensen–Mercer operator inequality, the Petrović operator inequality. For detailed discussion and introductory remarks see [189].

We start this section with our main result.

Theorem 9.48: *Let f be a continuous convex function on an interval J. Let $A, B, C, D \in \mathbb{B}(\mathscr{H})_h$ with spectra contained in J such that $A + D = B + C$ and $A \leq m \leq B, C \leq M \leq D$ for two real numbers $m < M$. If Φ is a unital positive linear map on $\mathbb{B}(\mathscr{H})$, then*

$$f(\Phi(B)) + f(\Phi(C)) \leq \Phi(f(A)) + \Phi(f(D)). \tag{9.54}$$

If f is concave on J, then inequality (9.54) is reversed.

Proof: For idea of the proof please see [189]. □

Remark 9.18: For examples and other related results we refer the reader to [189]. Here we state a couple of consequence of the above-stated result as under.

The Jensen–Mercer operator inequality follows directly from Theorem 9.48.

Corollary 9.16: *[168, Theorem 1] Let $\Phi_1, \cdots, \Phi_n$ be positive linear maps on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^{n} \Phi_i(I) = I$ and $B_1, \cdots, B_n \in \mathbb{B}(\mathscr{H})_h$ with spectra contained in $[m, M]$. If f is a continuous convex function on $[m, M]$, then*

$$f\left(m + M - \sum_{i=1}^{n} \Phi_i(B_i)\right) \leq f(m) + f(M) - \sum_{i=1}^{n} \Phi_i(f(B_i)).$$

The next result provides an extension of Petrović inequality.

Corollary 9.17: *Let $A, D, B_i \in \mathbb{B}(\mathscr{H})_h$ $(i = 1, \cdots, n)$ with spectra contained in an interval J such that $A + D = \sum_{i=1}^{n} B_i$ and $A \leq m \leq B_i \leq M \leq D$ $(i = 1, \cdots, n)$ for two real numbers $m < M$. If f is convex on J, then*

$$\sum_{i=1}^{n} f(B_i) \leq (n-1) f\left(\frac{1}{n-1} A\right) + f(D).$$

Remark 9.19: Some other important and similar results related to this section can also be found in [179] and [204, p. 154]).

Remark 9.20: For further related results we refer the reader to [198]

CHAPTER 10

On Niezgoda's Extension of Jensen–Mercer Inequality

This chapter presents Niezgoda's extension of the Jensen–Mercer inequality in discrete form. We first introduce an alternate form of Niezgoda's inequality, together with refinements and applications. The discussion then considers further generalizations and refinements of the Jensen–Mercer inequality, followed by the use of index set functions to obtain refined versions of the generalized Niezgoda inequality with applications. These results provide a systematic framework for discrete extensions and sharpenings of Mercer-type inequalities.

10.1 Niezgoda's Extension of Jensen–Mercer Inequality

The following extension of (1.3) is given in [196].

Theorem 10.1: *Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function. Suppose that $\boldsymbol{a} = (a_1, \ldots, a_m)$ with $a_j \in [a,b]$ and $\boldsymbol{X} = (x_{ij})$ is a real $n \times m$ matrix such that $x_{ij} \in [a,b]$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$.*

If $\boldsymbol{a}$ majorizes each row of $\boldsymbol{X}$, that is,

$$\boldsymbol{x}_{i.} = (x_{i1}, \ldots, x_{im}) \prec (a_1, \ldots, a_m) = \boldsymbol{a} \text{ for each } i \in \{1, \ldots, n\}, \tag{10.1}$$

then we have the inequality

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right) \leq \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f(x_{ij}), \tag{10.2}$$

where $\sum_{i=1}^{n} w_i = 1$ with $w_i \geq 0$.

Proof: Denote $\psi\mathbf{z} = \sum_{i=1}^{n} w_i z_i$ for $\mathbf{z} = (z_1, \ldots, z_n)^T$.

Evidently, $\psi \mathbf{1}_n = 1$. We can write

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \psi \mathbf{x}_{.j}\right) = f\psi\left(\sum_{j=1}^{m} a_j \mathbf{1}_n - \sum_{j=1}^{m-1} \mathbf{x}_{.j}\right) \leq \psi f\left(\sum_{j=1}^{m} a_j \mathbf{1}_n - \sum_{j=1}^{m-1} \mathbf{x}_{.j}\right). \tag{10.3}$$

By (10.1) we have $\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} x_{ij} = x_{im}$ for each $i \in \{1, \ldots, n\}$. Now applying f and using majorization theorem we get

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} x_{ij}\right) = f(x_{il}) \leq \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1} f(x_{ij}).$$

From this we obtain

$$f\left(\sum_{j=1}^{m} a_j \mathbf{1}_n - \sum_{j=1}^{m-1} \mathbf{x}_{.j}\right) \leq \sum_{j=1}^{m} f(a_j)\mathbf{1}_n - \sum_{j=1}^{m-1} f(\mathbf{x}_{.j}).$$

Hence

$$\psi f\left(\sum_{j=1}^{m} a_j \mathbf{1}_n - \sum_{j=1}^{m-1} \mathbf{x}_{.j}\right) \leq \psi\left(\sum_{j=1}^{m} f(a_j)\mathbf{1}_n - \sum_{j=1}^{m-1} f(\mathbf{x}_{.j})\right)$$
$$= \sum_{j=1}^{m} f(a_j)\psi\mathbf{1}_n - \sum_{j=1}^{m-1} \psi f(\mathbf{x}_{.j}) = \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1} \psi f(\mathbf{x}_{.j}).$$

Now combining above result with (10.3) yields (10.2). This completes the proof. □

Remark 10.1: Throughout this chapter, the inequality (10.2) would be referred as to Mercer-Niezgoda inequality.

10.1.1 Niezgoda's Inequality- an Alternate Form

The Niezgoda's inequality can be restated in an alternate form by using technique similar to article [196] (for complete proof see [132]).

Theorem 10.2: *Suppose that $\boldsymbol{a}$ be an m-tuple such that $a_i \in J$ and $\boldsymbol{X} = (\mathbf{x}_j) = (x_{ij})$ is a $n \times m$ matrix with $x_{ij} \in J$ $\forall$ $i \in I_n$ and $j \in I_m$. Let f be continuous convex function on J.*

If $\boldsymbol{a}$ majorizes each row of $\boldsymbol{X}$, then we have the inequality

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right)$$
$$\leq \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i f(x_{ij}) - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i f(x_{ij}) \tag{10.4}$$

where $\sum_{i=1}^{n} w_i = 1$ with $w_i \geq 0$ and $l \in \{1, 2, \ldots, m\}$.

Remark 10.2: (a) It is a straight forward observation that Theorem 10.1 and Theorem 10.2 are equivalent and from one we can deduce the other.

(b) Taking $l = m$, inequality (10.4) gives us inequality (10.2).

(c) The majorization preorder $\prec$ on $\mathbb{R}^n$ is permutation-invariant. That is, columns of the matrix $\mathbf{K}$ can be permuted, and Theorem 10.1 and consequentially the majorization preorder $\prec$ on $\mathbb{R}^n$ is permutation-invariant. Therefore columns of the matrix $\mathbf{K}$ can be permuted, and Theorem 10.1 still remain true.

(d) Although we have proved Theorem 10.2 by using Niezgoda technique from [196]. But ψ function is an extra assumption and this prove can be done without taking ψ function as we will show in our next chapter.

Remark 10.3: If $k = m = 2$, $a_1 = x_1, a_2 = x_n$ with $a_1 \leq a_2$ and $x_{i1} = x_1$ and $x_{i2} = a_1 + a_2 - x_i$ for $i \in \{1, \ldots, n\}$ then our result reduces to Jensen–Mercer inequality (1.3) (see also [151]).

10.1.2 Refinements

Refinements of inequalities through "D" functional remained a topic of interest for many mathematicians. Here we give a refinement to our previous result (10.2) by following the mode of [151]. This refinement will help us generating applications in our next section.

Let a convex function $f : J \to \mathbb{R}$. Suppose that $\mathbf{a} = (a_1, \ldots, a_m)$ with $a_j \in J$, and $\mathbf{X} = (x_{ij})$ is a real $n \times m$ matrix such that $x_{ij} \in J$ and $\mathbf{a}$ majorizes each row of $\mathbf{X}$ for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ with $\sum_{i=1}^{n} w_i = 1$. Then for any non-empty subset I of I_n we take $\overline{I} := I_n \backslash I \neq \emptyset$ and define $W_I = \sum_{i \in I} w_i$ and $W_{\overline{I}} = 1 - \sum_{i \in I} w_i$. For the convex function f and the matrix $\mathbf{K} = (x_{ij})$ for $i \in I$, $j \in I_m$ and $\mathbf{w} = (w_1, \ldots, w_n)$ as above, we can define following functional

$$
\begin{aligned}
D(\mathbf{w}, \mathbf{K}, f; I) := \quad & W_I f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i \in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i \in I} w_i x_{ij}\right) \\
& + W_{\overline{I}} f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_{\overline{I}}}\sum_{j=1}^{k-1}\sum_{i \in \overline{I}} w_i x_{ij} - \frac{1}{W_{\overline{I}}}\sum_{j=k+1}^{m}\sum_{i \in \overline{I}} w_i x_{ij}\right)
\end{aligned}
$$

Theorem 10.3: *Under the assumptions as stated above for any non empty subset I of I_n we have*

$$
\begin{aligned}
& f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right) \leq D(\mathbf{w}, \mathbf{K}, f; I) \\
& \qquad \leq \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i f(x_{ij}) - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i f(x_{ij})
\end{aligned}
$$

Remark 10.4: For proof of Theorem 10.3 and remarks and other related results we refer the reader to [132].

10.1.3 Applications

Investigating properties of means and establishing their relationship remained an area of interest for many mathematicians. For example, it is well known that

$$A_n \geq G_n \geq H_n,$$

$$\left(\frac{A_n}{G_n}\right)^{W_n} \geq \left(\frac{A_{n-1}}{G_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \left(\frac{A_1}{G_1}\right)^{W_1} \geq 1.$$

$$W_n(A_n - G_n) \geq W_{n-1}(A_{n-1} - G_{n-1}) \geq \cdots \geq W_1(A_1 - G_1) \geq 0.$$

Also we have the famous Ky Fan Inequality [35, p. 5] given by

$$\frac{A_n(\mathbf{x})}{A_n(\mathbf{1}-\mathbf{x})} \geq \frac{G_n(\mathbf{x})}{G_n(\mathbf{1}-\mathbf{x})}, \ 0 < x_{ij} \leq \frac{1}{2} \ \forall \ i, \ j.$$

It will be very interesting if we opt to give applications of our Theorem 10.2 in terms of weighted and power means. For this purpose, we need the following set of assumptions.

E: For $\emptyset \neq I \subseteq I_n$ A_I, G_I, H_I and $M_I^{[r]}$ are the arithmetic, geometric, harmonic means, and power mean of the order $r \in \mathbb{R}$, respectively of $x_{ij} \in [a, b] \in J$, where $0 < a < b$, formed along with the positive weights w_i for $i \in I$. Further, for $I = I_n$ denoting arithmetic, geometric, harmonic and power means as A_n, G_n, H_n and $M_n^{[r]}$ respectively.

If we define

$$\begin{aligned}
\tilde{A}_I : &= \sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij} \\
&= \sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} A_I(\mathbf{x}_j, \mathbf{w}) - \sum_{j=k+1}^{m} A_I(\mathbf{x}_j, \mathbf{w}) \\
\tilde{G}_I : &= \frac{\prod\limits_{i\in I} a_j}{\left(\prod\limits_{j=1}^{l-1}\prod\limits_{i\in I} x_{ij}^{w_i}\right)^{\frac{1}{W_I}} \left(\prod\limits_{j=l+1}^{m}\prod\limits_{i\in I} x_{ij}^{w_i}\right)^{\frac{1}{W_I}}} \\
\tilde{H}_I : &= \left(\sum_{j=1}^{m} {a_j}^{-1} - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij}^{-1} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}^{-1}\right)^{-1} \\
\tilde{M}_I^{[r]} : &= \begin{cases} \left(\sum_{j=1}^{m}(a_j)^r - M_I^{[r]}\right)^{\frac{1}{r}}, & r \neq 0, \\ \tilde{G}_I, & r = 0, \end{cases}
\end{aligned}$$

where

$$M_I^{[r]} := \begin{cases} \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij}^r + \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}^r, & r \neq 0, \\ \left(\prod_{i\in I} x_{ij}^{w_i}\right)^{\frac{1}{W_I}}, & r = 0. \end{cases}$$

It is worth mentioning that all the applicaitons as given in Chapter 2 related to refinement of Jensen–Mercer inequality can also be stated in this section. For detials see [151, 132].

10.2 Further Generalization and Refinements of Jensen–Mercer Inequality

Theorem 10.4: *Suppose that $\boldsymbol{a}$ be an m-tuple such that $a_j \in J$ for $j \in \{1,\ldots, m\}$ and $\boldsymbol{X} = (\mathbf{x}_j) = (x_{ij})$ is a $n \times m$ matrix such that $x_{ij} \in J$ $\forall i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,m\}$ and each sequence $\mathbf{x}_{i1}, \mathbf{x}_{i2}, \ldots, \mathbf{x}_{im}$ is nondecreasing. Let $\mathbf{w}$ be a n-tuple such that conditions on weights given in (A.6) are valid. Let f be continuous convex function on J. If*

$$\sum_{j=1}^{m} x_{ij} = \sum_{j=1}^{m} a_j \quad \forall i \in \{1,\ldots,n\} \tag{10.5}$$

and

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_{ij}) \leq f(a_j) \quad \forall j \in \{1,\ldots,m\}, \tag{10.6}$$

then we have the inequality

$$f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right)$$
$$\leq \sum_{j=1}^{m} f(a_j) - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i f(x_{ij}) - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i f(x_{ij}), \tag{10.7}$$

where $k \in \{1,\ldots,m\}$.

Proof: Fix $k \in \{1,\ldots,m\}$, using first Jensen-Steffensen's inequality and then using (10.6) we get,

$$\begin{aligned} & f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right) \\ & = f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^{m} x_{ij}\right)\right) \\ & \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^{m} x_{ij}\right) \\ & = \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(x_{ik}\right) \end{aligned}$$

$$\leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \left(\sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{k-1} f(x_{ij}) - \sum_{j=k+1}^{m} f(x_{ij})\right)$$
$$= \sum_{j=1}^{m} f(a_j) - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i f(x_{ij}) - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i f(x_{ij}).$$

□

Here we give a couple of remarks related to our first main result.

Remark 10.5: (a) It worth mentioning that Theorem 10.4 is stated for real weights w_i's satisfying conditions of Jensen-Steffensen (A.6) and we relaxed the condition of majorization (as given in the statement of Theorem 10.1) at the expense of conditions (10.5) and (10.6).

(b) Our main inequality (10.7) has close connection with the inequality (3) of [132] (see Theorem 4 there) which was proved for positive weights and we assumed weights to be real and not necessarily all positive.

(c) If $W_n = 1$ and $k = m$ in the inequality (10.7), then the sum $\sum_{j=k+1}^{m}$ becomes zero; in this case the inequality coincides with the inequality (10.2) for real weights. Here we do not claim that this is true generalization of Neizgoda's inequality (10.2) but it captures the inequality with slightly different assumptions involving real weights instead of nonnegative weights.

(d) If in the inequality (10.7) we set $k = m = 2$, $a_1 = a$, $a_2 = b$ and $x_{i1} = x_i$ for $i \in \{1, \ldots, n\}$, then this inequality reduces to the inequality (1.3) for real weights as stated in Theorem 1.2 and hence the part of results of Theorem 2 of [8] is a special case of Theorem 10.4. Further by imposing different conditions on weights we easily obtain Theorems 1.1 and 1.2.

(e) For further remarks see [132].

10.2.1 Index Set Functions and Refinements of Generalized Niezgoda's Inequality

In start of this section we give some construction which we would use throughout this section: Let I be a finite nonempty set of positive integers. Let $\mathbf{w} = (w_i)$, $i \in I$ be a real sequence and let $(\mathbf{x}_j) = (x_{ij})$ be a sequence of vectors such that $x_{ij} \in J$ $\forall i \in I$, $j \in \{1, \ldots, m\}$. Moreover we define $A_I(\mathbf{x}_j, \mathbf{w}) = \frac{1}{W_I}\sum_{i\in I} w_i x_{ij}$ where $W_I = \sum_{i\in I} w_i$. For the generalized Jensen–Mercer inequality (10.7), we define the index set function F as

$$F(I) = W_I\left[\sum_{j=1}^{m} f(a_j) - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i f(x_{ij}) - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i f(x_{ij}) - f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}\right)\right] \quad (10.8)$$

where $\mathbf{a}$ is an m-tuple such that $a_j \in J$ for $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, m\}$.

Here we state results analogues to results stated in Section 2.5. For proof kindly see [135]. Throughout this section $I_n = \{1, \ldots, n\}$.

Theorem 10.5: *Let $\boldsymbol{a}$ be an m-tuple such that $a_j \in J$ for $j \in \{1, \ldots, m\}$, I and I' be nonempty sets such that $I \bigcup I' = I_n$ and $I \cap I' = \emptyset$. Let $(\mathbf{x}_j) = (x_{ij})$ be a sequence of vectors such that $x_{ij} \in J\ \forall i \in I$, $j \in \{1, \ldots, m\}$ and $\mathbf{w} = (w_i)$, $i \in I \cup I'$ such that $W_{I\cup I'} > 0$. Let $A_S(\mathbf{x}_j, \mathbf{w}) \in J$ $(S \in \{I, I', I \cup I'\})$. If $W_I > 0$ and $W_{I'} > 0$, then under the assumptions of Theorem 10.4*

$$F(I \cup I') \geq F(I) + F(I'). \tag{10.9}$$

If $W_I \cdot W_{I'} < 0$, then the inequality (10.9) is reversed.

Corollary 10.1: *Let $\boldsymbol{a}$ be an m-tuple such that $a_j \in J$ for $j \in \{1, \ldots, m\}$. Let I_t, $t \in \{1, \ldots, l\}$ be finite nonempty sets of positive integers such that $I_s \cap I_t = \emptyset$ for all $s \neq t \in \{1, \ldots, l\}$. We further suppose that $(\mathbf{x}_j) = (x_{ij})$ be a real sequence of vectors such that $x_{ij} \in J\ \forall\, i \in \bigcup_{t=1}^{l} I_t$, $j \in \{1, \ldots, m\}$ and let $\mathbf{w} = (w_i)$, $i \in \bigcup_{t=1}^{l} I_t$ such that $W_{i\in\bigcup_{t=1}^{l} I_t} > 0$ and $A_S(\mathbf{x}_j, \mathbf{w}) \in J$ $(S \in \{I_1, \ldots, I_t, \bigcup_{t=1}^{r} I_t\})$ $(r \in \{2, \ldots, l\})$. Then under the assumptions of Theorem 10.4 we have*

(a) If $W_{I_t} > 0$ for $t \in \{1, \ldots, l\}$,

$$F\left(\bigcup_{t=1}^{l} I_t\right) \geq \sum_{t=1}^{l} F(I_t). \tag{10.10}$$

(b) If $W_{I_1} > 0$ and $W_{I_t} < 0$ for $t \in \{2, \ldots, l\}$, then the inequality (10.10) is reversed.

Following results give us refinements of Niezgoda's Inequality. For the rest of this section we assume $x_{ij} \in [a, b] \subseteq J$ for all values of i and j.

Corollary 10.2: *Let $\boldsymbol{a}$ be an m-tuple such that $a_j \in J$ for $j \in \{1, \ldots, m\}$. Let $I_k = \{1, \ldots, k\}$, $k \in \{1, \ldots, n\}$. We further suppose that $(\mathbf{x}_j) = (x_{ij})$ be a real sequence of vectors such that $x_{ij} \in J\ \forall\ i \in I_n$, $j \in \{1, \ldots, m\}$ and if $w_1 > 0$ and $w_i \geq 0$ for $i \in \{2, \ldots, n\}$, then under assumptions of Theorem 10.4 we have*

$$F(I_n) \geq F(I_{n-1}) \geq \cdots \geq F(I_2) \geq F(I_1) \geq 0.$$

If $w_i \leq 0$ for $i \in \{2, \ldots, n\}$, $W_{I_n} > 0$ and $A_{I_n}(\mathbf{x}_j, \mathbf{w}) \in [a, b] \subseteq J$, then

$$0 \leq F(I_n) \leq F(I_{n-1}) \leq \cdots \leq F(I_2) \leq F(I_1).$$

Corollary 10.3: *Let all the assumptions of Corollary 10.2 be valid. If $w_i > 0$ for $i \in \{1, \ldots, n\}$, then*

$$\begin{aligned} F(I_n) \geq \max_{1\leq s\leq t\leq n} &\left[(w_s + w_t)\left[\sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{k-1} \frac{w_s f(x_{sj}) + w_t f(x_{tj})}{w_s + w_t}\right.\right. \\ &\left.\left.- \sum_{j=k+1}^{m} \frac{w_s f(x_{sj}) + w_t f(x_{tj})}{w_s + w_t} - f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} \frac{w_s x_{sj} + w_t x_{tj}}{w_s + w_t} - \sum_{j=k+1}^{m} \frac{w_s x_{sj} + w_t x_{tj}}{w_s + w_t}\right)\right]\right] \end{aligned}$$

and

$$F(I_n) \geq \max_{1\leq t\leq n}\left[w_t\left[\sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{k-1} f(x_{tj}) - \sum_{j=k+1}^{m} f(x_{tj}) - f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} x_{tj} - \sum_{j=k+1}^{m} x_{tj}\right)\right]\right].$$

If $w_i \leq 0$ *for* $i \in \{2, \dots, n\}$ *with* $W_{I_n} > 0$ *and* $A_{I_n}(\mathbf{x}_j, \mathbf{w}) \in [a, b]$, *then*

$$F(I_n) \leq \min_{2\leq t\leq n}\left[(w_1+w_t)\left[\sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{k-1} \frac{w_1 f(x_{1j}) + w_t f(x_{tj})}{w_1+w_t} - \sum_{j=k+1}^{m} \frac{w_1 f(x_{1j}) + w_t f(x_{tj})}{w_1+w_t} - f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} \frac{w_1 x_{1j} + w_t x_{tj}}{w_1+w_t} - \sum_{j=k+1}^{m} \frac{w_1 x_{1j} + w_t x_{tj}}{w_1+w_t}\right)\right]\right].$$

Remark 10.6: If in Theorem 10.5 we set $k = m = 2$, $a_1 = L$, $a_2 = M$ and $x_{i1} = x_i$ for $i \in \{1, \dots, n\}$ and in its corollaries, then we obtain Theorem 2.1 and Corollaries 2.4, 2.5 and 2.6 of [167] as special case of our results.

Throughout this section we assume that $I \subseteq I_n$ unless stated otherwise. Now we give refinement of (10.7) as follows.

Theorem 10.6: *Let all the assumptions of Theorem 10.4 be valid. Then the following refinement hold:*

$$f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right) \leq D(\mathbf{w}, \mathbf{X}, f; I) \leq \sum_{j=1}^{m} f(a_j) - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i f(x_{ij}) - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i f(x_{ij}),$$

where $W_I = \sum_{i\in I} w_i$, $W_{\overline{I}} = \sum_{i\in \overline{I}} w_i$, $\overline{I} = I_n \backslash I$ *and* $k \in \{1, \dots, m\}$, *and*

$$\begin{aligned} D(\mathbf{w}, \mathbf{X}, f; I) &= \frac{W_I}{W_n} f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}\right) \\ &+ \frac{W_{\overline{I}}}{W_n} f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_{\overline{I}}}\sum_{j=1}^{k-1}\sum_{i\in \overline{I}} w_i x_{ij} - \frac{1}{W_{\overline{I}}}\sum_{j=k+1}^{m}\sum_{i\in \overline{I}} w_i x_{ij}\right). \end{aligned}$$

Remark 10.7: For other related results and remarks we refer the reader to [135].

10.2.2 Applications

H: For $\emptyset \neq I \subseteq I_n = \{1, \dots, n\}$, $x_{ij} \in [a, b] \subseteq J$ for $j \in \{1, \dots, m\}$ and $i \in I$, where $0 < a < b$, formed with weights w_i, $i \in I$ satisfies conditions stated in ($A.6$) we define the arithmetic, geometric, harmonic means and power mean of order $r \in \mathbb{R}$ as A_I, G_I, H_I and $M_I^{[r]}$ respectively. While For $I = I_n$ we denote the arithmetic, geometric, harmonic and power means by A_n, G_n, H_n and $M_n^{[r]}$ respectively.

If we define

$$\begin{aligned}
\tilde{A}_I : &= \sum_{j=1}^{m} a_j - \frac{1}{W_I} \sum_{j=1}^{k-1} \sum_{i \in I} w_i x_{ij} - \frac{1}{W_I} \sum_{j=k+1}^{m} \sum_{i \in I} w_i x_{ij} \\
&= \sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} A_I(\mathbf{x}_j, \mathbf{w}) - \sum_{j=k+1}^{m} A_I(\mathbf{x}_j, \mathbf{w}) \\
\tilde{G}_I : &= \frac{\prod_{j=1}^{m} a_j}{\left(\prod_{j=1}^{k-1} \prod_{i \in I} x_{ij}^{w_i}\right)^{\frac{1}{W_I}} \left(\prod_{j=k+1}^{m} \prod_{i \in I} x_{ij}^{w_i}\right)^{\frac{1}{W_I}}} \\
\tilde{H}_I : &= \left(\sum_{j=1}^{m} {a_j}^{-1} - \frac{1}{W_I} \sum_{j=1}^{k-1} \sum_{i \in I} w_i x_{ij}^{-1} - \frac{1}{W_I} \sum_{j=k+1}^{m} \sum_{i \in I} w_i x_{ij}^{-1}\right)^{-1}
\end{aligned}$$

$$\tilde{M}_I^{[r]} :$$

$$= \begin{cases} \left(\sum_{j=1}^{m} {a_j}^r - \frac{1}{W_I} \sum_{j=1}^{k-1} \sum_{i \in I} w_i x_{ij}^r - \frac{1}{W_I} \sum_{j=k+1}^{m} \sum_{i \in I} w_i x_{ij}^r\right)^{\frac{1}{r}} & r \neq 0, \\ \tilde{G}_I & r = 0, \end{cases}$$

Analogous applications and results as stated in previous sections and Chapters 2, 8 and 9 are valid for this section as well so we omit the details, interested readers can see [135].

10.3 Generalization of Niezgoda's Inequality Using Functions with Nondecreasing Increments.

Theorem 10.7: *Suppose that $f : \mathbf{U} \to \mathbb{R}$ be a continuous function with nondecreasing increments. Let there are m elements in $\acute{\mathbf{a}} = \left(\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(m)}\right)$ with $\boldsymbol{a}^{(j)} \in \boldsymbol{U}$ for $j \in \{1, \dots, m\}$. Also there are $n \times m$ elements represented as $\mathbf{x}^{(ij)}$ s.t. every $\mathbf{x}^{(ij)}$ is a k-tuple for $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$ and all these $\mathbf{x}^{(ij)}$ can be arranged in a $n \times m$ matrix $\boldsymbol{X} = (\mathbf{x}^{(ij)})$. Let $\mathbf{w}$ be a nonnegative n-tuple s. t $W_n > 0$ for $i \in \{1, \dots, n\}$ and If $\acute{\mathbf{a}}$ majorizes each row of $\boldsymbol{X}$, that is,*

$$\mathbf{x}^{(i.)} = \left(\mathbf{x}^{(i1)}, \dots, \mathbf{x}^{(im)}\right) \prec \left(\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(m)}\right) = \acute{\mathbf{a}} \text{ for each } i \in \{1, \dots, n\},$$

then we have the inequality

$$f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_i\mathbf{x}^{(ij)}\right)\leq\sum_{j=1}^{m}f\left(\mathbf{a}^{(j)}\right)-\frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_i f\left(\mathbf{x}^{(ij)}\right),\tag{10.11}$$

this inequality also holds for weights $\mathbf{w}$ *that meet the conditions stated in Theorem A.7.*

Remark 10.8: We can state similar results for this section as we stated in Subsection 10.2.1. For complete details, see [134].

10.3.1 Burkill-Mirsky-Pečarić's Result

The following result gives us a Jensen type inequality for functions with nondecreasing increments when the finite sequence of $k-$tuples $(\mathbf{X_1},\ldots,\mathbf{X_n})$ is monotone in means. This is in fact a Pečarić's generalization of Burkill-Mirsky's result which we refer as to Burkill-Mirsky-Pečarić's result (see [207]).

Theorem 10.8: *Let* $f:\mathbf{U}\to\mathbb{R}$ *be a continuous function with nondecreasing increments and* $\mathbf{w}=(w_1,\ldots,w_n)\in\mathbb{R}_+^n$. *If* $(\mathbf{X_1},\ldots,\mathbf{X_n})\in\mathbf{U}^n$ *is nondecreasing or nonincreasing in means with respect to weights* w_i *for* $i\in\{1,\ldots,n\}$, *then the inequality*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n}w_i\mathbf{X_i}\right)\leq\frac{1}{W_n}\sum_{i=1}^{n}w_i f(\mathbf{X_i})$$

holds.

Throughout this subsection, we will use the notation $\mathbf{X_i}=(x_{i1},\ldots,x_{ik})$ for $i\in\{1,\ldots,n\}$, where $\mathbf{X_i}\in\mathbb{R}^k$. The Burkill–Mirsky–Pečarić result given above is also generalized in following manners.

Theorem 10.9: *Let* $f:\mathbf{U}\to\mathbb{R}$ *be a continuous function with nondecreasing increments and* $\mathbf{w}=(w_1,\ldots,w_n)\in\mathbb{R}_+^n$. *If* $(\mathbf{X_1},\ldots,\mathbf{X_n})\in\mathbf{U}^n$ *is nondecreasing or nonincreasing in means with respect to weights* w_i *for* $i\in\{1,\ldots,n\}$, *then the inequality*

$$f\left(\boldsymbol{a}+\boldsymbol{b}-\frac{1}{W_n}\sum_{i=1}^{n}w_i\mathbf{X_i}\right)\leq f(\boldsymbol{a})+f(\boldsymbol{b})-\frac{1}{W_n}\sum_{i=1}^{n}w_i f(\mathbf{X_i})$$

holds. Where $\boldsymbol{a}=(a_1,\ldots,a_k)$ *and* $\boldsymbol{b}=(b_1,\ldots,b_k)$ *are two specially chosen k-tuples related to* $\boldsymbol{U}$.

Theorem 10.10: *Let* $(\mathbf{X_1},\ldots,\mathbf{X_n})\in[\mathbf{0},\mathbf{d}]^n$, $(\mathbf{d}>\mathbf{0})$ *be nondecreasing or nonincreasing in means with respect to positive weights* w_i *for* $i\in\{1,\ldots,n\}$. *If* f *is a continuous function with* *nondecreasing increments of order three on* $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$*, then the inequality*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(\mathbf{X_i}) - f\left(\boldsymbol{a} + \boldsymbol{b} - \frac{1}{W_n}\sum_{i=1}^{n} w_i \mathbf{X_i}\right)$$
$$\leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(2\mathbf{d} - \mathbf{X_i}\right) - f\left(\boldsymbol{a} + \boldsymbol{b} - \frac{1}{W_n}\sum_{i=1}^{n} w_i \left(2\mathbf{d} - \mathbf{X_i}\right)\right)$$

holds.

Theorem 10.11: *Let* $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \in [\mathbf{r}, \mathbf{d} - \mathbf{p}]^n$, $(\mathbf{0} < \mathbf{p} < \mathbf{d} - \mathbf{r})$ *be nondecreasing or nonincreasing in means with respect to positive weights* w_i *for* $i \in \{1, \ldots, n\}$*. If* f *is a continuous function with nondecreasing increments of order three on* $\mathbf{J} = [\mathbf{r}, \mathbf{d}]$*, then the following inequality holds*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(\mathbf{X_i}) - f\left(\boldsymbol{a} + \boldsymbol{b} - \frac{1}{W_n}\sum_{i=1}^{n} w_i \mathbf{X_i}\right)$$
$$\leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(\mathbf{p} + \mathbf{X_i}\right) - f\left(\boldsymbol{a} + \boldsymbol{b} - \frac{1}{W_n}\sum_{i=1}^{n} w_i \left(\mathbf{p} + \mathbf{X_i}\right)\right)$$

Remark 10.9: For proofs and corollaries of this subsection, see [134].

CHAPTER 11

On Integral Niezgoda Inequality

This chapter focuses on integral form of Niezgoda's inequality with applications. We begin with integral Niezgoda inequality derived using majorization and its applications to integral means. The discussion then extends to inequalities involving functions of bounded variation and generalizations using functions with nondecreasing increments. Finally, further extensions are developed through isotonic linear functionals, providing a comprehensive framework for integral refinements of Niezgoda's inequality.

11.1 Integral Niezgoda Inequality Using Majorization and Applications to Integral Means

Majorization, prior to the appearance of the celebrated volume "Inequalities: Theory of Majorization and its applications" by Marshall, Olkin & Arnold [164], many researchers were unaware of the rich body of literature related to majorization that was scattered in journals in a wide variety of fields. Many of important ideas, relative to majorization were already discussed in "Inequalities by Hardy, Littlewood & Polyá" [110].

In present section we have stated the brief introduction with some classical results related to integral majorization. In the second section, we would state integral version of Niezgoda's inequality with refinement and state plenty of corollaries as special cases of our proposed result. In second section we would use concept of majorization. The last section is based on applications of our main which deals with integral means of different types.

11.1.1 Niezgoda's Integral Inequality Using Majorization

Throughout this chapter we assume that $I_n = \{1, 2, \dots, n\}$, and J, K and $[a, b]$ are real intervals.

For our main result we need here a lemma which is stated as:

Lemma 11.1: *Let (M, Σ, μ) is measure space where $0 \leq \mu(E) < \infty$ for all $E \in \Sigma$ and let $f : M \times [a, b] \to K$ is measureable function such that for each $s \in X$, $f(s, \cdot) \prec g(\cdot)$. Further, let $a \leq a_1 \leq b_1 \leq b$ and denote $L = a_1 - a + b - b_1$. A function $h_s : [a, b] \to K$ is defined as*

$$h_s(t) = \begin{cases} \int_a^{a_1} \frac{f(s,t)}{a_1 - a} dt & , \quad a \leq t < a_1 \\ f(s,t) & , \quad a_1 < t < b_1 \\ \int_{b_1}^{b} \frac{f(s,t)}{b - b_1} dt & , \quad b_1 < t \leq b, \end{cases}$$

then $h_s \prec f(s, \cdot)$.

Proof: First we show that h_s is nonincreasing as follows: Let $t_1 \in [a, a_1]$, $t_2 \in (a_1, b_1)$, $t_3 \in [b_1, b]$. Since $f(s, \cdot)$ is nonincreasing, we have

$$h_s(t_1) = \int_a^{a_1} \frac{f(s,t)}{a_1 - a} dt \geq \int_a^{a_1} \frac{f(s,t_2)}{a_1 - a} dt = f(s, t_2)$$
$$= h_s(t_2) = \int_{b_1}^{d} \frac{f(s,t_2)}{b - b_1} dt \geq \int_{b_1}^{d} \frac{f(s,t_3)}{b - b_1} dt = h_s(t_3)$$

hence we have $h_s(t_1) \geq h_s(t_2) \geq h_s(t_3)$, therefore, h_s is nonincreasing.

Now we prove that $h_s \prec f(s, \cdot)$ as follows:

(a) Let $u \in [a, a_1]$. Then we proceed as follows

$$\begin{aligned}
&\int_a^u h_s(t) dt = \int_a^u \left(\int_a^{a_1} \frac{f(s,t)}{a_1 - a} dt \right) dt \\
&= \frac{u - a}{a_1 - a} \int_a^{a_1} f(s,t) dt \\
&= \int_a^u f(s,t) dt - \int_a^u f(s,t) dt + \frac{u - a}{a_1 - a} \int_a^{a_1} f(s,t) dt \\
&= \int_a^u f(s,t) dt - \int_a^u f(s,t) dt + \frac{u - a}{a_1 - a} \left[\int_a^u f(s,t) dt + \int_u^{a_1} f(s,t) dt \right] \\
&= \int_a^u f(s,t) dt - \int_a^u f(s,t) dt + \frac{u - a}{a_1 - a} \int_a^u f(s,t) dt + \frac{u - a}{a_1 - a} \int_u^{a_1} f(s,t) dt \\
&= \int_a^u f(s,t) dt - \frac{a_1 - u}{a_1 - a} \int_a^u f(s,t) dt + \frac{u - a}{a_1 - a} \int_u^{a_1} f(s,t) dt \\
&\leq \int_a^u f(s,t) dt - \frac{a_1 - u}{a_1 - a} (u - a) f(s,t) + \frac{u - a}{a_1 - a} (a_1 - u) f(s,t) \\
&= \int_a^u f(s,t) dt
\end{aligned}$$

(b) Let $u \in (a_1, b_1)$, therefore

$$\int_a^u h_s(t)dt = \int_a^{a_1} h_s(t)dt + \int_{a_1}^u h_s(t)dt = \int_a^{a_1} \left(\int_a^{a_1} \frac{f(s,t)}{a_1 - a}dt\right) dt + \int_{a_1}^u f(s,t)dt$$
$$= \int_a^{a_1} f(s,t)dt + \int_{a_1}^u f(s,t)dt = \int_a^u f(s,t)$$

(c) Finally for $u \in [b_1, b]$

$$\begin{aligned}
&\int_b^u h_s(t)dt = \int_b^u \left(\int_{b_1}^b \frac{f(s,t)}{b - b_1}dt\right) dt = \frac{u-b}{b-b_1}\int_{b_1}^d f(s,t)dt\\
&= \int_b^u f(s,t)dt - \int_b^u f(s,t)dt + \frac{u-b}{b-b_1}\int_{b_1}^d f(s,t)dt\\
&= \int_b^u f(s,t)dt - \int_b^u f(s,t)dt + \frac{u-b}{b-b_1}\left[\int_{b_1}^u f(s,t)dt + \int_b^u f(s,t)dt\right]\\
&= \int_b^u f(s,t)dt - \int_b^u f(s,t)dt + \frac{u-b}{b-b_1}\int_{b_1}^u f(s,t)dt\\
&+\frac{u-b}{b-b_1}\int_b^u f(s,t)dt\\
&= \int_b^u f(s,t)dt - \frac{u-b_1}{b-b_1}\int_b^u f(s,t)dt + \frac{u-b}{b-b_1}\int_{b_1}^u f(s,t)dt\\
&\leq \int_b^u f(s,t)dt - \frac{u-b_1}{b-b_1}(u-b)f(s,t) + \frac{u-b}{b-b_1}(u-b_1)f(s,t) = \int_b^u f(s,t)dt
\end{aligned}$$

Also we have

$$\begin{aligned}
&\int_a^b h_s(t)dt = \int_a^{a_1} h_s(t)dt + \int_{a_1}^{b_1} h_s(t)dt + \int_{b_1}^b h_s(t)dt\\
&= \int_a^{a_1}\int_a^{a_1} \frac{f(s,t)}{a_1-a}dt\,dt + \int_{a_1}^{b_1} f(s,t)dt + \int_{b_1}^b\int_{b_1}^b \frac{f(s,t)}{b-b_1}dt\,dt\\
&= \int_a^{a_1} f(s,t)dt + \int_{a_1}^{b_1} f(s,t)dt + \int_{b_1}^b f(s,t)dt = \int_a^b f(s,t)dt
\end{aligned}$$

Hence, we conclude that $h_s \prec f(s,\cdot)$.

□

Our next main results is given as follows.

Theorem 11.1: *Let $g : [a,b] \to K$ and let (M, Σ, μ) is measure space where $0 \leq \mu(E) < \infty$ for all $E \in \Sigma$, and consider the weight function $\omega : X \to [0, +\infty)$ such that $\int_M \omega d\mu = 1$ and let $f : M \times [a,b] \to K$ is measureable function such that for each $s \in X, f(s,\cdot) \prec g(\cdot)$. Further, let $a < a_1 < b_1 < b$ or $a = b_0 < a_1 < b_1 < a_2 = b$, $I = (a_1, b_1)$ and $I^c = [a, a_1] \cup [b_1, b]$ and denote $L = |I^c| = (a_1 - a) + (b - b_1)$. Then for every continuous convex function $\varphi : K \to \mathbb{R}$, we have*

$$\varphi\left(\frac{1}{L}\int_a^b g(t)dt - \frac{1}{L}\int_{a_1}^{b_1}\int_M \omega(s)f(s,t)d\mu(s)dt\right)$$
$$\leq \frac{1}{L}\int_a^b \varphi\left(g(t)\right)dt - \frac{1}{L}\int_{a_1}^{b_1}\int_M \omega(s)\varphi\left(f(s,t)\right)d\mu(s)dt.$$

Proof: Rearranging and then applying integral and discrete Jensen's inequality and using definition of Majorization, respectively, we get

$$
\begin{aligned}
&\varphi\left(\frac{1}{L}\int_a^b g(t)\,dt-\frac{1}{L}\int_{a_1}^{b_1}\int_M \omega(s)f(s,t)\,d\mu(s)\,dt\right)\\
&=\varphi\Bigg(\frac{1}{L}\int_a^b g(t)\Big(\int_M \omega(s)\,d\mu(s)\Big)dt\\
&\quad-\frac{1}{L}\int_{a_1}^{b_1}\int_M \omega(s)f(s,t)\,d\mu(s)\,dt\Bigg)\\
&=\varphi\left(\int_M \omega(s)\Big(\frac{1}{L}\int_a^b g(t)\,dt-\frac{1}{L}\int_{a_1}^{b_1} f(s,t)\,dt\Big)\,d\mu(s)\right)\\
&\leq \int_M \omega(s)\,\varphi\left(\frac{1}{L}\int_a^b g(t)\,dt-\frac{1}{L}\int_{a_1}^{b_1} f(s,t)\,dt\right)d\mu(s)\\
&=\int_M \omega(s)\,\varphi\left(\frac{1}{L}\int_a^b f(s,t)\,dt-\frac{1}{L}\int_{a_1}^{b_1} f(s,t)\,dt\right)d\mu(s)\\
&=\int_M \omega(s)\,\varphi\Bigg(\frac{1}{L}\Big(\int_a^{a_1} f(s,t)\,dt+\int_{a_1}^{b_1} f(s,t)\,dt\\
&\quad+\int_{b_1}^{b} f(s,t)\,dt-\int_{a_1}^{b_1} f(s,t)\,dt\Big)\Bigg)\,d\mu(s)\\
&=\int_M \omega(s)\,\varphi\left(\frac{1}{L}\int_a^{a_1} f(s,t)\,dt+\frac{1}{L}\int_{b_1}^{b} f(s,t)\,dt\right)d\mu(s)\\
&=\int_M \omega(s)\,\varphi\left(\frac{a_1-a}{L}\int_a^{a_1}\frac{f(s,t)}{a_1-a}\,dt+\frac{b-b_1}{L}\int_{b_1}^{b}\frac{f(s,t)}{b-b_1}\,dt\right)d\mu(s)\\
&\leq \int_M \omega(s)\Bigg(\frac{a_1-a}{L}\,\varphi\left(\int_a^{a_1}\frac{f(s,t)}{a_1-a}\,dt\right)\\
&\quad+\frac{b-b_1}{L}\,\varphi\left(\int_{b_1}^{b}\frac{f(s,t)}{b-b_1}\,dt\right)\Bigg)\,d\mu(s).
\end{aligned}
\tag{11.1}
$$

Now, let us define the function $h_s:[a,b]\to K$

$$
h_s(t)=\begin{cases}
\displaystyle\int_a^{a_1}\frac{f(s,t)}{a_1-a}dt & , \quad a\leq t<a_1\\
f(s,t) & , \quad a_1\leq t\leq b_1\\
\displaystyle\int_{b_1}^{b}\frac{f(s,t)}{b-b_1}dt & , \quad b_1<t\leq b.
\end{cases}
$$

It can be proved that $h_s\prec f(s,\cdot)$ and $f(s,\cdot)\prec g$, therefore, $h_s\prec g$. Hence by integral version of majorization theorem we have

$$
\int_a^b \varphi\left(h_s(t)\right)dt\leq \int_a^b \varphi\left(g(t)\right)dt.
$$

Since

$$\int_a^b \varphi\left(h_s(t)\right)dt = \int_a^{a_1} \varphi\left(h_s(t)\right)dt + \int_{a_1}^{b_1} \varphi\left(h_s(t)\right)dt + \int_{b_1}^{b} \varphi\left(h_s(t)\right)dt$$
$$= (a_1-a)\varphi\left(\int_a^{a_1} \frac{f(s,t)}{a_1-a}dt\right) + \int_{a_1}^{b_1} \varphi\left(f(s,t)\right)dt$$
$$+(b-b_1)\varphi\left(\int_{b_1}^{b} \frac{f(s,t)}{b-b_1}dt\right) \int_a^b \varphi\left(h_s(t)\right)dt - \int_{a_1}^{b_1} \varphi\left(f(s,t)\right)dt$$
$$= (a_1-a)\varphi\left(\int_a^{a_1} \frac{f(s,t)}{a_1-a}dt\right) + (b-b_1)\varphi\left(\int_{b_1}^{b} \frac{f(s,t)}{b-b_1}dt\right).$$

So by using majorization theorem we obtain $\int_a^b \varphi(h_s(t))dt \leq \int_a^b \varphi(g(t))dt$ and further we get

$$(a_1-a)\varphi\left(\int_a^{a_1} \frac{f(s,t)}{a_1-a}dt\right) + (b-b_1)\varphi\left(\int_{b_1}^{b} \frac{f(s,t)}{b-b_1}dt\right) \leq \int_a^b \varphi\left(g(t)\right)dt - \int_{a_1}^{b_1} \varphi\left(f(s,t)\right)dt.$$

Dividing both side by L

$$\frac{(a_1-a)}{L}\varphi\left(\int_a^{a_1} \frac{f(s,t)}{a_1-a}dt\right) + \frac{(b-b_1)}{L}\varphi\left(\int_{b_1}^{b} \frac{f(s,t)}{b-b_1}dt\right) \leq \frac{1}{L}\int_a^b \varphi\left(g(t)\right)dt - \frac{1}{L}\int_{a_1}^{b_1} \varphi\left(f(s,t)\right)dt.$$

Putting in last inequality of (11.1), we obtain

$$\varphi\left(\frac{1}{L}\int_a^b g(t)dt - \frac{1}{L}\int_{a_1}^{b_1}\int_E \omega(s)f(s,t)d\mu(s)dt\right)$$
$$\leq \int_M \omega(s)\left(\frac{1}{L}\int_a^b \varphi\left(g(t)\right)dt - \frac{1}{L}\int_{a_1}^{b_1} \varphi\left(f(s,t)\right)dt\right)d\mu(s)$$
$$= \frac{1}{L}\int_a^b \varphi\left(g(t)\right)\int_M \omega(s)d\mu(s)dt - \frac{1}{L}\int_{a_1}^{b_1}\int_E \omega(s)\varphi\left(f(s,t)\right)d\mu(s)dt$$
$$= \frac{1}{L}\int_a^b \varphi\left(g(t)\right)dt - \frac{1}{L}\int_{a_1}^{b_1}\int_M \omega(s)\varphi\left(f(s,t)\right)d\mu(s)dt.$$

Hence we get our required result. □

Now we further generalize this result as follows:

Theorem 11.2: *Let $g:[a,b]\to K$ and let (M,Σ,μ) is measure space where $0\leq\mu(E)<\infty$ for all $E\in\Sigma$, and consider the weight function $\omega: M\to[0,+\infty)$ such that $\int_M \omega d\mu = 1$ and let $f: M\times[a,b]\to K$ is measureable function such that for each $s\in X, f(s,\cdot)\prec g(\cdot)$. Further, let the following assumptions be valid $a=b_0\leq a_1<b_1<a_2<b_2<\cdots<a_k<b_k\leq a_{k+1}=b$, $J=\bigcup_{l=1}^{k}(a_l,b_l)$, $J^c=[a,b]\backslash J=\bigcup_{l=1}^{k+1}[b_{l-1},a_l]$ and $L=|J^c|=\sum_{l=1}^{k+1}(a_l-b_{l-1})$. Then for every continuous convex function $\varphi: K\to\mathbb{R}$, the following inequality holds*

$$\varphi\left(\frac{1}{L}\int_a^b g(t)dt - \frac{1}{L}\int_J\int_M \omega(s)f(s,t)d\mu(s)dt\right)$$
$$\leq \frac{1}{L}\int_a^b \varphi\left(g(t)\right)dt - \frac{1}{L}\int_J\int_M \omega(s)\varphi\left(f(s,t)\right)d\mu(s)dt. \quad (11.2)$$

Proof: By using the following function in proof of Theorem 11.1 we get what we wanted

$$h_s(t) = \begin{cases} f(s,t), & a_i < t < b_i, \\ \displaystyle\int_{b_i}^{a_{i+1}} \frac{f(s,t)}{a_{i+1}-b_i} dt, & b_i \leq t \leq a_{i+1}. \end{cases}$$

□

The following result is a direct consequence of Theorem 11.2.

Corollary 11.1: *Let the following assumptions be valid:* $\alpha = (\alpha_1, \ldots, \alpha_m) \in K^m$, $\mathbf{X} = (x_{ij})$ *is an* $n \times m$ *matrix such that* $x_{ij} \in K$ *for each* $i \in I_n$, $j \in I_m$ *and* α *majorizes each row of* $\mathbf{X}$. *Further, let* $a_l, b_l \in \mathbb{N}$ *for* $l \in K$ *be such that* $1 = b_0 \leq a_1 < b_1 < a_2 < b_2 < \ldots < a_k < b_k \leq a_{k+1} = m+1$ *and denote* $L = \sum_{l=1}^{k+1}(a_l - b_{l-1})$. *Then for every continuous convex function* $\varphi : K \to \mathbb{R}$, *the inequality*

$$\varphi\left(\frac{1}{L}\left(\sum_{j\in I_m}\alpha_j - \frac{1}{W_n}\sum_{l\in I_k}\sum_{j=a_l}^{b_l-1}\sum_{i\in I_n}\omega_i x_{ij}\right)\right) \leq \frac{1}{L}\left(\sum_{j\in I_m}\varphi(\alpha_j) - \frac{1}{W_n}\sum_{l\in I_k}\sum_{j=a_l}^{b_l-1}\sum_{i\in I_n}\omega_i\varphi(x_{ij})\right),$$

holds, where ω_i*'s are non-negative real weights with* $W_n > 0$.

Proof: The proof of corollary follows from Theorem 11.2 by taking step functions. Specifically, for $a = b_0 = 1$, $d = a_{k+1} = m+1$, with $X = [1, n+1)$, $g(t) = \sum\limits_{j\in I_m} a_j \chi_{[j,j+1)}(t)$ and

$$f(s,t) = \sum_{i\in I_n}\sum_{j\in I_m} x_{ij}\,\chi_{[i,i+1)}(s)\chi_{[j,j+1)}(t), d\mu(s) = \sum_{i\in I_n}\chi_{[i,i+1)}(s)d\mu_1(s),$$

where μ_1 is the Lebesgue measure. □

Remark 11.1: If in Corollary 11.1 we put $k = 1$, $a_1 = 1$ and $b_1 = m$ then we capture Niezgoda inequality (10.2).

11.1.2 Refinements

In remaining part of present section, some construction is needed: We define $E \subset M$ with $\mu(E), \mu(E^c) > 0$ and we define following notations

$$W_E = \frac{\mu(E)}{\mu(M)}, \quad W_{E^c} = \frac{\mu(E^c)}{\mu(M)} = 1 - W_E.$$

Under assumptions of Theorem 11.2, we define functional

$$\begin{aligned} F_1(f,g,\varphi;E) = W_E\varphi&\left(\frac{1}{L}\left(\int_a^b g(t)d(t) - \frac{1}{\mu(E)}\int_J\int_E \omega(s)f(s,t)d\mu(s)d(t)\right)\right) \\ &+ W_{E^c}\varphi\left(\frac{1}{L}\left(\int_a^b g(t)d(t) - \frac{1}{\mu(E^c)}\int_J\int_{E^c}\omega(s)f(s,t)d\mu(s)d(t)\right)\right). \end{aligned}$$

The following refinement of (11.2) is valid.

Theorem 11.3: *Let all assumptions of Theorem 11.2 be valid. Then $\forall \alpha$, $\phi \neq E \subset M$, the following inequality holds*

$$\varphi\left(\frac{1}{L}\left(\int_a^b g(t)d(t) - \frac{1}{\mu(M)}\int_J\int_X \omega(s)f(s,t)d\mu(s)d(t)\right)\right) \leq F_1(f,g,\varphi;E)$$
$$\leq \frac{1}{L}\left(\int_a^b \varphi(g(t))d(t) - \frac{1}{\mu(M)}\int_J\int_X \omega(s)\varphi(f(s,t))d\mu(s)d(t)\right) \quad (11.3)$$

Remark 11.2: For idea of the proof and other special cases and remarks we refer the reader to [14].

11.1.3 Applications: Integral Means

In [106], we can find the definition of integral weighted quasi-arithmetic means. Now we modify the definition a little bit as per our requirements.

Let $E \subset \mathbb{R}$ be an interval. Let $L_1(E)$ denote the vector space of all real Lebesgue measureable functions defined on the interval E with positive and finite Lebesgue measure μ and $L_1^+(E)$ denote the positive cone of $L_1(E)$. Now we define modified integral quasi-arithmetic means as follows.

Definition 11.1: Let $(w,f) \in L_1^+(E) \times L_1^+(E)$ and $\varphi : [0,\infty) \to \mathbb{R}$ be real continuous and strictly monotone function. The generalized quasi-arithmetic mean of a function f is a number $M_{E,\varphi}(f) \in \mathbb{R}$ where

$$M_{E,\varphi}(f) = \varphi^{-1}\left(\frac{1}{\mu(E)}\int_E w(t)\varphi(f(t))d\mu(t)\right)$$

where φ^{-1} stands for inverse function of the function φ.

Throughout the present section, we would consider φ as a real continuous and strictly monotone function. By substitution of suitable functions φ and f in $M_{E,\varphi}(f)$ we get different integral means. For instance,

(a) for $\varphi(t) = t = id(t)$ we get the generalized arithmetic mean

$$M_{E,\varphi}(f) = A_E(f) = \frac{1}{\mu(E)}\int_E w(t)f(t)d\mu(t).$$

(b) for $\varphi(t) = t^{-1}$ we obtain the generalized harmonic mean

$$M_{E,\varphi}(f) = H_E(f) = \left(\frac{1}{\mu(E)}\int_E w(t)(f(t))^{-1}d\mu(t)\right)^{-1}.$$

(c) for $\varphi(t) = t^r$ we botain generalized power mean of order r

$$M_{E,\varphi}(f) = M_E^{[r]}(f) = \begin{cases} \left(\frac{1}{\mu(E)}\int_E w(t)(f(t))^r d\mu(t)\right)^{\frac{1}{r}}, & r \neq 0 \\ \exp\left(\frac{1}{\mu(E)}\int_E w(t)lnf(t)dt\right), & r = 0. \end{cases}$$

For up coming results we also define a notation as follows:

$$\bar{M}_{E,\varphi}(g,f) = \varphi^{-1}\left(\frac{1}{L}\left(\int_a^b \varphi(g(t))d(t) - \frac{1}{\mu(E)}\int_J\int_E \omega(s)\varphi(f(s,t))d\mu(s)d(t)\right)\right). \quad (11.4)$$

Throughout this section we also assume that ln and exp have their natural domain. Under all assumptions of Theorem 11.2, we set following notations.

Arithmetic Mean

$$
\begin{aligned}
A'_g &= \frac{1}{L}\int_a^b g(t)d(t)\\
A_M &= \frac{1}{L}\frac{1}{\mu(M)}\int_J\int_M \omega(s)f(s,t)d\mu(s)d(t)\\
\tilde{A}_M &= A'_g - A_M.
\end{aligned}
$$

Geometric Mean

$$
\begin{aligned}
G'_g &= \exp\left(\frac{1}{L}\int_a^b ln\, g(t)d(t)\right)\\
G_M &= \exp\left(\frac{1}{L}\frac{1}{\mu(M)}\int_J\int_M \omega(s)\ln f(s,t)d\mu(s)d(t)\right)\\
\tilde{G}_M &= \frac{G'_g}{G_M}.
\end{aligned}
$$

Harmonic Mean

$$
\begin{aligned}
H'_g &= \left(\frac{1}{L}\int_a^b \frac{1}{g(t)}d(t)\right)^{-1}\\
H_M &= \left(\frac{1}{L}\frac{1}{\mu(M)}\int_J\int_M \frac{\omega(s)}{f(s,t)}d\mu(s)d(t)\right)^{-1}\\
\frac{1}{\tilde{H}_M} &= \frac{1}{H'_g} - \frac{1}{H_M}.
\end{aligned}
$$

Power Mean

$$
\begin{aligned}
M'^{[r]}_g &= \frac{1}{L}\int_a^b g(t)^r d(t)\\
M^{[r]}_M &= \frac{1}{L}\frac{1}{\mu(M)}\int_J\int_M \omega(s)f(s,t)^r d\mu(s)d(t)\\
\tilde{M}^{[r]}_M &= \left(M'^{[r]}_g - M^{[r]}_M\right)^{\frac{1}{r}}.
\end{aligned}
$$

where all the factors in denominator are non-zero.

Let assumptions of Theorem 11.2 are valid. Further let f and g are positive valued functions. Then following results hold:

Theorem 11.4: (i) $\tilde{G}_M \le \tilde{A}_E^{W_E}\tilde{A}_{E^c}^{W_{E^c}} \le \tilde{A}_M$
(ii) $\tilde{G}_M \le W_E\tilde{G}_E + W_{E^c}\tilde{G}_{E^c} \le \tilde{A}_M$.

Proof:

(i) It follows by using the convex function $\varphi(t) = -\ln(t)$ in (11.3).

(*ii*) We obtain required result by using convex function $\varphi(t) = \exp(t)$ in (11.3) and by replacing $g(t)$ and $f(s,t)$ with $\ln g(t)$ and $\ln f(s,t)$ respectively.

□

Corollary 11.2: (*i*) $\tilde{G}_M \leq \min_E \tilde{A}_E^{W_E} \tilde{A}_{E^c}^{W_{E^c}}$ *and* $\tilde{A}_M \geq \max_E \tilde{A}_E^{W_E} \tilde{A}_{E^c}^{W_{E^c}}$

(*ii*) $\tilde{G}_M \leq \min_E W_E \tilde{G}_E + W_{E^c} \tilde{G}_{E^c}$ *and* $\tilde{A}_M \leq \max_E W_E \tilde{G}_E + W_{E^c} \tilde{G}_{E^c}$.

Theorem 11.5: (*i*) $\dfrac{1}{\tilde{G}_M} \leq \dfrac{1}{\tilde{H}_E^{W_E} \tilde{H}_{E^c}^{W_{E^c}}} \leq \dfrac{1}{\tilde{H}_M}$

(*ii*) $\dfrac{1}{\tilde{G}_M} \leq \dfrac{W_E}{\tilde{G}_E} + \dfrac{W_{E^c}}{\tilde{G}_{E^c}} \leq \dfrac{1}{\tilde{H}_M}$.

Proof:

(*i*) By substituting $\varphi(t) = -\ln t$, $g(t) = \frac{1}{g(t)}$ and $f(s,t) = \frac{1}{f(s,t)}$ in (11.3) we get required result.

(*ii*) In (11.3), take $\varphi(t) = \exp(t)$, replace $g(t)$ with $\ln \frac{1}{g(t)}$ and $f(s,t)$ with $\ln \frac{1}{f(s,t)}$ to get our result.

□

Corollary 11.3: (*i*) $\dfrac{1}{\tilde{G}_M} \leq \min_E \dfrac{1}{\tilde{H}_E^{W_E} \tilde{H}_{E^c}^{W_{E^c}}}$ *and* $\frac{1}{\tilde{H}_M} \geq \max_E \frac{1}{\tilde{H}_E^{W_E} \tilde{H}_{E^c}^{W_{E^c}}}$

(*ii*) $\dfrac{1}{\tilde{G}_M} \leq \min_E \dfrac{W_E}{\tilde{G}_E} + \dfrac{W_{E^c}}{\tilde{G}_{E^c}}$ *and* $\dfrac{1}{\tilde{H}_M} \geq \min_E \dfrac{W_E}{\tilde{G}_E} + \dfrac{W_{E^c}}{\tilde{G}_{E^c}}$.

Remark 11.3: We can also obtain the Corollary 11.3 directly from Corollary 11.2 by the substitutions $g(t) \to \frac{1}{g(t)}$ and $f(s,t) \to \frac{1}{f(s,t)}$.

Theorem 11.6: (*i*) *For $r \leq 1$, we get $\tilde{M}_M^{[r]} \leq W_E \tilde{M}_E^{[r]} + W_{E^c}^{[r]} \tilde{M}_{E^c} \leq \tilde{A}_M$.*

(*ii*) *For $r \geq 1$, above inequalities are reversed.*

Proof: See [14]. □

Corollary 11.4: *For $r \leq 1$ we have*

$$\tilde{M}_M^{[r]} \leq \min_E W_E \tilde{M}_E^{[r]} + W_{E^c}^{[r]} \tilde{M}_{E^c} \quad \textit{and} \quad \tilde{A}_M \geq \max_E W_E \tilde{M}_E^{[r]} + W_{E^c}^{[r]} \tilde{M}_{E^c}. \tag{11.5}$$

If $1 \leq r$, then the inequalities in (11.5) are reversed.

Corollary 11.5: *Under all assumptions of Corollary 11.4, following inequalities are valid.*

$$\tilde{H}_M \leq \min_E \left[W_E \tilde{H}_E + W_{\overline{E}} \tilde{H}_{E^c} \right] \quad \textit{and} \quad \tilde{A}_M \geq \max_E \left[W_S \tilde{H}_E^c + W_{E^c} \tilde{H}_{E^c} \right].$$

Remark 11.4: Clearly, part (*ii*) of Corollary 11.2 is a direct consequences of Theorem 11.4.

Theorem 11.7: *Let $r, q \in \mathbb{R}$; $q \geq r$.*

(i) If $0 \leq q$*, then* $\left(\tilde{M}_M^{[r]}\right)^q \leq W_E(\tilde{M}_E)^q + W_{E^c}(\tilde{M}_{E^c})^q \leq (\tilde{A}_M)^q$.

(ii) For $q < 0$*, the above inequalities are reversed.*

Proof: See [14]. □

Corollary 11.6: *Let* $r, q \in \mathbb{R}$*,* $r \leq q$*.*

(a) For $q \geq 0$*, following inequalities hold*

$$\left(\tilde{M}_X^{[r]}\right)^q \leq \min_E \left[W_E \left(\tilde{M}_E^{[r]}\right)^q + W_{E^c} \left(\tilde{M}_{E^c}^{[r]}\right)^q\right], \tag{11.6}$$

$$\left(\tilde{A}_M\right)^q \geq \max_E \left[W_E \left(\tilde{M}_E^{[r]}\right)^q + W_{E^c} \left(\tilde{M}_{E^c}^{[r]}\right)^q\right]. \tag{11.7}$$

(b) For $s < 0$*, inequalities* (11.6) *and* (11.7) *are reversed.*

Theorem 11.8: *Let* $\varphi, \psi : [0, \infty] \to \mathbb{R}$ *be strictly monotonic continuous functions. If* $\psi \circ \varphi^{-1}$ *is convex, then inequalities*

$$\psi\left(\bar{M}_{X,\varphi}\right) \leq W_E \psi\left(\bar{M}_{E,\varphi}\right) + W_{E^c} \psi\left(\bar{M}_{E^c,\varphi}\right) \leq \psi\left(\bar{M}_{X,\psi}\right) \tag{11.8}$$

hold, where $\bar{M}_{X,\varphi}$ *is defined in* (11.4).

Corollary 11.7: *Let assumptions of Theorem* 11.8 *are valid. Then following inequalities hold*

$$\begin{aligned}
\psi\left(\bar{M}_{X,\varphi}\right) &\leq \min_E \left[W_E \psi\left(\bar{M}_{E,\varphi}\right) + W_{E^c} \psi\left(\bar{M}_{E^c,\varphi}\right)\right], \\
\psi\left(\bar{M}_{X,\psi}\right) &\geq \max_E \left[W_E \psi\left(\bar{M}_{E,\varphi}\right) + W_{E^c} \psi\left(\bar{M}_{E^c,\varphi}\right)\right].
\end{aligned}$$

Remark 11.5: Theorems 11.4, 11.6 and 11.7 follow from Theorem 11.8, by choosing appropriate functions φ, ψ and substitutions.

11.2 Integral Niezgoda Inequalities for Functions of Bounded Variation

Throughout chapter we assume that K and $[a, b]$ are intervals in $\mathbb{R}$. Here we need a result from [137].

Lemma 11.2: *Let* $H \in C[a, b]$ *or* $H \in BV[a, b]$*, let* $a \leq a_1 \leq b_1 \leq a_2 \leq \cdots \leq a_k \leq b_k \leq b$ *is partition of* $[a, b]$*,* $J = \bigcup_{i=1}^k [a_i, b_i]$ *and* $L = |J^c| = \int_I dH(t)$*. If*

$$H(a_i) \leq H(t) \leq H(b_i) \qquad \forall\, t \in (a_i, b_i) \text{ and } i \in I_k \tag{11.9}$$

and $L > 0$*, then* $\forall$ *function* $f : [a, b] \to K$ *which is continuous and monotonic* $\forall$ k *intervals* (a_i, b_i) *and* $\forall$ *continuous convex function* $\varphi : K \to \mathbb{R}$*, the following inequality holds*

$$\varphi\left(\frac{1}{L}\int_I f(t)\, dH(t)\right) \leq \frac{1}{L}\int_I \varphi(f(t))\, dH(t).$$

Proof: Denote $w_i = \int_{a_i}^{b_i} dH(t)$. Due to (11.9), if $H(a_i) = H(b_i)$ then dH is a null-measure on

$[a_i, b_i]$ and $w_i = 0$, while otherwise $w_i > 0$. Denote $S = \{i : w_i > 0\}$ and

$$x_i = \frac{1}{w_i}\int_{a_i}^{b_i} f(t)\, dH(t), \qquad \text{for } i \in S.$$

Notice that $L = \int_I dH(t) = \sum_{i\in S} w_i > 0$, $\int_I \varphi(f(t))\, dH(t) = \sum_{i\in S}\int_{a_i}^{b_i} \varphi(f(t))\, dH(t)$ and, due to Theorem A.12, $w_i\varphi(x_i) \le \int_{a_i}^{b_i}\varphi(f(t))\, dH(t)$, for $i \in S$. Therefore, taking into account the discrete Jensen's inequality,

$$\begin{aligned}\varphi\left(\frac{1}{L}\int_I f(t)\, dH(t)\right) &= \varphi\left(\frac{1}{L}\sum_{i\in S} w_i x_i\right) \le \frac{1}{L}\sum_{i\in S} w_i\varphi(x_i) \le \\ &\le \frac{1}{L}\sum_{i\in S}\int_{a_i}^{b_i}\varphi(f(t))\, dH(t) = \frac{1}{L}\int_I \varphi(f(t))\, dH(t).\end{aligned}$$

□

The following theorems are our main results of this section.

Theorem 11.9: *Let (M, Σ, μ) be a measure space with positive finite measure μ and consider the integrable weight function $\omega : M \to [0, +\infty)$. Let $g : [a, b] \to K$ and $f : M \times [a, b] \to K$ be nonincreasing continuous functions such that for each $s \in M$*

$$\begin{aligned}\int_a^x f(s,t)\, dH(t) &\le \int_a^x g(t) dH(t), \quad \forall \quad x \in (a, b), \\ \text{and} \quad \int_a^b f(s,t)\, dH(t) &= \int_a^b g(t)\, dH(t),\end{aligned}$$

hold, where $H \in BV[a, b]$. Further, let the following assumptions be valid $a = b_0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \le a_{k+1} = b$, $J = \bigcup_{l=1}^{k}(a_l, b_l)$, $J^c = [a, b]\backslash J = \bigcup_{l=1}^{k+1}[b_{l-1}, a_l]$ and $L = |J^c| = \sum_{l=1}^{k+1}(a_l - b_{l-1})$. Then for a continuous convex function $\varphi : J \to \mathbb{R}$ inequality

$$\begin{aligned}&\varphi\left(\frac{1}{L}\left(\int_c^b g(t)dH(t) - \frac{1}{\int_M \omega(s)d\mu(s)}\int_J\int_M \omega(s)f(s,t)d\mu(s)dH(t)\right)\right) \le \\ &\quad \frac{1}{L}\left(\int_c^b \varphi(g(t))dH(t) - \frac{1}{\int_M \omega(s)d\mu(s)}\int_J\int_M \omega(s)\varphi(f(s,t))d\mu(s)dH(t)\right) \qquad (11.10)\end{aligned}$$

holds, for each $s \in M$.

Proof: Rearranging and applying integral Jensen's inequality (*A*.12) twice and then using Lemma 11.2 and Theorem *A*.33, one gets

$$\begin{aligned}&\varphi\left(\frac{1}{L}\left(\int_c^b g(t)dH(t) - \frac{1}{\int_M \omega(s)d\mu(s)}\int_J\int_M \omega(s)f(s,t)d\mu(s)dH(t)\right)\right) \\ &= \varphi\left(\frac{1}{L}\left(\frac{\int_M \omega(s)d\mu(s)}{\int_M \omega(s)d\mu(s)}\int_c^b g(t)dH(t)\right.\right. \\ &\left.\left. - \frac{1}{\int_M \omega(s)d\mu(s)}\int_J\int_M \omega(s)f(s,t)d\mu(s)dH(t)\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \varphi\left(\frac{1}{\int_M \omega(s)d\mu(s)} \int_M \omega(s)\left[\frac{1}{L}\int_{J^c} f(s,t)dH(t)\right] d\mu(s)\right)\\
&= \varphi\left(\frac{1}{\int_M \omega(s)d\mu(s)} \int_M \omega(s)\left[\frac{1}{L}\int_{J^c} f(s,t)dH(t)\right] d\mu(s)\right)\\
&\le \frac{1}{\int_M \omega(s)d\mu(s)} \int_M \omega(s)\varphi\left(\frac{1}{L}\int_{J^c} f(s,t)dH(t)\right) d\mu(s)\\
&\le \frac{1}{L}\frac{1}{\int_M \omega(s)d\mu(s)} \int_M \omega(s)\int_{J^c} \varphi(f(s,t))dH(t)\, d\mu(s)\\
&\le \frac{1}{L}\frac{1}{\int_M \omega(s)d\mu(s)} \int_M \omega(s)\left(\int_c^b \varphi(g(t))dH(t) - \int_J \varphi(f(s,t))dH(t)\right) d\mu(s)\\
&\le \frac{1}{L}\left(\frac{1}{\int_M \omega(s)d\mu(s)} \int_M \omega(s)\left(\int_c^b \varphi(g(t))dH(t)\right) d\mu(s)\right.\\
&\left. - \frac{1}{\int_M \omega(s)d\mu(s)} \int_J\int_M \omega(s)\varphi(f(s,t))dH(t)\right) d\mu(s)\\
&= \frac{1}{L}\left(\int_c^b \varphi(g(t))dH(t) - \frac{1}{\int_M \omega(s)d\mu(s)} \int_J\int_M w(s)\varphi(f(s,t))d\mu(s)dH(t)\right).
\end{aligned}
$$

□

Remark 11.6: If we replace $dH(t)$ by dt in Theorem 11.9 we would get Theorem 11.2 and all its consequences as our main results special cases.

Remark 11.7: If we simply put $\omega(s) = 1\ \forall\ s \in M$ in Theorem 11.9, then we obtain Theorem 1 of [137] as our special case and in turn we also get all the corollaries and refinements of Theorem 1 of [137].

Remark 11.8: An alternate version of Theorem 11.9 can also be given for nondecreasing functions f and g see [15] for details.

Remark 11.9: We can also give refinements and applications of main result of this section as we have given in Sections 11.1.2 and 11.1.3 see [15], [141] for details.

11.3 Generalization of Integral Niezgoda Inequality Using Functions with Nondecreasing Increments.

At this stage we recall a result similar to Lemma 11.2 for functions with nondecreasing increments from [141] as follows.

Lemma 11.3: *Let $H \in C[a,b]$ or $H \in BV[a,b]$, let $c \le a_1 \le b_1 \le a_2 \le \cdots \le a_k \le b_k \le d$ is partition of $[a,b]$, $I = \bigcup_{i=1}^{k}[a_i,b_i]$ and $L = \int_I dH(t) > 0$. If*

$$H(a_i) \le H(t) \le H(b_i) \qquad \forall\ t \in (a_i,b_i) \text{ and } 1 \le i \le k,$$

then $\forall$ function $\mathbf{f} : [a,b] \to \mathbf{K}$ which is continuous and monotonic on (a_i,b_i) $\forall i \in I_k$ and every *continuous function with nondecreasing increments $\varphi : \mathbf{K} \to \mathbb{R}$, following inequality valid*

$$\varphi\left(\frac{1}{L}\int_I \mathbf{f}(t)\, dH(t)\right) \leq \frac{1}{L}\int_I \varphi(\mathbf{f}(t))\, dH(t).$$

Here comes next main result which generalizes Theorem 11.9.

Theorem 11.10: *Let (M, Σ, μ) is measure space where $0 \leq \mu(E) < \infty\ \forall\ E \in \Sigma$ and consider the weight function $\omega : M \to [0, +\infty)$. Let $g : [a, b] \to \mathbf{K}$ and $f : M \times [a, b] \to \mathbf{K}$ be nonincreasing continuous functions such that for each $s \in M$*

$$\begin{aligned} \int_a^x \mathbf{f}(t)\, dH(t) \ &\leq \ \int_a^x \mathbf{g}(t)\, dH(t), \quad \forall \quad x \in (a, b), \\ \text{and} \quad \int_a^b \mathbf{f}(t)\, dH(t) \ &= \ \int_a^b \mathbf{g}(t)\, dH(t), \end{aligned}$$

where $H \in BV[a, b]$. Moreover, let all the assumptions given in $\mathbf{S}$ are valid. Then for all continuous function $\varphi : \mathbf{K} \to \mathbb{R}$ with nondecreasing increments the following inequality holds

$$\begin{aligned} &\varphi\left(\frac{1}{L}\left(\int_c^b \mathbf{g}(t)dH(t) - \frac{1}{\int_M \omega(s)d\mu(s)}\int_J\int_M \omega(s)\mathbf{f}(s,t)d\mu(s)dH(t)\right)\right) \leq \\ &\qquad \frac{1}{L}\left(\int_c^b \varphi(\mathbf{g}(t))dH(t) - \frac{1}{\int_M \omega(s)d\mu(s)}\int_J\int_M \omega(s)\varphi(\mathbf{f}(s,t))d\mu(s)dH(t)\right) \end{aligned}$$

Remark 11.10: The proof of Theorem 11.10 is obtained easily by replacing Lemma 11.2 by Lemma 11.3 and Theorem *A*.33 by Theorem *A*.31 in proof of Theorem 11.9.

Remark 11.11: If in Theorem 11.10 we put $n = 1$, then φ becomes continuous Wright convex function which is in fact convex function and hence Theorem 11.9 becomes its special case.

Remark 11.12: If we simply put $\omega(s) = 1\ \forall\ s \in M$ in Theorem 11.10, then we obtain Theorem 15 of [141] as our special case and in turn we capture all the corollaries, refinements and applications of Theorem 15 of [141].

Remark 11.13: An alternate version of Theorem 11.10 can also be given for nondecreasing functions f and g see [15] for details.

Remark 11.14: We can also give refinements and applications of main result of this section as we have given in Sections 11.1.2 and 11.1.3 see [15], [141] for details.

11.3.1 Further Generalization of Niezgoda's Inequality Using Isotonic Linear Functionals

Now we present our main theorems of this section.

Theorem 11.11: *Let M be a non-empty set. Let $g : [a, b] \to K$ and $f : M \times [a, b] \to K$ be* *nonincreasing continuous functions such that for each* $s \in M$

$$\begin{aligned} \int_a^x f(s,t)\,dH(t) &\leq \int_a^x g(t)dH(t), \quad \forall \quad x \in (a,b), \\ \textit{and} \quad \int_a^b f(s,t)\,dH(t) &= \int_a^b g(t)\,dH(t), \end{aligned}$$

hold, where $H \in BV[a,b]$*. Further, let the following assumptions be valid* $a = b_0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq a_{k+1} = b$, $J = \bigcup_{l=1}^{k}(a_l, b_l)$, $J^c = [a,b]\backslash J = \bigcup_{l=1}^{k+1}[b_{l-1}, a_l]$ *and* $L = |J^c| = \sum_{l=1}^{k+1}(a_l - b_{l-1})$*. Further we suppose that* L *satisfies properties* $L1$ *and* $L2$ *on* X *and let* φ *is continuous convex function on* K*. If* A *is an isotonic linear functional on* L *with* $A(\mathbf{1}) = 1$*, then* $\forall\ f \in L$ *such that* $\int_{J^c} f(s,t)dH(t)$, $\varphi(f)$, $\varphi\left(\int_{J^c} f(s,t)dH(t)\right) \in L$ *we have*

$$\begin{aligned} &\varphi\left(\frac{1}{L}\left(\int_c^b g(t)dH(t) - \int_J A(f(s,t))dH(t)\right)\right) \\ &\leq \frac{1}{L}\left(\int_c^b \varphi(g(t))dH(t) - \int_J A(\varphi(f(s,t)))dH(t)\right). \end{aligned} \tag{11.11}$$

Proof: Starting with L.H.S. of (11.11), using linearity of isotonic linear functional, first applying Jessen version of Jensen's inequality and then applying integral version of Jensen's inequality and finally using Lemma 11.2 and Theorem A.33, one gets

$$\begin{aligned} &\varphi\left(\frac{1}{L}\left(\int_c^b g(t)dH(t) - \int_J A(f(s,t))dH(t)\right)\right) \\ &= \varphi\left(A\left(\frac{1}{L}\left(\int_c^b g(t)dH(t) - \int_J f(s,t)dH(t)\right)\right)\right) \\ &= \varphi\left(A\left(\frac{1}{L}\int_{J^c} f(s,t)dH(t)\right)\right) \\ &\leq A\left(\varphi\left(\frac{1}{L}\int_{J^c} f(s,t)dH(t)\right)\right) \\ &\leq A\left(\frac{1}{L}\int_{J^c} \varphi(f(s,t))dH(t)\right) \\ &\leq A\left(\frac{1}{L}\left(\int_c^b \varphi(g(t))dH(t) - \int_J \varphi(f(s,t))dH(t)\right)\right) \\ &= \frac{1}{L}\left(\int_c^b \varphi(g(t))dH(t) - \int_J A(\varphi(f(s,t)))dH(t)\right). \end{aligned}$$

□

Remark 11.15: If we choose $A(\mathbf{f}(s,t)) = \dfrac{1}{\int_M \omega(s)d\mu(s)} \displaystyle\int_M \omega(s)\mathbf{f}(s,t)d\mu(s)$ in Theorem 11.11, then we get Theorem 11.9 and all its consequences.

Remark 11.16: If in Theorem 11.11 we set $k = 1$, $a = 1$, $b = 2$, $f(s,t) = f(s)\chi_{[1,2)}(t)$, $g(t) = \sum_{j=1}^{2} \alpha_j\, \chi_{[j,j+1)}(t)$, $H = \lambda$, where λ is Lebesgue measure, we would get following result which is Theorem 2.1 of [76] which is stated as.

Corollary 11.8: *Let* L *satisfy properties* $L1$, $L2$ *on* $E \neq \phi$*, and let* φ *is convex function on* $I = [a,b]$ $(-\infty < a < b < \infty)$*. If* A *is an isotonic linear functional on* L *with* $A(\mathbf{1}) = 1$*, then*

$\forall\ g \in L$ such that $\varphi(g), \varphi(a+b-g) \in L$ (so that $a \leq g(t) \leq b\ \forall\ t \in E$), we have the following variant of Jessen's inequality

$$\varphi(a+b-A(g)) \leq \varphi(a)+\varphi(b)-A(\varphi(g)).$$

Clearly following series of inequalities hold

$$\begin{aligned}\varphi(m_1+m_2-A(g)) &\leq A(\varphi(m_1+m_2-g)) \\ &\leq \frac{m_2-A(g)}{m_2-m_1}\varphi(m_2)+\frac{A(g)-m_1}{m_2-m_1}\varphi(m_1) \\ &\leq \varphi(m_1)+\varphi(m_2)-A(\varphi(g)).\end{aligned}$$

Remark 11.17: By choose different isotonic linear functionals we get various results. For instance if we choose A such that it is defined on the class $L^1(\mu)$ as $A(f) = \int_M f d\mu$ we get a result and its refinements from [151].

Remark 11.18: An alternate version of Theorem 11.11 can also be given for nondecreasing functions f and g see [15] for details.

Remark 11.19: We can also give refinements and applications of main result of this section as we have given in Sections 11.1.2 and 11.1.3 see [15] for details.

Remark 11.20: For other related results we refer the reader to [134].

CHAPTER 12

Generalized Niezgoda Inequality for Similarly Separable Vectors

This chapter investigates generalized Niezgoda inequalities for similarly separable vectors. We begin by defining similarly separable vectors and highlighting their key properties. The discussion then presents generalized Niezgoda's inequalities for similarly separable vectors in $\mathbb{R}^n$, incorporating index set functions and associated refinements. Finally, these results are extended to separable Hilbert spaces, providing further generalizations and sharper bounds. The chapter offers a unified approach to Mercer-type inequalities in both finite- and infinite-dimensional settings.

12.1 Generalized Niezgoda Inequality for Similarly Separable Vectors in $\mathbb{R}^n$

12.1.1 Similarly Separable Vectors

In current section we provide generalization of Niezgoda's inequality (10.2). For this purpose, some notations are required here. We also quote some relevant definitions from [196].

Throughout the paper, we assume that $\mathbf{p} = (p_1, \ldots, p_m)$ is a positive m-tuple.

Here we introduce inner product on $\mathbb{R}^m$ for $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j \in I_m} p_j a_j b_j. \tag{12.1}$$

For $\lambda \in I_m$, we denote $P_\lambda = \sum_{j \in I_\lambda} p_j, \quad \hat{P}_\lambda = \sum_{j \in I_\lambda} j p_j, \ \tilde{P}_\lambda = \sum_{j \in I_\lambda} j^2 p_j.$

Unless specified elsewise, we take $\varepsilon = \{\mathbf{e_1}, \ldots, \mathbf{e_m}\}$ as an ordered basis in $\mathbb{R}^m$ and $D = \{\mathbf{d_1}, \ldots, \mathbf{d_m}\}$ as the dual basis of ε , that is, $\langle \mathbf{e_i}, \mathbf{d_j} \rangle = \delta_{ij}$ (Kronecker delta) for $i, j \in I_m$.

Definition 12.1: We define a vector $\mathbf{v} \in \mathbb{R}^m$ to be ε-positive if $\langle \mathbf{e_i}, \mathbf{v} \rangle > 0$ for all $i \in I_m$. Let J_1 and J_2 be two sets of indices such that $J_1 \cup J_2 = J$. Given $\mu \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^m$, a vector $\mathbf{z} \in \mathbb{R}^m$ is known as $\mu, \mathbf{v}$-separable on J_1 and J_2 (with respect to basis ε), if

$$\langle \mathbf{e_i}, \mathbf{z} - \mu \mathbf{v} \rangle \geq 0 \quad \text{for} \quad i \in J_1 \quad \text{and} \quad \langle \mathbf{e_j}, \mathbf{z} - \mu \mathbf{v} \rangle \leq 0 \quad \text{for} \quad j \in J_2.$$

We say $\mathbf{z}$ is $\mu, \mathbf{v}$-separable on J_1 and J_2 with respect to the basis ε if and only if

$$\max_{i \in J_2} \frac{\langle \mathbf{e_i}, \mathbf{z} \rangle}{\langle \mathbf{e_i}, \mathbf{v} \rangle} \leq \mu \leq \min_{j \in J_1} \frac{\langle \mathbf{e_j}, \mathbf{z} \rangle}{\langle \mathbf{e_j}, \mathbf{v} \rangle}$$

where $\mathbf{v}$ is ε-positive.

Definition 12.2: A vector $\mathbf{z} \in \mathbb{R}^m$ is $\mathbf{v}$-separable on J_1 and J_2 (with respect to the basis ε), if for some $\mu \in \mathbb{R}$, $\mathbf{z}$ is $\mu, \mathbf{v}$-separable on J_1 and J_2.

Definition 12.3: A map $\varphi : J \to \mathbb{R}$ is said to preserve $\mathbf{v}$-separability on J_1 and J_2 with respect to ε, if $\varphi(\mathbf{z}) = (\varphi(z_1), \ldots, \varphi(z_m))$ is $\mathbf{v}$-separable on J_1 and J_2 with respect to ε, whenever $\mathbf{z} = (z_1, \ldots, z_m) \in J^m$ is $\mathbf{v}$-separable on J_1 and J_2 with respect to ε.

Remark 12.1: In case where $\mathbf{v}$ is ε-positive, $J_1 = \{\lambda_0\}$ and $J_2 = J \backslash \{\lambda_0\}$, the $\mathbf{v}$-separability of $\mathbf{z}$ is implied by:

$$\frac{\langle \mathbf{e}_\lambda, \mathbf{z} \rangle}{\langle \mathbf{e}_\lambda, \mathbf{v} \rangle} \leq \frac{\langle \mathbf{e}_{\lambda_0}, \mathbf{z} \rangle}{\langle \mathbf{e}_{\lambda_0}, \mathbf{v} \rangle}$$

for $\lambda \in I_m$.

Definition 12.4: [224, pp. 32, 110] Given a real convex function Ψ on J, $\partial \Psi$ denotes the subdifferential of Ψ. It is the set of all functions $\varphi : J \to [-\infty, \infty]$ such that $\varphi(J^o) \subseteq \mathbb{R}$ and

$$\Psi(x) \geq \Psi(a) + (x - a)\varphi(a) \quad \text{for any } x,\, a \in J.$$

Using the notations defined until now, we now present our main result:

Theorem 12.1: *Define $\Psi : J \to \mathbb{R}$ to be a convex function on an open interval $J \subseteq \mathbb{R}$. Suppose $\mathbf{a} = (a_1, \ldots, a_m) \in J^m$ and $\boldsymbol{X} = (\mathbf{x}_j) = (x_{ij})$ is an $n \times m$ matrix such that $x_{ij} \in J$ and $(\mathbf{x}_j)$ is a monotonic m-tuple for all $i \in I_n$, $j \in I_m$. Also assume that the weight w_i for $i \in I_n$ satisfying the conditions as in (A.9) and $\frac{1}{W_n} \sum_{i \in I_n} w_i \Psi(x_{ij}) \leq \Psi(a_j)$ for $j \in I_m$. We further let that $\langle \mathbf{a} - \mathbf{x_{i.}}, \mathbf{v} \rangle = 0$ for $i \in I_n$. Then the following inequality holds:*

$$\begin{aligned}
&\Psi \left(\sum_{j \in I_m} \epsilon p_j v_j a_j - \frac{1}{W_n} \sum_{j=1}^{k-1} \epsilon p_j v_j \sum_{i \in I_n} w_i x_{ij} - \frac{1}{W_n} \sum_{j=k+1}^{m} \epsilon p_j v_j \sum_{i \in I_n} w_i x_{ij} \right) \\
&\qquad \leq \frac{1}{p_k} \sum_{j \in I_m} p_j \Psi(a_j) - \frac{1}{p_k} \frac{1}{W_n} \sum_{j=1}^{k-1} p_j \sum_{i \in I_n} w_i \Psi(x_{ij}) - \frac{1}{p_k} \frac{1}{W_n} \sum_{j=k+1}^{m} p_j \sum_{i \in I_n} w_i \Psi(x_{ij}),
\end{aligned}$$

where $\epsilon = \frac{1}{p_k v_k}$ with $v_k \neq 0$ for $k \in I_m$.

Proof: First, we use the assumption that for each $j \in I_m$

$$\frac{1}{W_n}\sum_{i\in I_n} w_i \Psi(x_{ij}) \leq \Psi(a_j)$$

multiplying it by p_j and taking sum over $j \in I_m$, we get

$$\frac{1}{W_n}\sum_{i\in I_n}\sum_{j\in I_m} p_j w_i \Psi(x_{ij}) \leq \sum_{j\in I_m} p_j \Psi(a_j). \tag{12.2}$$

Since $\langle \mathbf{a}-\mathbf{x_{i.}}, \mathbf{v}\rangle = 0$ for all $i \in I_n$ by (12.1) we have for each $i \in I_n$.

$$\epsilon\left(\sum_{j\in I_m} p_j v_j a_j - \sum_{j=1}^{k-1} p_j v_j x_{ij} - \sum_{j=k+1}^{m} p_j v_j x_{ij}\right) = x_{ik} \tag{12.3}$$

Now, using Jensen-Steffensen inequality for weights w_i and then using (12.3) and inequality (12.2) for weights p_j we obtain our required result as follows

$$\begin{aligned}
&p_k\Psi\left(\sum_{j\in I_m}\epsilon p_j v_j a_j - \frac{1}{W_n}\sum_{j=1}^{k-1}\epsilon p_j v_j \sum_{i\in I_n} w_i x_{ij} - \frac{1}{W_n}\sum_{j=k+1}^{m}\epsilon p_j v_j \sum_{i\in I_n} w_i x_{ij}\right)\\
&= p_k\Psi\left(\frac{1}{W_n}\sum_{i\in I_n} w_i\left(\sum_{j\in I_m}\epsilon p_j v_j a_j - \sum_{j=1}^{k-1}\epsilon p_j v_j x_{ij} - \sum_{j=k+1}^{m}\epsilon p_j v_j x_{ij}\right)\right)\\
&\leq \frac{1}{W_n}\sum_{i\in I_n} w_i p_k \Psi\left(\sum_{j\in I_m}\epsilon p_j v_j a_j - \sum_{j=1}^{k-1}\epsilon p_j v_j x_{ij} - \sum_{j=k+1}^{m}\epsilon p_j v_j x_{ij}\right)\\
&= \frac{1}{W_n}\sum_{i\in I_n} w_i p_k \Psi\left(x_{ik}\right)\\
&\leq \frac{1}{W_n}\sum_{i\in I_n} w_i\left(\sum_{j\in I_m} p_j\Psi(a_j) - \sum_{j=1}^{k-1} p_j\Psi(x_{ij}) - \sum_{j=k+1}^{m} p_j\Psi(x_{ij})\right)\\
&= \sum_{j\in I_m} p_j\Psi(a_j) - \frac{1}{W_n}\sum_{j=1}^{k-1} p_j\sum_{i\in I_n} w_i\Psi(x_{ij}) - \frac{1}{W_n}\sum_{j=k+1}^{m} p_j\sum_{i\in I_n} w_i\Psi(x_{ij}).
\end{aligned}$$

□

Remark 12.2: Here we observe that the proof of Theorem 12.1 is much more simpler than proof of Theorem 3.1 of [196]. It should be noted that in Theorem 3.1 of [196] the author has used positive weights while we have used real weights satisfying assumptions as stated in (*A*.9) with monotonic $m-$tuples $(\mathbf{x}_j)$. It is also worth mentioning that we did not use bifractional inequality [197] or concept of similarly separable vectors, we removed all these assumptions at the expense of the assumption that $\frac{1}{W_n}\sum_{i\in I_n} w_i\Psi(x_{ij}) \leq \Psi(a_j)$ for $j \in I_m$.

Now we state a corollary and other special case of Theorem 12.1 as under:

Corollary 12.1: *Define $\Psi : J \to \mathbb{R}$ to be a convex function on an open interval $J \subseteq \mathbb{R}$. Let*

$\partial\Psi : J \to \mathbb{R}$ be the subdifferential of Ψ and $\varphi \in \partial\Psi$. Suppose $\mathbf{a} = (a_1, \ldots, a_m) \in J^m$ and $\boldsymbol{X} = (\mathbf{x}_j) = (x_{ij})$ is an $n \times m$ matrix such that $x_{ij} \in J$ for all $i \in I_n$, $j \in I_m$. Further suppose that the weight w_i are positive real weights for $i \in I_n$ Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ such that $\langle \mathbf{u}, \mathbf{v} \rangle > 0$. Let there exist index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that for each $i \in I_n$ we have:

- *(i)* $\mathbf{x_{i.}}$ *is* $\mathbf{v}$*-separable on* J_1 *and* J_2 *with respect to* ε,
- *(ii)* $\mathbf{a} - \mathbf{x_{i.}}$ *is* $0, \mathbf{u}$*-separable on* J_1 *and* J_2 *with respect to* D,
- *(iii)* $\langle \mathbf{a} - \mathbf{x_{i.}}, \mathbf{v} \rangle = 0$,
- *(iv)* φ *preserves* $\mathbf{v}$*-separability on* J_1 *and* J_2 *with respect to* ε.

Then the following inequality holds:

$$\Psi\left(\sum_{j\in I_m} \epsilon p_j v_j a_j - \frac{1}{W_n}\sum_{j=1}^{k-1} \epsilon p_j v_j \sum_{i\in I_n} w_i x_{ij} - \frac{1}{W_n}\sum_{j=k+1}^{m} \epsilon p_j v_j \sum_{i\in I_n} w_i x_{ij}\right)$$
$$\le \frac{1}{p_k}\sum_{j\in I_m} p_j \Psi(a_j) - \frac{1}{p_k}\frac{1}{W_n}\sum_{j=1}^{k-1} p_j \sum_{i\in I_n} w_i \Psi(x_{ij}) - \frac{1}{p_k}\frac{1}{W_n}\sum_{j=k+1}^{m} p_j \sum_{i\in I_n} w_i \Psi(x_{ij}), \tag{12.4}$$

where $\epsilon = \frac{1}{p_k v_k}$ with $v_k \neq 0$ for $k \in I_m$.

Remark 12.3: If we simply put $k = m$ and $W_n = 1$ in Corollary 12.1 we will get Theorem 3.1 of [196] (for further remarks see [132]) and consequently we capture all its corollaries and special cases. For other corollaries and remarks we refer the reader to [142].

Theorem 12.2: *If in Corollary 12.1, Ψ is a differential function with $\mathbf{a} = (a, a, \ldots, a) \in J^m$, $\mathbf{p} = (p_1, \ldots, p_m)$ is a real m-tuple satisfying the conditions given in (A.9) and (12.1), then inequality (12.4) can be written as:*

$$p_m \Psi\left(a \sum_{j\in I_m} \epsilon p_j v_j - \frac{1}{W_n}\sum_{j=1}^{m-1} \epsilon p_j v_j \sum_{i\in I_n} w_i x_{ij}\right) \le P_m \Psi(a) - \frac{1}{W_n}\sum_{j=1}^{m-1} p_j \sum_{i\in I_n} w_i \Psi(x_{ij})),$$

Proof: In start of this proof, it is worth mentioning that here m-tuple $\mathbf{p}$ is real satisfying $0 \le P_j \le P_m$ for $j \in \{1, \ldots, m-1\}$ with $P_m > 0$ and clearly $P_{m-1} \le P_m$ which implies that $0 \le P_m - P_{m-1} = p_m$. For an $n \times m$ matrix $\mathbf{X} = (\mathbf{x}_j) = (x_{ij})$ such that $x_{ij} \in J$ and $(\mathbf{x}_j)$ is a monotonic m-tuple for all $i \in I_n$, $j \in I_m$, then for $a_i \equiv a \in J$, if Ψ is a differential function then, for m-tuple $\mathbf{p} = (p_1, \ldots, p_m)$ satisfying conditions (A.9) and (12.1), [27, Theorem 2.1] can be written as

$$\frac{1}{P_m}\left[\sum_{j\in I_m} p_j \Psi(x_{ij}) - \sum_{j\in I_m} p_j \Psi(a)\right] \le \frac{1}{P_m}\sum_{j\in I_m} p_j \Psi'(x_{ij})(x_{ij} - a)$$

which implies that

$$\sum_{j\in I_m} p_j \left(\Psi(x_{ij}) - \Psi(a)\right) \le \sum_{j\in I_m} p_j \Psi'(x_{ij})(x_{ij} - a)$$

multiplying both sides by (-1)

$$\sum_{j\in I_m} p_j\left(\Psi(a)-\Psi(x_{ij})\right) \geq \sum_{j\in I_m} p_j\Psi'(x_{ij})(a-x_{ij})$$

that follows to inequality (12.4) in the following form, that is, fixing $k=m$

$$p_m\Psi\left(a\sum_{j\in I_m}\epsilon p_j v_j - \frac{1}{W_n}\sum_{j=1}^{m-1}\epsilon p_j v_j\sum_{i\in I_n} w_i x_{ij}\right) \leq P_m\Psi(a) - \frac{1}{W_n}\sum_{j=1}^{m-1} p_j\sum_{i\in I_n} w_i\Psi(x_{ij})).$$

□

Remark 12.4: Similar results can be produced for concave functions by making use of the definition of concave functions, i.e, Ψ is concave if and only if $-\Psi$ is convex.

Remark 12.5: Some other results related to similarly separable vectors and Sequences can be found in [199] and [197].

12.1.2 Index Set Functions and Refinements of Generalized Niezgoda's Inequality For Similarly Separable Vectors

In start of this section we give some construction which we will use throughout this section: Let I be a finite nonempty set of positive integers. Let $\mathbf{w}=(w_i),\ i\in I$ be a real sequence and let $(\mathbf{x}_j)=(x_{ij})$ be a sequence of vectors such that $x_{ij}\in J$ for all $i\in I$, $j\in I_m$. Moreover we define $A_I(\mathbf{x}_j,\mathbf{w})=\frac{1}{W_I}\sum_{i\in I} w_i x_{ij}$ where $W_I=\sum_{i\in I} w_i$. For a convex function $\Psi: J\to\mathbb{R}$. Also if assumptions of Theorem 12.1 are valid we define the index set function F as

$$\begin{aligned}F(I)=W_I\Bigg[&\sum_{j\in I_m}\tilde{p}_j\Psi(a_j)-\frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I}\tilde{p}_j w_i\Psi(x_{ij})-\frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I}\tilde{p}_j w_i\Psi(x_{ij})\\&-\Psi\left(\sum_{j\in I_m}\varepsilon p_j v_j a_j-\frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I}\varepsilon p_j v_j w_i x_{ij}-\frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I}\varepsilon p_j v_j w_i x_{ij}\right)\Bigg]\end{aligned}$$

where $\mathbf{a}=(a_1,a_2,\ldots,a_m)$ is an m-tuple such that $a_j\in J$ for $j\in I_m$ and $\tilde{p}_j=\frac{p_j}{p_k}$ and $k\in I_m$.

Now we state some results here. Since proving techniques are similar to that of results given in previous chapters so we omit the details, interested readers can see [142].

Theorem 12.3: *Let $\mathbf{a}=(a_1,a_2,\ldots,a_m)$ be an m-tuple such that $a_j\in J$ for $j\in I_m$, I and $\bar{I}$ be nonempty sets such that $I\cup\bar{I}=I_n$ and $I\cap\bar{I}=\emptyset$. Let $(\mathbf{x}_j)=(x_{ij})$ be a sequence of vectors such that $x_{ij}\in J$ for all $i\in I$, $j\in I_m$ and $\mathbf{w}=(w_i),\ i\in I_n$ such that $W_{I_n}>0$. Let Ψ be a convex function on J and $A_S(\mathbf{x}_j,\mathbf{w})\in J$ $(S\in\{I,\bar{I},I\cup\bar{I}\})$. If $W_I>0$ and $W_{\bar{I}}>0$, then*

$$F(I\cup\bar{I})\geq F(I)+F(\bar{I}). \tag{12.5}$$

If $W_I\cdot W_{\bar{I}}<0$, then inequality (12.5) is reversed.

Corollary 12.2: *Let $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ be an m-tuple such that $a_j \in J$ for $j \in I_m$. Let I_t, $t \in I_\lambda$ be finite nonempty sets of positive integers such that $I_s \cap I_t = \emptyset$ for all $s \neq t \in I_\lambda$. We further suppose that $(\mathbf{x}_j) = (x_{ij})$ is a real sequence of vectors such that $x_{ij} \in J$ for all $i \in \bigcup_{t=1}^{\lambda} I_t$, $j \in I_m$ and let $\mathbf{w} = (w_i)$, $i \in \bigcup_{t=1}^{\lambda} I_t$ such that $W_{i \in \bigcup_{t=1}^{\lambda} I_t} > 0$ and $A_S(\mathbf{x}_j, \mathbf{w}) \in J$ $(S \in \{I_1, \ldots, I_t, \bigcup_{t=1}^{r} I_t\})$ $(r \in \{2, \ldots, \lambda\})$.*

(a) If $W_{I_t} > 0$ for $t \in I_\lambda$, we have

$$F\left(\bigcup_{t=1}^{\lambda} I_t\right) \geq \sum_{t=1}^{\lambda} F(I_t). \tag{12.6}$$

(b) If $W_{I_1} > 0$ and $W_{I_t} < 0$ for $t \in \{2, \ldots, \lambda\}$, then inequality (12.6) is reversed.

The following results give us refinements of Niezgoda's Inequality. For the remaining part of this section we assume $x_{ij} \in [a, b] \subseteq J$ for all i and j.

Corollary 12.3: *Let $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ be an m-tuple such that $a_j \in J$ for $j \in I_m$. We further suppose that $(\mathbf{x}_j) = (x_{ij})$ is a real sequence of vectors such that $x_{ij} \in J$ for all $i \in I_n$, $j \in I_m$ and let Ψ be a continuous convex function on J. If $w_1 > 0$ and $w_i \geq 0$ for $i \in \{2, \ldots, n\}$, then under the assumptions in Corollary 12.1 we have*

$$F(I_n) \geq F(I_{n-1}) \geq \cdots \geq F(I_2) \geq F(I_1) \geq 0.$$

If $w_i \leq 0$ for $i \in \{2, \ldots, n\}$, $W_{I_n} > 0$ and $A_{I_n}(\mathbf{x}_j, \mathbf{w}) \in [a, b] \subseteq J$, then

$$0 \leq F(I_n) \leq F(I_{n-1}) \leq \cdots \leq F(I_2) \leq F(I_1).$$

Remark 12.6: For main result of this section we can also state results analogous to Theorem 12.3 and its corollaries. For further refinements, remarks and corollaries and applications of this section see [142].

12.2 Generalization and Refinements of Niezgoda's Inequality Using Similarly Separable Vectors in Separable Hilbert Spaces

12.2.1 Similary Separable Vectors in Separable Hilbert Spaces

Throughout this section, H is a separable Hilbert space with suitable inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ and U is an open subset of H. It is a known fact that, every separable Hilbert space has a countable orthonormal basis [160]. Separable Hilbert spaces possess many interesting properties (for comprehensive discussion on this topic, we refer the reader to [160] and [220]).

Here we recall some definitions from [196]: Let $E = \{e_i : i \in \bar{\mathbb{N}}\}$ be an ordered basis of H and $D = \{d_i : i \in \bar{\mathbb{N}}\}$ be the dual basis of H, that is $\langle e_i, d_j \rangle = \delta_{ij}$ (Kronecker delta) for $i, j \in \bar{\mathbb{N}}$; where $\bar{\mathbb{N}} \subseteq \mathbb{N}$ depending on whether dimension of H is finite or infinite and Kronecker delta is defined as

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Definition 12.5: We say that a vector $v \in H$ is $E-$positive if $\langle e_i, v\rangle > 0$ for all $i \in \bar{\mathbb{N}}$.

We denote $J = \bar{\mathbb{N}}$. Let J_1 and J_2 be two sets of indices such that $J_1 \bigcup J_2 = J$.

Definition 12.6: Given $\alpha \in \mathbb{R}$ and $v \in H$, a vector $z \in H$ is said to be $\alpha, v-$separable on J_1 and J_2 with respect to the basis E, if $\langle e_i, z - \alpha\, v\rangle \geq 0$ for $i \in J_1$ and $\langle e_j, z - \alpha v\rangle \leq 0$ for $j \in J_2$.

Definition 12.7: A vector $z \in H$ is said to be v-separable on J_1 and J_2 with respect to E, if z is $\alpha, v-$separable on J_1 and J_2 for some $\alpha \in \mathbb{R}$.

Definition 12.8: A map $\Phi : U \to \mathbb{R}$ is said to preserve $v-$separability on J_1 and J_2 with respect to E, if $\Phi(z)$ is $v-$separable on J_1 and J_2 with respect to $\mathbf{E}$, whenever $z \in U$ is $v-$separable on J_1 and J_2 with respect to E.

12.2.2 Generalization of Niezgoda's Inequality Using Similarly Separable Vectors in Separable Hilbert Spaces

Here we recall Theorem 3.5 of [197]:

Theorem 12.4: *Assume that E is a basis of a separable Hilbert space H with inner product $\langle .,.\rangle : H \times H \to \mathbb{R}$ and D is the dual basis of E. Let f, g, v and y be vectors in H with $\langle y, v\rangle \neq 0$. Denote $\lambda = \langle f, v\rangle / \langle y, v\rangle$. If there exist index sets J_1 and J_2 with $J_1 \bigcup J_2 = J$, then the following two statements are equivalent:*

(i) The vector g is $\lambda, y-$separable on J_1 and J_2 with respect to D if $\langle y, v\rangle > 0$ (or on J_2 and J_1 with respect to D if $\langle y, v\rangle < 0$).

(ii) The inequality $\langle f, y\rangle\langle g, v\rangle \leq \langle f, g\rangle\langle y, v\rangle$ holds for every vector f which is $v-$separable on J_1 and J_2 with respect to E.

Remark 12.7: This result has many important consequences as stated by Niezgoda in [197]. Niezgoda chose $H = \mathbb{R}^k$ with standard inner product and $\bar{\mathbb{N}} = \{1, \ldots, k\}$ for some fixed $k \in \mathbb{N}$ in Theorem 12.4 and stated all the related results and corollaries for the discrete version in [197]. Here we are interested in its integral version.

For the next corollary we need some notations and definitions: Let us consider a measure space $([a, b], \Sigma, \nu)$. Let $w : [a, b] \to [0, \infty)$ be a measurable function with $w \not\equiv 0$ on a set of nonzero measure. The $w-$weighted L_2 space is defined as $L_2([a, b], wd\nu)$, where $wd\nu$ means the measure M defined by $M(A) \equiv \int_A w(x)\mathrm{d}\nu(x)$ and $A \in \Sigma$. The inner product for $L_2([a, b], wd\nu)$ is

$$\langle f, g\rangle := \langle f, g\rangle_{L_2([a,b],w\,\mathrm{d}\nu)} = \int_a^b w(t)f(t)g(t)d\nu(t). \tag{12.7}$$

Corollary 12.4: *Assume that E is an ordered basis of $L_2([a,b], wd\nu)$ and D is the dual basis of E. Let f, g, v and y be vectors in $L_2([a,b], wd\nu)$ with $\langle y, v\rangle > 0$.*
Denote $\lambda = \langle f, v\rangle / \langle y, v\rangle$.
If there exist index sets J_1 and J_2 with $J_1 \bigcup J_2 = J$ such that

(i) f is $v-$separable on J_1 and J_2 with respect to E and
(ii) g is $\lambda, y-$separable on J_1 and J_2 with respect to D,

then the following inequality holds

$$\int_a^b w(t)f(t)y(t)d\nu(t) \int_a^b w(t)v(t)g(t)d\nu(t) \leq \int_a^b w(t)f(t)g(t)d\nu(t) \int_a^b w(t)v(t)y(t)d\nu(t). \quad (12.8)$$

Remark 12.8: If we simply choose $y \equiv 1$ and $v \equiv 1$ in (12.8) then using Lebesgue measure we obtain the well-known Čebyšev inequality [204, p. 197]:

Corollary 12.5: *Let $f, g : [a,b] \to \mathbb{R}$ and $w : [a,b] \to (0, \infty)$ be integrable functions. If f and g are monotonic in the same direction, then we get*

$$\int_a^b w(t)f(t)dt \int_a^b w(t)g(t)dt \leq \int_a^b w(t)dt \int_a^b w(t)f(t)g(t)dt \quad (12.9)$$

provided that the integrals exist. If f and g are monotonic in opposite directions, then the reverse of inequality (12.9) holds. In both cases, equality in (12.9) holds if and only if either g or f is constant almost everywhere.

Remark 12.9: The inequality (12.9) still holds under different assumptions. For detailed discussion on inequality (12.9) we refer [204, p. 198-199].

We now introduce some terminologies which will help us in understanding the upcoming results. If φ is a convex function on an open convex subset U of a normed linear space V, then φ has a supporting hyperplane at each point $z \in U$. This means the existence of a continuous linear functional φ on V (the support of φ at z) such that

$$\varphi(x) \geq \varphi(z) + \varphi(x - z) \quad \text{for all} \quad x \in U. \quad (12.10)$$

The set $\partial\varphi(z)$ of all such functionals φ constitutes the subdifferential of φ at the point z. When V is an $\mathbb{R}^k$ the inequality (12.10) becomes $\varphi(x) \geq \varphi(z) + \langle x - z, \Phi(z)\rangle$ for all $x \in U$ where $\Phi(z) = (\varphi(z_1), \ldots, \varphi(z_k))$ for $z = (z_1, \ldots, z_k) \in \mathbb{R}^k$ and the subdifferential $\partial\varphi(a)$ is the set of all functions φ (usually called subgradients) (see e.g. [194]).

We now present first result of this section:

Theorem 12.5: *Let $\varphi : U \to \mathbb{R}$ be a convex function defined on an open subset U of H. Let $\partial\varphi : U \to \mathbb{R}$ be the subdifferential of φ and let $\varphi \in \partial\varphi$. Assume that E is a basis of a separable Hilbert space H with inner product $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{R}$ and D is the dual basis of E. Let f, g, v and y be vectors in H with $\langle y, v\rangle \neq 0$. Denote $\lambda = \langle g - f, v\rangle / \langle y, v\rangle$ with $\langle y, v\rangle > 0$.*
If there exist index sets J_1 and J_2 with $J_1 \bigcup J_2 = J$ such that

(i) *f is $v-$separable on J_1 and J_2 with respect to E,*
(ii) *$g-f$ is $\lambda, y-$separable on J_1 and J_2 with respect to D and*
(iii) *Φ preserves $v-$separability on J_1 and J_2 with respect to E.*

Then the following statements are valid:

(a) *If $\langle g-f, v\rangle = 0$, then the following inequality holds*

$$\langle \varphi(g) - \varphi(f), 1\rangle \geq 0. \tag{12.11}$$

(b) *If $\langle g-f, v\rangle \geq 0$ and $\langle \Phi(f), y\rangle \geq 0$, then the inequality (12.11) holds.*

Proof:

(a) Using the definition of subdifferential we have

$$\varphi(g) - \varphi(f) \geq \langle g-f, \Phi(f)\rangle, \tag{12.12}$$

Combining (i) and (iii), we deduce that the vector $\Phi(f)$ is $v-$separable on J_1 and J_2 with respect to E. Using Theorem 12.4, we get

$$\langle g-f, \Phi(f)\rangle \geq \frac{1}{\langle y, v\rangle}\langle g-f, v\rangle\langle \Phi(f), y\rangle \tag{12.13}$$

since $\langle y, v\rangle > 0$. So, if $\langle g-f, v\rangle = 0$, then (12.11) follows from (12.12) and (12.13).

(b) Clearly, (12.11) holds whenever $\langle g-f, v\rangle \geq 0$ and $\langle \Phi(f), y\rangle \geq 0$ by using inequalities (12.12) and (12.13).

$\square$

Remark 12.10: By choosing $H = \mathbb{R}^k$ with usual weighted inner product defined on $\mathbb{R}^k$, Theorem 2.2 of [199] becomes special case of our result. Also, we can easily obtain its corollaries and examples. Here we are interested in one of its consequences in integral version.

Corollary 12.6: *Let $\varphi : I \to \mathbb{R}$ be a convex function on an open interval I of $\mathbb{R}$. Let $\partial\varphi : I \to \mathbb{R}$ be the subdifferential of φ and let $\varphi \in \partial\varphi$. Define*

Let $([a,b], \Sigma, \nu)$ be a measure space with positive finite measure ν. Let f, $g : [a,b] \to I$ be two functions such that $g, f \in L_2([a,b], wd\nu)$ where w be a non-negative measurable function on $[a,b]$ with $w \not\equiv 0$ on a set of nonzero measure.

Assume that E is an ordered basis in $L_2([a,b], wd\nu)$ and D is the dual basis of E. Let v and y be vectors in $L_2([a,b], wd\nu)$and inner product is given by (12.7). Denote $\lambda = \langle g-f, v\rangle / \langle y, v\rangle$ with $\langle y, v\rangle > 0$. Suppose that there exist index sets J_1 and J_2 with $J_1 \bigcup J_2 = J$ such that

(i) *f is $v-$separable on J_1 and J_2 with respect to E,*
(ii) *$g-f$ is $\lambda, y-$separable on J_1 and J_2 with respect to D and*
(iii) *Φ preserves $v-$separability on J_1 and J_2 with respect to E.*

Then the following statements are valid:

(a) If $\langle g-f, v\rangle = 0$, then the following inequality holds

$$\int_a^b w(t)\varphi(f(t))d\nu(t) \leq \int_a^b w(t)\varphi(g(t))d\nu(t). \tag{12.14}$$

(b) If $\langle g-f, v\rangle \geq 0$ and $\langle \varphi(f), w\rangle \geq 0$ then the inequality (12.14) holds.

Let us introduce some notations here. We denote this set of assumptions by **S**.

S: $I = \bigcup_{i=1}^{k}(a_i, b_i)$, $I^c = [a,b]\backslash I = \bigcup_{i=1}^{k+1}[b_{i-1}, a_i]$ and $|I^c| = \sum_{i=1}^{k+1}(a_i - b_{i-1})$ where $a = b_0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq a_{k+1} = b$ is a partition of the interval $[a,b]$.

Theorem 12.6: *Let $\varphi : U \to \mathbb{R}$ be a convex function on open interval $U \subset H$. Let $\partial\varphi : U \to \mathbb{R}$ be the subdifferential of φ and let $\varphi \in \partial\varphi$. Let $([a,b], \Sigma, \nu)$ and (X, Ω, μ) be two measure space with positive finite measures ν and μ respectively. Let $g : [a,b] \to U$ and $f : X \times [a,b] \to U$ be two functions such that $g, f \in L_2([a,b], wd\nu)$ where w be a non-negative measurable function on $[a,b]$ with $w \not\equiv 0$ on set of measure nonzero. Moreover let all the assumptions given in* **S** *be valid. Further we assume that E, D, y and v be as in Theorem 12.5 and inner product is given by (12.7). Denote $\lambda = \langle g - f(s), v\rangle / \langle y, v\rangle$ for $s \in X$ with $\langle y, v\rangle > 0$. Suppose that there exist index sets J_1 and J_2 with $J_1 \bigcup J_2 = J$ such that for $s \in X$*

(i) $f(s)$ is $v-$separable on J_1 and J_2 with respect to. E,
(ii) $g - f(s)$ is $0, u-$separable on J_1 and J_2 with respect to. D,
(iii) $\langle g - f(s), v\rangle = 0$,
(iv) Φ preserves $v-$separability on J_1 and J_2 with respect to. E.
(v) $v(t) = K$ for all $t \in I^c$ where K is a non-zero constant.

Then the following inequality holds:

$$\begin{aligned}\varphi\left(\frac{1}{K}\Omega\left[\int_a^b w(t)v(t)g(t)d\nu(t) - \frac{1}{\mu(X)}\int_I w(t)v(t)\int_X f(s,t)d\mu(s)d\nu(t)\right]\right) \\ \leq \frac{1}{\Omega}\left(\int_a^b w(t)\varphi\left(g(t)\right)d\nu(t) - \frac{1}{\mu(X)}\int_I w(t)\int_X \varphi\left(f(s,t)\right)d\mu(s)d\nu(t)\right),\end{aligned} \tag{12.15}$$

where $\mu(X)$, $\Omega = \int_{I^c} w(t)d\nu(t) > 0$.

Proof: For $s \in X$, by *(iii)* we have $\langle g - f(s), v\rangle / \langle y, v\rangle = 0$. So, under the assumptions of the theorem, it follows from Corollary 12.6 that the following inequality holds for each $s \in X$

$$\int_a^b w(t)\varphi(f(s,t))d\nu(t) \leq \int_a^b w(t)\varphi(g(t))d\nu(t). \tag{12.16}$$

Also, we consider the fact that, since $\langle g - f(s), v\rangle = 0$ for each $s \in X$, we have

$$\begin{aligned}\left[\int_a^b w(t)v(t)g(t)d\nu(t) - \int_I w(t)v(t)f(s,t)d\nu(t)\right] &= \int_{I^c} w(t)v(t)f(s,t)d\nu(t) \\ &= K\int_{I^c} w(t)f(s,t)d\nu(t).\end{aligned}$$

Now, we consider L.H.S. of the inequality (12.15) and applying Integral Jensen's inequality twice and using the aforementioned fact with (12.16) we get,

$$\begin{aligned}
&\int_{I^c} w(t)d\nu(t)\varphi\left(\frac{1}{K\int_{I^c} w(t)d\nu(t)}\left[\int_a^b w(t)v(t)g(t)d\nu(t)\right.\right.\\
&\left.\left.-\frac{1}{\mu(X)}\int_I w(t)v(t)\int_X f(s,t)d\mu(s)d\nu(t)\right]d\mu(s)\right)\\
&=\int_{I^c} w(t)d\nu(t)\varphi\left(\frac{1}{\mu(X)}\int_X \frac{K}{K\int_{I^c} w(t)d\nu(t)}\int_{I^c} w(t)f(s,t)d\nu(t)d\mu(s)\right)\\
&\leq \int_{I^c} w(t)d\nu(t)\frac{1}{\mu(X)}\int_X \varphi\left(\frac{1}{\int_{I^c} w(t)d\nu(t)}\int_{I^c} w(t)f(s,t)d\nu(t)\right)d\mu(s)\\
&\leq \frac{\int_{I^c} w(t)d\nu(t)}{\int_{I^c} w(t)d\nu(t)}\frac{1}{\mu(X)}\int_X\int_{I^c} w(t)\varphi\left(f(s,t)\right)d\nu(t)d\mu(s)\\
&\leq \frac{1}{\mu(X)}\int_X\left(\int_a^b w(t)\varphi\left(g(t)\right)d\nu(t)-\int_I w(t)\varphi\left(f(s,t)\right)d\nu(t)\right)d\mu(s)\\
&=\int_a^b w(t)\varphi\left(g(t)\right)d\nu(t)-\frac{1}{\mu(X)}\int_I w(t)\int_X \varphi\left(f(s,t)\right)d\mu(s)d\nu(t).
\end{aligned}$$

□

The analogous discrete version of inequality (12.15), discussed in [142], is stated below.

Corollary 12.7: *Define $\varphi : I \to \mathbb{R}$ to be a convex function on an open interval $I \subseteq \mathbb{R}$. Let $\partial\varphi : I \to \mathbb{R}$ be the subdifferential of φ and $\varphi \in \partial\varphi$. Suppose $\mathbf{a} = (a_1, \ldots, a_m) \in I^m$ and $X = (\mathbf{x}_j) = (x_{ij})$ is an $n \times m$ matrix such that $x_{ij} \in I$ and $(\mathbf{x}_j)$ is a monotonic m-tuple $\forall i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ such that $\langle \mathbf{u}, \mathbf{v}\rangle > 0$. For each $i \in J_n$, suppose $\exists$ index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that*

(i) $\mathbf{x_{i.}}$ *is* $\mathbf{v}$*-separable on* J_1 *and* J_2 *with respect to* E,
(ii) $\mathbf{a} - \mathbf{x_{i.}}$ *is* $0, \mathbf{u}$*-separable on* J_1 *and* J_2 *with respect to* D,
(iii) $\langle \mathbf{a} - \mathbf{x_{i.}}, \mathbf{v}\rangle = 0$,
(iv) Φ *preserves* $\mathbf{v}$*-separability on* J_1 *and* J_2 *with respect to* E.

Then the following inequality holds:

$$\begin{aligned}
&\varphi\left(\sum_{j=1}^{m}\epsilon p_j v_j a_j-\frac{1}{W_n}\sum_{j=1}^{k-1}\epsilon p_j v_j\sum_{i=1}^{n} w_i x_{ij}-\frac{1}{W_n}\sum_{j=k+1}^{m}\epsilon p_j v_j\sum_{i=1}^{n} w_i x_{ij}\right)\\
&\qquad\leq \frac{1}{p_k}\sum_{j=1}^{m} p_j\varphi(a_j)-\frac{1}{p_k}\frac{1}{W_n}\sum_{j=1}^{k-1} p_j\sum_{i=1}^{n} w_i\varphi(x_{ij})-\frac{1}{p_k}\frac{1}{W_n}\sum_{j=k+1}^{m} p_j\sum_{i=1}^{n} w_i\varphi(x_{ij}),
\end{aligned}$$

where $\epsilon = \frac{1}{p_k v_k}$ with $v_k \neq 0$ for $k \in \{1, \ldots, m\}$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a real $n-$tuple such that w_i represents the weights satisfy the conditions given in ($A.9$).

Remark 12.11: For refinements and applications to generalized integral means, interested reader can see [142].

CHAPTER 13

A Variant and Extensions of Gabler Inequality

This chapter presents a variant and extensions of Gabler's inequality. We begin by defining Gabler's inequality and reviewing related foundational results. The discussion then introduces a Jensen–Mercer type variant and explores its applications to generalized means. Further, a Niezgoda-type extension is developed, followed by considerations of Gabler's inequality for functions with nondecreasing increments. These results highlight the versatility of Mercer-type refinements in extending classical inequalities.

13.1 The Gabler's Inequality

In [101] Gabler defined a special case of convex functions namely sequentially convex functions by employing the following double index function.

Definition 13.1: For $x = (x_1, \ldots, x_n) \in J^n$ and a real-valued function $f : J \to \mathbb{R}$, define

$$f_{k,n} = f_{k,n}(x) = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} f\left(\frac{1}{k}\left(x_{i_1} + \cdots + x_{i_k}\right)\right). \tag{13.1}$$

Gabler termed this double index function as an arithmetic mean of all possible convex functions generated by arithmetic means of any k values chosen from $(x_1, \ldots, x_n)$. Gabler then defined sequentially convex functions as follows:

Definition 13.2: Let $f : J \to \mathbb{R}$ and let $f_{k,n}$ be defined as in (13.1), then f is said to be sequentially convex if $(f_{k,n})$ is a convex sequence in k for all $n > 2$ and all $x_1, \ldots, x_n \in J$.

While investigating sequentially convex functions Gabler also made the following important observation which we will call as Gabler inequality.

Theorem 13.1: *For a sequentially convex function f of type (13.1) the following inequality holds where $k \in \{1, \dots, n-1\}$*

$$f_{k,n}(x) \geq f_{k+1,n}(x). \tag{13.2}$$

Through the proof in [101] it is interesting to notice that (13.2) is also true for midconvex functions see [182, 203].

It was 1994 when J. Pečarić upgraded the double index function (13.1) and came up with the following weighted version see [202].

Definition 13.3: For $x = (x_1, \dots, x_n) \in J^n$ and a real-valued function $f : J \to \mathbb{R}$, define

$$f_{k,n} = f_{k,n}(x, w) = \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (w_{i_1} + \cdots + w_{i_k}) f\left(\frac{w_{i_1}x_{i_1} + \cdots + w_{i_k}x_{i_k}}{w_{i_1} + \cdots + w_{i_k}}\right) \tag{13.3}$$

where w_i's are positive weights for $i \in \{1, \dots, n\}$.

In the same article the Gabler result was further strengthen by defining it for convex functions in the following way.

Theorem 13.2: *For a convex function f of type (13.3) the following inequality holds where $k \in \{1, \dots, n-1\}$*

$$f_{k,n}(x, w) \geq f_{k+1,n}(x, w). \tag{13.4}$$

Furthermore it was proved that the inequality (13.4) is an interpolating inequality for Jensen's inequality as can be seen in [182, 196].

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) = f_{n,n} \leq \cdots \leq f_{k+1,n} \leq f_{k,n} \leq \cdots \leq f_{1,n} = \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i). \tag{13.5}$$

In the present chapter, a variant of the Gabler inequality in terms of Jensen–Mercer inequality and its extension via Niezgoda's inequality will be stated along with some refinements similar to (13.5).

13.2 Jensen–Mercer Type Variant of Gabler Inequality

Theorem 13.3: *Under the assumptions of Theorem 13.2, if we define*

$$\begin{aligned} f_{k,n} &= f_{k,n}(x, w) \\ &= \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (w_{i_1} + \cdots + w_{i_k}) f\left(a + b - \frac{w_{i_1}x_{i_1} + \cdots + w_{i_k}x_{i_k}}{w_{i_1} + \cdots + w_{i_k}}\right), \end{aligned} \tag{13.6}$$

then the inequality (13.4) holds.

Proof: By using the definition of convex functions and rearrangements, we have

$$
\begin{aligned}
&\left(w_{i_1}+\cdots+w_{i_{k+1}}\right) f\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_{k+1}}x_{i_{k+1}}}{w_{i_1}+\cdots+w_{i_{k+1}}}\right)\\
&=\left(w_{i_1}+\cdots+w_{i_{k+1}}\right) f\left[\frac{1}{\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)}\right.\\
&\times\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)\\
&\times\left.\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_{k+1}}x_{i_{k+1}}-w_{i_l}x_{i_l}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right)\right]\\
&\leq\left(w_{i_1}+\cdots+w_{i_{k+1}}\right)\left[\frac{1}{\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)}\right.\\
&\times\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)\\
&\times f\left.\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_{k+1}}x_{i_{k+1}}-w_{i_l}x_{i_l}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right)\right]\\
&=\frac{1}{k}\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)\\
&\times f\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_{k+1}}x_{i_{k+1}}-w_{i_l}x_{i_l}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right).
\end{aligned}
$$

In order to use the above result, we consider

$$
\begin{aligned}
&f_{k+1,n}=\frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n}\left(w_{i_1}+\cdots+w_{i_{k+1}}\right)\\
&f\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_{k+1}}x_{i_{k+1}}}{w_{i_1}+\cdots+w_{i_{k+1}}}\right)\\
&\leq\frac{1}{k\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n}\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)\\
&\times f\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_{k+1}}x_{i_{k+1}}-w_{i_l}x_{i_l}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right)\\
&=\frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n}\left(w_{i_1}+\cdots+w_{i_k}\right)\\
&\times f\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_k}x_{i_k}}{w_{i_1}+\cdots+w_{i_k}}\right)=f_{k,n}.
\end{aligned}
$$

This concludes our proof. □

Similar to (13.5), the following refinement for (1.3) using (13.6) can be defined.

Corollary 13.1: *Under the assumptions of Theorem 13.3, we have*

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right)=f_{n,n}\leq\cdots$$
$$\leq f_{k+1,n}\leq f_{k,n}\leq\cdots\leq f_{1,n}\leq f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i).$$

Proof: For $k=n$, double index function (13.6) yields to

$$f_{n,n}=\frac{1}{\binom{n-1}{n-1}W_n}\left(w_1+\cdots+w_n\right)f\left(a+b-\frac{w_1x_1+\cdots+w_nx_n}{w_1+\cdots+w_n}\right)=f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right).$$

Similarly for $k=1$, the double index function (13.6) and Lemma 1.1 give us

$$f_{1,n}=\frac{1}{\binom{n-1}{1-1}W_n}\sum_{i_1=1}^{n}w_{i_1}f\left(a+b-\frac{w_{i_1}x_{i_1}}{w_{i_1}}\right)=\frac{1}{W_n}\sum_{i=1}^{n}w_if\left(a+b-x_i\right).$$
$$\leq\frac{1}{W_n}\sum_{i=1}^{n}w_i\left[f(a)+f(b)-f(x_i)\right]=f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i).$$

Above calculations in accordance to inequality (13.4) concludes our proof. □

13.3 Niezgoda Type Extension of Gabler Inequality

In [69] the author established the double index function in terms of Neizgoda's inequality satisfying Gabler inequality as follow:

Theorem 13.4: *Under the assumptions of Theorem 10.1, if we define*

$$\begin{aligned}f_{k,n}&=f_{k,n}(x,a,w)\\&=\frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n}\left(w_{i_1}+\cdots+w_{i_k}\right)f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1}+\cdots+w_{i_k}x_{i_k}}{w_{i_1}+\cdots+w_{i_k}}\right),\end{aligned}\qquad(13.7)$$

where $W_n=\sum_{i=1}^{n}w_i>0$ for $w_i\geq 0$ for all $i\in\{1,\ldots,n\}$, then the inequality (13.4) holds.

Proof: By using the definition of convex functions and rearrangements, we have

$$\begin{aligned}&\left(w_{i_1}+\cdots+w_{i_{k+1}}\right)f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1j}+\cdots+w_{i_{k+1}}x_{i_{k+1}j}}{w_{i_1}+\cdots+w_{i_{k+1}}}\right)\\&=\left(w_{i_1}+\cdots+w_{i_{k+1}}\right)f\left[\frac{1}{\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)}\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)\right.\\&\left.\times\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1j}+\cdots+w_{i_{k+1}}x_{i_{k+1}j}-w_{i_l}x_{i_lj}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\leq \left(w_{i_1}+\cdots+w_{i_{k+1}}\right)\left[\frac{1}{\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)}\right.\\
&\times \sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)\\
&\left.\times f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1 j}+\cdots+w_{i_{k+1}}x_{i_{k+1}j}-w_{i_l}x_{i_l j}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right)\right]\\
&=\frac{1}{k}\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)\\
&\times f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1 j}+\cdots+w_{i_{k+1}}x_{i_{k+1}j}-w_{i_l}x_{i_l j}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right).
\end{aligned}
$$

In order to use the above result we consider

$$
\begin{aligned}
f_{k+1,n}&=\frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n}\left(w_{i_1}+\cdots+w_{i_{k+1}}\right)\\
&\times f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1 j}+\cdots+w_{i_{k+1}}x_{i_{k+1}j}}{w_{i_1}+\cdots+w_{i_{k+1}}}\right)\\
&\leq \frac{1}{k\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n}\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)\\
&\times f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1 j}+\cdots+w_{i_{k+1}}x_{i_{k+1}j}-w_{i_l}x_{i_l j}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right)\\
&=\frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n}\left(w_{i_1}+\cdots+w_{i_k}\right)\times\\
&\times f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1 j}+\cdots+w_{i_k}x_{i_k j}}{w_{i_1}+\cdots+w_{i_k}}\right)=f_{k,n}.
\end{aligned}
$$

This concludes our proof. □

Similar to (13.5), the following refinement for (10.2) using (13.7) can be given.

Corollary 13.2: *Under the assumptions of Theorem 13.4, we have*

$$
\begin{aligned}
f\left(\sum_{j=1}^{m} a_j-\frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right)&=f_{n,n}\leq\cdots\leq f_{k+1,n}\\
&\leq f_{k,n}\leq\cdots\leq f_{1,n}\leq\sum_{j=1}^{m} f(a_j)-\frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f(x_{ij}).
\end{aligned}
$$

Proof: For $k=n$, double index function (13.7) yields to

$$f_{n,n} = \frac{1}{\binom{n-1}{n-1} W_n} (w_1 + \cdots + w_n) f \left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \frac{w_1 x_{1j} + \cdots + w_n x_{nj}}{w_1 + \cdots + w_n} \right)$$
$$= f \left(\sum_{j=1}^{m} a_j - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i x_{ij} \right)$$

Now by using similar technique for majorization theorem as in Theorem 2.1 of [196] and double index function (13.7) for $k = 1$, we have

$$\begin{aligned} f_{1,n} \; &= \; \frac{1}{\binom{n-1}{1-1} W_n} \sum_{i_1=1}^{n} w_{i_1} f \left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \frac{w_{i_1} x_{i_1 j}}{w_{i_1}} \right) = \sum_{i=1}^{n} w_i f \left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} x_{ij} \right) \\ &\leq \sum_{j=1}^{m} f(a_j) - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i f(x_{ij}). \end{aligned}$$

Above calculations in accordance to inequality (13.4) concludes our proof. □

Remark 13.1: we set $k = m = 2$, $a_1 = a$, $a_2 = b$ and $x_{i1} = x_i$ for $i \in \{1, \ldots, n\}$, then Theorem 13.3 and related results will become special cases of Theorem 13.4.

13.3.1 Applications of Jensen–Mercer Type Variant of Gabler Inequality to Generalized Means

F: Let all the assumptions of Corollary 13.1 be valid. For $J \in [a, b], 0 < a < b$ and positive weights w_i for $i \in \{1, \ldots, \alpha\}$, where $\alpha \in \{1, \ldots, n\}$, we define following arithmetic, geometric and harmonic means along with power mean of the order $r \in \mathbb{R}$ for all $x_i \in [a, b]$.

$$A(x_1, \ldots, x_\alpha; w_1, \ldots, w_\alpha) = a + b - \frac{1}{W_\alpha} \sum_{i=1}^{\alpha} w_i x_i,$$

$$G(x_1, \ldots, x_\alpha; w_1, \ldots, w_\alpha) = \frac{ab}{\left(\prod_{i=1}^{\alpha} x_i^{w_i} \right)^{\frac{1}{W_\alpha}}},$$

$$H(x_1, \ldots, x_\alpha; w_1, \ldots, w_\alpha) = \left(a + b - \frac{1}{W_\alpha} \sum_{i=1}^{\alpha} w_i \frac{1}{x_i} \right)^{-1},$$

$$M^{[r]} = (x_1, \ldots, x_\alpha; w_1, \ldots, w_\alpha) = \begin{cases} \left(a^r + b^r - \frac{1}{W_\alpha} \sum_{1=1}^{\alpha} w_i x_i^r \right)^{\frac{1}{r}}, & r \neq 0, \\ G(x_1, \ldots, x_\alpha; w_1, \ldots, w_\alpha), & r = 0. \end{cases}$$

Also for $i \in \{1, \ldots, n\}$ and $p \in \{1, \ldots, k, k+1, \ldots, n\}$, we define

$$A_{p,n} = A(x_{i_1}, \ldots, x_{i_p}; w_{i_1}, \ldots, w_{i_p}),$$

$$G_{p,n} = G(x_{i_1}, \ldots, x_{i_p}; w_{i_1}, \ldots, w_{i_p}),$$

$$H_{p,n} = H(x_{i_1}, \ldots, x_{i_p}; w_{i_1}, \ldots, w_{i_p}),$$
$$M_{p,n}^{[r]} = M_{[r]}\left(x_{i_1}, \ldots, x_{i_p}; w_{i_1}, \ldots, w_{i_p}\right).$$

For $x_{i_p} \mapsto \left(1 - x_{i_p}\right) \in (0, \frac{1}{2}]$, we define following means

$$A'_{p,n} = A(1 - x_{i_1}, \ldots, 1 - x_{i_p}; w_{i_1}, \ldots, w_{i_p}),$$
$$G'_{p,n} = G(1 - x_{i_1}, \ldots, 1 - x_{i_p}; w_{i_1}, \ldots, w_{i_p}),$$
$$H'_{p,n} = H(1 - x_{i_1}, \ldots, 1 - x_{i_p}; w_{i_1}, \ldots, w_{i_p}).$$

Clearly for $n = p$,

$$A_{n,n} = a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i = A_n,$$
$$G_{n,n} = \frac{ab}{\left(\prod_{i=1}^{n} x_i^{w_i}\right)^{\frac{1}{W_n}}} = G_n,$$
$$H_{n,n} = \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{W_n}\sum_{i=1}^{n} w_i \frac{1}{x_i}\right)^{-1} = H_n,$$

$$M_{n,n}^{[r]} = \begin{cases} \left(a^r + b^r - \frac{1}{W_n}\sum_{1=1}^{n} w_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0, \\ G_{n,n}, & r = 0, \end{cases}$$
$$= M^{[n]}$$
$$A'_{n,n} = (1-a) + (1-b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i \left(1 - x_i\right) = A'_n,$$
$$G'_{n,n} = \frac{(1-a)(1-b)}{\left(\prod_{i=1}^{n} (1 - x_i)^{w_i}\right)^{\frac{1}{W_n}}} = G'_n,$$
$$H'_{n,n} = \left(\frac{1}{1-a} + \frac{1}{1-b} - \frac{1}{W_n}\sum_{i=1}^{l} w_i \frac{1}{(1 - x_i)}\right)^{-1} = H'_n,$$

where $a = \min\limits_{1 \leq i \leq n} \{x_i\}$ and $b = \max\limits_{1 \leq i \leq n} \{x_i\}$.

We now introduce mixed symmetric means as follows:

$$M_{k,n}^{[s,r]} = \begin{cases} \left(\frac{1}{\binom{n-1}{k-1}W_n} \sum_{1\le i_1<\cdots<i_k\le n} \left(\sum_{l=1}^{k} w_{i_l}\right) \left(M_{k,n}^{[r]}\right)^s \right)^{\frac{1}{s}}, & s\neq 0,\\ \left(\prod_{1\le i_1<\cdots<i_k\le n} M_{k,n}^{[r]} \right)^{\frac{1}{\binom{n-1}{k-1}W_n}}, & s=0. \end{cases}$$

We start our applications to Theorem 13.3, with following refinement series of the arithmetic-geometric and Ky Fan inequalities (see [72] and reference therein):

Theorem 13.5: *Let all the assumptions stated in* **F** *be valid. Then*

(*i*)

$$A_n \le \cdots \le \left(\prod_{1\le i_1<\cdots<i_{k+1}\le n} (A_{k+1,n})^{W_{i_{k+1}}} \right)^{\frac{1}{\binom{n-1}{k}W_n}}$$
$$\le \left(\prod_{1\le i_1<\cdots<i_k\le n} (A_{k,n})^{W_{i_k}} \right)^{\frac{1}{\binom{n-1}{k-1}W_n}} \le \cdots \le \left(\prod_{i=1}^{n} (A_{1,n})^{w_i} \right)^{\frac{1}{W_n}} \le G_n.$$

(*ii*)

$$\frac{A'_n}{A_n} \le \cdots \le \left(\prod_{1\le i_1<\cdots<i_{k+1}\le n} \left(\frac{A'_{k+1,n}}{A_{k+1,n}}\right)^{W_{i_{k+1}}} \right)^{\frac{1}{\binom{n-1}{k}W_n}}$$
$$\le \left(\prod_{1\le i_1<\cdots<i_k\le n} \left(\frac{A'_{k,n}}{A_{k,n}}\right)^{W_{i_k}} \right)^{\frac{1}{\binom{n-1}{k-1}W_n}} \le \cdots \le \left(\prod_{i=1}^{n} \left(\frac{A'_{1,n}}{A_{1,n}}\right)^{w_i} \right)^{\frac{1}{W_n}} \le \frac{G'_n}{G_n},$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}$.

Proof:

(*i*) By apply convex function $f(x) = -\ln(x), x \in (0, \frac{1}{2}]$, to Corollary 13.1 we obtain required result.

(*ii*) For $x \in (0, \frac{1}{2}]$, applying convex function $f(x) = \ln\left(\frac{1-x}{\mathbf{x}}\right)$, to the Corollary 13.1 we get,

$$\ln\left(\frac{1-a-b+\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i}{a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i} \right) \le \cdots$$
$$\le \frac{1}{\binom{n-1}{k}W_n} \sum_{1\le i_1<\cdots<i_{k+1}\le n} W_{i_{k+1}} \ln\left(\frac{1-a-b+\frac{1}{W_{i_{k+1}}}\sum_{l=1}^{k+1} w_{i_l} x_{i_l}}{a+b-\frac{1}{W_{i_{k+1}}}\sum_{l=1}^{k+1} w_{i_l} x_{i_l}} \right)$$
$$\le \frac{1}{\binom{n-1}{k-1}W_n} \sum_{1\le i_1<\cdots<i_k\le n} W_{i_k} \ln\left(\frac{1-a-b+\frac{1}{W_{i_k}}\sum_{l=1}^{k} w_{i_l} x_{i_l}}{a+b-\frac{1}{W_{i_k}}\sum_{l=1}^{k} w_{i_l} x_{i_l}} \right)$$

$$\leq \cdots \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \ln\left(\frac{1-a-b+x_i}{a+b-x_i}\right)$$
$$\leq \ln\left(\frac{1-a}{a}\right) + \ln\left(\frac{1-b}{b}\right) - \frac{1}{W_n}\sum_{i=1}^{n} w_i \ln\left(\frac{1-x_i}{x_i}\right).$$

Consequently,

$$\ln\left(\frac{A'_n}{A_n}\right) \leq \cdots \leq \ln\left(\prod_{1\leq i_{k+1}<\cdots<i_1\leq n}\left(\frac{A'_{k+1}}{A_{k+1}}\right)^{W_{i_{k+1}}}\right)^{\frac{1}{\binom{n-1}{k}W_n}}$$
$$\leq \ln\left(\prod_{1\leq i_k<\cdots<i_1\leq n}\left(\frac{A'_k}{A_k}\right)^{W_{i_k}}\right)^{\frac{1}{\binom{n-1}{k-1}W_n}} \leq \cdots \leq \ln\left(\prod_{i=1}^{n}\left(\frac{A'_{1,n}}{A_{1,n}}\right)^{w_i}\right)^{\frac{1}{W_n}} \leq \ln\left(\frac{G'_n}{G_n}\right).$$

Finally, by taking exponential we deduced our result.

□

The refinements series of the variant of arithmetic-geometric inequality is given as:

Theorem 13.6: *Let all the assumptions stated in* **F** *be valid. Then*

$$\frac{G_n}{G_n+G'_n} \leq \cdots \leq \frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n} W_{i_{k+1}}\left(\frac{G_{k+1,n}}{G_{k+1,n}+G'_{k+1,n}}\right)$$
$$\leq \frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n} W_{i_k}\left(\frac{G_{k,n}}{G_{k,n}+G'_{k,n}}\right) \leq \cdots \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\frac{G_{1,n}}{G_{1,n}+G'_{1,n}}\right) \leq A_n,$$

where $\quad W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}.$

Proof: By applying the convex function $f(x) = \frac{1}{1+\exp \mathbf{x}}, x \in [0,\infty)$ to Corollary 13.1 and replacing a by $\ln\frac{1-a}{a}$, b by $\ln\frac{1-b}{b}$ and x_{i_l} by $\ln\frac{1-x_{i_l}}{x_{i_l}}$ we obtain the result. □

Now, we present refinement series of the arithmetic and harmonic mean as follow:

Theorem 13.7: *Let all the assumptions stated in* **F** *be valid. Then*

(i)

$$\frac{1}{A_n} \leq \cdots \leq \frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n} W_{i_{k+1}}\left(\frac{1}{A_{k+1,n}}\right)$$
$$\leq \frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n} W_{i_k}\left(\frac{1}{A_{k,n}}\right) \leq \cdots \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\frac{1}{A_{1,n}}\right) \leq \frac{1}{H_n}.$$

(ii)

$$\frac{1}{A'_n} \le \cdots \le \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \le i_1 < \cdots < i_{k+1} \le n} W_{i_{k+1}} \left(\frac{1}{A'_{k+1,n}} \right)$$
$$\le \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \le i_1 < \cdots < i_k \le n} W_{i_k} \left(\frac{1}{A'_{k,n}} \right) \le \cdots \le \frac{1}{W_n} \sum_{1}^{n} w_i \left(\frac{1}{A'_{1,n}} \right) \le \frac{1}{H'_n}$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}$ *and so on.*

Proof:

(i) By applying convex function $f(x) = \frac{1}{\mathbf{x}}, x \in (0, \frac{1}{2}]$ to Corollary 13.1 we obtain required result.

(ii) By applying convex function $f(x) = \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Corollary 13.1 we obtain required result.

□

In the following theorem we establish a refinement series of the difference of the arithmetic and harmonic mean.

Theorem 13.8: *Let all the assumptions stated in* **F** *be valid. Then*

$$\frac{1}{A_n} - \frac{1}{A'_n} \le \cdots \le \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \le i_1 < \cdots < i_{k+1} \le n} W_{i_{k+1}} \left(\frac{1}{A_{k+1,n}} - \frac{1}{A'_{k+1,n}} \right)$$
$$\le \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \le i_1 < \cdots < i_k \le n} W_{i_k} \left(\frac{1}{A_{k,n}} - \frac{1}{A'_{k,n}} \right) \le \cdots \le \frac{1}{W_n} \sum_{i=1}^{n} w_i \left(\frac{1}{A_{1,n}} - \frac{1}{A'_{1,n}} \right) \le \frac{1}{H_n} - \frac{1}{H'_n}$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}$ *and so on.*

Proof: By applying convex function $f(x) = \frac{1}{\mathbf{x}} - \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Corollary 13.1 we obtain required result. □

We now prove some results in terms of mixed symmetric means.

Theorem 13.9: *Under the assumptions stated in* **F**, *if* $r, s \in \mathbb{R}$ *such that* $s \le r$, *then we have*

$$M_s = M^{[r,s]}_{n,n} \le \cdots \le M^{[r,s]}_{k+1,n} \le M^{[r,s]}_{k,n} \le \cdots \le M^{[r,s]}_{1,n} \le M_r. \tag{13.8}$$

$$M_s \le M^{[s,r]}_{1,n} \le \cdots \le M^{[s,r]}_{k,n} M^{[s,r]}_{k+1,n} \le \cdots \le M^{[s,r]}_{n,n} = M_r. \tag{13.9}$$

Proof: Let $r, s \ne 0$. By applying the function $\varphi(x) = x^{\frac{s}{r}}$, in the Corollary 13.1 and replacing a, b and x_{i_l} by a^r, b^r and $(x_{i_l})^r$ respectively, and then raising the power $\frac{1}{s}$, we get (13.8). Similarly, using the function $\varphi(x) = x^{\frac{r}{s}}$ in the Corollary 13.1 and replacing a, b and x_{i_l} by a^s, b^s and $(x_{i_l})^s$ respectively then raising the power $\frac{1}{r}$, we get (13.9). For $s = 0$ or $r = 0$, we obtain the required result by taking limit. □

Let $\varphi, \psi : J \to \mathbb{R}$ be continuous strictly monotonic functions. We define the quasi-arithmetic means with respect to Theorem 13.3, as follow:

$$\hat{M}_{k,n}^{[\varphi,\psi]} = \varphi^{-1}\left[\frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n}\sum_{l=1}^{k} w_{i_l}\left(\varphi\circ\psi^{-1}\right)\left(\psi(a)+\psi(b)-\frac{\sum_{l=1}^{k} w_{i_l}\psi(x_{i_l})}{\sum_{l=1}^{k} w_{i_l}}\right)\right].$$

Where $\left(\varphi\circ\psi^{-1}\right)$ is convex function.

Corollary 13.3: *Let all the assumptions stated in* **F** *be valid. If we define a continuous and strictly monotonic function* $\varphi : J \to \mathbb{R}$ *as*

$$\hat{M}^{[\varphi]} = \varphi^{-1}\left[\varphi(a)+\varphi(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i\varphi(x_i)\right].$$

Then following monotonicity of generalized quasi-arithmetic means holds

$$\hat{M}^{[\varphi]} \geq \hat{M}_{1,n}^{[\varphi,\psi]} \geq \cdots \geq \hat{M}_{k,n}^{[\varphi,\psi]} \geq \hat{M}_{k+1,n}^{[\varphi,\psi]} \geq \cdots \geq \hat{M}_{n,n}^{[\varphi,\psi]} = \hat{M}^{[\psi]} \tag{13.10}$$

Proof: Setting $f \mapsto \varphi\circ\psi^{-1}$ and replacing a, b and x_{i_l} by $\psi(a), \psi(b)$ and $\psi(x_{i_l})$ respectively in Corollary 13.1 and then applying φ^{-1} we get (13.10). □

Remark 13.2: In similar manner, applications for Niezgoda Type Extension of Gabler Inequality to generalized means can be given, we refer the reader to [133].

13.4 Gabler Inequality for Functions with Nondecreasing Increments

Throughout this chapter we consider that **U** is an interval in a k-dimensional vector space $\mathbb{R}^k$ and $w_1, \ldots, w_n$ are wights such that $W_i = \sum_{j=1}^{i} w_j, i \in \{1, \ldots, n\}$ with $W_n = \sum_{j=1}^{n} w_j$ and $H \subset \mathbb{R}$.

In [214] Pecaric established a result which we would use as our lemma:

Lemma 13.1: *For* $y_i \in H, i \in \{1, \ldots, n\}$, *let* $f : H \to \mathbb{R}$ *be a mid-convex function then*

$$f\left(\frac{1}{n}\sum_{j=1}^{n} y_j\right) \leq \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{y_1+\cdots+\hat{y}_j+y_n}{n-1}\right),$$

for all $j \in \{1, \ldots, n\}$, *where* $\hat{y}_j$ *means that* y_j *is omitted.*

In present section we establish Gabler inequality for Jensen inequality, Mercer and Niezdoga inequality for continuous function with nondecreasing increments of convex type.

Theorem 13.10: *Let $f : \mathbf{U} \to \mathbb{R}$ be a continuous functions with decreasing increments of convex type defined on $\mathbf{U}$ and $\mathbf{x}^{(i)} \in \mathbf{U}, i \in \{1, \ldots, n\}$ if we suppose*

$$f_{k,n} = \binom{n}{k}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{\mathbf{x}^{(i_1)} + \cdots + \mathbf{x}^{(i_k)}}{k}\right)$$

then

$$f_{k,n} \geq f_{k+1,n}$$

holds, for $k \in \{1, \ldots, n-1\}$.

Proof: See [204, p-96]. □

Theorem 13.11: *let $f : \mathbf{U} \to \mathbb{R}$ be a continuous function with nondecreasing increments of convex type defined on $\mathbf{U}$ and $\mathbf{x}^{(i)} \in \mathbf{U}, i \in \{1, \ldots, n\}$ with $\boldsymbol{w} = (w_1, \ldots, w_n)$ is a nonnegative n tuple such that $W_n > 0$. Let*

$$f^w{}_{k,n} = \binom{n-1}{k-1}^{-1} W_n \sum_{1 \leq i_1 < \cdots < i_k \leq n} (w_{i_1} + \cdots + w_{i_k})\, f\left(\frac{w_{i_1}\mathbf{x}^{(i_1)} + \cdots + w_{i_k}\mathbf{x}^{(i_k)}}{w_{i_1} + \cdots + w_{i_k}}\right),$$

then

$$f^w{}_{k,n}(\mathbf{x}, w) \geq f^w{}_{k+1,n}(\mathbf{x}, w),$$

holds for $k \in \{1, \ldots, n-1\}$.

Proof: Since any function with nondecreasing increments of convex type follows the Theorem A.4, we have

$$\begin{aligned}
&(w_{i_1} + \cdots + w_{i_{k+1}})\, f\left(\frac{w_{i_1}\mathbf{x}^{(i_1)} + \cdots + w_{i_{k+1}}\mathbf{x}^{(i_{k+1})}}{w_{i_1} + \cdots + w_{i_{k+1}}}\right) \\
&= (w_{i_1} + \cdots + w_{i_{k+1}}) \\
&\times f\left(\frac{\sum_{j=1}^{k+1}(w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_j})\frac{w_{i_1}\mathbf{x}^{(i_1)} + \cdots + w_{i_{k+1}}\mathbf{x}^{(i_{k+1})} - w_{i_j}\mathbf{x}^{(i_j)}}{w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_j}}}{\left(\sum_{j=1}^{k+1} w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_j}\right)}\right) \\
&\leq (w_{i_1} + \cdots + w_{i_{k+1}}) \\
&\times \frac{\sum_{j=1}^{k+1}(w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_j})\, f\left(\frac{w_{i_1}\mathbf{x}^{(i_1)} + \cdots + w_{i_{k+1}}\mathbf{x}^{(i_{k+1})} - w_{i_j}\mathbf{x}^{(i_j)}}{w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_j}}\right)}{\sum_{j=1}^{k+1}(w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_j})} \\
&= \frac{1}{k}\sum_{j=1}^{k+1}(w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_j})\, f\left(\frac{w_{i_1}\mathbf{x}^{(i_1)} + \cdots + w_{i_{k+1}}\mathbf{x}^{(i_{k+1})} - w_{i_j}\mathbf{x}^{(i_j)}}{w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_j}}\right)
\end{aligned}$$

therefore

$$\begin{aligned}
{f^w}_{k+1,n} &= \binom{n-1}{k}^{-1} w_n \sum_{1\le i_1<\cdots<i_{k+1}\le n} (w_{i_1}+\cdots+w_{i_{k+1}}) \\
&\times f\left(\frac{w_{i_1}\mathbf{x}^{(i_1)}+\cdots+w_{i_{k+1}}\mathbf{x}^{(i_{k+1})}}{w_{i_1}+\cdots+w_{i_{k+1}}}\right) \\
&\le \frac{1}{k\binom{n-1}{k}w_n} \sum_{1\le i_1<\cdots<i_{k+1}\le n} \sum_{j=1}^{k+1} (w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j}) \\
&\times f\left(\frac{w_{i_1}\mathbf{x}^{(i_1)}+\cdots+w_{i_{k+1}}\mathbf{x}^{(i_{k+1})}-w_{i_j}\mathbf{x}^{(i_j)}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j}}\right) \\
&= \frac{1}{k\binom{n-1}{k}w_n} \sum_{1\le i_1<\cdots<i_k\le n} (w_{i_1}+\cdots+w_{i_k}) \\
&\times f\left(\frac{w_{i_1}\mathbf{x}^{(i_1)}+\cdots+w_{i_k}\mathbf{x}^{(i_k)}}{w_{i_1}+\cdots+w_{i_k}}\right) \\
&= {f^w}_{k,n}.
\end{aligned}$$

□

Theorem 13.12: *Let $f:\mathbf{U}\to\mathbb{R}$ be a continuous functions with nondecreasing increments of convex type with $\mathbf{x}^{(i)}\in\mathbf{U}, i\in\{1,\ldots,n\}$ and $\boldsymbol{w}=(w_1,\ldots,w_n)$ is a nonnegative n-tuple such that $w_n>0$.*

$$\begin{aligned}
{f^w}_{k,n} = f_{k,n}(\mathbf{x},w) &= \frac{1}{\binom{n-1}{k-1}w_n} \\
\times \sum_{1\le i_1<\cdots<i_k\le n} &(w_{i_1}+\cdots+w_{i_k})\, f\left(\mathbf{a}+\mathbf{b}-\frac{w_{i_1}\mathbf{x}^{(i_1)}+\cdots+w_{i_k}\mathbf{x}^{(i_k)}}{w_{i_1}+\cdots+w_{i_k}}\right) \qquad (13.11)
\end{aligned}$$

then for $k\in\{1,\ldots,n-1\}$

$${f^w}_{k,n}(\mathbf{x},w) \ge {f^w}_{k+1,n}(\mathbf{x},w)$$

holds, where $\boldsymbol{a}=(a_1,\ldots,a_n)$ and $\boldsymbol{b}=(b_1,\ldots,b_n)$ are chosen from $\mathbf{U}$ with $\boldsymbol{a}\le\boldsymbol{x}^{(i)}\le\boldsymbol{b}$.

Proof: Since any function with nondecreasing increments of convex type follows the Theorem 1.1 under Remark 1.2, we have

$$\begin{aligned}
&(w_{i_1}+\cdots+w_{i_{k+1}})\, f\left(\mathbf{a}+\mathbf{b}-\frac{w_{i_1}\mathbf{x}^{(i_1)}+\cdots+w_{i_{k+1}}\mathbf{x}^{(i_{k+1})}}{w_{i_1}+\cdots+w_{i_{k+1}}}\right) \\
&= (w_{i_1}+\cdots+w_{i_{k+1}})\, f\left[\frac{1}{\sum_{j=1}^{k+1}(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j})}\sum_{j=1}^{k+1}(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j})\right. \\
&\times \left(\mathbf{a}+\mathbf{b}-\frac{w_{i_1}\mathbf{x}^{(i_1)}+\cdots+w_{i_{k+1}}\mathbf{x}^{(i_{k+1})}-w_{i_j}\mathbf{x}^{(i_j)}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(w_{i_1}+\cdots+w_{i_{k+1}}\right)\left[\frac{1}{\sum_{j=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j}\right)}\right.\\
&\times \sum_{j=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j}\right)\\
&\left.\times f\left(\mathbf{a}+\mathbf{b}-\frac{w_{i_1}\mathbf{x}^{(i_1)}+\cdots+w_{i_{k+1}}\mathbf{x}^{(i_{k+1})}-w_{i_j}\mathbf{x}^{(i_j)}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j}}\right)\right]\\
&=\frac{1}{k}\sum_{j=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j}\right)\times\\
&f\left(\mathbf{a}+\mathbf{b}-\frac{w_{i_1}\mathbf{x}^{(i_1)}+\cdots+w_{i_{k+1}}\mathbf{x}^{(i_{k+1})}-w_{i_j}\mathbf{x}^{(i_j)}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j}}\right)
\end{aligned}
$$

in order to use above result we consider

$$
\begin{aligned}
&f^{w}{}_{k+1,n}=\frac{1}{\binom{n-1}{k}w_n}\times\\
&\sum_{1\leq i_1<\cdots<i_{k+1}\leq n}\left(w_{i_1}+\cdots+w_{i_{k+1}}\right)f\left(\mathbf{a}+\mathbf{b}-\frac{w_{i_1}\mathbf{x}^{(i_1)}+\cdots+w_{i_{k+1}}\mathbf{x}^{(i_{k+1})}}{w_{i_1}+\cdots+w_{i_{k+1}}}\right)\\
&\leq\frac{1}{k\binom{n-1}{k}w_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n}\sum_{j=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j}\right)\\
&\times f\left(\mathbf{a}+\mathbf{b}-\frac{w_{i_1}\mathbf{x}^{(i_1)}+\cdots+w_{i_{k+1}}\mathbf{x}^{(i_{k+1})}-w_{i_j}\mathbf{x}^{(i_j)}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_j}}\right)\\
&=\frac{1}{\binom{n-1}{k-1}w_n}\sum_{1\leq i_1<\cdots<i_k\leq n}\left(w_{i_1}+\cdots+w_{i_k}\right)\\
&\times f\left(\mathbf{a}+\mathbf{b}-\frac{w_{i_1}\mathbf{x}^{(i_1)}+\cdots+w_{i_k}\mathbf{x}^{(i_k)}}{w_{i_1}+\cdots+w_{i_k}}\right)\\
&=f^{w}{}_{k,n}
\end{aligned}
$$

which concludes our proof. □

Remark 13.3: Similar to (13.5), the following refinement for (13.11) can be defined.

$$
f\left(\mathbf{a}+\mathbf{b}-\frac{1}{w_n}\sum_{i=1}^{n}w_i\mathbf{x}_i\right)=f^{w}{}_{n,n}\leq\cdots\leq f^{w}{}_{k+1,n}\leq
$$

$$
f^{w}{}_{k,n}\leq\cdots\leq f^{w}{}_{1,n}\leq f(\mathbf{a})+f(\mathbf{b})-\frac{1}{w_n}\sum_{i=1}^{n}w_if(\mathbf{x}_i).
$$

Now, we establish Gabler inequality for Neizgoda's inequality using functions with nondecreasing increments of convex type.

Theorem 13.13: *Suppose that $f:\mathbf{U}\to\mathbb{R}$ be a continuous function with nondecreasing in-*

crements of convex type. Let there are m elements in $\mathbf{\acute{a}} = (\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(m)})$ with $\boldsymbol{a}^{(j)} \in \boldsymbol{U}$ for $j \in \{1, \ldots, m\}$. Also there are $n \times m$ elements represented as $\mathbf{x}^{(ij)}$ such that every $\mathbf{x}^{(ij)}$ is a k-tuple for $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$ and all these $\mathbf{x}^{(ij)}$ can be arranged in a $n \times m$ matrix $\boldsymbol{X} = (\mathbf{x}^{(ij)})$ and $\mathbf{w}$ be a nonnegative n-tuple with $W_n > 0$ for $i \in \{1, \ldots, n\}$. Now if $\mathbf{\acute{a}}$ majorizes each row of $\boldsymbol{X}$, that is,

$$\mathbf{x}^{(i.)} = \left(\mathbf{x}^{(i1)}, \ldots, \mathbf{x}^{(im)}\right) \prec \left(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(m)}\right) = \mathbf{\acute{a}} \text{ for each } i \in \{1, \ldots, n\},$$

and we define

$$f^w{}_{k,n} = f^w{}_{k,n}(\mathbf{x}, \acute{a}, w) = \frac{1}{\binom{n-1}{k-1} w_n} \sum_{1 \le i_1 < \cdots < i_k \le n} (w_{i_1} + \cdots + w_{i_k}) \times f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \frac{w_{i_1}\mathbf{x}^{(i_1)} + \cdots + w_{i_k}\mathbf{x}^{(i_k)}}{w_{i_1} + \cdots + w_{i_k}}\right) \tag{13.12}$$

then the following inequality holds for $k \in \{1, \ldots, n-1\}$

$$f^w{}_{k,n}(\mathbf{x}, \acute{a}, w) \ge f^w{}_{k+1,n}(\mathbf{x}, \acute{a}, w)$$

Proof: See [69] □

Remark 13.4: Similar to (13.5), the following refinement for (13.12) can be given.

$$f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{1}{w_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i \mathbf{x}^{(i_j)}\right) = f^w{}_{n,n} \le \cdots \le f^w{}_{k+1,n} \le$$

$$f^w{}_{k,n} \le \cdots \le f^w{}_{1,n} \le \sum_{j=1}^{m} f(\mathbf{a}^{(j)}) - \frac{1}{w_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i f(\mathbf{x}^{(i_j)}).$$

CHAPTER 14

Refinement of Jensen–Mercer Inequality and Cyclic Mixed Symmetric Means

This chapter focuses on refinements of the Jensen–Mercer inequality through cyclic mixed symmetric means. We begin with the cyclic refinement of Jensen's inequality and then extend these ideas to refine Jensen, Jensen–Mercer, and Niezgoda inequalities for real weights. Applications are presented to illustrate the effectiveness of cyclic refinements in improving classical bounds and analyzing inequalities under weighted settings. The cyclic refinement of the Jensen's inequality in paper [45] is given as follow:

Theorem 14.1: *Let $f : I \to \mathbb{R}$ be a convex function and I be an interval in $\mathbb{R}, \mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a nonnegative n-tuple such that $\sum_{i=1}^k \lambda_i = 1$ for some k, $2 \le k \le n$. Then*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \frac{1}{n}\sum_{i=1}^{n} f\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \le \frac{1}{n}\sum_{i=1}^{n} f\left(x_i\right),$$

where $i+j$ means $i+j-n$ in case of $i+j>n$.

In this chapter we are going to use some of the following assumptions:

(A_1) : Let I be a real interval, $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ be a real nondecreasing n-tuple such that $x_{i+n} = x_i$ for $i \in \{1, 2, \ldots, n\}$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a non-negative n-tuple such that $\sum\limits_{i=1}^{k} \lambda_i = 1$ for some k, $2 \le k \le n$.

(A_2) : Let $a = \min\limits_{1\le i\le n}\{x_i\}$ and $b = \max\limits_{1\le i\le n}\{x_i\}$.

(A_3) : Let conditions stated for weight in ($A.9$) be valid.

(A_4) : Let $\mathbf{a}$ be a m-tuple such that $a_j \in I^n$ and a $n \times m$ matrix $X = (x_{ij}) \in I^n, \forall i \in \{1, \ldots, n\}$ and $\forall j \in \{1, \ldots, m\}$ such that $(x_{i1+k}, \ldots, x_{im+k}) = (x_{i1}, \ldots, x_{im}), \forall i \in \{1, \ldots, n\})$ and λ :

$(\lambda_1, \ldots, \lambda_n)$ ba a n-tuple such that $\sum_{p=1}^{q} \lambda_p = 1, q \in \{2, \ldots, n\}$.

(A_5) : Let $f : I \to \mathbb{R}$ be a convex function.

(A_6) : Let $\varphi, \psi : I \to \mathbb{R}$ be continuous and strictly monotone functions.

Under the assumptions stated above it should be noted that:

$$\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}, a+b-\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j} \in I,$$

and

$$\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1}\sum_{p=0}^{q-1} \lambda_{p+1} x_{ij+p} - \sum_{j=k+1}^{m}\sum_{p=0}^{q-1} \lambda_{p+1} x_{ij+p}\right) \in I.$$

14.1 Refinement of Jensen's Inequality for Real Weights

Now we establish refinement of the Theorem A.6 as follow:

Theorem 14.2: *Let the assumptions stated in $(A_1), (A_3)$ and (A_5) be true. Then the following inequalities hold*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(x_i\right), \tag{14.1}$$

for all $x_i \in I$ for $1 \leq i \leq n$.

Proof: We prove first inequality since f is convex function and $\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j} \in I$. Therefore by Jensen-Steffensen's inequality we have,

$$\begin{aligned}
\frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) &\geq f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \sum_{j=0}^{k-1} \lambda_{j+1} w_i x_{i+j}\right) \\
&= f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \sum_{j=0}^{k-1} \lambda_{j+1} w_i x_i\right) = f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \sum_{j=1}^{k} \lambda_j\right) = f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right).
\end{aligned}$$

To prove second inequality, for each $i \in \{1, 2, \ldots, n\}$ we consider $\frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right)$. Since for each $i \in \{1, 2, \ldots, n\}$, $\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j} \in I$, therefore by Jensen-Steffensen's inequality we have: $\frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \sum_{j=0}^{k-1} \lambda_{j+1} f\left(x_{i+j}\right)$ equivalent to,

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i \sum_{j=0}^{k-1} \lambda_{j+1} f\left(x_{i+j}\right) = \frac{1}{W_n}\sum_{i=1}^{n} w_i \sum_{j=1}^{k} \lambda_j f\left(x_i\right) = \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(x_i\right),$$

which conclude our proof. □

14.2 Refinement of Jensen–Mercer Inequality for Real Weights

Before we further proceed we recall here a lemma from [174] stated as under:

Lemma 14.1: *Let the assumptions stated in* $(A_1), (A_2)$ *and* (A_5) *be true. Then*

$$f\left(a+b-x_i\right) \leq f\left(a\right)+f\left(b\right)-f\left(x_i\right), \quad 1 \leq i \leq n.$$

Theorem 14.3: *Let the assumptions stated in* $(A_1), (A_2), (A_3)$ *and* (A_5) *be true. Then the following inequalities hold*

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right) \leq f\left(a\right)+f\left(b\right)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(x_i\right), \quad (14.2)$$

for all $x_i \in I$ *for* $1 \leq i \leq n$.

Proof: To prove first inequality, since f is convex function and $a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j} \in I$, therefore by reverse Jensen's inequality we have,

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right) \geq f\left(\frac{1}{W_n}\left(\sum_{i=1}^{n} w_i a+\sum_{i=1}^{n} w_i b-\sum_{i=1}^{n}\sum_{j=0}^{k-1}\lambda_{j+1}w_i x_{i+j}\right)\right)$$

which is equivalent to $f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a+\frac{1}{W_n}\sum_{i=1}^{n} w_i b-\frac{1}{W_n}\sum_{i=1}^{n}\sum_{j=0}^{k-1}\lambda_{j+1}w_i x_i\right)$

consequently, $f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \sum_{j=1}^{k}\lambda_j\right) = f\left(c+d-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)$.

To prove second inequality, for each $i \in \{1,2,\ldots,n\}$ we consider $f\left(a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right)$. Since for each $i \in \{1,2,\ldots,n\}$, $\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j} \in I^n$, therefore by Lemma 14.1 we have:

$$f\left(a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right) \leq f\left(a\right)+f\left(b\right)-f\left(\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right)$$

$$= f\left(a\right)+f\left(b\right)-f\left(\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right) = f\left(a\right)+f\left(b\right)-f\left(\sum_{j=1}^{k}\lambda_j x_i\right) = f\left(a\right)+f\left(b\right)-f\left(x_i\right),$$

or we can write it as for each $i \in \{1,2,\ldots,n\}$,

$$f\left(a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right)\le f(a)+f(b)-f(x_i). \tag{14.3}$$

Now multiplying inequality (14.3) with $\frac{1}{W_n}\sum_{i=1}^{n}w_i$, we get our required result. □

14.3 Refinement of Extension of Niezgoda's Inequality for Real Weights

Theorem 14.4: *Let the assumptions stated in Theorem 10.4 be true. In addition we suppose that the assumptions given in* $(A_3), (A_4)$ *and* (A_5) *are also valid. Then we have*

$$\begin{aligned} f&\left(\sum_{j=1}^{m}a_j-\frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n}w_ix_{ij}-\frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n}w_ix_{ij}\right)\\ &\le\frac{1}{W_n}\sum_{i=1}^{n}w_if\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{k-1}\sum_{p=0}^{q-1}\lambda_{p+1}x_{ij+p}-\sum_{j=k-1}^{m}\sum_{p=0}^{q-1}\lambda_{p+1}x_{ij+p}\right)\\ &\le\sum_{j=1}^{m}f(a_j)-\frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n}w_if(x_{ij})-\frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n}w_if(x_{ij}). \end{aligned}\tag{14.4}$$

Proof: Since proving techniques is similar to previous theorem so we omit the details. □

Remark 14.1: If we set $m=2$, $a_1=a$, $a_2=b$ and $x_{i1}=x_i$ for $i\in\{1,\ldots,n\}$, then Theorem 14.4 will become special cases of Theorem 14.3.

14.3.1 Applications: For Cyclic Refinement of Jensen Inequality Via Real Weights

Throughout this section, let the assumptions stated in $(A_1), (A_3)$ and (A_5) be valid. We define generalized (or modified) arithmetic, geometric and harmonic mean respectively as follow:

$$\begin{aligned} A_n&=\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i,\\ G_n&=\left(\prod_{i=1}^{n}x_i^{w_i}\right)^{\frac{1}{W_n}},\\ H_n&=\left(\frac{1}{W_n}\sum_{i=1}^{n}w_i\frac{1}{x_i}\right)^{-1}. \end{aligned}$$

Further for $x_i\in[0,\frac{1}{2}]$ we have

$$A'_n=\frac{1}{W_n}\sum_{i=1}^{n}w_i(1-x_i),$$

$$G'_n = \left(\prod_{i=1}^{n}(1-x_i)^{w_i}\right)^{\dfrac{1}{W_n}},$$

$$H'_n = \left(\frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\frac{1}{1-x_i}\right)\right)^{-1}$$

We also define new notations $A(\lambda, \mathbf{x})$ and $G(\lambda, \mathbf{x})$ as under:

$$A(\lambda, \mathbf{x}) = \sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j},$$

$$G(\lambda, \mathbf{x}) = \prod_{j=0}^{k-1}(x_{i+j})^{\lambda_{j+1}},$$

Here also $x_{i+j} \in [0, \frac{1}{2}]$

$$A'(\lambda, \mathbf{x}) = \sum_{j=0}^{k-1}\lambda_{j+1}(1-x_{i+j}),$$

$$G'(\lambda, \mathbf{x}) = \prod_{j=0}^{k-1}(1-x_{i+j})^{\lambda_{j+1}}.$$

Now, we present the refinement of the Ky Fan type inequality. For Ky Fan inequality and related results see [17] and [72] and references given therein.

Theorem 14.5: *Let assumptions $(A_1), (A_3)$ and (A_5) be true. Then following inequality holds:*

$$\frac{A_n}{A'_n} \le \left(\prod_{i=1}^{n}\left(\frac{A(\lambda, \mathbf{x})}{A'(\lambda, \mathbf{x})}\right)^{w_i}\right)^{\frac{1}{W_n}} \le \frac{G_n}{G'_n}.$$

Proof: By applying the convex function $f(x) = ln\left(\frac{x}{1-x}\right)$ for all $x \in (0, \frac{1}{2}]$, to the inequality (A.17), we get,

$$\ln\left(\frac{\dfrac{1}{W_n}\displaystyle\sum_{i=1}^{n} w_i x_i}{1-\dfrac{1}{W_n}\displaystyle\sum_{i=1}^{n} w_i x_i}\right) \le \frac{1}{W_n}\sum_{i=1}^{n} w_i \ln\left(\frac{\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}}{1-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}}\right) \le \frac{1}{W_n}\sum_{i=1}^{n} w_i \ln\left(\frac{x_i}{1-x_i}\right)$$

consequently, $\ln\left(\dfrac{A_n}{A'_n}\right) \le \dfrac{1}{W_n}\ln\displaystyle\prod_{i=1}^{n}\left(\frac{\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}}{1-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}}\right)^{w_i} \le \ln\left(\dfrac{G_n}{G'_n}\right)$

Finally, we obtain $\left(\dfrac{A_n}{A'_n}\right) \le \left(\displaystyle\prod_{i=1}^{n}\left(\frac{A(\lambda, \mathbf{x})}{A'(\lambda, \mathbf{x})}\right)^{w_i}\right)^{\frac{1}{W_n}} \le \left(\dfrac{G_n}{G'_n}\right)$,
which completes the proof. □

Now, we present refinement of arithmetic-geometric mean type inequality as follow:

Corollary 14.1: *Let the assumptions* $(A_1), (A_3)$ *and* (A_5) *be true. Then following inequality holds:*

$$A_n \geq \left(\prod_{i=1}^{n} (A(\lambda, \mathbf{x}))^{w_i}\right)^{\frac{1}{W_n}} \geq G'_n.$$

Proof: By applying the convex function $f(x) = -\ln(x), x \in (0, \frac{1}{2}]$ to Theorem 14.2 we obtain required result. □

Now, we present refinement of harmonic and geometric means inequality as follow:

Corollary 14.2: *Let the assumptions* $(A_1), (A_3)$ *and* (A_5) *be true. Then the following inequalities hold:* $\left(G'_n\right)^{-1} \leq \dfrac{1}{W_n} \displaystyle\sum_{i=1}^{n} w_i \left(G'(\lambda, \mathbf{x})\right)^{-1} \leq \left(H'_n\right)^{-1}$.

We would also establish refinements related to arithmetic-harmonic means inequalities as follow:

Corollary 14.3: *Let the assumptions* $(A_1), (A_3)$ *and* (A_5) *be true. Then the following inequalities hold:*

$$\frac{1}{A_n} \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \left(\frac{1}{A(\lambda, \mathbf{x})}\right) \leq \frac{1}{H_n}, \qquad \frac{1}{A'_n} \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \left(\frac{1}{A'(\lambda, \mathbf{x})}\right) \leq \frac{1}{H'_n}.$$

We establish a refinement of the difference of the arithmetic and harmonic mean.

Corollary 14.4: *Let the assumptions* $(A_1), (A_3)$ *and* (A_5) *be true. Then following inequalities hold:*

$$\frac{1}{A_n} - \frac{1}{A'_n} \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \left(\frac{1}{A(\lambda, \mathbf{x})} - \frac{1}{A'(\lambda, \mathbf{x})}\right) \leq \frac{1}{H_n} - \frac{1}{H'_n}.$$

Proof: By applying convex function $f(x) = \frac{1}{x} - \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Theorem 14.2 we obtain required result. □

The Jensen's inequality and Jensen–Mercer inequality are much fertile to study about mixed means (see [114] and [144]). Let the assumptions stated in $(A_1), (A_3)$ and (A_5) be true. Then we define power mean of the order $r \in \mathbb{R}$, for positive n-tuple $\mathbf{x}$ as follow:

$$M_r\left(x_i, \ldots, x_{i+k-1}; \lambda_1, \ldots, \lambda_k\right) = \begin{cases} \left(\displaystyle\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}^{r}\right)^{\frac{1}{r}}, & r \neq 0, \\ \displaystyle\prod_{j=0}^{k-1} \left(x_{i+j}\right)^{\lambda_{j+1}}, & r = 0, \end{cases}$$

and cyclic mixed symmetric means corresponding to (14.1) is given as:

$$M_{r,s}(\mathbf{x},\lambda)=\begin{cases}\left(\dfrac{1}{W_n}\displaystyle\sum_{i=1}^{n}w_iM_r^s\left(x_i,\ldots,x_{i+k-1},\lambda_1,\ldots,\lambda_k\right)\right)^{\frac{1}{s}}, & s\neq 0,\\ \left(\displaystyle\prod_{i=1}^{n}M_r\left(x_i,\ldots,x_{i+k-1},\lambda_1,\ldots,\lambda_k\right)^{w_i}\right)^{\frac{1}{W_n}}, & s=0.\end{cases}$$

The standard power mean of order $r\in\mathbb{R}$ for the positive n-tuple $\mathbf{x}$ are define as follow:

$$M_r\left(\mathbf{x}\right)=\begin{cases}\left(\dfrac{1}{W_n}\displaystyle\sum_{i=1}^{n}w_ix_i^r\right)^{\frac{1}{r}}, & r\neq 0,\\ \left(\displaystyle\prod_{i=1}^{n}x_i^{w_i}\right)^{\frac{1}{W_n}}, & r=0.\end{cases}$$

Corollary 14.5: *For $r\leq 1$ and by considering the assumptions state in (A_1) for positive n-tuple $\mathbf{x}$, then the following inequality hold.*

$$M_r\left(\mathbf{x}\right)\leq M_r\left(\mathbf{x},\lambda\right)\leq A_n. \tag{14.5}$$

For $r\geq 0$, the inequality (14.5) is reversed.

Now, we define the bounds for power mean and cyclic mixed symmetric means as follow:

Corollary 14.6: *Let $r,s\in\mathbb{R}$ such that $r\leq s$ and considering the assumption stated in A_1 for positive n-tuple $\mathbf{x}$, then following inequalities holds.*

$$M_r\left(\mathbf{x}\right)\leq M_{r,s}\left(\mathbf{x},\lambda\right)\leq M_s\left(\mathbf{x}\right),\qquad M_r\left(\mathbf{x}\right)\leq M_{r,s}\left(\mathbf{x},\lambda\right)\leq A_n.$$

Let $f,\psi:I\to\mathbb{R}$ be a strictly monotonic and continuous functions and under the assumptions stated in (A_1) and (A_3), we define generalized means with respect to (14.2) as follow:

$$M_{f,\psi}\left(\mathbf{x},\lambda\right)=\varphi^{-1}\left[\frac{1}{W_n}\sum_{i=1}^{n}w_i\left(\varphi\circ\psi^{-1}\right)\left(\sum_{j=0}^{k-1}\lambda_{j+1}\psi\left(x_{i+j}\right)\right)\right].$$

Now, we establish the relation among generalized means and quasi-arithmetic means as follow:

Corollary 14.7: *Let assumptions (A_1) and (A_3) be true. Then*

$$\tilde{M}_\psi\left(\mathbf{x}\right)\leq\tilde{M}_{f,\psi}\left(\mathbf{x},\lambda\right)\leq\tilde{M}_f\left(\mathbf{x}\right),$$

if either $\varphi\circ\psi^{-1}$ is convex and ψ is strictly increasing or $\varphi\circ\psi^{-1}$ is concave and ψ is strictly decreasing.

Remark 14.2: (i) All the refinements (applications to generalized means) as stat-ed in this section can also be stated for Jensen-Seffensen's inequality of Mercer type (14.2) and Niezgoad's inequality (14.4), interested readers can see [70, 68].

(ii) For further remarks and related results we refer the reader to [70, 68].

CHAPTER 15

Improvement of Jensen–Mercer Type Inequality for Functions with Nondecreasing Increments and Related Results

This chapter presents improvements of Jensen–Mercer type inequalities for functions with non-decreasing increments and related results. We begin with a generalized result in normed linear and Hilbert spaces and explore its consequences. The discussion then considers integral Jensen–Mercer inequalities for functions with nondecreasing increments, followed by improvements of the Jensen–Boas, Jensen–Mercer, integral Niezgoda and Levinson inequalities for functions with nondecreasing increments are examined, providing a unified framework for enhanced bounds in integral settings.

15.1 A Generalized Result on Normed Linear and Hilbert Spaces and its Consequences

The results of present chapters are extracted from [16].

We start with some results from [156] that will help us in our main constructions:

Theorem 15.1: *A linear functional, having its domain contained in a normed linear space, is continuous $\Leftrightarrow$ it is bounded.*

Following is the well known Riesz Representation Theorem:

Theorem 15.2: *There is an inner product representation for each linear functional f which is bounded on a Hilbert space H and is written as $f(x) = \langle x, \vartheta \rangle$ here ϑ depending upon f has unique value that is evaluated through f. The norm of ϑ is $\|\vartheta\| = \|f\| = \sup\limits_{0 \neq x \in D(f)} \dfrac{|f(x)|}{\|x\|}$.*

In [194, p. 128] we see an interesting result that if V is a normed linear space and U be an open convex subset in V then a convex function f on U generates a supporting hyperplane at every point say $a_0 \in U$. This implies the presence of a linear functional h which in continuous on V and is characterized as

$$f(x) \geq f(a_0) + h(x - a_0) \; \forall \; x \in U. \tag{15.1}$$

Such functionals h are said to be the support of f at a_0 and the subdifferential of f at the point a_0 is established through the set $\partial f(a_0)$ of all these functionals h.

Now if V is a Hilbert space then we have a unique representation of all such functionals h as $h(x) = \langle x, \vartheta \rangle$ for $x \in V$ such that $\|f\| = \|\vartheta\|$ by using Riesz representation theorem.

In this case inequality (15.1) becomes

$$f(x) \geq f(a_0) + \langle x - a_0, \vartheta \rangle \quad \text{for all} \quad x \in U \tag{15.2}$$

all these vectors ϑ are usually termed as subgradients and the set of all vectors ϑ constitute the subdifferential $\partial f(a_0)$. Using these facts we state our first main result:

Theorem 15.3: *If f is a convex function on an open convex subset U of a Hilbert space H, then for all x, $a_0 \in U$*

$$f(x) - f(a_0) - \langle x - a_0, \vartheta \rangle \geq \big| |f(x) - f(a_0)| - \|f\| \, \|x - a_0\| \big|,$$

where $\vartheta \in \partial f(a_0)$.

Proof: Using (15.2) we have that for all $x, a_0 \in U$

$$\begin{aligned} f(x) - f(a_0) - \langle x - a_0, \vartheta \rangle &= |f(x) - f(a_0) - \langle x - a_0, \vartheta \rangle| \\ &\geq \big| |f(x) - f(a_0)| - |\langle x - a_0, \vartheta \rangle| \big| \end{aligned}$$

Using Cauchy-Schwartz inequality $|\langle x, y \rangle\| \leq \|x\| \|y\|$, we have

$$f(x) - f(a_0) - \langle x - a_0, \vartheta \rangle \geq \big| |f(x) - f(a_0)| - \|x - a_0\| \|\vartheta\| \big|$$

Since $\|\vartheta\| = \|f\|$ by Riesz representation theorem, so we are done. □

Let U be an open subset of a Banach space E. Then the real valued function f, defined on U, have one-sided directional derivatives (left(right) Gâteaux derivatives) at $a_0 \in U$ relative to (or may say in the direction of) v. These Gâteaux derivatives are defined through limits as:

$$\nabla_{+(-)} f(a_0; v) := \lim_{t \to 0+(-)} \frac{f(a_0 + tv) - f(a_0)}{t}$$

It is worth mentioning that both of the one-sided directional derivatives $\nabla_{-}f(a_0;\ v)$ and $\nabla_{+}f\,(a_0;v)$ are subadditive as well as positively homogenous and their existence and equivalence gives us the directional derivative of f relative to v at a_0, written as $f'(a_0;v)$, which is nothing but the common value of $\nabla_{-}f\,(a_0;v)$ and $\nabla_{+}f\,(a_0;v)$. We infer that $\nabla_{-}f\,(a_0;v)$ and $\nabla_{+}f\,(a_0;v)$ are linear when they exist and obey the expression $\nabla_{+}f\,(a_0;v) = -\nabla_{-}f\,(a_0;-v)$ To be more specific, the directional derivatives are actually the partial derivatives when taken in connection to canonical basis of $\mathbb{R}^n$. For the sake of simplicity we will denote right directional derivative by f'_{+} and left directional derivative by f'_{-} and for directional derivative by f' if exist.

In the rest of this chapter, we assume that f is a continuous nondecreasing map from the real interval $[a,b]$ to the interval $\mathbf{I} \subset \mathbb{R}^k$ of the form $f(t) = \mathbf{a}t + \mathbf{b}$ where $\mathbf{0} \leq \mathbf{a}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$. Hence a functions with nondecreasing increments f along positively oriented lines with positive direction cosines is convex over this domain $\mathbf{U}$ and we will call this function a functions with nondecreasing increments of convex type.

Corollary 15.1: *Let $f : \mathbf{U} \to \mathbb{R}$ be a functions with nondecreasing increments of convex type then $\forall\ \mathbf{x}, \mathbf{x_0} \in \mathbf{U}$ we have*

$$f(\mathbf{x}) - f(\mathbf{x_0}) - \langle f'_{+}(\mathbf{x_0}), \mathbf{x} - \mathbf{x_0}\rangle \geq \big| |f(\mathbf{x}) - f(\mathbf{x_0})| - \|f'_{+}(\mathbf{x_0})\| \|\mathbf{x} - \mathbf{x_0}\| \big|$$

where $\langle\cdot,\cdot\rangle$ represent usual inner product on $\mathbb{R}^n$ and $f'_{+}(\mathbf{x_0})$ represents the right-directional derivative of f at $\mathbf{x_0}$.

Remark 15.1: For further corollaries and related results we refer the reader to [12].

Remark 15.2: For further study related to some results of this section and applications we refer the readers to [116], [146], [150] and [131]. and references given therein.

15.2 Integral Jensen–Mercer Inequality for Functions with Nondecreasing Increments

In order to avoid repetition again and again, from now onwards we let $\overline{\mathbf{f}} = \dfrac{\int_a^b f(t)\,dH(t)}{\int_a^b dH(t)}$ and $\overline{\varphi(\mathbf{f})} = \dfrac{\int_a^b \varphi\,(f(t))\,dH(t)}{\int_a^b dH(t)}$ where $\int_a^b f\,dH$ is the vector $\left(\int_a^b f_1 d\,H, \ldots, \int_a^b f_k\,dH\right)$.

The integral version of Jensen–Mercer inequality for functions with nondecreasing increments under the condition of Jensen-Boas inequality ($A.20$) is given below:

Theorem 15.4: *Let $H : [a,b] \to \mathbb{R}$ be a function of bounded variation satisfying*

$$H(a) \leq H(a_1) \leq H(b_1) \leq H(a_2) \leq \cdots \leq H(b_{l-1}) \leq H(a_l) \leq H(b), \tag{15.3}$$

with $H(b) > H(a)$. Further let for all $a_i \in (b_{i-1}, b_i)$ ($b_0 = a,\ b_l = b$) and let f be a continuous *nondecreasing map from the real interval* $[a, b]$ *to the interval* $\mathbf{U}$. *If* $\varphi : \mathbf{I} \to \mathbb{R}$ *is a continuous functions with nondecreasing increments, then*

$$\varphi\left(\boldsymbol{a}+\boldsymbol{b}-\overline{\mathbf{f}}\right) \leq \varphi(\boldsymbol{a})+\varphi(\boldsymbol{b})-\overline{\varphi(\mathbf{f})} \tag{15.4}$$

where $\boldsymbol{a} = (a_1, \dots, a_k)$ *and* $\boldsymbol{b} = (b_1, \dots, b_k)$ *are two* k*-tuples related to* $\boldsymbol{U}$ *such that* $\boldsymbol{a} \leq f(t) \leq \boldsymbol{b}$ *for all* $t \in [a, b]$.

Also if $\overline{\mathbf{f}} \in \mathbf{U}$ *and* $\forall x \in a_i$ *such that* $(b_{i-1}, b_i)(b_0 = a, b_l = b)$ *and we have either* $H(x) \geq H(b)$ *or* $H(x) \leq H(a)$, *then the inequality* (15.4) *remains valid.*

Proof: Under the conditions (15.3) for a continuous nondecreasing map f from the real interval $[a, b]$ to the interval $\mathbf{U}$, we have $\varphi\left(\mathbf{a}+\mathbf{b}-\overline{\mathbf{f}}\right) = \varphi\left[\frac{1}{\int_a^b dH(t)}\int_a^b \left(\mathbf{a}+\mathbf{b}-f(t)\right) dH(t)\right]$ now if f is a continuous nondecreasing map from the real interval $[a, b]$ to the interval $\mathbf{U}$, then $\mathbf{a}+\mathbf{b}-f(t)$ is also a continuous nondecreasing map for for all $t \in [a, b]$.

Therefore by using Theorem $A.24$ we have

$$\varphi\left(\mathbf{a}+\mathbf{b}-\overline{\mathbf{f}}\right) \leq \frac{1}{\int_a^b dH(t)}\int_a^b \varphi\left(\mathbf{a}+\mathbf{b}-f(t)\right) dH(t)$$

now by using Lemma 1 of [26] we get $\varphi\left(\mathbf{a}+\mathbf{b}-\overline{\mathbf{f}}\right) \leq \varphi(\mathbf{a})+\varphi(\mathbf{b})-\overline{\varphi(\mathbf{f})}$ now if $\overline{\mathbf{f}} \in \mathbf{U}$ and $\forall x \in a_i$ such that $(b_{i-1}, b_i)(b_0 = a, b_l = b)$ and we have either $H(x) \geq H(b)$ or $H(x) \leq H(a)$, then by using Lemma 1 of [26] and Remark $A.8$ we have

$$\varphi\left(\mathbf{a}+\mathbf{b}-\overline{\mathbf{f}}\right) \leq \varphi(\mathbf{a})+\varphi(\mathbf{b})-\varphi\left(\overline{\mathbf{f}}\right) \leq \varphi(\mathbf{a})+\varphi(\mathbf{b})-\overline{\varphi(\mathbf{f})}$$

which completes the proof. □

We now state the integral Jensen–Mercer inequality under the condition of Jensen-Steffensen inequality (15.6) is given below:

Corollary 15.2: *Let* $H : [a, b] \to \mathbb{R}$ *be a function of bounded variation satisfying*

$$H(a) \leq H(x) \leq H(b), \quad H(b) > H(a)$$

and let f *be a continuous nondecreasing map from the real interval* $[a, b]$ *to the interval* $\mathbf{U}$. *If* $\varphi : \mathbf{I} \to \mathbb{R}$ *is a continuous functions with nondecreasing increments, then*

$$\varphi\left(\boldsymbol{a}+\boldsymbol{b}-\overline{\mathbf{f}}\right) \leq \varphi(\boldsymbol{a})+\varphi(\boldsymbol{b})-\overline{\varphi(\mathbf{f})} \tag{15.5}$$

where $\int_a^b f\,dH$ *is the vector* $(\int_a^b f_1 d\,H, \dots, \int_a^b f_k\,dH)$, *and* $\boldsymbol{a} = (a_1, \dots, a_k)$ *and* $\boldsymbol{b} = (b_1, \dots, b_k)$ *are two* k*-tuples related to* $\boldsymbol{U}$ *such that* $\boldsymbol{a} \leq f(t) \leq \boldsymbol{b}$ *for all* $t \in [a, b]$.

Also if $\overline{\mathbf{f}} \in \mathbf{U}$ *and we have either* $H(x) \geq H(b)$ *or* $H(x) \leq H(a)$, *then the inequality* (15.5) *still remains valid.*

Proof: It is nothing but a straight forward case of Theorem 15.4 for $l = 1$. □

In the rest of this chapter we assume that f is a continuous nondecreasing map from the real interval $[a,b]$ to the interval $\mathbf{I} \subset \mathbb{R}^k$ of the form $f(t) = \mathbf{a}t + \mathbf{b}$ where $\mathbf{0} \leq \mathbf{a}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$. Hence a functions with nondecreasing increments φ along positively oriented lines with positive direction cosines is convex over this domain $\mathbf{U}$ and we will call this function a functions with nondecreasing increments of convex type.

Before going onto our main results we need the following important lemma with respect to the Corollary 15.1.

Lemma 15.1: *Let $\varphi : \mathbf{U} \to \mathbb{R}$ be a functions with nondecreasing increments of convex type then $\forall\ \mathbf{x}, \mathbf{x_0} \in \mathbf{U}$ we have*

$$\varphi(\mathbf{x}) - \varphi(\mathbf{x_0}) - \langle \varphi'_+(\mathbf{x_0}), \mathbf{x} - \mathbf{x_0} \rangle \geq \big| |\varphi(\mathbf{x}) - \varphi(\mathbf{x_0})| - \|\varphi'_+(\mathbf{x_0})\| \|\mathbf{x} - \mathbf{x_0}\| \big|$$

where $\langle \cdot, \cdot \rangle$ represent usual inner product on $\mathbb{R}^n$ and $\varphi'_+(\mathbf{x_0})$ represents the right-directional derivative of φ at $\mathbf{x_0}$.

15.3 Improvements of the Jensen-Boas Inequality

Theorem 15.5: *Let all the assumptions of Theorem A.24 be valid for the functions with nondecreasing increments of convex type. Then we have*

$$\overline{\varphi(\mathbf{f})} - \varphi\left(\bar{\mathbf{f}}\right) \geq \left| \frac{\int_a^b |\varphi(\mathbf{f}(t)) - \varphi(\bar{\mathbf{f}})| dH(t)}{\int_a^b dH(t)} - \|\varphi'_+(\bar{\mathbf{f}})\| \frac{\int_a^b \|\mathbf{f}(t) - \bar{\mathbf{f}}\| dH(t)}{\int_a^b dH(t)} \right|,$$

Proof: Setting $\mathbf{x} = \mathbf{f}(t)$ and $\mathbf{x_0} = \bar{\mathbf{f}}$ for all $t \in [a,b]$ in Lemma 15.1 we get

$$\varphi(\mathbf{f}(t)) - \varphi(\bar{\mathbf{f}}) - \langle \varphi'_+(\bar{\mathbf{f}}), \mathbf{f}(t) - \bar{\mathbf{f}} \rangle \geq \big| |\varphi(\mathbf{f}(t)) - \varphi(\bar{\mathbf{f}})| - \|\varphi'_+(\bar{\mathbf{f}})\| \|\mathbf{f}(t) - \bar{\mathbf{f}}\| \big|$$

By integrating with respect to function of bounded variation H over interval $[a,b]$ we get

$$\begin{aligned}\int_a^b \varphi(\mathbf{f}(t)) dH(t) - \varphi(\bar{\mathbf{f}}) \int_a^b dH(t) &\geq \int_a^b \big| |\varphi(\mathbf{f}(t)) - \varphi(\bar{\mathbf{f}})| - \|\varphi'_+(\bar{\mathbf{f}})\| \|\mathbf{f}(t) - \bar{\mathbf{f}}\| \big| dH(t) \\ &\geq \left| \int_a^b |\varphi(\mathbf{f}(t)) - \varphi(\bar{\mathbf{f}})| dH(t) - \|\varphi'_+(\bar{\mathbf{f}})\| \int_a^b \|\mathbf{f}(t) - \bar{\mathbf{f}}\| dH(t) \right|\end{aligned}$$

and after dividing by $\int_a^b dH(t) > 0$ we get what we wanted. □

Remark 15.3: Under the assumptions of Remark *A.8* the inequality in Theorem 15.5 get reversed.

The following result gives the improvement of the Jensen-Steffensen inequality.

Corollary 15.3: *Let all the assumptions of Theorem A.23 be valid the functions with nonde-*

creasing increments of convex type. Then

$$\overline{\varphi(\mathbf{f})} - \varphi\left(\bar{\mathbf{f}}\right) \geq \left| \frac{\int_a^b |\varphi(\mathbf{f}(t)) - \varphi(\bar{\mathbf{f}})| dH(t)}{\int_a^b dH(t)} - \|\varphi'_+(\bar{\mathbf{f}})\| \frac{\int_a^b \|\mathbf{f}(t) - \bar{\mathbf{f}}\| dH(t)}{\int_a^b dH(t)} \right|$$

Proof: The Theorem 15.5 generates the desired result by using (A.18) for $l = 1$. □

Remark 15.4: (i) Under the assumptions of Remark A.7 the inequality in Corollary 15.3 get reversed.

(ii) If we take H as an increasing and bounded function with $H(a) \neq H(b)$ then the integral inequality of Theorem 2.1 of [116] becomes special case of our result for $k = 1$.

15.4 Improvements of the Jensen–Mercer Inequality

Theorem 15.6: *Let all the assumptions of Theorem 15.4 be valid for the functions with nondecreasing increments of convex type. Then we have*

$$\varphi(\mathbf{a}) + \varphi(\mathbf{b}) - \overline{\varphi(\mathbf{f})} - \varphi\left(\overline{\mathbf{f_1}}\right) \geq \left| \frac{\int_a^b |\varphi(\mathbf{a}+\mathbf{b}-\mathbf{f}(t)) - \varphi(\overline{\mathbf{f_1}})| dH(t)}{\int_a^b dH(t)} - \|\varphi'_+(\overline{\mathbf{f_1}})\| \frac{\int_a^b \|\mathbf{a}+\mathbf{b}-\mathbf{f}(t) - \overline{\mathbf{f_1}}\| dH(t)}{\int_a^b dH(t)} \right|$$

where $\mathbf{a}, \mathbf{b}$ *are two* $k-$*tuples in* $\mathbf{U}$ *such that* $\boldsymbol{a} \leq \mathbf{f}(t) \leq \boldsymbol{b}$ *for all* $t \in [a, b]$ *with* $\overline{\mathbf{f_1}} = \mathbf{a} + \mathbf{b} - \bar{\mathbf{f}}$.

Proof: See [16]. □

15.5 Improvements of Integral Neizgoda's Inequality

Theorem 15.7: *Let all the assumptions of Theorem 11.10 hold for functions with nondecreasing increments of convex type, then we have*

$$\begin{aligned}
&\frac{1}{|I^c|}\left(\int_a^b \varphi(h(t))dH(t) - \frac{1}{\int_f \omega(s)d\mu(s)} \int_I \int_f \omega(s)\varphi(f(s,t))d\mu(s)dH(t)\right) \\
&\qquad - \varphi\left(\frac{1}{|I^c|}\left(\int_a^b h(t)dH(t) - \frac{1}{\int_f \omega(s)d\mu(s)} \int_I \int_f \omega(s)g(s,t)d\mu(s)dH(t)\right)\right) \\
&\qquad \geq \left| \frac{\int_f \omega(s)|\varphi\left(\frac{1}{|I^c|}\left(\int_a^b h(t)dH(t) - \int_I g(s,t)dH(t)\right)\right)\varphi(\overline{\mathbf{f_2}})|d\mu(s)}{\int_f \omega(s)d\mu(s)} \right. \\
&\qquad\qquad \left. - \|\varphi'_+(\overline{\mathbf{f_2}})\| \frac{\int_f \omega(s)\|\left(\frac{1}{|I^c|}\left(\int_a^b h(t)dH(t) - \int_I g(s,t)dH(t)\right)\right) - \overline{\mathbf{f_2}}\|d\mu(s)}{\int_f \omega(s)d\mu(s)} \right|
\end{aligned}$$

where

$$\overline{\mathbf{f_2}} = \left(\frac{1}{|I^c|}\left(\int_a^b h(t)dH(t) - \frac{1}{\int_f \omega(s)d\mu(s)} \int_I \int_f \omega(s)g(s,t)d\mu(s)dH(t)\right)\right).$$

Proof: See [16]. □

15.6 Improvements of the Levinson Inequality for Functions with Nondecreasing Increments

Now we state a result namely generalized Levinson inequality from [139].

Theorem 15.8: *Let $H:[a,b]\to\mathbb{R}$ be a function of bounded variation such that (A.18) holds and let f be a continuous and nondecreasing map from $[a,b]\subset\mathbb{R}$ to an interval $\mathbf{I}=[\mathbf{0},\mathbf{d}]\subset\mathbb{R}^k$, $\mathbf{d}>\mathbf{0}$. If φ is a continuous functions with nondecreasing increments of order 3 on $\mathbf{J}=[\mathbf{0},2\mathbf{d}]$, then*

$$\overline{\varphi(\mathbf{f})}-\varphi\left(\bar{\mathbf{f}}\right)\leq\frac{\int_a^b\varphi\left(2\mathbf{d}-f(t)\right)\,dH(t)}{\int_a^b\,dH(t)}-\varphi\left(\frac{\int_a^b(2\mathbf{d}-f(t))\,dH(t)}{\int_a^b\,dH(t)}\right).$$

Theorem 15.9: *Let $H:[a,b]\to\mathbb{R}$ be a function of bounded variation such that (A.18) holds, and let φ be a continuous functions with nondecreasing increments of order 3 on $[\mathbf{c},\mathbf{d}]\subset\mathbb{R}^k$. Let $\mathbf{0}<\mathbf{a}<\mathbf{d}-\mathbf{c}$. If $f(t):[a,b]\to[\mathbf{c},\mathbf{d}-\mathbf{a}]$ is a continuous and nondecreasing map, then*

$$\overline{\varphi(\mathbf{f})}-\varphi\left(\bar{\mathbf{f}}\right)\leq\frac{\int_a^b\varphi\left(\mathbf{a}+f(t)\right)\,dH(t)}{\int_a^b\,dH(t)}-\varphi\left(\frac{\int_a^b(\mathbf{a}+f(t))\,dH(t)}{\int_a^b\,dH(t)}\right).$$

We now prove another important result related to functions with nondecreasing increments.

Here we first state the improvement of the Levinson inequality by using Jesen-Boas Inequality.

Theorem 15.10: *Let $H:[a,b]\to\mathbb{R}$ be a function of bounded variation such that (A.18) holds, and let f be a continuous and nondecreasing map from $[a,b]\subset\mathbb{R}$ to an interval $\mathbf{I}=[\mathbf{0},\mathbf{d}]\subset\mathbb{R}^k$, $\mathbf{d}>\mathbf{0}$. If φ is a continuous functions with nondecreasing increments of convex type of order 3 on $\mathbf{J}=[\mathbf{0},2\mathbf{d}]$, then*

$$\begin{aligned}&\frac{\int_a^b\varphi(2\mathbf{d}-\mathbf{f}(t))dH(t)}{\int_a^b dH(t)}-\varphi\left(2\mathbf{d}-\bar{\mathbf{f}}\right)-\frac{\int_a^b\varphi(\mathbf{f}(t))dH(t)}{\int_a^b dH(t)}+\varphi\left(\bar{\mathbf{f}}\right)\\&\qquad\geq\left|\frac{\int_a^b|\varphi(2\mathbf{d}-\mathbf{f}(t))-\varphi(\mathbf{f}(t))-\varphi(2\mathbf{d}-\bar{\mathbf{f}})+\varphi(\bar{\mathbf{f}})|dH(t)}{\int_a^b dH(t)}\right.\\&\qquad\qquad\left.-\|\varphi'_+(2\mathbf{d}-\bar{\mathbf{f}})\|+\|\varphi'_+(\bar{\mathbf{f}})\|\frac{\int_a^b\|\mathbf{f}(t)-\bar{\mathbf{f}}\|dH(t)}{\int_a^b dH(t)}\right|.\end{aligned}$$

Proof: If φ is a functions with nondecreasing increments of order 3 on $\mathbf{J}$, then

$$\Delta_{\mathbf{h}}\Delta_{\mathbf{t}}\Delta_{\mathbf{s}}\varphi(\mathbf{x})\geq 0\quad(\mathbf{x},\mathbf{x}+\mathbf{h}+\mathbf{t}+\mathbf{s}\in\mathbf{J},\ \mathbf{0}\leq\mathbf{h},\mathbf{t},\mathbf{s}\in\mathbb{R}^k),$$

$$\text{i.e.,}\quad\Delta_{\mathbf{h}}\Delta_{\mathbf{t}}(\varphi(\mathbf{x}+\mathbf{s})-f(\mathbf{x}))\geq 0.\tag{15.6}$$

If $\mathbf{x}\in\mathbf{I}$ and $\mathbf{s}=2\mathbf{d}-2\mathbf{x}$, we have $\Delta_{\mathbf{h}}\Delta_{\mathbf{t}}(\varphi(2\mathbf{d}-\mathbf{x})-\varphi(\mathbf{x}))\geq 0$, i.e. the function $\mathbf{x}\mapsto\varphi(2\mathbf{d}-\mathbf{x})-\varphi(\mathbf{x})$ is a functions with nondecreasing increments of order 2, i.e., it is a functions with nondecreasing increments. Now, by replacing $\varphi(\mathbf{x})$ by $\varphi(2\mathbf{d}-\mathbf{x})-\varphi(\mathbf{x})$ in Theorem 15.5, we obtain Theorem 15.10. □

Theorem 15.11: *Let $H : [a,b] \to \mathbb{R}$ be a function of bounded variation such that ([A.18]) holds, and let φ be a continuous functions with nondecreasing increments of convex type of order 3 on $[\mathbf{r},\mathbf{d}] \subset \mathbb{R}^k$. Let $\mathbf{0} < \mathbf{p} < \mathbf{d}-\mathbf{r}$. If $f(t) : [a,b] \to [\mathbf{r},\mathbf{d}-\mathbf{p}]$ is a continuous and nondecreasing map, then*

$$\begin{aligned}\frac{\int_a^b \varphi(\mathbf{p}+\mathbf{f}(t))dH(t)}{\int_a^b dH(t)} &- \varphi\left(\mathbf{p}+\bar{\mathbf{f}}\right) - \overline{\varphi(\mathbf{f})} + \varphi\left(\bar{\mathbf{f}}\right) \\ &\geq \left| \frac{\int_a^b |\varphi(\mathbf{p}+\mathbf{f}(t)) - \varphi(\mathbf{f}(t)) - \varphi(\mathbf{p}+\bar{\mathbf{f}}) + \varphi(\bar{\mathbf{f}})|\mathbf{dH(t)}}{\int_a^b dH(t)} \right. \\ &\qquad \left. - \|\varphi'_+(\mathbf{p}+\bar{\mathbf{f}})\| + \|\varphi'_+(\bar{\mathbf{f}})\| \frac{\int_{\mathbf{a}}^{\mathbf{b}} \|\mathbf{f(t)}-\bar{\mathbf{f}}\|\mathbf{dH(t)}}{\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{dH(t)}} \right|,\end{aligned}$$

Proof: Using (15.6) for $\mathbf{s} = \mathbf{c} =$ constant $\in \mathbb{R}^k$, we have $\varphi(\mathbf{p}+\mathbf{x}) - \varphi(\mathbf{x})$ is a functions with nondecreasing increments, now applying Theorem 15.5 to get the Theorem 15.11. □

Corollary 15.4: *Let $H : [a,b] \to \mathbb{R}$ be a function of bounded variation such that ([A.18]) holds, and let f be a continuous and nondecreasing map from $[a,b] \subset \mathbb{R}$ to an interval $\mathbf{I} = [\mathbf{0},\mathbf{d}] \subset \mathbb{R}^k$, $\mathbf{d} > \mathbf{0}$. If φ is a continuous functions with nondecreasing increments of convex type of order 3 on $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, then we have*

$$\begin{aligned}\frac{\int_a^b \varphi(2\mathbf{d}-\mathbf{f}(t))dH(t)}{\int_a^b dH(t)} &- \varphi\left(2\mathbf{d}-\bar{\mathbf{f}}\right) - \overline{\varphi(\mathbf{f})} + \varphi\left(\bar{\mathbf{f}}\right) \\ &\geq \left| \frac{\int_a^b |\varphi(2\mathbf{d}-\mathbf{f}(t)) - \varphi(\mathbf{f}(t)) - \varphi(2\mathbf{d}-\bar{\mathbf{f}}) + \varphi(\bar{\mathbf{f}})|dH(t)}{\int_a^b dH(t)} \right. \\ &\qquad \left. - \|\varphi'_+(2\mathbf{d}-\bar{\mathbf{f}})\| + \|\varphi'_+(\bar{\mathbf{f}})\| \frac{\int_a^b \|\mathbf{f}(t)-\bar{\mathbf{f}}\|dH(t)}{\int_a^b dH(t)} \right|\end{aligned}$$

Proof: It can be directly obtained from Theorem 15.10 by taking $l = 1$ in ([A.18]). □

Corollary 15.5: *Let $H : [a,b] \to \mathbb{R}$ be a function of bounded variation such that ([A.18]) holds, and let φ be a continuous functions with nondecreasing increments of order 3 on $[\mathbf{r},\mathbf{d}] \subset \mathbb{R}^k$. Let $\mathbf{0} < \mathbf{p} < \mathbf{d}-\mathbf{r}$. If $f(t) : [a,b] \to [\mathbf{r},\mathbf{d}-\mathbf{p}]$ is a continuous and nondecreasing map, then*

$$\begin{aligned}\frac{\int_a^b \varphi(\mathbf{p}+\mathbf{f}(t))dH(t)}{\int_a^b dH(t)} &- \varphi\left(\mathbf{p}+\bar{\mathbf{f}}\right) - \overline{\varphi(\mathbf{f})} + \varphi\left(\bar{\mathbf{f}}\right) \\ &\geq \left| \frac{\int_a^b |\varphi(\mathbf{p}+\mathbf{f}(t)) - \varphi(\mathbf{f}(t)) - \varphi(\mathbf{p}+\bar{\mathbf{f}}) + \varphi(\bar{\mathbf{f}})|dH(t)}{\int_a^b dH(t)} \right. \\ &\qquad \left. - \|\varphi'_+(\mathbf{p}+\bar{\mathbf{f}})\| + \|\varphi'_+(\bar{\mathbf{f}})\| \frac{\int_a^b \|\mathbf{f}(t)-\bar{\mathbf{f}}\|dH(t)}{\int_a^b dH(t)} \right|,\end{aligned}$$

Proof: It is a straight forward result from Theorem 15.11 with $l = 1$ in ([A.18]). □

Remark 15.5: (i) For other related results of Levinson type we refer the reader to [16].

(ii) For analogous discrete cases of all the results presented in this chapter, related corollaries and remarks we refer the reader to [16].

CHAPTER 16

The Hermite–Hadamard–Mercer Type Inequalities for Convex, Strongly Convex Functions and Majorization

This chapter develops Hermite–Hadamard–Mercer type inequalities for convex and strongly convex functions, with applications of majorization techniques. We begin with Hermite –Hadamard–Jensen–Mercer type inequalities for Riemann–Liouville fractional integrals and extend these to fractional Hermite–Hadamard–Mercer inequalities. Bounds for the differences of these inequalities are analyzed, followed by results for strongly convex functions. The chapter further presents generalized inequalities via majorization and unifies continuous and discrete fractional Hermite–Hadamard–Mercer type inequalities, providing a comprehensive framework for both classical and fractional settings.

16.1 The Hermite–Hadamard–Jensen–Mercer Type Inequalities for Riemann–Liouville Fractional Integrals

The Hermite-Hadamard inequality is one of the most investigated inequality in the theory of convex functions due to its geometrical significance and applications. Because of the importance of Hermite-Hadamard inequality, there is an ample amount of research work dedicated to the extensions, generalizations, refinements and applications of the Hermite-Hadamard inequality.

Here we recall the definition of Riemann-Liouville fractional operator stated as under:

Definition 16.1 *([180])*: Let f be an integrable function defined on $[a, b]$. Then the integrals $J^{\alpha}_{a^+}f(x)$ and $J^{\alpha}_{b^-}f(x)$ defined by

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)ds, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s) ds, \quad x < b$$

are called the left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ respectively. Here Γ represents gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-s}\, s^{\alpha-1} ds$.

In this chapter, we establish fractional Hermite-Hadamard-Jensen–Mercer type inequalities. We give identities involving fractional integrals and from these identities, we derive trapezoidal and midpoint type inequalities.

Throughout this chapter α represents a positive real number.

16.1.1 Fractional Hermite-Hadamard-Mercer Type Inequalities

We begin this section with following result.

Theorem 16.1: *Suppose $f : [a,b] \to \mathbb{R}$ is a convex function and $x, y \in [a,b]$ such that $x < y$. Then*

$$f\left(a+b-\frac{\alpha x+y}{\alpha+1}\right) \leq f(a)+f(b)-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J_{x+}^{\alpha} f(y) \leq f(a)+f(b)-f\left(\frac{\alpha x+y}{\alpha+1}\right)$$

and

$$\begin{aligned} f\left(a+b-\frac{\alpha x+y}{\alpha+1}\right) &\leq \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J_{(a+b-x)-}^{\alpha} f(a+b-y) \\ &\leq \frac{\alpha f(a+b-x)+f(a+b-y)}{\alpha+1} \leq f(a)+f(b)-\frac{\alpha f(x)+f(y)}{\alpha+1}. \end{aligned} \tag{16.1}$$

Proof: See [239] □

Remark 16.1: If we put $\alpha = 1$ in Theorem 16.1 and in the obtained expressions substitute $u = sx + (1-s)y$ and $u = a+b-v$ respectively, we obtain

$$f\left(a+b-\frac{x+y}{2}\right) \leq f(a)+f(b)-\int_0^1 f\left(sx+(1-s)y\right) ds \leq f(a)+f(b)-f\left(\frac{x+y}{2}\right) \tag{16.2}$$

and

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{1}{y-x}\int_x^y f\left(a+b-v\right) dv \\ &\leq \frac{f(a+b-x)+f(a+b-y)}{2} \leq f(a)+f(b)-\frac{f(x)+f(y)}{2} \end{aligned} \tag{16.3}$$

respectively. The inequalities (16.2) and (16.3) have been proved in [90].

Remark 16.2: Substituting $\alpha = 1$, $x = a$ and $y = b$ in (16.1), one can obtain Hermite-Hadamard inequality.

16.1.2 Bounds for the Difference of Hermite-Hadamard-Jensen–Mercer Type Inequalities

Throughout this section, we consider $f : [a,b] \to \mathbb{R}$ is a differentiable function.

Theorem 16.2: *Let $|f'|$ be a convex function defined on $[a,b]$ and let $x,y \in [a,b]$ such that $x<y$. Then*

$$\begin{aligned}&\left|\frac{\alpha f(a+b-x)+f(a+b-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(a+b-x)^-}f(a+b-y)\right|\\&\leq \frac{y-x}{\alpha+1}\left[P_1(\alpha)\left|f'(a)\right|+P_1(\alpha)\left|f'(b)\right|-P_2(\alpha)\left|f'(x)\right|-P_3(\alpha)\left|f'(y)\right|\right],\end{aligned} \tag{16.4}$$

where

$$P_1(\alpha)=\frac{2\alpha}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}},\ P_2(\alpha)=\frac{\alpha\left[2+(\alpha+1)^{\frac{2}{\alpha}}\right]}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}}\ \text{ and}$$
$$P_3(\alpha)=\frac{\alpha\left[4(\alpha+2)(\alpha+1)^{\frac{1}{\alpha}-1}-(\alpha+1)^{\frac{2}{\alpha}}-2\right]}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}}.$$

Remark 16.3: If we put $\alpha=1$, $x=a$ and $y=b$ in (16.4), we get the inequality given in Theorem 2.2 of [85].

Theorem 16.3: *Let $|f'|^q$ be a convex function for $q \geq 1$ and let $x,y \in [a,b]$ such that $x<y$. Then*

$$\begin{aligned}&\left|\frac{\alpha f(a+b-x)+f(a+b-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(a+b-x)^-}f(a+b-y)\right|\\&\leq \frac{y-x}{\alpha+1}\left(P_1(\alpha)\right)^{1-\frac{1}{q}}\left(P_1(\alpha)\left|f'(a)\right|^q+P_1(\alpha)\left|f'(b)\right|^q-P_2(\alpha)\left|f'(x)\right|^q-P_3(\alpha)\left|f'(y)\right|^q\right)^{\frac{1}{q}},\end{aligned} \tag{16.5}$$

where $P_1(\alpha)$, $P_2(\alpha)$, and $P_3(\alpha)$ are same as defined in Theorem 16.2.

Remark 16.4: If we put $\alpha=1$, $x=a$ and $y=b$ in (16.5), we obtain the inequality proved in Theorem 1 of [201].

In the following theorem, we derive trapezoidal type inequality.

Theorem 16.4: *Let $p,\ q>1$, $|f'|^q$ be a convex function and let $x,y \in [a,b]$ such that $x<y$. Then*

$$\begin{aligned}&\left|\frac{\alpha f(a+b-x)+f(a+b-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(a+b-x)^-}f(a+b-y)\right|\\&\qquad\leq \frac{y-x}{\alpha+1}\left(L_4(\alpha,p)\right)^{\frac{1}{p}}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q-\frac{|f'(x)|^q+|f'(y)|^q}{2}\right)^{\frac{1}{q}},\end{aligned}$$

where $L_4(\alpha,p)=\int_0^1|(\alpha+1)s^{\alpha}-1|^p\,ds$ such that $\frac{1}{p}+\frac{1}{q}=1$.

Remark 16.5: Substituting $\alpha=1$, $x=a$ and $y=b$ in Theorem 16.4, we obtain Theorem 2.3

of [85].

Theorem 16.5: *Let $|f'|$ be a convex function and let $x, y \in [a, b]$ such that $x < y$. Then*

$$\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J^{\alpha}_{(a+b-x)^-} f(a+b-y) - f\left(a+b-\frac{\alpha x+y}{\alpha+1}\right)\right| \leq (y-x)\left(P_5(\alpha)\left|f'(x)\right| + P_6(\alpha)\left|f'(y)\right|\right),$$

$$\text{where} \quad P_5(\alpha) = \frac{-\alpha}{2(\alpha+2)(\alpha+1)^2} \text{ and } P_6(\alpha) = \frac{\alpha}{2(\alpha+2)(\alpha+1)^2}.$$

In next theorem we use power mean inequality and derive midpoint type inequality.

Theorem 16.6: *Let $|f'|^q$ be a convex function for $q \geq 1$ and let $x, y \in [a, b]$ such that $x < y$. Then*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J^{\alpha}_{(a+b-x)^-} f(a+b-y) - f\left(a+b-\frac{\alpha x+y}{\alpha+1}\right)\right| \\
&\leq (y-x)\left(L_8(\alpha)\right)^{1-\frac{1}{q}}\left(\left(L_8(\alpha)\left|f'(a)\right|^q + L_8(\alpha)\left|f'(b)\right|^q - L_5(\alpha)\left|f'(x)\right|^q - L_6(\alpha)\left|f'(y)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad \left. - \left(L_8(\alpha)\left|f'(a)\right|^q + L_8(\alpha)\left|f'(b)\right|^q - N_5(\alpha)\left|f'(x)\right|^q - N_6(\alpha)\left|f'(y)\right|^q\right)^{\frac{1}{q}}\right),
\end{aligned}$$

where

$$L_8(\alpha) = \int_{\frac{\alpha}{\alpha+1}}^{1}(1-s^{\alpha})ds = \int_0^{\frac{\alpha}{\alpha+1}} s^{\alpha}ds = \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}$$

and $L_5(\alpha)$, $L_6(\alpha)$, $N_5(\alpha)$ and $N_6(\alpha)$ are given in the proof of Theorem 16.5.

Remark 16.6: Substituting $\alpha = 1$, $x = a$ and $y = b$ in Theorem 16.6, we obtain the following midpoint type inequality.

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left(\left(\frac{\left|f'(a)\right|^q + 2\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}} - \left(\frac{2\left|f'(a)\right|^q + \left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}\right).$$

Another midpoint type inequality is presented in the following theorem.

Theorem 16.7: *Let $p,\ q > 1$, $|f'|^q$ be convex and let $x, y \in [a, b]$ such that $x < y$. Then*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J^{\alpha}_{(a+b-x)^-} f(a+b-y) - f\left(a+b-\frac{\alpha x+y}{\alpha+1}\right)\right| \leq (y-x)\left(L_9(\alpha,p)\right)^{\frac{1}{p}} \\
&\times\left(\frac{1}{\alpha+1}\left|f'(a)\right|^q + \frac{1}{\alpha+1}\left|f'(b)\right|^q - \frac{2\alpha+1}{2(\alpha+1)^2}\left|f'(x)\right|^q - \frac{1}{2(\alpha+1)^2}\left|f'(y)\right|^q\right)^{\frac{1}{q}} - (y-x) \\
&\times\left(L_{10}(\alpha,p)\right)^{\frac{1}{p}}\left(\frac{\alpha}{\alpha+1}\left|f'(a)\right|^q + \frac{\alpha}{\alpha+1}\left|f'(b)\right|^q - \frac{\alpha^2}{2(\alpha+1)^2}\left|f'(x)\right|^q - \frac{\alpha(\alpha+2)}{2(\alpha+1)^2}\left|f'(y)\right|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where $L_9(\alpha,p) = \int_{\frac{\alpha}{\alpha+1}}^{1}(1-s^{\alpha})^p ds$ and $L_{10}(\alpha,p) = \int_0^{\frac{\alpha}{\alpha+1}} s^{\alpha p}ds$ such that $\frac{1}{p}+\frac{1}{q} = 1$.

Remark 16.7: Substituting $\alpha = 1$, $x = a$ and $y = b$ in Theorem 16.7, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{\frac{1}{q}} - \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

Theorem 16.8: *Let $x, y \in [a, b]$ such that $x < y$. If $|f'|$ is concave on $[a, b]$, then*

$$\left|\frac{\alpha f(a+b-x) + f(a+b-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J^\alpha_{(a+b-x)^-} f(a+b-y)\right|$$
$$\leq \frac{y-x}{\alpha+1}\Big(L_1(\alpha)\left|f'\left(a+b-L_{11}(\alpha)x - L_{12}(\alpha)y\right)\right|$$
$$+N_1(\alpha)\left|f'\left(a+b-L_{13}(\alpha)x - L_{14}(\alpha)y\right)\right|\Big),$$

where L_1 and N_1 are given in the proof of Theorem 16.2 and

$$L_{11}(\alpha) = \frac{(\alpha+1)^{1-\frac{1}{\alpha}}}{2(\alpha+2)},\ L_{12}(\alpha) = \frac{(\alpha+1)^{1-\frac{1}{\alpha}}\left[2(\alpha+2)(\alpha+1)^{\frac{1}{\alpha}-1} - 1\right]}{2(\alpha+2)},$$
$$L_{13}(\alpha) = \frac{(\alpha+1)^{1-\frac{1}{\alpha}}\left[(\alpha+1)^{\frac{2}{\alpha}} + 1\right]}{2(\alpha+2)} \quad \textit{and}$$
$$L_{14}(\alpha) = \frac{(\alpha+1)^{1-\frac{1}{\alpha}}\left[2(\alpha+2)(\alpha+1)^{\frac{1}{\alpha}-1} - (\alpha+1)^{\frac{2}{\alpha}} - 1\right]}{2(\alpha+2)}.$$

Theorem 16.9: *Let $|f'|$ be a concave function and let $x, y \in [a, b]$ such that $x < y$. Then*

$$\left|\frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J^\alpha_{(a+b-x)^-} f(a+b-y) - f\left(a+b-\frac{\alpha x + y}{\alpha+1}\right)\right|$$
$$\leq (y-x)\left(\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\right)\left(\left|f'\left(a+b-L_{15}(\alpha)x - L_{16}(\alpha)y\right)\right|\right.$$
$$\left. - \left|f'\left(a+b-L_{17}(\alpha)x - L_{18}(\alpha)y\right)\right|\right),$$

where

$$L_{15}(\alpha) = \frac{(\alpha+1)^\alpha + 2\alpha^{\alpha+1}}{2\alpha^\alpha(\alpha+2)},\quad L_{16}(\alpha) = \frac{4\alpha^\alpha - (\alpha+1)^\alpha}{2\alpha^\alpha(\alpha+2)},$$
$$L_{17}(\alpha) = \frac{\alpha}{\alpha+2} \quad \textit{and} \quad L_{18}(\alpha) = \frac{2}{\alpha+2}.$$

16.2 Hermite–Hadamard–Mercer Type Inequalities for Strongly Convex Functions

In this work, the authors see use of Jensen–Mercer inequality for convex and strongly convex functions and also the well known Hölder's inequality which plays a significant role in functional analysis, mathematical analysis, complex analysis, numerical analysis, statistics and probability, and partial differential equations etc.

First we recall definition of strongly convex functions (see [219]) as follows:

Definition 16.2: A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus $c > 0$ if

$$f(\tau m + (1-\tau)M) \leq \tau f(m) + (1-\tau)f(M) - c\tau(1-\tau)(m-M)^2$$

holds for all $m, M \in I$ and $\tau \in [0,1]$.

Hermite-Hadamard inequality for strongly convex functions (see [176]) can be stated as follows:

Theorem 16.10: *Let $f : [m, M] \to \mathbb{R}$ be a strongly convex function with modulus c. Then*

$$f\left(\frac{m+M}{2}\right) + \frac{c}{12}(M-m)^2 \leq \frac{1}{M-m}\int_m^M f(\tau)d\tau \leq \frac{f(m)+f(M)}{2} - \frac{c}{6}(M-m)^2.$$

First Hermite–Hadamard–Mercer inequalities for fractional integrals was presented in [200].

For strongly convex functions, the Jensen–Mercer inequality is proved in [188] as follows:

Theorem 16.11: *Let $f : [m, M] \to \mathbb{R}$ be a strongly convex function with modulus c. Then*

$$\begin{aligned} f\left(x_1 + x_n - \sum_{i=1}^n w_i x_i\right) &\leq f(x_1) + f(x_n) - \sum_{i=1}^n w_i f(x_i) \\ &\quad - c\left(2\sum_{i=1}^n w_i \lambda_i (1-\lambda_i)(x_1 - x_n)^2 + \sum_{i=1}^n w_i \left(x_i - \sum_{i=1}^n w_i x_i\right)^2\right), \end{aligned} \tag{16.6}$$

where $\sum_{i=1}^n w_i = 1$, $\lambda_i \in [0,1]$, $x_1 = \min_{1\leq i\leq n} x_i$, $x_n = \max\limits_{1\leq i\leq n} x_i$ *and* $x_i \in [m, M]$.

Substitute $n = 2$ in (16.6), we obtain Jensen–Mercer inequality for strongly convex functions as follows:

$$\begin{aligned} &f\left(m + M - \frac{a+b}{2}\right) \\ &\qquad \leq f(m) + f(M) - \frac{f(a)+f(b)}{2} - c\Big[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)\Big](M-m)^2 - \frac{c}{4}(a-b)^2, \end{aligned}$$

where $w_1 = w_2 = \frac{1}{2}$, $\lambda_1, \lambda_2 \in [0,1]$ and $a, b \in [m, M]$.

In this section, we state the Hermite-Hadamard-Mercer type inequality for strongly convex functions by using the Jensen–Mercer inequality for strongly convex functions. We derive some new inequalities related to the right and left sides of the Hermite-Hadamard-Mercer type inequalities for differentiable functions whose derivatives in absolute value are convex. All the results are stated without proof, for idea of the proof we refer the reader to [147].

Theorem 16.12: *Let $f : [m, M] \to \mathbb{R}$ be a strongly convex function with modulus c. Then for* *all* $x, y \in [m, M]$ *with* $x < y$*,* $m \geq 0$ *and* $\alpha > 0$*, we have*

$$f\left(m+M-\frac{x+y}{2}\right) \leq f(m)+f(M)-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J_{x^+}^{\alpha}f(y)+J_{y^-}^{\alpha}f(x)\right]$$
$$-c(m-M)^2\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right]-\frac{c(\alpha^2-\alpha+2)}{4(\alpha+1)(\alpha+2)}(y-x)^2$$
$$\leq f(m)+f(M)-f\left(\frac{x+y}{2}\right)-c(m-M)^2\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right]-\frac{c(\alpha^2-\alpha+2)}{2(\alpha+1)(\alpha+2)}(y-x)^2.$$

Theorem 16.13: *Let all the assumptions of Theorem 16.12 hold. Then*

$$\begin{aligned}
& f\left(m+M-\frac{x+y}{2}\right) \\
& \leq \frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J_{(m+M-x)^-}^{\alpha}f(m+M-y)+J_{(m+M-y)^+}^{\alpha}f(m+M-x)\right] \\
& -\frac{c(\alpha^2-\alpha+2)}{4(\alpha+1)(\alpha+2)}(y-x)^2 \\
& \leq \frac{f(m+M-x)+f(m+M-y)}{2}-\frac{c}{4}(y-x)^2 \\
& \leq f(m)+f(M)-\frac{f(x)+f(y)}{2}-\frac{c}{4}(y-x)^2.
\end{aligned}$$

Corollary 16.1: *If we take* $\alpha = 1$ *in Theorem 5, then we have*

$$\begin{aligned}
& f\left(M+m-\frac{x+y}{2}\right) \leq f(m)+f(M)-\int_0^1 f(\tau x+(1-\tau)y)d\tau \\
& -c(M-m)^2\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right]-\frac{c}{12}(y-x)^2 \\
& \leq f(m)+f(M)-f\left(\frac{x+y}{2}\right)-c(M-m)^2\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right]-\frac{c}{6}(y-x)^2.
\end{aligned}$$

Corollary 16.2: *If we take* $\alpha = 1$ *in Theorem 6, then we have*

$$\begin{aligned}
f\left(m+M-\frac{x+y}{2}\right) & \leq \frac{1}{y-x}\int_x^z f(m+M-\tau)d\tau-\frac{c}{12}(y-x)^2 \\
& \leq f(m)+f(M)-\frac{f(x)+f(y)}{2}-\frac{c}{4}(y-x)^2.
\end{aligned} \tag{16.7}$$

Remark 16.8: If we take $x = m$ and $y = M$ in (16.7), we obtain the Hermite-Hadamard inequality for strongly convex function given in [176].

Theorem 16.14: *Let all the assumptions of Theorem 16.12 hold. Then*

$$\begin{aligned}
& f\left(m+M-\frac{x+y}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J_{(m+M-\frac{x+y}{2})^-}^{\alpha}f(m+M-y)\right. \\
& \left.+J_{(m+M-\frac{x+y}{2})^+}^{\alpha}f(m+M-x)\right]-\frac{c}{2(\alpha+1)(\alpha+2)}(y-x)^2 \\
& \leq f(m)+f(M)-\frac{f(x)+f(y)}{2}-c\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right](m-M)^2-\frac{c}{4}(y-x)^2.
\end{aligned}$$

Theorem 16.15: *Let all the assumptions of Theorem 16.12 hold. Then*

$$\begin{aligned}
& f\left(m+M-\frac{x+y}{2}\right) \leq \\
& f(m)+f(M)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(\frac{x+y}{2}\right)^-}f(x)+J^{\alpha}_{\left(\frac{x+y}{2}\right)^+}f(y)\right] \\
& -c\Big[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\Big](m-M)^2-\frac{c}{2(\alpha+1)(\alpha+2)}(y-x)^2 \\
& \leq f(m)+f(M)-f\left(\frac{x+y}{2}\right)-c\Big[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\Big](m-M)^2 \\
& -\frac{c}{(\alpha+1)(\alpha+2)}(y-x)^2.
\end{aligned}$$

Theorem 16.16: *Let $f:[m,M]\to\mathbb{R}$ be a differentiable function such that $|f'|$ is convex on $[m,M]$. Then for all $x,y\in[m,M]$ with $x<y$ and $\alpha>0$, we have*

$$\begin{aligned}
& \Bigg|\frac{f(m+M-x)+f(m+M-y)}{2}- \\
& \frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J^{\alpha}_{(m+M-y)^+}f(m+M-x)+J^{\alpha}_{(m+M-x)^-}f(m+M-y)\right]-\frac{\alpha c}{(\alpha+1)(\alpha+2)}(y-x)^2\Bigg| \\
& \leq \frac{y-x}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(M)|+|f'(m)|-\frac{|f'(x)|+|f'(y)|}{2}+\frac{2^{\alpha}\alpha c(y-x)}{(2^{\alpha}-1)(\alpha+2)}\right].
\end{aligned}$$

Theorem 16.17: *Let $f:[m,M]\to\mathbb{R}$ be a differentiable function such that $|f'|^q$ is convex on $[m,M]$, where $q>0$ and $\frac{1}{p}+\frac{1}{q}=1$. Then for $\alpha>0$ and for all $x,y\in[m,M]$ such that $x<y$ and, we have*

$$\begin{aligned}
& \Bigg|\frac{f(m+M-x)+f(m+M-y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}} \\
& \times\left[J^{\alpha}_{(m+M-y)^+}f(m+M-x)+J^{\alpha}_{(m+M-x)^-}f(m+M-y)\right]-\frac{\alpha c}{(\alpha+1)(\alpha+2)}(y-x)^2\Bigg| \\
& \leq \frac{y-x}{2}\left\{[L(\alpha,p)]^{\frac{1}{p}}\left[|f'(m)|^q+|f'(M)|^q-\frac{|f'(x)|^q+|f'(y)|^q}{2}\right]^{\frac{1}{q}}+\frac{2\alpha c(y-x)}{(\alpha+1)(\alpha+2)}\right\},
\end{aligned}$$

where $L(\alpha,p)=\int_0^1|\tau^{\alpha}-(1-\tau)^{\alpha}|^p d\tau$.

Theorem 16.18: *If $f:[m,M]\to\mathbb{R}$ is differentiable function and $|f'|$ is convex on $[m,M]$, then for all $x,y\in[m,M]$ with $x<y$ and $\alpha>0$ we have*

$$\begin{aligned}
& \Bigg|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(m+M-\frac{x+y}{2}\right)^+}f(m+M-x)+J^{\alpha}_{\left(m+M-\frac{x+y}{2}\right)^-}f(m+M-y)\right] \\
& -f\left(m+M-\frac{x+y}{2}\right)-\frac{c}{2(\alpha+1)(\alpha+2)}(y-x)^2\Bigg| \\
& \leq \frac{y-x}{2(\alpha+1)}\left\{|f'(m)|+|f'(M)|-\frac{|f'(x)|+|f'(y)|}{2}+\frac{c}{\alpha+2}(y-x)\right\}.
\end{aligned}$$

Theorem 16.19: *Let all the assumptions of Theorem 16.17 hold. Then*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(m+M-\frac{x+y}{2}\right)^{+}}f(m+M-x)+J^{\alpha}_{\left(m+M-\frac{x+y}{2}\right)^{-}}f(m+M-y)\right]\right.\\&\left.-f\left(m+M-\frac{x+y}{2}\right)-\frac{c}{2(\alpha+1)(\alpha+2)}(y-x)^2\right|\\&\le\frac{y-x}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\left[|f'(m)|^q+|f'(M)|^q-\frac{3|f'(x)|^q+|f'(y)|^q}{4}\right]^{\frac{1}{q}}\right.\\&\left.+\left[|f'(m)|^q+|f'(M)|^q-\frac{|f'(x)|^q+3|f'(y)|^q}{4}\right]^{\frac{1}{q}}+\frac{2c(y-x)}{(q+1)^{\frac{1}{q}}}\right\}.\end{aligned}$$

Remark 16.9: We can find other results related to the generalization of Hermite–Hadamard–Mercer type results for strongly convex generalized functions on fractal sets in the article [66]. For further work on fractal Hadamard–Mercer-type inequalities with applications we refer the reader to [64]. Interested readers can also see [65] for the results related to generalized fractal Jensen and Jensen–Mercer inequalities for harmonic convex function with applications.

16.3 Generalized Hermite-Hadamard-Mercer Type Inequalities Via Majorization

In the underlying theorems we give Hermite–Hadamard inequality of the Jensen–Mercer type by using majorization concept.

Theorem 16.20: *Suppose that f is a real valued convex function on I and $\mathbf{a}=(a_1,\dots,a_m)$, $\mathbf{x}=(x_1,\dots,x_m)$, $\mathbf{y}=(y_1,\dots,y_m)$ are three m-tuples such that $a_j,x_j,\ y_j\in I$, $x_j\neq y_j$ for all $j\in\{1,\cdots,m\}$. If $\mathbf{x}\prec\mathbf{a}$ and $\mathbf{y}\prec\mathbf{a}$, then*

$$\begin{aligned}f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{m-1}\left(\frac{x_j+y_j}{2}\right)\right)&\le\sum_{j=1}^{m}f(a_j)-\sum_{j=1}^{m-1}\frac{1}{y_j-x_j}\int_{x_j}^{y_j}f(u)du \qquad (16.8)\\&\le\sum_{j=1}^{m}f(a_j)-\sum_{j=1}^{m-1}f\left(\frac{x_j+y_j}{2}\right).\end{aligned}$$

Proof: See [94]. □

Remark 16.10: In the above theorem, if $x_j=y_j$ for all $j\in\{1,2,\dots,m\}$ then by following the proof of Theorem 16.20, we obtain

$$f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{m-1}x_j\right)\le\sum_{j=1}^{m}f(a_j)-\sum_{j=1}^{m-1}f(x_j).$$

If $x_j=y_j$ for some j, then (16.8) reduces to the form

$$f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{m-1}\left(\frac{x_j+y_j}{2}\right)\right)\le\sum_{j=1}^{m}f(a_j)-\sum_{j\in I}f(x_j)-\sum_{j\in I^c}\frac{1}{y_j-x_j}\int_{x_j}^{y_j}f(u)du,$$

where $I=\{j\in\{1,2,\dots,m\}:x_j=y_j\}$ and $I^c=\{1,2,\dots,m\}\setminus I$.

Theorem 16.21: *If all the hypotheses of Theorem 16.20 are valid and $x_m \neq y_m$, then*

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\left(\frac{x_j+y_j}{2}\right)\right) \leq \frac{1}{\sum_{j=1}^{m-1}(y_j-x_j)} \int_{\sum_{j=1}^{m-1} x_j}^{\sum_{j=1}^{m-1} y_j} f\left(\sum_{j=1}^{m} a_j - u\right) du$$
$$\leq \sum_{j=1}^{m} f(a_j) - \frac{1}{2}\left\{\sum_{j=1}^{m-1} f(x_j) + \sum_{j=1}^{m-1} f(y_j)\right\}. \quad (16.9)$$

Proof: See [94]. □

Remark 16.11: Following the suppositions of Theorem 16.20 and Theorem 16.21, if $m = 2$, then we get [154].

In the following theorems we prove Hermite–Hadamard inequalities of the Jensen–Mercer–type for one arbitrary and two monotonic tuples.

Theorem 16.22: *Suppose that f is a real valued convex function on I and $\mathbf{a} = (a_1, \ldots, a_m)$, $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$, $\mathbf{w} = (w_1, \ldots, w_m)$ are four $m-$ tuples such that $a_j, x_j, y_j \in I$, $x_j \neq y_j$, $w_j \geq 0$ with $w_m \neq 0$ for all $j \in \{1, \ldots, m\}$ and $\eta = \frac{1}{w_m}$. If $\mathbf{x}$ and $\mathbf{y}$ are decreasing m-tuples and*

$$\sum_{j=1}^{k} w_j x_j \leq \sum_{j=1}^{k} w_j a_j, \ \sum_{j=1}^{k} w_j y_j \leq \sum_{j=1}^{k} w_j a_j \ \textit{for } k = 1, \ldots, m-1,$$

$$\sum_{j=1}^{m} w_j a_j = \sum_{j=1}^{m} w_j x_j, \ \sum_{j=1}^{m} w_j a_j = \sum_{j=1}^{m} w_j y_j,$$

then

$$f\left(\sum_{j=1}^{m} \eta w_j a_j - \eta \sum_{j=1}^{m-1}\left(\frac{w_j x_j + w_j y_j}{2}\right)\right)$$
$$\leq \sum_{j=1}^{m} \eta w_j f(a_j) - \eta \sum_{j=1}^{m-1} \frac{w_j}{y_j - x_j} \int_{x_j}^{y_j} f(u) du$$
$$\leq \sum_{j=1}^{m} \eta w_j f(a_j) - \eta \sum_{j=1}^{m-1} w_j f\left(\frac{x_j+y_j}{2}\right). \quad (16.10)$$

Proof: See [94] □

Remark 16.12: In the above theorem, if $x_j = y_j$ for all $j \in \{1, 2, \ldots, m\}$ then by following the proof of Theorem 16.22, we obtain

$$f\left(\sum_{j=1}^{m} \eta w_j a_j - \eta \sum_{j=1}^{m-1} w_j x_j\right) \leq \sum_{j=1}^{m} \eta w_j f(a_j) - \eta \sum_{j=1}^{m-1} w_j f(x_j).$$

If $x_j = y_j$ for some j, then (16.10) reduces to the form

$$f\left(\sum_{j=1}^{m}\eta w_j a_j - \eta\sum_{j=1}^{m-1}\left(\frac{w_jx_j + w_jy_j}{2}\right)\right)$$
$$\leq \sum_{j=1}^{m}\eta w_j f(a_j) - \sum_{j\in I}\eta w_j f(x_j) - \eta\sum_{j\in I^c} w_j\frac{1}{y_j - x_j}\int_{x_j}^{y_j} f(u)du,$$

where $I = \{j \in \{1, 2, \ldots, m\} : x_j = y_j\}$ and $I^c = \{1, 2, \ldots, m\} \setminus I$.

Theorem 16.23: *If all the hypotheses of Theorem 16.22 are valid and $x_m \neq y_m$, then*

$$f\left(\sum_{j=1}^{m}\eta w_j a_j - \eta\sum_{j=1}^{m-1}\left(\frac{w_jx_j + w_jy_j}{2}\right)\right)$$
$$\leq \frac{1}{\sum\limits_{j=1}^{m-1}(\eta w_j y_j - \eta w_j x_j)}\int_{\sum\limits_{j=1}^{m-1}\eta w_j x_j}^{\sum\limits_{j=1}^{m-1}\eta w_j y_j} f\left(\sum_{j=1}^{m}\eta w_j a_j - u\right)du$$
$$\leq \sum_{j=1}^{m}\eta w_j f(a_j) - \frac{1}{2}\left\{\sum_{j=1}^{m-1}\eta w_j f(x_j) + \sum_{j=1}^{m-1}\eta w_j f(y_j)\right\}.$$

The following theorems give generalized Hermite–Hadamard–type inequalities by imposing relax condition on weights.

Theorem 16.24: *Suppose that f is a real valued convex function on I and $\mathbf{a} = (a_1, \ldots, a_m)$, $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$, $\mathbf{w} = (w_1, \ldots, w_m)$ are four $m-$ tuples such that $a_j, x_j, y_j \in I$, $x_j \neq y_j$, $w_j \geq 0$ with $w_m \neq 0$ for all $j \in \{1, \ldots, m\}$ and $\eta = \frac{1}{w_m}$. If $\mathbf{a}-\mathbf{x}$, $\mathbf{x}$, $\mathbf{a}-\mathbf{y}$ and $\mathbf{y}$ are monotonic in the same sense and $\sum\limits_{j=1}^{m} w_j a_j = \sum\limits_{j=1}^{m} w_j x_j$, $\sum\limits_{j=1}^{m} w_j a_j = \sum\limits_{j=1}^{m} w_j y_j$, then*

$$f\left(\sum_{j=1}^{m}\eta w_j a_j - \eta\sum_{j=1}^{m-1}\left(\frac{w_jx_j + w_jy_j}{2}\right)\right)$$
$$\leq \sum_{j=1}^{m}\eta w_j f(a_j) - \eta\sum_{j=1}^{m-1}\frac{w_j}{y_j - x_j}\int_{x_j}^{y_j} f(u)du \leq \sum_{j=1}^{m}\eta w_j f(a_j) - \eta\sum_{j=1}^{m-1} w_j f\left(\frac{x_j + y_j}{2}\right).$$

Theorem 16.25: *If all the hypotheses of Theorem 16.24 are valid and $x_m \neq y_m$, then*

$$f\left(\sum_{j=1}^{m}\eta w_j a_j - \eta\sum_{j=1}^{m-1}\left(\frac{w_jx_j + w_jy_j}{2}\right)\right)$$
$$\leq \frac{1}{\sum\limits_{j=1}^{m-1}(\eta w_j y_j - \eta w_j x_j)}\int_{\sum\limits_{j=1}^{m-1}\eta w_j x_j}^{\sum\limits_{j=1}^{m-1}\eta w_j y_j} f\left(\sum_{j=1}^{m}\eta w_j a_j - u\right)du$$
$$\leq \sum_{j=1}^{m}\eta w_j f(a_j) - \frac{1}{2}\left\{\sum_{j=1}^{m-1}\eta w_j f(x_j) + \sum_{j=1}^{m-1}\eta w_j f(y_j)\right\}.$$

Remark 16.13: It may be noted that Theorem 16.20 and Theorem 16.21 have been proved for three $m-$tuples without monotonicity conditions while Theorem 16.22 and Theorem 16.23 have been proved by considering a decreasing $m-$tuple with non-negative weights. Moreover, Theorem 16.24 and Theorem 16.25 have been proved for monotonic $m-$tuples with strict conditions of monotonicity and relax conditions on weights.

16.4 Unifications of Continuous and Discrete Fractional Hermite-Hadamard–Mercer Type Inequalities Via Majorization

There are multiple number of fractional operators but due to our interest we limit ourselves to the well-known Caputo fractional operators. Their definition is given as below.

Definition 16.3: Consider a function $f \in C^n[a,b]$ $\big($the space of functions whose n^{th}-derivative exist and continuous on $[a,b]\big)$, $\alpha > 0$, such that $\alpha \notin \{1,2,3,\dots\}$ and $n = [\alpha]+1$. Then the α ordered, Caputo fractional derivative operators are defined as [155]:

$$
\begin{aligned}
{}^cD^{\alpha}_{a+}f(z) &= \frac{1}{\Gamma(n-\alpha)}\int_a^z \frac{f^{(n)}(u)}{(z-u)^{\alpha-n+1}}du,\ z > a,\\
{}^cD^{\alpha}_{b-}f(z) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_z^b \frac{f^{(n)}(u)}{(u-z)^{\alpha-n+1}}du,\ z < b,
\end{aligned}
$$

where ${}^cD^{\alpha}_{a+}f(z)$ and ${}^cD^{\alpha}_{b-}f(z)$ stand for the left- and right-sided, Caputo fractional derivative operators respectively.

It may be noted that if the usual derivative of the function f of order n exists, then it coincides with ${}^cD^{\alpha}_{a+}f(z)$ for $\alpha = n \in \{1,2,3,\dots\}$. Also, for $n = 1$ and $\alpha = 0$, we have:

$$
\big({}^cD^{0}_{a+}f\big)(z) = \big({}^cD^{0}_{b-}f\big)(z) = f(z).
$$

The associated Hermite-Jensen–Mercer inequality in terms of Caputo fractional operators is defined as follows [245]:

Theorem 16.26: *Consider a function f defined on the interval $[a,b]$ such that $f \in C^n[a,b]$ and $f^{(n)}$ is convex on $[a,b]$ with $[x,y] \subset [a,b]$, $\alpha > 0$, then we have:*

$$
\begin{aligned}
f^{(n)}\left(a+b-\frac{x+y}{2}\right) &\le \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\Big\{\big({}^cD^{\alpha}_{\left(a+b-(x+y/2)\right)^+}f\big)(a+b-x)\\
&\quad +(-1)^n\big({}^cD^{\alpha}_{\left(a+b-(x+y/2)\right)^-}f\big)(a+b-y)\Big\} \le f^{(n)}(a)+f^{(n)}(b)-\frac{f^{(n)}(x)+f^{(n)}(y)}{2}. \quad (16.11)
\end{aligned}
$$

The inequality (9.50) can be obtained from (16.11) when $n = 1$, $\alpha = 0$, $x = a$ and $y = b$. Some more work related to Hermite-Jensen–Mercer inequalities via fractional operators can be traced in [96, 97, 192, 52, 244, 55].

Remark 16.14: For other related results, remarks, associated bounds, discrete and weighted versions we refer the reader to [55, 81, 95, 191, 222, 228, 243].

Remark 16.15: Further we see that some other results related to Hermite–Hadamard–Mercer and Fejér–Pachpatte–Mercer type can be found in [63, 58] [57, 53, 51, 50, 54, 93, 159, 200]. To be more specific:

1. If we replace $k = 1$ in the results of [54], then we capture the results of [53].
2. If we replace $k = 1$ in the results of [51], then we capture the results of [50].
3. The results obtained in [58] is related to fractional Hermite-Hadamard-Mercer type inequalities for Harmonic convex function.
4. The results obtained in [63] is related to Fejér−Pachpatte−Mercer type inequalities for Harmonically convex functions involving exponential functions.
5. The results obtained in [93] is related to Jensen–Mercer and Hermite–Hadamard–Mercer type inequalities for $GA - h-$convex functions.

Remark 16.16: The readers who are interested in latest work on Hermite–Hadamard–Jessen–Mercer and Hermite–Hadamard–Jensen–Mercer type inequalities for $h-$convex functions may study [128] and [80] respectively.

CHAPTER 17

New Estimates for Csiszár Divergence and Zipf–Mandelbrot Entropy Via Jensen–Mercer Inequality

This chapter applies Jensen–Mercer's inequality to derive new results in information theory. Using this approach, we establish bounds for Csiszár divergence and other divergence measures through appropriate convex functions. The discussion further presents bounds for Zipf-Mandelbrot entropy by employing the Zipf-Mandelbrot law in place of probability distributions within Kullback–Leibler and Jeffreys divergences. Additionally, new estimates for Zipf- Mandelbrot entropy are obtained for various parametric forms of the Zipf–Mandelbrot law, highlighting the utility of Mercer-type inequalities in information-theoretic contexts. The contents of this chapter are extracted from [149].

17.1 Divergence Measure

Divergence measure is basically the measure of distance between two probability distributions. The concept of divergence measure is working efficiently to resolve different problems related to probability theory. A few divergence measures are reasonable relying upon the idea of the issue. Recently, Dragomir [89, 88], Jain [119, 120] and Taneja [233] have made many contributions in this field, they introduced different divergence measures, obtained their bounds, presented relations with other divergences and discussed their properties.

Divergence measures have gigantic applications in assortment of fields, such as: approximation of probability distributions [78, 129], biology [218], economics and political science [235, 236], analysis of contingency tables [104], signal processing [126, 127], color image segmentation [195], pattern recognition [73, 125], magnetic resonance image analysis [238] etc.

Some specific operational facts about divergence measure have been introduced, characterized and applied in different fields, such as: Bregman's f-divergence [44], Csiszár's f-divergence [82], Burbea Rao's f-divergence [49] and Renyi's like f-divergence [223]. By suitable choice of the function f many divergence measures can be obtained from these generalized divergence. Because of its compact nature Csiszár f- divergence is one of the most important divergence which is given below:

$$I_f(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{17.1}$$

where $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$ are positive real n-tuples and $f : \mathbb{R}^+ \to \mathbb{R}$ is a convex function. Here, convexity of the function f ensures that $I_f(\boldsymbol{p},\boldsymbol{q}) \geq 0$.

By choosing appropriate convex function instead of function f, many well known divergences or distance functions can be acquired from (17.1), such as: Hellinger, Rényi, Bhattacharyya, Chi-square, Kullback-Leibler, triangular discrimination, Jeffery's divergences etc.

Zipf law is one of the mandatory laws in information science and is often utilized in linguistics as well. In 1932 George Zipf [246] found that one can tally how often each word shows up in the text. Therefore, if p is the rank of the word, and q is the frequency of occurrence of that word, then $q.p = c$, where c is a constant. There are several utilizations of Zipf law, which can be used in city populations [103], geology [177] and solar flare intensity [193]. For more details see [145, 178].

In 1966 Benoit Mandelbrot [163] gave generalization of Zipf law, known as Zipf-Mandelbrot law, which gave improvement on account of the low-rank words in corpus [187] which is given as $g(i) = \dfrac{c}{(i+\tau)^u}$, for $i < 1000$, $u, c > 0$ and $\tau \geq 0$. If $\tau = 0$, we get Zipf law.

If $n \in \mathbb{Z}^+$, $\tau \in \mathbb{R}^+ \cup \{0\}$, $u > 0$, $i \in \{1, 2, \ldots, n\}$ and $W_{n,\tau,u} = \sum_{i=1}^{n} \frac{1}{(i+\tau)^u}$, than the probability mass function for Zipf-Mandelbrot law is given by $f(i, n, \tau, u) = \dfrac{1/(i+\tau)^u}{W_{n,\tau,u}}$.

The formula for Zipf-Mandelbrot entropy is given below:

$$Z(W, \tau, u) = \frac{u}{W_{n,\tau,u}} \sum_{i=1}^{n} \frac{\log(i+\tau)}{(i+\tau)^u} + \log W_{n,\tau,u}.$$

17.2 Bounds for Csiszár and Related Divergences

Theorem 17.1: *Let $f : [a,b] \to \mathbb{R}$ be a convex function. If p_i and q_i for $i \in \{1, 2, \ldots, n\}$ are positive real numbers with $\frac{p_i}{q_i} \in [a,b]$ such that $\sum\limits_{i=1}^{n} p_i = P_n$ and $\sum\limits_{i=1}^{n} q_i = Q_n$, then*

$$f\left(a + b - \frac{P_n}{Q_n}\right) \leq f(a) + f(b) - \frac{1}{Q_n} I_f(\boldsymbol{p},\boldsymbol{q}). \tag{17.2}$$

Proof: Replace p_i by q_i and put $x_i = \frac{p_i}{q_i}$ in (1.3), we get (17.2). $\square$

Theorem 17.2: *Let a and b be positive real numbers with $a < b$. If p_i and q_i $(i \in \{1, 2, \ldots, n\})$ are positive real numbers with $\frac{q_i}{p_i} \in [a, b]$ such that $\sum\limits_{i=1}^{n} p_i = P_n$ and $\sum\limits_{i=1}^{n} q_i = Q_n$, then*

$$\log\left(a + b - \frac{Q_n}{P_n}\right)^{-1} \leq \log\left(\frac{1}{ab}\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right). \tag{17.3}$$

Proof: Let $f(x) = -\log x$, $x > 0$, so clearly $-\log x$ is a convex function. Therefore using (17.2) and interchange q_i and p_i, we get

$$-\log\left(a + b - \frac{Q_n}{P_n}\right) \leq -\log a - \log b + \frac{1}{P_n}\sum_{i=1}^{n} p_i \log\left(\frac{q_i}{p_i}\right).$$

Now using the definition of Kullback-Leibler divergence, we obtain (17.3). $\square$

Corollary 17.1: *Let a and b be positive real numbers with $a < b$. If p_i for $i \in \{1, 2, \ldots, n\}$ are positive real numbers such that $\frac{1}{p_i} \in [a, b]$ and $\sum\limits_{i=1}^{n} p_i = 1$, then*

$$\log(a + b - n)^{-1} \leq \log\frac{1}{ab} + \sum_{i=1}^{n} p_i \log\left(\frac{1}{p_i}\right). \tag{17.4}$$

Proof: By choosing $q_i = 1$, $i \in \{1, 2, \ldots, n\}$ in (17.3), we get (17.4). $\square$

Theorem 17.3: *Let a and b be positive real numbers with $a < b$. If p_i and q_i $(i \in \{1, 2, \ldots, n\})$ are positive real numbers such that $\frac{p_i}{q_i} \in [a, b]$ for $i \in \{1, 2, \ldots, n\}$, $\sum\limits_{i=1}^{n} p_i = P_n$ and $\sum\limits_{i=1}^{n} q_i = Q_n$, then*

$$\left(a + b - \frac{P_n}{Q_n}\right)\log\left(a + b - \frac{P_n}{Q_n}\right) \leq \log a^a b^b - \frac{1}{Q_n}\sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right). \tag{17.5}$$

Proof: Let $f(x) = x\log x$, $x > 0$, so clearly $x \log x$ is a convex function. Therefore using (17.2), we get (17.5). $\square$

Corollary 17.2: *Let a and b be positive real numbers with $a < b$. If p_i for $i \in \{1, 2, \ldots, n\}$ are positive real numbers such that $p_i \in [a, b]$ and $\sum\limits_{i=1}^{n} p_i = 1$, then*

$$\left(a + b - \frac{1}{n}\right)\log\left(a + b - \frac{1}{n}\right) \leq \log a^a b^b + \frac{1}{n}\sum_{i=1}^{n} p_i \log\left(\frac{1}{p_i}\right). \tag{17.6}$$

Proof: By choosing $q_i = 1, i \in \{1, 2, \ldots, n\}$ in (17.5), we obtain (17.6). $\square$

Theorem 17.4: *Let all the assumptions of Theorem 17.3 hold, then we have the following inequality*

$$-\sqrt{\left(a + b - \frac{P_n}{Q_n}\right)} \leq -\left(\sqrt{a} + \sqrt{b}\right) + \frac{1}{Q_n}\sum_{i=1}^{n} \sqrt{p_i q_i}. \tag{17.7}$$

Proof: *If $f(x) = -\sqrt{x}$, $x > 0$, then $f''(x) = \frac{1}{4x^{\frac{3}{2}}} > 0$, so $f(x)$ is a convex function for $x > 0$. Therefore using $f(x) = -\sqrt{x}$, in Theorem 17.1, we get (17.7).* □

Theorem 17.5: *Let all the assumptions of Theorem 17.3 hold, then we have the following inequality*

$$\left(a+b-\frac{P_n+Q_n}{Q_n}\right)\log\left(a+b-\frac{P_n}{Q_n}\right) \leq \log a^{a-1}b^{b-1} - \frac{1}{Q_n}\sum_{i=1}^{n}(p_i-q_i)\log\left(\frac{p_i}{q_i}\right). \tag{17.8}$$

Proof: *If $f(x) = (x-1)\log x$, $x > 0$, then $f''(x) = \frac{x+1}{x^2} > 0$, so $f(x)$ is a convex function for $x > 0$. Therefore using $f(x) = (x-1)\log x$, in Theorem 17.1, we get (17.8).* □

Corollary 17.3: *Let a and b be positive real numbers with $a < b$. If p_i for $i \in \{1, 2, \ldots, n\}$ are positive real numbers such that $p_i \in [a, b]$ and $\sum\limits_{i=1}^{n} p_i = 1$, then*

$$\left(a+b-\frac{1+n}{n}\right)\log\left(a+b-\frac{1}{n}\right) \leq \log a^{a-1}b^{b-1} + \frac{1}{n}\sum_{i=1}^{n}\log p_i + \frac{1}{n}\sum_{i=1}^{n}p_i\log\left(\frac{1}{p_i}\right). \tag{17.9}$$

Proof: By choosing $q_i = 1$, $i \in \{1, 2, \ldots, n\}$ in (17.8), we obtain

$$\left(a+b-\frac{1+n}{n}\right)\log\left(a+b-\frac{1}{n}\right) \leq \log a^{a-1}b^{b-1} + \frac{1}{n}\sum_{i=1}^{n}\log p_i - \frac{1}{n}\sum_{i=1}^{n}p_i\log p_i.$$

Now using the definition of Shannon entropy, we get (17.9). □

Theorem 17.6: *Let all the assumptions of Theorem 17.3 hold, then we have the following inequality*

$$\left(1-\sqrt{a+b-\frac{P_n}{Q_n}}\right)^2 \leq a^2+b^2+2(1-(\sqrt{a}+\sqrt{b})) - \frac{1}{Q_n}\sum_{i=1}^{n}(\sqrt{q_i}-\sqrt{p_i})^2. \tag{17.10}$$

Proof: If $f(x) = (1-\sqrt{x})^2$, $x > 0$, then $f''(x) = \frac{1}{2x} - \frac{\sqrt{x}-1}{2x^{\frac{3}{2}}} \geq 0$, so $f(x)$ is a convex function for $x > 0$. Therefore using $f(x) = (1-\sqrt{x})^2$, in Theorem 17.1, we get (17.10). □

Theorem 17.7: *Let all the assumptions of Theorem 17.3 hold. Then we have the following inequality for $s > 1$*

$$\left(a+b-\frac{P_n}{Q_n}\right)^s \leq a^s+b^s - \frac{1}{Q_n}\sum_{i=1}^{n}p_i^s q_i^{1-s}. \tag{17.11}$$

Proof: For $s > 1$, the function $f(x) = x^s$, $x > 0$, is clearly a convex function. Therefore using $f(x) = x^s$, in Theorem 17.1, we get (17.11). □

Theorem 17.8: *Let all the assumptions of Theorem 17.3 hold. Then we have the following inequality*

$$\left|a+b-\frac{P_n+Q_n}{Q_n}\right| \leq |a-1|+|b-1| - \frac{1}{Q_n}\sum_{i=1}^{n}|p_i-q_i|. \tag{17.12}$$

Proof: If $f(x) = |x-1|$, $x \in \mathbb{R}$, then we know that absolute function is always convex on $\mathbb{R}$.

Therefore using $f(x) = |x-1|$, in Theorem 17.1, we get (17.12). $\square$

Theorem 17.9: *Let all the assumptions of Theorem 17.3 hold, Then we have the following inequality*

$$\left(a+b-\frac{P_n+Q_n}{Q_n}\right)^2 \leq a^2+b^2-2(a+b-1)-\frac{1}{Q_n}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i}. \tag{17.13}$$

Proof: If $f(x) = (x-1)^2$, $x>0$, then $f''(x) = 2 > 0$, so $f(z)$ is a convex function for $x>0$. Therefore using $f(x) = (x-1)^2$, in Theorem 17.1, we get (17.13). $\square$

Theorem 17.10: *Let all the assumptions of Theorem 17.3 hold, Then we have the following inequality*

$$\frac{\left(a+b-\frac{1}{Q_n}(P_n+Q_n)\right)^2}{\left(a+b-\frac{1}{Q_n}(P_n-Q_n)\right)} \leq \frac{(a-1)^2}{(a+1)}+\frac{(b-1)^2}{(b+1)}-\frac{1}{Q_n}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{(p_i+q_i)}. \tag{17.14}$$

Proof: If $f(x) = \frac{(x-1)^2}{(x+1)}$, $x>0$, then $f''(x) = \frac{8}{(x+1)^3} \geq 0$, so $f(x)$ is a convex function for $x>0$. Therefore using $f(x) = \frac{(x-1)^2}{(x+1)}$, in Theorem 17.1, we get (17.14). $\square$

Theorem 17.11: *Let all the assumptions of Theorem 17.3 hold, Then we have the following inequality*

$$\begin{aligned}\frac{1}{2}\left(a+b-\frac{1}{Q_n}(P_n-Q_n)\right)\log\left(\frac{a+b-\frac{1}{Q_n}(P_n-Q_n)}{2(a+b-\frac{P_n}{Q_n})}\right)&\\ \leq \frac{a+1}{2}\log\frac{1+a}{2a}+\frac{b+1}{2}\log\frac{1+b}{2b}-\frac{1}{Q_n}\sum_{i=1}^{n}\frac{p_i+q_i}{2}\log\left(\frac{p_i+q_i}{2p_i}\right).&\end{aligned} \tag{17.15}$$

Proof: If $f(x)=\frac{x+1}{2}\log\frac{1+x}{2x}$, $x>0$, then $f''(x) = \frac{1}{2x^2(x+1)} > 0$, so $f(x)$ is a convex function for $x>0$. Therefore using $f(x) = \frac{(x-1)^2}{(x+1)}$, in Theorem 17.1, we get (17.15). $\square$

17.3 Bounds for Zipf–Mandelbrot Entropy

In this section we present some bounds for Zipf–Mandelbrot entropy.

Theorem 17.12: *Let a and b be positive real numbers with $a<b$. If $n \in \{1,2,3,\ldots\}$, $\tau \geq 0$, $u>0$, $q_i>0$, $i \in \{1,2,\ldots,n\}$ such that $q_i(i+\tau)^u W_{n,\tau,u} \in [a,b]$ and $\sum_{i=1}^{n} q_i = Q_n$, then*

$$\log(a+b-Q_n)^{-1} \leq \log\left(\frac{1}{ab}\right)+\sum_{i=1}^{n}\frac{\log q_i}{(i+\tau)^u W_{n,\tau,u}}+Z(W,\tau,u). \tag{17.16}$$

Proof: Let $p_i = \frac{1}{(i+\tau)^u W_{n,\tau,u}}$, $i \in \{1,2,\ldots,n\}$, then

$$\begin{aligned}\sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} &= \sum_{i=1}^{n} \frac{1}{(i+\tau)^u W_{n,\tau,u}} \log\left(\left(\frac{1}{q_i}\right)\left(\frac{1}{(i+\tau)^u W_{n,\tau,u}}\right)\right)\\ &= -\sum_{i=1}^{n} \frac{\log q_i}{(i+\tau)^u W_{n,\tau,u}} - \sum_{i=1}^{n} \frac{\log\left((i+\tau)^u W_{n,\tau,u}\right)}{(i+\tau)^u W_{n,\tau,u}}\\ &= -\sum_{i=1}^{n} \frac{\log q_i}{(i+\tau)^u W_{n,\tau,u}} - \sum_{i=1}^{n} \frac{u}{(i+\tau)^u W_{n,\tau,u}} \log(i+\tau) - \sum_{i=1}^{n} \frac{\log W_{n,\tau,u}}{(i+\tau)^u W_{n,\tau,u}}\\ &= -\sum_{i=1}^{n} \frac{\log q_i}{(i+\tau)^u W_{n,\tau,u}} - Z(W,\tau,u).\end{aligned}$$

As $W_{n,\tau,u} = \sum_{i=1}^{n} \frac{1}{(i+\tau)^u}$, therefore $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \frac{1}{(i+\tau)^u W_{n,\tau,u}} = 1$. Hence, using (17.3) for $p_i = \frac{1}{(i+\tau)^u W_{n,\tau,u}}, i \in \{1, 2, \ldots, n\}$, we obtain (17.16). □

The following corollary is the special case of Theorem 17.12.

Corollary 17.4: *Let a and b be positive real numbers with $a < b$. If $n \in \{1, 2, 3, \ldots\}$, $\tau \geq 0$, $u > 0$ and $(i+\tau)^u W_{n,\tau,u} \in [a, b]$, then*

$$\log(a+b-n)^{-1} \leq \log\left(\frac{1}{ab}\right) + Z(W,\tau,u). \tag{17.17}$$

Proof: By choosing $q_i = 1, i \in \{1, 2, \ldots, n\}$ in (17.16), we obtain (17.17). □

The next result gives the inequality for Zipf–Mandelbrot entropy using two different parameters.

Theorem 17.13: *Let a and b be positive real numbers with $a < b$. Let $n \in \{1, 2, \ldots, \}$ $\tau_1, \tau_2 \geq 0$, $u_1, u_2 > 0$ and $\left((i+\tau_1)^{u_1} W_{n,\tau_1,u_1}\right) / \left((i+\tau_2)^{u_2} W_{n,\tau_2,u_2}\right) \in [a, b]$, then*

$$\log(a+b-1)^{-1} \leq \log\left(\frac{1}{ab}\right) + Z(W,\tau_1,u_1) - \sum_{i=1}^{n} \frac{\log(i+\tau_2)^{u_2} W_{n,\tau_2,u_2}}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}}. \tag{17.18}$$

Proof: Let $p_i = \frac{1}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}}$ and $q_i = \frac{1}{(i+\tau_2)^{u_2} W_{n,\tau_2,u_2}}$, $i \in \{1, 2, \ldots, n\}$, then

$$\begin{aligned}\sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} &= \sum_{i=1}^{n} \frac{1}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}} \log\left(\frac{(i+\tau_2)^{u_2} W_{n,\tau_2,u_2}}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}}\right)\\ &= -\sum_{i=1}^{n} \frac{\log(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}} + \sum_{i=1}^{n} \frac{\log(i+\tau_2)^{u_2} W_{n,\tau_2,u_2}}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}}\\ &= -Z(W,\tau_1,u_1) + \sum_{i=1}^{n} \frac{\log(i+\tau_2)^{u_2} W_{n,\tau_2,u_2}}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}}.\end{aligned}$$

Also $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \frac{1}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}} = 1$ and $\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} \frac{1}{(i+\tau_2)^{u_2} W_{n,\tau_2,u_2}} = 1$, therefore, using (17.3) for $p_i = \frac{1}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}}$ and $q_i = \frac{1}{(i+\tau_2)^{u_2} W_{n,\tau_2,u_2}}$, $i \in \{1, 2, \ldots, n\}$, we get (17.18). □

Theorem 17.14: *Let a and b be positive real numbers with $a < b$. If $n \in \{1, 2, 3, \ldots\}$, $\tau \geq 0$,*

$u > 0$, $q_i > 0$, $i \in \{1, 2, \ldots, n\}$ *such that* $\frac{1}{q_i(i+\tau)^u W_{n,\tau,u}} \in [a, b]$ *and* $\sum_{i=1}^{n} q_i = Q_n$, *then*

$$\left(a+b-\frac{1}{Q_n}\right)\log\left(a+b-\frac{1}{Q_n}\right) \le \log a^a b^b + \frac{1}{Q_n}\sum_{i=1}^{n}\frac{\log q_i}{(i+\tau)^u W_{n,\tau,u}} + \frac{1}{Q_n}Z(W,\tau,u). \quad (17.19)$$

Proof: Put $p_i = \frac{1}{(i+\tau)^u W_{n,\tau,u}}$, $i \in \{1, 2, \ldots, n\}$, in (17.5), and using similar method which we use in the proof Theorem 17.12, we get (17.19). □

The following corollary is the special case of Theorem 17.14.

Corollary 17.5: *Let a and b be positive real numbers with $a < b$. If $n \in \{1, 2, 3, \ldots\}$, $\tau \ge 0$, $u > 0$ and $\frac{1}{(i+\tau)^u W_{n,\tau,u}} \in [a, b]$, then*

$$\left(a+b-\frac{1}{n}\right)\log\left(a+b-\frac{1}{n}\right) \le \log a^a b^b + \frac{1}{n}Z(W,\tau,u). \quad (17.20)$$

Proof: By choosing $q_i = 1, i \in \{1, 2, \ldots, n\}$ in (17.19), we obtain (17.20). □

The next result gives the inequality for Zipf–Mandelbrot entropy using two different parameters.

Theorem 17.15: *Let a and b be positive real numbers with $a < b$. Let $n \in \{1, 2, \ldots, \}$ $\tau_1, \tau_2 \ge 0$, $u_1, u_2 > 0$ and $((i+\tau_2)^{u_2} W_{n,\tau_2,u_2}) \,/\, ((i+\tau_1)^{u_1} W_{n,\tau_1,u_1}) \in [a, b]$, then*

$$(a+b-1)\log(a+b-1) \le \log a^a b^b + Z(W,\tau_1,u_2) - \sum_{i=1}^{n}\frac{\log(i+\tau_2)^{u_2} W_{n,\tau_2,u_2}}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}}. \quad (17.21)$$

Proof: Let $p_i = \frac{1}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}}$ and $q_i = \frac{1}{(i+\tau_2)^{u_2} W_{n,\tau_2,u_2}}$, $i \in \{1, 2, \ldots, n\}$, in (17.5), and using similar method which we use in the proof Theorem 17.13, we get (17.21). □

Theorem 17.16: *Let all the assumptions of Theorem 17.14 hold, then we have the following inequality:*

$$\begin{aligned}
&\left(a+b-\frac{1+Q_n}{Q_n}\right)\log\left(a+b-\frac{1}{Q_n}\right)\\
&\le \log a^{a-1}b^{b-1} + \frac{1}{Q_n}\left[\sum_{i=1}^{n}\frac{\log q_i}{(i+\tau)^u W_{n,\tau,u}} + Z(W,\tau,u)\right.\\
&\left.-\sum_{i=1}^{n} q_i \log q_i - u\sum_{i=1}^{n} q_i \log(i+\tau)\right] - \log W_{n,\tau,u}.
\end{aligned} \quad (17.22)$$

Proof: Let $p_i = \frac{1}{(i+\tau)^u W_{n,\tau,u}}$, $i \in \{1, 2, \ldots, n\}$, then

$$\sum_{i=1}^{n} p_i \log\frac{p_i}{q_i} = -\sum_{i=1}^{n}\frac{\log q_i}{(i+\tau)^u W_{n,\tau,u}} - Z(W,\tau,u), \quad (17.23)$$

$$\begin{aligned}
\sum_{i=1}^{n} q_i \log\frac{p_i}{q_i} &= -\sum_{i=1}^{n} q_i \log q_i (i+\tau) W_{n,\tau,u}\\
&= -\sum_{i=1}^{n} q_i \log q_i - u\sum_{i=1}^{n} q_i \log(i+\tau) - Q_n \log W_{n,\tau,u}.
\end{aligned} \quad (17.24)$$

Also $\sum\limits_{i=1}^{n} p_i = \sum\limits_{i=1}^{n} \frac{1}{(i+\tau)^u W_{n,\tau,u}} = 1$, therefore, using (17.23) and (17.24) in (17.8) for $p_i = \frac{1}{(i+\tau)^u W_{n,\tau,u}}$, $i \in \{1,2,\ldots,n\}$, we get (17.22). □

The following corollary is the particular case of Theorem 17.16.

Corollary 17.6: *Let a and b be positive real numbers with $a < b$. If $n \in \{1,2,3,\ldots\}$, $\tau \geq 0$, $u > 0$ and $\frac{1}{(i+\tau)^u W_{n,\tau,u}} \in [a,b]$. Then*

$$\left(a+b-\frac{1+n}{n}\right)\log\left(a+b-\frac{1}{n}\right) \leq \log a^{a-1}b^{b-1} + \frac{1}{n}\left[Z(W,\tau,u) - u\sum_{i=1}^{n}\log(i+\tau)\right] - \log W_{n,\tau,u}. \quad (17.25)$$

Proof: By choosing $q_i = 1, i \in \{1,2,\ldots,n\}$ in (17.22), we obtain (17.25). □

Theorem 17.17: *Let all the assumptions of Theorem 17.15 hold. Then we have the following inequality*

$$(a+b-2)\log(a+b-1) \leq \log a^{a-1}b^{b-1} + Z(W,\tau_1,u_1) - Z(W,\tau_2,u_2) + \sum_{i=1}^{n}\left[\frac{\log(i+\tau_1)^{u_1}W_{n,\tau_1,u_1}}{(i+\tau_2)^{u_2}W_{n,\tau_2,u_2}} - \frac{\log(i+\tau_2)^{u_2}W_{n,\tau_2,u_2}}{(i+\tau_1)^{u_1}W_{n,\tau_1,u_1}}\right]. \quad (17.26)$$

Proof: Let $p_i = \frac{1}{(i+\tau_1)^{u_1} W_{n,\tau_1,u_1}}$ and $q_i = \frac{1}{(i+\tau_2)^{u_2} W_{n,\tau_2,u_2}}$, $i \in \{1,2,\ldots,n\}$, then

$$\sum_{i=1}^{n} p_i \log\frac{p_i}{q_i} = -Z(W,\tau_1,u_1) + \sum_{i=1}^{n}\frac{\log(i+\tau_2)^{u_2}W_{n,\tau_2,u_2}}{(i+\tau_1)^{u_1}W_{n,\tau_1,u_1}}. \quad (17.27)$$

And

$$\begin{aligned}\sum_{i=1}^{n} q_i \log\frac{p_i}{q_i} &= \sum_{i=1}^{n}\frac{1}{(i+\tau_2)^{u_2}W_{n,\tau_2,u_2}}\log\left(\frac{(i+\tau_2)^{u_2}W_{n,\tau_2,u_2}}{(i+\tau_1)^{u_1}W_{n,\tau_1,u_1}}\right)\\ &= \sum_{i=1}^{n}\frac{\log(i+\tau_2)^{u_2}W_{n,\tau_2,u_2}}{(i+\tau_2)^{u_2}W_{n,\tau_2,u_2}} - \sum_{i=1}^{n}\frac{\log(i+\tau_2)^{u_2}W_{n,\tau_2,u_2}}{(i+\tau_1)^{u_1}W_{n,\tau_1,u_1}}\\ &= Z(W,\tau_2,u_2) - \sum_{i=1}^{n}\frac{\log(i+\tau_1)^{u_1}W_{n,\tau_1,u_1}}{(i+\tau_2)^{u_2}W_{n,\tau_2,u_2}}.\end{aligned}$$

Also $\sum\limits_{i=1}^{n} p_i = \sum\limits_{i=1}^{n}\frac{1}{(i+\tau_1)^{u_1}W_{n,\tau_1,u_1}} = 1$ and $\sum\limits_{i=1}^{n} q_i = \sum\limits_{i=1}^{n}\frac{1}{(i+\tau_2)^{u_2}W_{n,\tau_2,u_2}} = 1$, therefore, using (17.27) and (17.28) in (17.8) for $p_i = \frac{1}{(i+\tau_1)^{u_1}W_{n,\tau_1,u_1}}$ and $q_i = \frac{1}{(i+\tau_2)^{u_2}W_{n,\tau_2,u_2}}$, $i \in \{1,2,\ldots,n\}$, we get (17.26). □

CHAPTER 18

Inequalities of Simpson–Mercer Type Including Atangana–Baleanu Fractional Operators and their Applications

This chapter presents Simpson–Mercer type inequalities and their applications, including extensions involving Atangana–Baleanu fractional operators. We begin by reviewing s-convex functions in the first and second senses and related foundational results. The discussion then develops Simpson–Mercer type inequalities, introduces the Atangana–Baleanu derivative operator, and explores applications involving modified Bessel functions, special means, and the q-digamma function. These results illustrate the versatility of Mercer-type refinements in fractional calculus and special function contexts. The content of this chapter is extracted from [234].

18.1 Simpson–Mercer Type Inequalities

Hudzik et al. [115], introduced the notion of s-convex functions in the first and second sense and also discussed some of their notable features.

sectionSome Basic Results and Definitions

Definition 18.1: Let $s \in (0,1]$ and f be a real valued function on an interval $\mathcal{I} = [0,\infty)$. Then f is said to be s-convex in the first sense if

$$f(w_1x_1 + w_2x_2) \leq w_1^s f(x_1) + w_2^s f(x_2),$$

holds true for all $x_1, x_2 \in \mathcal{I}$, $w_1, w_2 \geq 0$ with $w_1^s + w_2^s = 1$.

Definition 18.2: Let $s \in (0,1]$ and f be a real valued function on an interval $\mathcal{I} = [0,\infty)$. Then f is said to be s-convex in the second sense if

$$f(w_1x_1 + w_2x_2) \leq w_1^s f(x_1) + w_2^s \phi(x_2),$$

holds true for all $x_1, x_2 \in \mathcal{I}$, $w_1, w_2 \geq 0$ with $w_1 + w_2 = 1$.

The class of s-convex function in the first and second sense are denoted by ($f \in K_s^1$) and ($f \in K_s^2$) respectively. Moreover, they also studied that the class of s-convex function in the second sense is more stronger than the s-convexity in first sense for $s \in (0,1)$. Several features of s-convex function in both senses are discussed taking some examples into consideration. It is interesting to see that if $s \in (0,1)$ and $f \in K_s^2$, then f is non negative.

For more detail on s-convex function (see [115, 75, 48]) and references cited therein. Chen [75] gave a nice relationship between convex functions and s-convex functions as:

Lemma 18.1: *Suppose $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ is a convex function. Then*
(i) If f is non-negative, then it is s-convex for $s \in (0,1]$.
(ii) If f is non-positive, then it is s-convex for $s \in [1,\infty)$.

Butt et al. [56] investigated the J-M inequality for s-convex functions in Breckner sense for $s > 0$ as follows:

Theorem 18.1: *Let $w_1, w_2, ..., w_n$ be positive probability distributions and $\sum\limits_{i=1}^{n} w_i^s \leq 1$. If $f : [\mathrm{a}, \mathrm{b}] \subset (0,\infty) \to \mathbb{R}$ is s-Breckner sense convex function, then the following inequality is valid*

$$f\left(\mathrm{a}+\mathrm{b}-\sum_{i=1}^{n} w_i x_i\right) \leq f(\mathrm{a}) + f(\mathrm{b}) - \sum_{i=1}^{n} w_i^s f(x_i), \tag{18.1}$$

for any increasing sequence $\{x_i\}_{i=1}^{n} \in [\mathrm{a}, \mathrm{b}]$.

The following conditions are necessary:

(i) By considering $s = 1$ in (18.1), we get the Jensen–Mercer inequality (1.3).

(ii) By considering $s = 1$ and $w_1 = \mathrm{a}$ and $w_2 = \mathrm{b}$, in (18.1) it gives the definition of the classical convex function.

The following inequality named as Simpson's inequality, is given in the literature as (see [19, 227, 74]).

$$\left|\frac{1}{3}\left\{\frac{f(\mathrm{a}) + f(\mathrm{b})}{2} + 2f\left(\frac{\mathrm{a}+\mathrm{b}}{2}\right) - \frac{1}{\mathrm{b}-\mathrm{a}}\int_{\mathrm{a}}^{\mathrm{b}} f(x)\,dx\right\}\right| \leq \frac{1}{2880}\left|\left|f^4\right|\right|_{\infty}(\mathrm{b}-\mathrm{a})^4,$$

where $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ is a four times continuously differentiable mapping on (a, b) and $\left|\left|f^4\right|\right|_{\infty} = \sup_{x \in (\mathrm{a},\mathrm{b})}\left|f^4(x)\right| < \infty$.

The introduction of new notions of fractional operators is a journey to succeed momentum to the fractional calculus and to gain the most efficient operators to the discussion. Now, we recall the notion of the Caputo-Fabrizio fractional operator:

Definition 18.3: Let $p \in [1,\infty)$ and $[x_1, x_2]$ be an open subset of $\mathbb{R}$, the Sobolev space $\mathcal{H}^p(x_1, x_2)$ is defined by

$$\mathcal{H}^p(x_1,x_2) = \left\{ f \in \mathcal{L}^2(x_1,x_2) \ : \ D^\mu f \in \mathcal{L}^2(x_1,x_2), \quad for \ \ all \ |\mu| \leq p \right\}.$$

Definition 18.4: [67] Let $f \in \mathcal{H}^1(0,t)$, $t > \omega$, $\alpha \in [0,1]$ then, the definition of the new Caputo fractional derivative is given as:

$$^{CF}D^\alpha f(t) = \frac{\mathcal{M}(\alpha)}{1-\alpha} \int_\omega^t f'(s) \exp\left[-\frac{\alpha}{(1-\alpha)}(t-s) \right] ds,$$

where $\mathcal{M}(\alpha)$ is normalization function with $\mathcal{M}(0) = \mathcal{M}(1) = 1$.

Moreover, the corresponding Caputo-Fabrizio fractional integral operator is given as:

Definition 18.5: [2] Let $f \in \mathcal{H}^1(0,t)$, $t > \omega$, $\alpha \in [0,1]$. Then, the definition of Caputo-Fabrizio fractional integrals are given as:

$$\left({}^{CF}_{\ \omega}\mathcal{I}^\alpha f \right)(t) = \frac{1-\alpha}{\mathcal{M}(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \int_\omega^t f(y)dy,$$

and

$$\left({}^{CF}\mathcal{I}^\alpha_t f \right)(t) = \frac{1-\alpha}{\mathcal{M}(\alpha)} f(t) + \frac{\alpha}{\mathcal{M}(\alpha)} \int_t^t f(y)dy,$$

where $\mathcal{M}(\alpha)$ is normalization function with $\mathcal{M}(0) = \mathcal{M}(1) = 1$.

Recently, Atangana & Baleanu introduced a new fractional operator containing the Mittag-Leffler function in the kernel, that solves the problem of retrieving the original function (a clear advantage over the Caputo-Fabrizio operator). Mittag-Leffler's function is more suitable than a power law in modeling nature and physical phenomena. This attribute has made the A-B operator more effective and helpful. As a result, many researchers have shown keen interest in utilizing this operator. Atangana & Baleanu introduced the derivative operator both in Caputo and Riemann-Liouville sense:

Definition 18.6: [23] Let $t > \omega$, $\alpha \in [0,1]$ and $f \in \mathcal{H}^1(\omega, t)$. The new fractional derivative is given as:

$$^{ABC}_{\ \ \omega}D^\alpha_t \left[f(t) \right] = \frac{\mathcal{M}(\alpha)}{1-\alpha} \int_\omega^t f'(x) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{(1-\alpha)} \right] dx,$$

where E_α is the Mittag-Leffler function and is defined as

$$E_\alpha(-t^\alpha) = \sum_{k=0}^{\infty} \frac{(-t)^{\alpha k}}{\Gamma(\alpha k + 1)}.$$

Definition 18.7: [23] Let $f \in \mathcal{H}^1(\omega, t)$, $\omega > t$, $\alpha \in [0,1]$. The new fractional derivative is given as:

$$^{ABR}_{\ \ \omega}D^\alpha_t \left[f(t) \right] = \frac{\mathcal{M}(\alpha)}{1-\alpha} \frac{d}{dt} \int_\omega^t f(x) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{(1-\alpha)} \right] dx,$$

where E_α is the Mittag-Leffler function.

However, in the same paper the authors also presented corresponding A-B fractional integral operator as:

Definition 18.8: [23] The fractional integral operator with non-local kernel of a function $f \in \mathcal{H}^1(\omega, t)$ is defined as:

$$ {}^{AB}_{\omega}\mathcal{I}_t^{\alpha}\left\{f(t)\right\} = \frac{1-\alpha}{\mathcal{M}(\alpha)}f(t) + \frac{\alpha}{\mathcal{M}(\alpha)\Gamma(\alpha)}\int_{\omega}^{t} f(y)(t-y)^{\alpha-1}dy, $$

where $t > \omega, \alpha \in [0, 1]$.

In [3], the right hand side of A-B fractional integral operator is presented as follows:

$$ {}^{AB}_{t}\mathcal{I}_t^{\alpha}\left\{f(t)\right\} = \frac{1-\alpha}{\mathcal{M}(\alpha)}f(t) + \frac{\alpha}{\mathcal{M}(\alpha)\Gamma(\alpha)}\int_{t}^{t} f(y)(y-t)^{\alpha-1}dy. $$

Here, $\Gamma(\alpha)$ is the Gamma function. The positivity of the normalization function $\mathcal{M}(\alpha)$ implies that the fractional A-B integral of a positive function is positive. It is worth noticing that the case when the order $\alpha \to 1$ yields the classical integral and the case when $\alpha \to 0$ provides the initial function.

In this section, we study several Simpson-Mercer's like inequalities via the A-B fractional integral operator for differentiable functions on (a, b). For this, we establish a new A-B integral identity that will serve as an auxiliary result to produce subsequent results for improvements. Throughout the paper, let us assume that $\Omega := \left(\frac{1+t}{2}\right)^s + \left(\frac{1-t}{2}\right)^s \leq 1$, for $t \in [0, 1]$.

Lemma 18.2: *Suppose the mapping $f : I = [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ is differentiable on (a, b) with $\mathrm{b} > \mathrm{a}$. If $f' \in \mathcal{L}[\mathrm{a}, \mathrm{b}]$, then for all $x_1, x_2 \in [\mathrm{a}, \mathrm{b}]$ and $\alpha > 0$, the following A-B fractional identity of Simpson-Mercer type holds true:*

$$ \begin{aligned} &\frac{1}{6}\{f(\mathrm{a}+\mathrm{b}-x_1) + f(\mathrm{a}+\mathrm{b}-x_2)\} + \frac{2}{3}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right) \\ &+ \frac{2^{\alpha}(1-\alpha)\Gamma(\alpha)}{(x_2-x_1)^{\alpha}}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right) \\ &- \frac{2^{\alpha-1}\mathcal{M}(\alpha)\Gamma(\alpha)}{(x_2-x_1)^{\alpha}}\left\{ {}^{AB}_{(\mathrm{a}+\mathrm{b}-x_1)}\mathcal{I}^{\alpha}_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)} f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right. \\ &\left. + {}^{AB}_{(\mathrm{a}+\mathrm{b}-x_2)}\mathcal{I}^{\alpha}_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)} f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right\} \\ &= \frac{x_2-x_1}{2}\left[\int_0^1 \left(\frac{t^{\alpha}}{2}-\frac{1}{3}\right) f'\left(\mathrm{a}+\mathrm{b}-\left[\frac{1+t}{2}x_1+\frac{1-t}{2}x_2\right]\right)dt\right. \\ &\left. + \int_0^1 \left(\frac{1}{3}-\frac{t^{\alpha}}{2}\right) f'\left(\mathrm{a}+\mathrm{b}-\left[\frac{1-t}{2}x_1+\frac{1+t}{2}x_2\right]\right)dt\right], \end{aligned} $$

for $t \in [0, 1]$.

Theorem 18.2: *Taking all the conditions as defined in Lemma 18.2 and assumption Ω into consideration and let $|f'|$ be s-convex function on $[\mathrm{a},\mathrm{b}]$, for $s>0$. Then, for all $\alpha>0$, the following Simpson-Mercer type inequality for A-B fractional integral holds true for $t\in[0,1]$:*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\{f(\mathrm{a}+\mathrm{b}-x_1)+f(\mathrm{a}+\mathrm{b}-x_2)\}+\frac{2}{3}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\\
&+\frac{2^{\alpha}(1-\alpha)\Gamma(\alpha)}{(x_2-x_1)^{\alpha}}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\\
&-\frac{2^{\alpha-1}\mathcal{M}(\alpha)\Gamma(\alpha)}{(x_2-x_1)^{\alpha}}\left\{{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_1)}\mathcal{I}^{\alpha}_{\left(\mathrm{a}+\mathrm{b}-\frac{\omega_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.\\
&\left.+{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_2)}\mathcal{I}^{\alpha}_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right\}\Bigg|\\
&\leq\frac{x_2-x_1}{2}\Bigg[2\Bigg\{\left(\frac{2}{3}\right)^{1+\frac{1}{\alpha}}\left(\frac{\alpha}{\alpha+1}\right)+\frac{1-2\alpha}{6(\alpha+1)}\Bigg\}\Big[|f'(\mathrm{a})|+|f'(\mathrm{b})|\Big]\\
&-\frac{1}{2^s}\Big[|f'(x_1)|+|f'(x_2)|\Big]\Big\{\mathcal{J}_3(\alpha)+\mathcal{J}_4(\alpha)\Big\}\Bigg],
\end{aligned}$$

where

$$\mathcal{J}_3(\alpha)=\int_0^{\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{3}-\frac{t^{\alpha}}{2}\right)[(1+t)^s+(1-t)^s]dt,$$

and

$$\mathcal{J}_4(\alpha)=\int_{\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{t^{\alpha}}{2}-\frac{1}{3}\right)[(1+t)^s+(1-t)^s]dt.$$

Remark 18.1: If we consider $x_1=\mathrm{a}$ and $x_2=\mathrm{b}$ and $\alpha=1$ in Theorem 18.2, we obtain Theorem 7 in [227].

Corollary 18.1: *Taking the same assumptions as defined in Theorem 18.2 with $s=1$ into consideration, we get the following Simpson-Mercer type inequality*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\{f(\mathrm{a}+\mathrm{b}-x_1)+f(\mathrm{a}+\mathrm{b}-x_2)\}+\frac{2}{3}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\\
&+\frac{2^{\alpha}(1-\alpha)\Gamma(\alpha)}{(x_2-x_1)^{\alpha}}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\\
&-\frac{2^{\alpha-1}\mathcal{M}(\alpha)\Gamma(\alpha)}{(x_2-x_1)^{\alpha}}\left\{{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_1)}\mathcal{I}^{\alpha}_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.\\
&\left.+{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_2)}\mathcal{I}^{\alpha}_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right\}\Bigg|\\
&\leq\frac{x_2-x_1}{2}\Bigg[2\Bigg\{\left(\frac{2}{3}\right)^{1+\frac{1}{\alpha}}\left(\frac{\alpha}{\alpha+1}\right)+\frac{1-2\alpha}{6(\alpha+1)}\Bigg\}\Big[|f'(\mathrm{a})|+|f'(\mathrm{b})|\Big]\\
&-\frac{1}{2}\Big[|f'(x_1)|+|f'(x_2)|\Big]\Big\{\mathcal{J}_3(\alpha)+\mathcal{J}_4(\alpha)\Big\}\Bigg].
\end{aligned}$$

Corollary 18.2: *If we consider $\alpha=1$ in Theorem 18.2, then we have the following Simpson-Mercer type inequality*

$$\left|\frac{1}{6}\left\{f(\mathrm{a}+\mathrm{b}-x_1)+4f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)+f(\mathrm{a}+\mathrm{b}-x_2)\right\}-\frac{1}{x_2-x_1}\int_{\mathrm{a}+\mathrm{b}-x_2}^{\mathrm{a}+\mathrm{b}-x_1}f(x)\,dx\right|$$

$$\leq \frac{x_2-x_1}{2}\left[\frac{5}{18}\left(\left|f'(\mathrm{a})\right|+\left|f'(\mathrm{b})\right|\right)-\frac{1}{2^s}\left(\left|f'(x_1)\right|+\left|f'(x_2)\right|\right)\right.$$
$$\left\{\frac{1}{6(s+1)}\left(\frac{5^{s+2}-1}{3^{s+1}}-6\right)-\frac{1}{2(s+2)}\left(\frac{5^{s+2}+1}{3^{s+2}}-2\right)\right.$$
$$\left.\left.+\ \frac{5^{s+2}+1}{2(s+1)(s+2)3^{s+2}}-\frac{2^s(2s+7)}{3(s+1)(s+2)}\right\}\right].$$

In the results to follow hereon, we will give the new upper bounds for the right hand side of the Simpson-Mercer type inequality via s-convexity involving A-B fractional integrals operator.

Theorem 18.3: *Taking all the conditions as defined in Lemma 18.2 and assumption Ω into consideration, if $|f'|^q$ is s-convex function on $[\mathrm{a},\mathrm{b}]$, for $s>0$, then for all $\alpha>0$, the following inequality for A-B fractional integral holds true for $t\in[0,1]$:*

$$\left|\frac{1}{6}\{f(\mathrm{a}+\mathrm{b}-x_1)+f(\mathrm{a}+\mathrm{b}-x_2)\}+\frac{2}{3}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.$$
$$+\frac{2^\alpha(1-\alpha)\Gamma(\alpha)}{(x_2-x_1)^\alpha}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)$$
$$-\ \frac{2^{\alpha-1}\mathcal{M}(\alpha)\Gamma(\alpha)}{(x_2-x_1)^\alpha}\left\{{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_1)}\mathcal{I}^\alpha_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.$$
$$\left.\left.+\ {}^{AB}_{(\mathrm{a}+\mathrm{b}-x_2)}\mathcal{I}^\alpha_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right\}\right|$$
$$\leq\ \frac{x_2-x_1}{2}\left(\int_0^1\left|\frac{t^\alpha}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\left|f'(\mathrm{a})\right|^q+\left|f'(\mathrm{b})\right|^q-\left[\frac{\left|f'(x_1)\right|^q+\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'(\mathrm{a})\right|^q+\left|f'(\mathrm{b})\right|^q-\left[\frac{\left|f'(x_2)\right|^q+\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $q>1$.

Remark 18.2: If we set $x_1=\mathrm{a}$, $x_2=\mathrm{b}$ and $\alpha=1$ in Theorem 18.3, we obtain Theorem 8 in [227].

Corollary 18.3: *Under the same assumptions as defined in Theorem 18.3 with $s=1$, we get the following A-B fractional inequality of Simpson-Mercer type:*

$$\left|\frac{1}{6}\{f(\mathrm{a}+\mathrm{b}-x_1)+f(\mathrm{a}+\mathrm{b}-x_2)\}+\frac{2}{3}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.$$
$$+\frac{2^\alpha(1-\alpha)\Gamma(\alpha)}{(x_2-x_1)^\alpha}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)$$
$$-\ \frac{2^{\alpha-1}\mathcal{M}(\alpha)\Gamma(\alpha)}{(x_2-x_1)^\alpha}\left\{{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_1)}\mathcal{I}^\alpha_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.$$
$$\left.\left.+\ {}^{AB}_{(\mathrm{a}+\mathrm{b}-x_2)}\mathcal{I}^\alpha_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right\}\right|$$

$$\leq \frac{x_2-x_1}{2}\left(\int_0^1\left|\frac{t^\alpha}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left\{\left(\left|f'(\mathrm{a})\right|^q+\left|f'(\mathrm{b})\right|^q-\left[\frac{\left|f'(x_1)\right|^q+\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q}{2}\right]\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'(\mathrm{a})\right|^q+\left|f'(\mathrm{b})\right|^q-\left[\frac{\left|f'(x_2)\right|^q+\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q}{2}\right]\right)^{\frac{1}{q}}\right\}.$$

Corollary 18.4: *Under the same assumptions as defined in Theorem 18.3 with $\alpha=1$, we get the following Simpson-Mercer type inequality*

$$\left|\frac{1}{6}\left\{f(\mathrm{a}+\mathrm{b}-x_1)+4f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)+f(\mathrm{a}+\mathrm{b}-x_2)\right\}-\frac{1}{x_2-x_1}\int_{\mathrm{a}+\mathrm{b}-x_2}^{\mathrm{a}+\mathrm{b}-x_1}f(x)\,dx\right|$$
$$\leq \frac{x_2-x_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\left|f'(\mathrm{a})\right|^q+\left|f'(\mathrm{b})\right|^q-\left[\frac{\left|f'(x_1)\right|^q+\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'(\mathrm{a})\right|^q+\left|f'(\mathrm{b})\right|^q-\left[\frac{\left|f'(x_2)\right|^q+\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right\}.$$

Theorem 18.4: *Taking all the conditions as defined in Lemma 18.2 and assumption Ω into consideration, if $|f'|^q$ is s-convex function on $[\mathrm{a},\mathrm{b}]$, for $s>0$ and $q>1$, then for all $\alpha>0$, the following inequality for A-B fractional integral holds true for $t\in[0,1]$:*

$$\left|\frac{1}{6}\{f(\mathrm{a}+\mathrm{b}-x_1)+f(\mathrm{a}+\mathrm{b}-x_2)\}+\frac{2}{3}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.$$
$$+\frac{2^\alpha(1-\alpha)\Gamma(\alpha)}{(x_2-x_1)^\alpha}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)$$
$$-\frac{2^{\alpha-1}\mathcal{M}(\alpha)\Gamma(\alpha)}{(x_2-x_1)^\alpha}\left\{{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_1)}\mathcal{I}^\alpha_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_2)}\mathcal{I}^\alpha_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right\}\right|$$
$$\leq \frac{x_2-x_1}{2}\left(\int_0^1\left|\frac{t^\alpha}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\left|f'(\mathrm{a})\right|^q+\left|f'(\mathrm{b})\right|^q-\left[\frac{2^{s+1}-1}{(s+1)2^s}\left|f'(x_1)\right|^q+\frac{1}{(s+1)2^s}\left|f'(x_2)\right|^q\right]\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'(\mathrm{a})\right|^q+\left|f'(\mathrm{b})\right|^q-\left[\frac{1}{(s+1)2^s}\left|f'(x_1)\right|^q+\frac{2^{s+1}-1}{(s+1)2^s}\left|f'(x_2)\right|^q\right]\right)^{\frac{1}{q}}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $q>1$.

Remark 18.3: If we set $x_1=\mathrm{a}$, $x_2=\mathrm{b}$ and $\alpha=1$ in Theorem 18.4, we obtain Theorem 9 in [227].

Corollary 18.5: *Under the same assumptions as defined in Theorem 18.4 with $s=1$, we get* *the following A-B fractional inequality of Simpson-Mercer type:*

$$
\begin{aligned}
&\left|\frac{1}{6}\{f(\mathrm{a}+\mathrm{b}-x_1)+f(\mathrm{a}+\mathrm{b}-x_2)\}+\frac{2}{3}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.\\
&+\frac{2^{\alpha}(1-\alpha)\Gamma(\alpha)}{(x_2-x_1)^{\alpha}}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\\
&-\frac{2^{\alpha-1}\mathcal{M}(\alpha)\Gamma(\alpha)}{(x_2-x_1)^{\alpha}}\left\{{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_1)}\mathcal{I}^{\alpha}_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.\\
&\left.\left.+{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_2)}\mathcal{I}^{\alpha}_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right\}\right|\\
&\leq\frac{x_2-x_1}{2}\left(\int_0^1\left|\frac{t^{\alpha}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\\
&\times\left\{\left(|f'(\mathrm{a})|^q+|f'(\mathrm{b})|^q-\left[\frac{3|f'(x_1)|^q+|f'(x_2)|^q}{4}\right]\right)^{\frac{1}{q}}\right.\\
&\left.+\left(|f'(\mathrm{a})|^q+|f'(\mathrm{b})|^q-\left[\frac{|f'(x_1)|^q+3|f'(x_2)|^q}{4}\right]\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Corollary 18.6: *Under the same assumptions as defined in Theorem 18.4 with $\alpha=1$, we get the following Simpson-Mercer type inequality:*

$$
\begin{aligned}
&\left|\frac{1}{6}\left\{f(\mathrm{a}+\mathrm{b}-x_1)+4f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)+f(\mathrm{a}+\mathrm{b}-x_2)\right\}\right.\\
&\left.-\frac{1}{x_2-x_1}\int_{\mathrm{a}+\mathrm{b}-x_2}^{\mathrm{a}+\mathrm{b}-x_1}f(x)\,dx\right|\\
&\leq\frac{x_2-x_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\\
&\times\left\{\left(|f'(\mathrm{a})|^q+|f'(\mathrm{b})|^q-\left[\frac{2^{s+1}-1}{(s+1)2^s}|f'(x_1)|^q+\frac{1}{(s+1)2^s}|f'(x_2)|^q\right]\right)^{\frac{1}{q}}\right.\\
&\left.+\left(|f'(\mathrm{a})|^q+|f'(\mathrm{b})|^q-\left[\frac{1}{(s+1)2^s}|f'(x_1)|^q+\frac{2^{s+1}-1}{(s+1)2^s}|f'(x_2)|^q\right]\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Theorem 18.5: *Taking all the conditions as defined in Lemma 18.2 and assumption Ω into consideration, if $|f'|^q$ is s-convex function on $[\mathrm{a},\mathrm{b}]$, for $s>0$, $q\geq 1$, then for all $\alpha>0$, the following Simpson- Mercer type inequality for A-B fractional integral holds true for $t\in[0,1]$:*

$$
\begin{aligned}
&\left|\frac{1}{6}\{f(\mathrm{a}+\mathrm{b}-x_1)+f(\mathrm{a}+\mathrm{b}-x_2)\}+\frac{2}{3}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.\\
&+\frac{2^{\alpha}(1-\alpha)\Gamma(\alpha)}{(x_2-x_1)^{\alpha}}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\\
&-\frac{2^{\alpha-1}\mathcal{M}(\alpha)\Gamma(\alpha)}{(x_2-x_1)^{\alpha}}\left\{{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_1)}\mathcal{I}^{\alpha}_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.\\
&\left.\left.+{}^{AB}_{(\mathrm{a}+\mathrm{b}-x_2)}\mathcal{I}^{\alpha}_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)}f\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right\}\right|\\
&\leq\frac{x_2-x_1}{2}\mathcal{J}_5(\alpha)\times\left\{\mathcal{J}_6(\alpha)^{\frac{1}{q}}+\mathcal{J}_7(\alpha)^{\frac{1}{q}}\right\},
\end{aligned}
$$

where

$$\mathcal{J}_5(\alpha) = \left(\int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right| dt\right)^{1-\frac{1}{q}} = \left(\int_0^1 \left|\frac{1}{3} - \frac{t^\alpha}{2}\right| dt\right)^{1-\frac{1}{q}},$$

$$\mathcal{J}_6(\alpha) = \int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right| \Bigg\{ |f'(\mathrm{a})|^q + |f'(\mathrm{b})|^q - \left[\left(\frac{1+t}{2}\right)^s |f'(x_1)|^q + \left(\frac{1-t}{2}\right)^s |f'(x_2)|^q\right]\Bigg\} dt,$$

and

$$\mathcal{J}_7(\alpha) = \int_0^1 \left|\tfrac{t^\alpha}{2} - \tfrac{1}{3}\right| \Bigg\{ |f'(\mathrm{a})|^q + |f'(\mathrm{b})|^q - \left[\left(\tfrac{1-t}{2}\right)^s |f'(x_1)|^q + \left(\tfrac{1+t}{2}\right)^s |f'(x_2)|^q\right]\Bigg\} dt.$$

Remark 18.4: If we consider $x_1 = \mathrm{a}$, $x_2 = \mathrm{b}$ and $\alpha = 1$ in Theorem 18.5, we obtain Theorem 10 in [227].

Corollary 18.7: *Under the same assumptions as defined in Theorem 18.5 with $s = 1$, we get the following A-B fractional inequality of Simpson-Mercer type:*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\{f(\mathrm{a}+\mathrm{b}-x_1) + f(\mathrm{a}+\mathrm{b}-x_2)\} \\
&+ \frac{2}{3} f\left(\mathrm{a}+\mathrm{b} - \frac{x_1+x_2}{2}\right) + \frac{2^\alpha (1-\alpha)\Gamma(\alpha)}{(x_2-x_1)^\alpha} f\left(\mathrm{a}+\mathrm{b} - \frac{x_1+x_2}{2}\right) \\
&- \frac{2^{\alpha-1}\mathcal{M}(\alpha)\Gamma(\alpha)}{(x_2-x_1)^\alpha} \Bigg\{ {}^{AB}_{(\mathrm{a}+\mathrm{b}-x_1)}\mathcal{I}^\alpha_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)} f\left(\mathrm{a}+\mathrm{b} - \frac{x_1+x_2}{2}\right) \\
&+ {}^{AB}_{(\mathrm{a}+\mathrm{b}-x_2)}\mathcal{I}^\alpha_{\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)} f\left(\mathrm{a}+\mathrm{b} - \frac{x_1+x_2}{2}\right)\Bigg\}\Bigg| \\
&\leq \frac{x_2-x_1}{2}\left(\mathcal{J}_5(\alpha) \times \left\{\mathcal{J}_6(\alpha)^{\frac{1}{q}} + \mathcal{J}_7(\alpha)^{\frac{1}{q}}\right\}\right).
\end{aligned}$$

18.2 Examples and Applications

18.2.1 Modified Bessel Functions

Example 18.1: *Let the function $\mathcal{J}_\alpha : \mathbb{R} \to [1,\infty)$ be defined [240] as*

$$\mathcal{J}_\alpha(u) = 2^\alpha \Gamma(\alpha+1) u^{-\alpha} \mathrm{I}_\alpha(u), \quad u \in \mathbb{R}.$$

Here, we consider the modified Bessel function of first kind given in

$$\mathcal{J}_\alpha(u) = \sum_{n=0}^{\infty} \frac{\left(\frac{u}{2}\right)^{\alpha+2n}}{n!\Gamma(\alpha+n+1)}.$$

The first and second order derivative are given as

$$\mathcal{J}'_\alpha(u) = \frac{u}{2(\alpha+1)} \mathcal{J}_{\alpha+1}(u).$$

$$\mathcal{J}''_{\alpha}(u) = \frac{1}{4(\alpha+1)}\left[\frac{u^2}{(\alpha+1)}\mathcal{J}_{\alpha+2}(u) + 2\mathcal{J}_{\alpha+1}(u)\right].$$

If we use, $f(u) = \mathcal{J}'_{\alpha}(u)$ and the above functions in Corollary 18.2, we have

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Bigg\{\frac{(a+b-x_1)}{2(\alpha+1)}\mathcal{J}_{\alpha+1}(a+b-x_1)+\\
&4\left(\frac{a+b-\frac{x_1+x_2}{2}}{2(\alpha+1)}\mathcal{J}_{\alpha+1}\left(a+b-\frac{x_1+x_2}{2}\right)\right)\\
&+\frac{a+b-x_2}{2(\alpha+1)}\mathcal{J}_{\alpha+1}(a+b-x_2)\Bigg\} - \frac{1}{x_2-x_1}\left[\mathcal{J}_{\alpha}(a+b-x_1) - \mathcal{J}_{\alpha}(a+b-x_2)\right]\Bigg|\\
&\leq \frac{x_2-x_1}{2}\Bigg[\frac{5}{18}\Bigg\{\left|\frac{1}{4(\alpha+1)}\left[\frac{d^2}{(\alpha+1)}\mathcal{J}_{\alpha+2}(a) + 2\mathcal{J}_{\alpha+1}(a)\right]\right|+\\
&\left|\frac{1}{4(\alpha+1)}\left[\frac{D^2}{(\alpha+1)}\mathcal{J}_{\alpha+2}(b) + 2\mathcal{J}_{\alpha+1}(b)\right]\right|\Bigg\}\\
&-\frac{1}{2^s}\Bigg[\left|\frac{1}{4(\alpha+1)}\left[\frac{x_1^2}{(\alpha+1)}\mathcal{J}_{\alpha+2}(x_1) + 2\mathcal{J}_{\alpha+1}(x_1)\right]\right|\\
&+\left|\frac{1}{4(\alpha+1)}\left[\frac{x_2^2}{(\alpha+1)}\mathcal{J}_{\alpha+2}(x_2) + 2\mathcal{J}_{\alpha+1}(x_2)\right]\right|\Bigg]\\
&\times\left(\frac{1}{6(s+1)}\left(\frac{5^{s+2}-1}{3^{s+1}} - 6\right) - \frac{1}{2(s+2)}\left(\frac{5^{s+2}+1}{3^{s+2}} - 2\right)\right.\\
&\left.+\frac{5^{s+2}+1}{2(s+1)(s+2)3^{s+2}} - \frac{2^s(2s+7)}{3(s+1)(s+2)}\right)\Bigg].
\end{aligned}$$

Remark 18.5: If we use, $f(u) = \mathcal{J}'_{\alpha}(u)$ and the above functions in Corollary 18.4, then we will get another example as stated above.

18.2.2 Special Means

Consider the following two special means for $0 < \mathrm{a} < \mathrm{b}$.

The arithmetic mean:

$$A(\mathrm{a},\mathrm{b}) = \frac{\mathrm{a}+\mathrm{b}}{2}.$$

The generalized logarithmic-mean:

$$L_k(\mathrm{a},\mathrm{b}) = \left[\frac{\mathrm{b}^{k+1} - \mathrm{a}^{k+1}}{(k+1)(\mathrm{b}-\mathrm{a})}\right]^{\frac{1}{k}};\ \ k \in \mathbb{R}\setminus\{-1,0\}.$$

Theorem 18.6: *Suppose that* $\mathrm{a},\mathrm{b} \in \mathbb{R}$, $s \in (0,1]$, $0 < \mathrm{a} < \mathrm{b}$, $0 \notin [\mathrm{a},\mathrm{b}]$. *Then,*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Big\{2A\left((\mathrm{a}+\mathrm{b}-x_1)^s, (\mathrm{a}+\mathrm{b}-x_2)^s\right)\Big\}\\
&+4\left(2A(\mathrm{a},\mathrm{b}) - A(x_1,x_2)\right)^s - L_s^s(\mathrm{a}+\mathrm{b}-x_1, \mathrm{a}+\mathrm{b}-x_2)\Bigg|\\
&\leq \frac{s(x_2-x_1)}{2}\left[\left(\frac{5}{9}A\left(\mathrm{a}^{s-1},\mathrm{b}^{s-1}\right) - \frac{1}{2^{s-1}}A\left(x_1^{s-1},x_2^{s-1}\right)\right)\Big(\mathcal{J}_3(1) + \mathcal{J}_4(1)\Big)\right].
\end{aligned}$$

Proof: *The proof follows an immediate consequences from Corollary 18.2 by considering $f(u) =$*

u^s. □

Theorem 18.7: *Suppose that* $\mathrm{a}, \mathrm{b} \in \mathbb{R}$, $s \in (0,1]$, $0 < \mathrm{a} < \mathrm{b}$, $0 \notin [\mathrm{a}, \mathrm{b}]$. *Then for all* $q > 1$, *the following inequality holds true:*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\left\{2A\left((\mathrm{a}+\mathrm{b}-x_1)^{-s}, (\mathrm{a}+\mathrm{b}-x_2)^{-s}\right)\right\} \\
&+4\left(2A(\mathrm{a},\mathrm{b})-A(x_1,x_2)\right)^{-s}-L_{-s}^{-s}(\mathrm{a}+\mathrm{b}-x_1, \mathrm{a}+\mathrm{b}-x_2)\Bigg| \\
&\leq \frac{s(x_2-x_1)}{12}\left(\frac{1+2^{p+1}}{(3p+3)}\right)^{\frac{1}{p}} \\
&\left\{\left(2A\left(\mathrm{a}^{-(s+1)q}, \mathrm{b}^{-(s+1)q}\right)-\left[\frac{x_1^{-(s+1)q}+A(x_1,x_2)^{-(s+1)q}}{s+1}\right]\right)^{\frac{1}{q}}\right. \\
&\left.+\left(2A\left(\mathrm{a}^{-(s+1)q}, \mathrm{b}^{-(s+1)q}\right)-\left[\frac{x_2^{-(s+1)q}+A(x_1,x_2)^{-(s+1)q}}{s+1}\right]\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof: *The proof follows an immediate consequence from Corollary 18.4 by considering* $f(u) = u^{-s}$. □

Theorem 18.8: *Suppose that* $\mathrm{a}, \mathrm{b} \in \mathbb{R}$, $s \in (0,1]$, $0 < \mathrm{a} < \mathrm{b}$, $0 \notin [\mathrm{a}, \mathrm{b}]$. *Then,*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\left\{2A\left((\mathrm{a}+\mathrm{b}-x_1)^{s}, (\mathrm{a}+\mathrm{b}-x_2)^{s}\right)\right\} \\
&+4\left(2A(\mathrm{a},\mathrm{b})-A(x_1,x_2)\right)^{s}-L_{s}^{s}(\mathrm{a}+\mathrm{b}-x_1, \mathrm{a}+\mathrm{b}-x_2)\Bigg| \\
&\leq \frac{s(x_2-x_1)}{12}\left(\frac{1+2^{p+1}}{(3p+3)}\right)^{\frac{1}{p}} \\
&\left\{\left(2A\left(\mathrm{a}^{-(s+1)q}, \mathrm{b}^{-(s+1)q}\right)-\left[\frac{(2^{s+1}-1)x_1^{(s-1)q}+x_2^{(s-1)q}}{2^s(s+1)}\right]\right)^{\frac{1}{q}}\right. \\
&\left.+\left(2A\left(\mathrm{a}^{-(s+1)q}, \mathrm{b}^{-(s+1)q}\right)-\left[\frac{(2^{s+1}-1)x_2^{(s-1)q}+x_1^{(s-1)q}}{(s+1)2^s}\right]\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof: *The proof follows an immediate consequences from Corollary 18.6 by considering* $f(u) = u^s$. □

18.2.3 Q-Digamma Function

The q-digamma(psi) function α_ρ, is the ρ-analogue of the digamma function α (see [240, 121]) given as:

$$\alpha_\rho = -\ln(1-\rho) + \ln\rho\sum_{k=0}^{\infty}\frac{\rho^{k+x}}{1-\rho^{k+x}} = -\ln(1-\rho) + \ln\rho\sum_{k=0}^{\infty}\frac{\rho^{kx}}{1-\rho^{kx}}.$$

For $\rho > 1$ and $x > 0$, ρ-digamma function α_ρ can be given as:

$$\alpha_\rho = -\ln(\rho-1) + \ln\rho\left[x - \frac{1}{2} - \sum_{k=0}^{\infty}\frac{\rho^{-(k+x)}}{1-\rho^{-(k+x)}}\right] = -\ln(\rho-1) + \ln\rho\left[x - \frac{1}{2} - \sum_{k=0}^{\infty}\frac{\rho^{-kx}}{1-\rho^{-kx}}\right].$$

Theorem 18.9: *Suppose* $\mathrm{a}, \mathrm{b}, \rho$ *be real numbers such that* $0 < \mathrm{a} < \mathrm{b}$, $0 < \rho < 1$, $s \in (0,1]$ *and* $\frac{1}{p} + \frac{1}{q} = 1, q > 1$. *Then the following inequality holds true:*

$$\begin{aligned}
&\left|\frac{1}{6}\left\{\alpha_\rho(\mathrm{a}+\mathrm{b}-x_1) + \alpha_\rho(\mathrm{a}+\mathrm{b}-x_2)\right\} + \frac{2}{3}\alpha_\rho\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)\right.\\
&\left.-\frac{1}{x_2-x_1}\int_{\mathrm{a}+\mathrm{b}-x_2}^{\mathrm{a}+\mathrm{b}-x_1}\alpha_\rho(x)\,dx\right|\\
&\leq \frac{x_2-x_1}{12}\left(\frac{1+2^{p+1}}{(3p+3)}\right)^{\frac{1}{p}}\\
&\left\{\left(\left|\alpha'_\rho(\mathrm{a})\right|^q + \left|\alpha'_\rho(\mathrm{b})\right|^q - \left[\frac{2^{s+1}-1}{(s+1)2^s}\left|\alpha'_\rho(x_1)\right|^q + \frac{1}{(s+1)2^s}\left|\alpha'_\rho(x_2)\right|^q\right]\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left|\alpha'_\rho(\mathrm{a})\right|^q + \left|\alpha'_\rho(\mathrm{b})\right|^q - \left[\frac{1}{(s+1)2^s}\left|\alpha'_\rho(x_1)\right|^q + \frac{2^{s+1}-1}{(s+1)2^s}\left|\alpha'_\rho(x_2)\right|^q\right]\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof: By employing the definition of ρ-digamma function $\alpha_\rho(x)$, it is easy to notice that ρ -digamma function $f(x) = \alpha_\rho(x)$ is completely monotonic on $(0,\infty)$. This ensures that the function $f'(x) = \alpha'_\rho(x)$ is s-convex on $(0,\infty)$ for each $\rho \in (0,1)$. Now by applying Corollary 18.6, we get the desired result. □

Theorem 18.10: *Suppose* $\mathrm{a}, \mathrm{b}, q, \rho$ *be real numbers such that* $0 < \mathrm{a} < \mathrm{b}$, $\frac{1}{p} + \frac{1}{q} = 1, q > 1$ *and* $0 < \rho < 1$, $s \in (0,1]$. *Then the following inequality holds true:*

$$\begin{aligned}
&\left|\frac{1}{6}\left\{\alpha_\rho(\mathrm{a}+\mathrm{b}-x_1) + \alpha_\rho(\mathrm{a}+\mathrm{b}-x_2)\right\}\right.\\
&\left.+\frac{2}{3}\alpha_\rho\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right) - \frac{1}{x_2-x_1}\int_{\mathrm{a}+\mathrm{b}-x_2}^{\mathrm{a}+\mathrm{b}-x_1}\alpha_\rho(x)\,dx\right|\\
&\leq \frac{x_2-x_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\\
&\left\{\left(\left|\alpha'_\rho(\mathrm{a})\right|^q + \left|\alpha'_\rho(\mathrm{b})\right|^q - \left[\frac{\left|\alpha'_\rho(x_1)\right|^q + \left|\alpha'_\rho\left(\frac{x_1+x_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left|\alpha'_\rho(\mathrm{a})\right|^q + \left|\alpha'_\rho(\mathrm{b})\right|^q - \left[\frac{\left|\alpha'_\rho(x_2)\right|^q + \left|\alpha'_\rho\left(\frac{x_1+x_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof: By employing the definition of ρ-digamma function $\alpha_\rho(x)$, it is easy to notice that ρ -digamma function $f(x) = \alpha_\rho(x)$ is completely monotonic on $(0,\infty)$. This ensures that the function $f'(x) = \alpha'_\rho(x)$ is s-convex on $(0,\infty)$ for each $\rho \in (0,1)$. Now by applying Corollary 18.4, we get the desired result. □

Theorem 18.11: *Suppose* $\mathrm{a}, \mathrm{b}, \rho$ *be real numbers such that* $0 < \mathrm{a} < \mathrm{b}$, *and* $0 < \rho < 1$, $s \in (0,1]$.

Then the following inequality holds true:

$$\begin{aligned}
&\left|\frac{1}{6}\left\{\alpha_\rho\left(\mathrm{a}+\mathrm{b}-x_1\right)+\ \alpha_\rho\left(\mathrm{a}+\mathrm{b}-x_2\right)\right\}\right.\\
&\left.+\frac{2}{3}\alpha_\rho\left(\mathrm{a}+\mathrm{b}-\frac{x_1+x_2}{2}\right)-\frac{1}{x_2-x_1}\int_{\mathrm{a}+\mathrm{b}-x_2}^{\mathrm{a}+\mathrm{b}-x_1}\alpha_\rho\left(x\right)dx\right|\\
&\leq\frac{x_2-x_1}{2}\left[\frac{5}{18}\left(\left|\alpha_\rho'\left(\mathrm{a}\right)\right|+\left|\alpha_\rho'\left(\mathrm{b}\right)\right|\right)-\frac{1}{2^s}\left(\left|\alpha_\rho'\left(x_1\right)\right|+\left|\alpha_\rho'\left(x_2\right)\right|\right)\right.\\
&\left\{\frac{1}{6\left(s+1\right)}\left(\frac{5^{s+2}-1}{3^{s+1}}-6\right)-\frac{1}{2\left(s+2\right)}\left(\frac{5^{s+2}+1}{3^{s+2}}-2\right)\right.\\
&\left.\left.+\ \frac{5^{s+2}+1}{2\left(s+1\right)\left(s+2\right)3^{s+2}}-\frac{2^s(2s+7)}{3\left(s+1\right)\left(s+2\right)}\right\}\right].
\end{aligned}$$

Proof: Employing the definition of ρ-digamma function $\alpha_\rho(x)$, it is easy to notice that ρ -digamma function $f(x)=\alpha_\rho(x)$ is completely monotonic on $(0,\infty)$. This ensures that the function $f'(x)=\alpha_\rho'(x)$ is s-convex on $(0,\infty)$ for each $\rho\in(0,1)$. Now by applying Corollary 18.2, we get the desired result. □

Remark 18.6: We can also find further results related to quantum Mercer estimates of Simpson–Newton-like inequalities via convexity in the article [62].

CHAPTER 19

On Generalized Ostrowski–Mercer Type Inequalities

This chapter develops generalized Ostrowski–Mercer type inequalities, extending the classical Ostrowski inequality through $(\alpha, \beta, \gamma, \delta)$–convexity in mixed form. We also establish Ostrowski–type inequalities for functions whose derivatives in absolute value at certain powers satisfy $(\alpha, \beta, \gamma, \delta)$–convexity, employing techniques such as Hölder's inequality and the improved power mean inequality. Special cases and applications, including midpoint inequalities for special means, are discussed to illustrate the practical impact of these generalizations

19.1 Ostrowski–Mercer Inequality

The well-known Ostrowski-inequality is extracted from [91].

Theorem 19.1: *Let $f : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (where I^o is the interior of I) such that $f \in L[a, b]$, where $a, b \in I$ and $a < b$. If $|f'(t)| \leq M$ valid $\forall\ t \in (a, b)$ where M is positive real constant. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] = \left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] M. \quad (19.1)$$

We recall a variant of the Ostrowski inequality, the Ostrowski–Mercer inequality from [230].

Theorem 19.2: *Let $f \in AC\,[a, b]$, $f \in L\,[a, b]$, and $|f'(x)| \leq M$ for $x \in [a, b]$, then*

$$\begin{aligned} &\left| (x-y)\, f\,(x+a-y) + (z-x)\, f\,(x+b-z) - \left[\int_a^{x+a-y} f(t)dt + \int_{x+b-z}^b f(t)dt \right] \right| \\ &\leq \frac{M}{2} \left((x-y)^2 + (z-x)^2 \right), \end{aligned} \quad (19.2)$$

$\forall x \in [y, z]$, with $y, z \in [a, b]$, $x \in [a, b]$.

Remark 19.1: If $y = a$ and $z = b$ in Theorem 19.2, then inequality (19.2) reduces into (19.1).

19.2 Ostrowski–Mercer Type Inequalities Via $\alpha, \beta, \gamma, \delta-$ Convex Functions

Throughout this section we denote,

$$Y_f(x,y,z,a,b) = (x-y)f(x+a-y) + (z-x)f(x+b-z) - \left[\int_a^{x+a-y} f(t)dt + \int_{x+b-z}^{b} f(t)dt\right].$$

The Beta function, defined for $x, y > 0$ as:

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}du = \frac{\gamma(x)\gamma(y)}{\gamma(x+y)},$$

where $\gamma(x) = \displaystyle\int_0^\infty e^{-u}u^{x-1}du$. Here we state an important result related to $(\alpha, \beta, \gamma, \delta)$–convex functions, for detailed discussion on $(\alpha, \beta, \gamma, \delta)$–convexity, we refer the reader to [111, 112]:

Theorem 19.3: *Let f be an absolutely continuous and integrable function defined on $[a, b]$ whose first derivative is bounded, i.e., $|f'(x)| \leq M$ for all x. If f is $(\alpha, \beta, \gamma, \delta)$ convex function in the mixed kind, then for all $x \in [y, z]$ such that $y, z \in [a, b]$ we have*

$$|Y_f(x,y,z,a,b)| \leq M\left[1 - \frac{1}{\alpha\gamma+2} - \frac{1}{\beta}B\left(\frac{2}{\beta}, \delta+1\right)\right]\kappa_y^z(x). \tag{19.3}$$

where $\kappa_y^z(x) = (x-y)^2 + (z-x)^2$.

Proof: By using Lemma 1 of [230]

$$\begin{aligned}|Y_f(x,y,z,a,b)| \;&\leq\; (x-y)^2\int_0^1 t|f'(x+a-(ty+(1-t)x))|dt \\ &\quad +(z-x)^2\int_0^1 t|f'(x+b-(tz+(1-t)x))|dt.\end{aligned}$$

Using (1.3), we get

$$\begin{aligned}|Y_f(x,y,z,a,b)| &\leq (x-y)^2\int_0^1 t\left[|f'(x)| + |f'(a)| - t^{\alpha\gamma}|f'(y)| - (1-t^{\beta})^{\delta}|f'(x)|\right]dt \\ &+(z-x)^2\int_0^1 t\left[|f'(x)| + |f'(b)| - t^{\alpha\gamma}|f'(z)| - (1-t^{\beta})^{\delta}|f'(x)|\right]dt.\end{aligned}$$

Hence by $|f'(x)| \leq M$, we have

$$\begin{aligned}|Y_f(x,y,z,a,b)| \;&\leq\; M(x-y)^2\left[1 - \frac{1}{\alpha\gamma+2} - \frac{1}{\beta}B\left(\frac{2}{\beta}, \delta+1\right)\right] \\ &\quad +M(z-x)^2\left[1 - \frac{1}{\alpha\gamma+2} - \frac{1}{\beta}B\left(\frac{2}{\beta}, \delta+1\right)\right].\end{aligned}$$

□

Remark 19.2: In Theorem 11.1, if $\alpha = \beta = \gamma = \delta = 1$, then (19.3) reduces to the inequality

(19.2), which is the Corollary 1 of [230].

Remark 19.3: In Theorem 11.1, if $y = a$ and $z = b$, then for $\alpha = \beta = \gamma = \delta = 1$, (19.3) reduces to the inequality (19.1).

Theorem 19.4: *Let all the assumptions of Theorem 19.3 hold. Let $|f'|^q$ be $(\alpha, \beta, \gamma, \delta)-$convex in mixed kind for $q \geq 1$ with $|f'| \leq M$, then*

$$\begin{aligned} |Y_f(x,y,z,a,b)| \leq& \frac{M}{[(p+1)(p+2)]^{\frac{1}{p}}} \\ &\times \left(1 - \frac{1}{\alpha\gamma+1} + \frac{1}{\alpha\gamma+2} - \frac{1}{\beta}B\left(\frac{1}{\beta}, \delta+1\right) + \frac{1}{\beta}B\left(\frac{2}{\beta}, \delta+1\right)\right)^{\frac{1}{q}} \kappa_y^z(x) \\ &+ \frac{M}{(p+2)^{\frac{1}{p}}}\left(1 - \frac{1}{\alpha\gamma+2} - \frac{1}{\beta}B\left(\frac{2}{\beta}, \delta+1\right)\right)^{\frac{1}{q}} \kappa_y^z(x). \end{aligned} \tag{19.4}$$

Theorem 19.5: *Let all the assumptions of Theorem 19.3 hold. If $|f'|^q$ be a $(\alpha, \beta, \gamma, \delta)-$convex function in the mixed kind for $q \geq 1$ with $|f'| \leq M$, then*

$$\begin{aligned} &|Y_f(x,y,z,a,b)| \leq M^{2-\frac{1}{q}} \kappa_y^z(x) \\ &\times \Bigg[\left(1 - \frac{1}{\alpha\gamma+1} + \frac{1}{\alpha\gamma+2} - \frac{1}{\beta}B\left(\frac{1}{\beta}, \delta+1\right) + \frac{1}{\beta}B\left(\frac{2}{\beta}, \delta+1\right)\right)^{1-\frac{1}{q}} \\ &\left(\frac{1}{3} - \frac{1}{\alpha\gamma+2} + \frac{1}{\alpha\gamma+3} - \frac{1}{\beta}B\left(\frac{2}{\beta}, \delta+1\right) + \frac{1}{\beta}B\left(\frac{3}{\beta}, \delta+1\right)\right)^{\frac{1}{q}} \\ &+ \left(1 - \frac{1}{\alpha\gamma+2} - \frac{1}{\beta}B\left(\frac{2}{\beta}, \delta+1\right)\right)^{1-\frac{1}{q}} \left(\frac{2}{3} - \frac{1}{\alpha\gamma+3} - \frac{1}{\beta}B\left(\frac{3}{\beta}, \delta+1\right)\right)^{\frac{1}{q}}\Bigg]. \end{aligned} \tag{19.5}$$

Remark 19.4: In the last two results stated above, we can identify various special cases corresponding to different values α, β, γ and δ.

19.2.1 Applications to Special Means

Let A, G, H, L, I and L_p represent arithmetic, geometric, harmonic, logarithmic, identric and p-logarithmic means, respectively.

Remark 19.5: In Theorem 19.4, we put $y = a$ and $z = b$, then

1. Let $x = \frac{a+b}{2}$, $0 < a < b$, $q \geq 1$, $f : \mathbb{R} \to \mathbb{R}^+$, $f(x) = x^n$ then (19.4), reduces

$$\begin{aligned} |A^n(a,b) - L_n^n(a,b)| \leq& \frac{M(b-a)}{2[(p+1)(p+2)]^{\frac{1}{p}}} \\ &\times \left(1 - \frac{1}{\alpha\gamma+1} + \frac{1}{\alpha\gamma+2} - \frac{1}{\beta}B\left(\frac{1}{\beta}, \delta+1\right) + \frac{1}{\beta}B\left(\frac{2}{\beta}, \delta+1\right)\right)^{\frac{1}{q}} \\ &+ \frac{M(b-a)}{2(p+2)^{\frac{1}{p}}}\left(1 - \frac{1}{\alpha\gamma+2} - \frac{1}{\beta}B\left(\frac{2}{\beta}, \delta+1\right)\right)^{\frac{1}{q}}. \end{aligned}$$

2. Let $x = \frac{a+b}{2}$, $0 < a < b$, $q \geq 1$, $f : (0,1] \to \mathbb{R}$, and $f(x) = -\ln x$ then (19.4) reduces

$$\begin{aligned}\left|\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right| &\leq \frac{M(b-a)}{2[(p+1)(p+2)]^{\frac{1}{p}}}\\ &\times\left(1-\frac{1}{\alpha\gamma+1}+\frac{1}{\alpha\gamma+2}-\frac{1}{\beta}B\left(\frac{1}{\beta},\delta+1\right)+\frac{1}{\beta}B\left(\frac{2}{\beta},\delta+1\right)\right)^{\frac{1}{q}}\\ &+\frac{M(b-a)}{2(p+2)^{\frac{1}{p}}}\left(1-\frac{1}{\alpha\gamma+2}-\frac{1}{\beta}B\left(\frac{2}{\beta},\delta+1\right)\right)^{\frac{1}{q}}.\end{aligned}$$

Remark 19.6: In Theorem 19.5, we put $y = a$ and $z = b$, then

1. Let $x = \frac{a+b}{2}$, $0 < a < b$, $q \geq 1$, $f : \mathbb{R} \to \mathbb{R}^+$, and $f(x) = x^n$, then (19.5) reduces,

$$\begin{aligned}&|A^n(a,b) - L_n^n(a,b)| \leq \frac{M^{2-\frac{1}{q}}(b-a)}{2}\\ &\left[\left(1-\frac{1}{\alpha\gamma+1}+\frac{1}{\alpha\gamma+2}-\frac{1}{\beta}B\left(\frac{1}{\beta},\delta+1\right)+\frac{1}{\beta}B\left(\frac{2}{\beta},\delta+1\right)\right)^{1-\frac{1}{q}}\right.\\ &\left(\frac{1}{3}-\frac{1}{\alpha\gamma+2}+\frac{1}{\alpha\gamma+3}-\frac{1}{\beta}B\left(\frac{2}{\beta},\delta+1\right)+\frac{1}{\beta}B\left(\frac{3}{\beta},\delta+1\right)\right)^{\frac{1}{q}}\\ &\left.+\left(1-\frac{1}{\alpha\gamma+2}-\frac{1}{\beta}B\left(\frac{2}{\beta},\delta+1\right)\right)^{1-\frac{1}{q}}\left(\frac{2}{3}-\frac{1}{\alpha\gamma+3}-\frac{1}{\beta}B\left(\frac{3}{\beta},\delta+1\right)\right)^{\frac{1}{q}}\right].\end{aligned}$$

2. Let $x = \frac{a+b}{2}$, $0 < a < b$, $q \geq 1$, $f : (0,1] \to \mathbb{R}$, and $f(x) = -\ln x$, then (19.5), reduces

$$\begin{aligned}&\left|\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right| \leq \frac{M^{2-\frac{1}{q}}(b-a)}{2}\\ &\left[\left(1-\frac{1}{\alpha\gamma+1}+\frac{1}{\alpha\gamma+2}-\frac{1}{\beta}B\left(\frac{1}{\beta},\delta+1\right)+\frac{1}{\beta}B\left(\frac{2}{\beta},\delta+1\right)\right)^{1-\frac{1}{q}}\right.\\ &\left(\frac{1}{3}-\frac{1}{\alpha\gamma+2}+\frac{1}{\alpha\gamma+3}-\frac{1}{\beta}B\left(\frac{2}{\beta},\delta+1\right)+\frac{1}{\beta}B\left(\frac{3}{\beta},\delta+1\right)\right)^{\frac{1}{q}}\\ &\left.+\left(1-\frac{1}{\alpha\gamma+2}-\frac{1}{\beta}B\left(\frac{2}{\beta},\delta+1\right)\right)^{1-\frac{1}{q}}\left(\frac{2}{3}-\frac{1}{\alpha\gamma+3}-\frac{1}{\beta}B\left(\frac{3}{\beta},\delta+1\right)\right)^{\frac{1}{q}}\right].\end{aligned}$$

Remark 19.7: We can also find some results related to the Ostrowski–Mercer inequality in [60], [61] and [230].

CHAPTER 20

Extension of the Jensen–Mercer Inequality on Time Scales

This chapter presents a generalization of the Jensen–Mercer inequality via the $\diamondsuit_\alpha-$integral on time scales, encompassing both discrete and continuous cases. Applications are given to the Ky Fan inequality and the arithmetic–geometric and arithmetic–harmonic mean inequalities, together with refinements and potential implications in economics. The work builds upon Hilger's (1988) introduction of time scale calculus, which unifies discrete and continuous analysis and has since grown significantly with numerous applications across science [113]. For foundational material, readers may consult Bohner and Peterson's monograph on single-variable time scale calculus and its applications [40]. For multivariate time scale analysis one can see [42].

The chapter assumes readers already have a basic understanding of time scale theory.

Definition 20.1: A time scale is an arbitrary nonempty closed subset of the real numbers.

We will denote time scale by T. Examples of time scales are $\mathbb{R}, \mathbb{Z}$ and $q^{\mathbb{N}_0} := \{q^k | k \in \mathbb{N}_0\}$. For more details and basics of time scale, we refer the reader to [40].

20.1 Generalization of Jensen–Mercer Inequality for Δ- Integral

In [71], generalization of the Jensen Mercer Inequality for $\Delta-$Integral is presented as follows:

Theorem 20.1: *If $g, \omega \in C([c,d]_T, [a,b])$ where ω is a probability density function and $\varphi \in C([a,b], \mathbb{R})$ is convex, then*

$$\varphi\left(a + b - \int_c^d \omega(t)g(t)\Delta t\right) \leq \varphi(a) + \varphi(b) - \int_c^d \omega(t)\varphi(g(\eta))\Delta t.$$

Corollary 20.1: *Let $T = \mathbb{R}$ and let assumptions of Theorem 20.3 be valid. Then*

$$\varphi\left(a+b-\int_c^d \omega(t)g(t)dt\right) \leq \varphi(a)+\varphi(b)-\int_c^d \omega(t)\varphi(g(t))\,dt.$$

Remark 20.1: For other related results, special cases and applications, we refer the reader to [71]. Furthermore, the results given in this section can also be given for $\nabla-$integral in similar way. One can find economic application of $\nabla-$integral in [98].

20.2 Generalization of Jensen–Mercer Inequality for $\diamondsuit_\alpha$ Integral

Let $g : T \to \mathbb{R}$. The $\diamondsuit_\alpha$-integral of g over $c, d \in T$ is defined as

$$\int_c^d g(t)\diamondsuit_\alpha(t) = \alpha\int_c^d g(t)\,\Delta t + (1-\alpha)\int_c^d g(t)\,\nabla t, \quad \alpha \in [0,1],$$

provided that both the delta and nabla integrals of g exist on T. When g is continuous, the $\diamondsuit_\alpha$-integral of g clearly exists.

The $\diamondsuit_\alpha$ integral of Jensen's inequality has been proposed in [20] as follow:

Theorem 20.2: *If $g \in C([c,d]_T,[a,b])$ and $\varphi \in C([a,b],\mathbb{R})$ is convex and $\omega \in C([c,d]_T,[a,b])$ is a probability density function, then*

$$\varphi\left(\int_c^d g(t)\omega(t)\diamondsuit_t\right) \leq \int_c^d \varphi(g(t))\,\omega(t)\diamondsuit_\alpha t. \tag{20.1}$$

We establish a Jensen Mercer $\diamondsuit_\alpha$ integral inequality.

Theorem 20.3: *If $g \in C([c,d]_T,[a,b])$ and $\omega \in C([c,d]_T,[a,b])$ is a probability density function and also $\varphi \in C([a,b],\mathbb{R})$ is convex, then*

$$\varphi\left(a+b-\int_c^d \omega(t)g(t)\diamondsuit_\alpha t\right) \leq \varphi(a)+\varphi(b)-\int_c^d \omega(t)\varphi(g(t))\diamondsuit_\alpha t. \tag{20.2}$$

Proof:

Since φ is convex function and $\left(a+b-\int_c^d \omega(t)g(t)\diamondsuit_\alpha t\right) \in [a,b]$, therefore by inequality (20.1) and Lemma 14.1, we have

$$\begin{aligned}\varphi\left(a+b-\int_c^d \omega(t)g(t)\diamondsuit_\alpha t\right) &= \varphi\left(\int_c^d \omega(t)(a+b-g(t))\diamondsuit_\alpha t\right)\\ &\leq \int_c^d \omega(t)\left[\varphi(a+b-g(t))\right]\diamondsuit_\alpha t \leq \varphi(a)+\varphi(b)-\int_c^d \omega(t)\varphi(g(t))\diamondsuit_\alpha t.\end{aligned}$$

□

Remark 20.2: If $\alpha = 1$, then we get the obtained inequality in Δ integral [71]:

$$\varphi\left(a+b-\int_{c}^{d}\omega(t)g(t)\Delta t\right)\leq\varphi\left(a\right)+\varphi\left(b\right)-\int_{c}^{d}\omega(t)\varphi\left(g(t)\right)\Delta t.$$

Remark 20.3: If $\alpha=0$, then we get the obtained inequality in ∇ integral:

$$\varphi\left(a+b-\int_{c}^{d}\omega(t)g(t)\nabla t\right)\leq\varphi\left(a\right)+\varphi\left(b\right)-\int_{c}^{d}\omega(t)\varphi\left(g(t)\right)\nabla t.$$

Corollary 20.2: *Let $T=\mathbb{R}$ and by considering the assumptions of Theorem 20.3. Then*

$$\varphi\left(a+b-\int_{c}^{d}\omega(t)g(t)dt\right)\leq\varphi\left(a\right)+\varphi\left(b\right)-\int_{c}^{d}\omega(t)\varphi\left(g(t)\right)dt.$$

Corollary 20.3: *Let $T=\mathbb{Z}$, Fix $a=1$ and $b=k+1$, let $g:\{1,\ldots,k+1\}\to(0,\infty)$ and by considering the assumptions of Theorem 20.3. Then*

$$\varphi\left(a+b-\alpha\sum_{t=1}^{k}\omega(t)g(t)-(1-\alpha)\sum_{t=1}^{k+2}\omega(t)g(t)\right)$$
$$\leq\varphi\left(a\right)+\varphi\left(b\right)-\alpha\sum_{t=1}^{k}\omega(t)\varphi\left(g(t)\right)-(1-\alpha)\sum_{t=1}^{k+2}\omega(t)\varphi\left(g(t)\right).$$

20.3 Refinement

In this section we are going to establish refinement of Jensen Mercer inequality in time scale. Throughout this section we let $X=[c,d]_T$ and $E,E^c\subset X$ such that $E\cup E^c=X$ and $E\cap E^c=\emptyset$, and also we suppose $W_E=\int_E\omega(t)\Diamond_\alpha t>0$, and $W_{E^c}=1-W_E$.

Theorem 20.4: *Let $X=[c,d]_T$ and $g\in C(X,[a,b])$ and $\omega\in C(X,[a,b])$ is a probability density function and also $\varphi\in C([a,b],\mathbb{R})$ is convex. Then*

$$\varphi\left(a+b-\int_{X}\omega(t)g(t)\Diamond_\alpha t\right)\leq D(\omega,g,\varphi,E)\leq\varphi\left(a\right)+\varphi\left(b\right)-\int_{X}\omega(t)\varphi\left(g(t)\right)\Diamond_\alpha t,$$

where

$$D(\omega,g,\varphi;E)=W_E\varphi\left(a+b-\frac{1}{W_E}\int_E\omega(t)g(t)\Diamond_\alpha t\right)+W_{E^c}\varphi\left(a+b-\frac{1}{W_{E^c}}\int_{E^c}\omega(t)g(t)\Diamond_\alpha t\right).$$

20.4 Applications to Ky Fan Inequality and related results

In 1961, the Ky Fan inequality was presented in the renowned monograph 'Inequalities' in [35] as follow: Let

$$A_n:=\frac{1}{n}\sum_{i=1}^{n}x_i,\qquad G_n:=\left(\prod_{i=1}^{n}x_i\right)^{1/n}$$

denote the arithmetic and geometric mean, respectively, of $x_1, \ldots, x_n \in (0, 1/2]$, and let

$$A'_n := \frac{1}{n}\sum_{i=1}^{n}(1-x_i), \qquad G'_n := \left(\prod_{i=1}^{n}(1-x_i)\right)^{1/n}.$$

Then

$$\frac{G_n}{G'_n} \le \frac{A_n}{A'_n},$$

equality holds iff $x_1 = \cdots = x_n$, which magnetize the attention of several mathematician. For generalization and refinement of the Ky Fan inequality see paper [72] and references therein.

In this section we are improving Ky Fan inequality and related results for time scale calculus. Further, throughout this section, we assume that the interval $[a, b]$ contains the interval $(0, 1/2]$.

Definition 20.2: Let the assumptions of Theorem 20.3 be valid. We define the generalized weighted arithmetic-, geometric- and harmonic-means of $g \in C([c, d]_T, [a, b])$ with weight ω respectively, as under:

$$\begin{aligned} A_{[c,d]}(g, \omega) &= a + b - \int_c^d \omega(t)g(t)\diamondsuit_\alpha t, \\ G_{[c,d]}(g, \omega) &= exp\left[\ln(ab) - \int_c^d \omega(t)\ln(g(t))\diamondsuit_\alpha t\right], \\ H_{[c,d]}(g, \omega) &= \left(\frac{1}{a} + \frac{1}{b} - \int_c^d \omega(t)\frac{1}{(g(t))\diamondsuit_\alpha t}\right)^{-1}. \end{aligned}$$

Remark 20.4: The generalized AM-GM-HM inequality is also true for the generalized weighted means. We have,

$$H_{[c,d]}(g, \omega) \le G_{[c,d]}(g, \omega) \le A_{[c,d]}(g, \omega,) \tag{20.3}$$

To establish the right-hand side of this inequality, we take the concave function $f(t) = \ln(t)$ and set $g(x) = x$, leveraging Theorem 20.2. On the contrary, to proof the left side, we employ the identical function f and set g as $\frac{1}{x}$.

Now, we establish generalized Ky Fan inequality for time scale. Under the assumptions of Theorem 20.3, following results are valid:

Corollary 20.4:

$$\frac{A_{[c,d]}(g, \omega)}{G_{[c,d]}(g, \omega)} \ge \frac{A_{[c,d]}(1-g, \omega)}{G_{[c,d]}(1-g, \omega)}.$$

Proof: By applying $\varphi(x) = \ln\frac{1-x}{x}, x \in (0, \frac{1}{2}]$ to the Theorem 20.3, we obtain required results. □

In next result, we provide refinement of the Ky Fan inequality as follow:

Corollary 20.5:

$$\frac{A_{[c,d]}(g,\omega)}{G_{[c,d]}(g,\omega)} \geq \left[\frac{A_{[c,d]}(g,\omega)}{G_{[c,d]}(g,\omega)}\right]^{\frac{N^2}{(1-N)^2}} \geq \frac{A_{[c,d]}(1-g,\omega)}{G_{[c,d]}(1-g,\omega)} \geq \left[\frac{A_{[c,d]}(g,\omega)}{G_{[c,d]}(g,\omega)}\right]^{\frac{n^2}{(1-n)^2}} \geq 1,$$

where $0 < n \leq g(t) \leq N = \frac{1}{2}$.

Corollary 20.6:

$$A_{[c,d]}(1-g,\omega) \geq G_{[c,d]}(1-g,\omega).$$

Proof: By applying $\varphi(x) = x - \ln(1-x)$ for all $x \in (0, \frac{1}{2}]$ to the Theorem 20.3, we get requires results. □

Now, we present refinement of Ky Fan inequality via convexity.

Corollary 20.7:

$$\frac{A_{[c,d]}(1-g,\omega)}{G_{[c,d]}(1-g,\omega)} \leq \frac{1}{G_{[c,d]}(g,\omega) + G_{[c,d]}(1-g,\omega)} \leq \frac{A_{[c,d]}(g,\omega)}{G_{[c,d]}(g,\omega)}.$$

Here we state some of the results as under:

Corollary 20.8: *Let the assumptions of Theorem 20.3, be valid also let* $g(t) \in (0, \frac{1}{2}] \subset [a,b]$. *Then*

(*i*) $A_{[c,d]}(g,\omega) \geq H_{[c,d]}(g,\omega)$.
(*ii*) $A_{[c,d]}(1-g,\omega) \geq H_{[c,d]}(1-g,\omega)$.

Now, we present arithmetic and harmonic mean inequality.

Corollary 20.9:

$$\frac{1}{A_{[c,d]}(g,\omega)} - \frac{1}{A_{[c,d]}(1-g,\omega)} \leq \frac{1}{H_{[c,d]}(g,\omega)} - \frac{1}{H_{[c,d]}(1-g,\omega)}.$$

Now, we establish geometric and harmonic mean inequality for time scale.

Corollary 20.10:

$$H_{[c,d]}(g,\omega) \leq G_{[c,d]}(g,\omega).$$

Corollary 20.11:

$$H_{[c,d]}(1-g,\omega) \leq G_{[c,d]}(1-g,\omega).$$

Corollary 20.12:

$$\frac{H_{[c,d]}(1-g,\omega)}{H_{[c,d]}(g,\omega)} \leq \frac{G_{[c,d]}(1-g,\omega)}{G_{[c,d]}(g,\omega)}.$$

Corollary 20.13:

$$\frac{H_{[c,d]}(1-g,\omega)}{G_{[c,d]}(1-g,\omega)} \leq \frac{1}{G_{[c,d]}(g,\omega)+G_{[c,d]}(1-g,\omega)} \leq \frac{H_{[c,d]}(g,\omega)}{G_{[c,d]}(g,\omega)}.$$

Remark 20.5: For idea of the proof of some of the results we refer the reader to visit previous chapter.

20.5 Application to Economic Model

Many economic models are dynamic and can be naturally formulated on time scales, encompassing both discrete and continuous frameworks. In particular, consumer behavior can be analyzed through a utility function $u(C)$, where C represents consumption. The utility function satisfies $u'(C) > 0$ and $u''(C) < 0$, reflecting the Law of Diminishing Marginal Utility-each additional unit of consumption provides less satisfaction than the previous one [98].

By framing consumption as an optimal control problem, one can determine the optimal consumption pattern over time, maximizing lifetime utility as the sum of instantaneous utilities.

$$U = \int_0^{\sigma(T)} u(C(\rho(s)))e_\delta(0,\rho(s))\diamond_\alpha s,$$

where, σ and ρ are forward and backward jump operators respectively, and the $\diamond_\alpha-$ exponential function is defined, as in [41], by $e_\delta(c,d) := \exp\left(\int_c^d \xi_{v(u)}(\delta(u))\diamond_\alpha u\right)$, with internal rate of preference δ for every $c,d \in T$ and the cylinder transformation ξ_v (for basic assumptions, proper definitions and other details we refer the reader to [41]). By denoting $\omega(t) := \frac{1}{\lambda}e_\delta(0,\rho(t))$ where $\lambda = \int_0^{\sigma(T)} e_\delta(0,\rho(s))\diamond_\alpha s$ and $g(t) := u(C(\rho(t)))$ and using the notation in Definition 20.2, we get $A_{[0,\sigma(T)]}(g,\omega) = a+b-\frac{U}{\lambda}$. With the above notations, the generalized inequality (20.3), gives an estimation for the upper bound of the utility function, as under:

$$G_{[0,\sigma(T)]}(g,\omega) \leq A_{[0,\sigma(T)]}(g,\omega) = a+b-\frac{U}{\lambda}$$

which gives us

$$U \leq \lambda[a+b-G_{[0,\sigma(T)]}(g,\omega)]$$

Remark 20.6: For other remarks and further discussion on the topic we refer the reader to [84].

Remark 20.7: For more results on time scale related to Jensen–Mercer inequality we refer the reader to [34].

Concluding Remarks

The present volume marks the culmination of six years of work in collecting, systematizing, and analyzing developments related to the Jensen–Mercer inequality and its extended family of results. The pace of evolution in this field has been remarkable. Despite our sustained effort to maintain comprehensive coverage, the research activity in 2024 and 2025 has accelerated to a point where continuous tracking would necessitate perpetual revision. This chapter, therefore, provides a reflective synthesis of the most impactful recent advances and establishes a natural endpoint for this edition.

Recent Developments (2024–2025): A Paradigm of Expansion and Refinement

The literature from the last two years reveals a vibrant field characterized by both deep theoretical refinements and a significant expansion into new mathematical territories. The dominant trends can be summarized as follows:

Proliferation of Fractional and Generalized Operators

The integration of fractional calculus with Mercer-type inequalities has solidified into a major research paradigm. Recent works have extensively developed Hermite-Hadamard-Mercer (HHM) and Jensen–Mercer-type inequalities involving a diverse set of non-local operators. The year 2024 saw significant contributions using **Atangana-Baleanu operators in the Katugampola sense** {2}, **generalized fractional operators** {4}, and **conformable derivatives** {1} and {3}. This trend continued into 2025 with further explorations via ψ-Hilfer operators {13} and Riemann-Liouville fractional integrals {18}. These approaches systematically extend the classical theory to nonlocal contexts, demonstrating how memory-dependent and generalized operators interact with various convexity classes.

Refinements via Strong Convexity and Curvature Assumptions

A parallel and equally significant development has been the strengthening of Jensen–Mercer inequalities under more restrictive curvature conditions. Several articles from 2024 provide improved formulations for **strongly convex functions** {5} and {6}, establishing tighter bounds that have direct consequences for information-theoretic measures like Csiszár divergence, Jensen-Shannon divergence, and Shannon entropy. This line of research demonstrates the inequality's enduring effectiveness as a tool in mathematical statistics and information theory.

Refinements of Jessen–Mercer Inequality Using Time Scale Calculus

The work of Saïd and Torres {12} represents a significant unification of the theory, extending the Jessen–Mercer framework to the domain of time scale calculus. By establishing refined versions of the inequality for the Δ.

Δ-integral, their work provides a powerful tool that seamlessly bridges continuous and discrete analysis. This generalization not only encapsulates classical integral and discrete sum inequalities as special cases but also opens new avenues for applications in dynamic equations and economic models where processes evolve in a mixed continuous-discrete time setting.

PhD Thesis of Shaista Alamgir on the topic "Linear Inequalities Via Interpolation Polynomial and Weighted Hermite-Hadamard Inequalities"

In this dissertation {19}, a significant portion of the work has been devoted to extending the theory of weighted Hermite–Hadamard inequalities to the **Mercer-type framework**. Chapters 5 and 6 systematically explore linear inequalities derived via various interpolation polynomials—such as those based on the extensions of Montgomery, Taylor, Fink identities, as well as Hermite and Abel–Gontscharoff polynomials—under Mercer-type transformations. These results not only generalize classical and weighted Hermite–Hadamard inequalities but also provide explicit expressions for the unknown terms P and λ in both discrete and integral forms. The incorporation of Green functions further enriches the analysis, offering new insights and broader applicability. These Mercer-type inequalities open avenues for future research in convex analysis, optimization, and applied mathematics, particularly in contexts where symmetric or transformed domains are natural.

Generalizations to Broader Convexity Classes and Settings

The expansion of Mercer-type inequalities to broader frameworks continues unabated. Recent publications have explored:

- **Coordinated convexity** on rectangles, including refinements for coordinated h-convex functions {7}.
- **Interval-valued analysis**, with new versions of Jensen's inequality for interval-valued coordinated convex functions {11}.
- **Operator-theoretic extensions**, including generalizations of Mercer inequalities in Hilbert space settings {9}.
- **Measure-theoretic perspectives**, such as integral Jensen–Mercer inequalities for signed measures {15}.

These works often include computational demonstrations, confirming the practical utility and tightness of the derived theoretical bounds.

Applications in Information Theory and Statistics

The applied reach of these inequalities has been notably emphasized in recent works. Several 2024–2025 papers highlight the utility of refined Jensen–Mercer inequalities in deriving sharp bounds for entropy and divergence measures. Applications now encompass:

- Shannon entropy
- Kullback-Leibler divergence
- Jensen-Shannon divergence
- Applications in probability and statistics {17}

These contributions reinforce the inequality's versatility far beyond the realms of classical convex analysis.

Interpretation and Thematic Summary

A unified analysis of the recent literature yields several key insights:

- **Maturation into a Framework:** The Jensen–Mercer inequality has evolved from a singular result into a comprehensive methodological framework for generating new inequalities across multiple mathematical disciplines.

- **Interdisciplinary Fusion:** The inequality now sits at the crossroads of information theory, operator theory, fractional calculus, numerical analysis, and statistical modeling, demonstrating remarkable interdisciplinary reach.
- **Dual Research Trajectories:** The field exhibits two complementary directions: highly general frameworks (interval-valued, operator-theoretic) alongside specific refinements under stronger assumptions (strong convexity, higher-order differentiability).
- **Computational Validation:** An increasing number of articles supplement theoretical results with numerical evaluations, indicating a growing emphasis on practical applications and algorithmic implementations.
- **Fractional Dominance:** The union of Mercer-type inequalities with fractional calculus has emerged as the most dynamically expanding research direction, with continued growth expected in the coming years.

Open Questions and Future Research Directions

While this volume concludes with the current landscape, the literature suggests several promising avenues for future investigation:

1. Sharpness and precise equality conditions for Jensen–Mercer-type inequalities under strong convexity and other generalized convexity assumptions.
2. Development of stochastic and probabilistic versions involving mean-square fractional integrals and their applications to random processes.
3. Exploration of non-commutative and quantum versions in operator algebras and quantum probability frameworks.
4. Implementation of these inequalities in numerical algorithms, optimization protocols, and machine learning models.
5. Further generalizations to preinvexity, pseudo-convexity, and their interplay with time-scale calculus.
6. Deeper interconnections with majorization theory, Schur-convexity, and optimal transport problems.
7. Systematic exploration of the relationship between different fractional operators and their efficacy in producing sharp bounds for various function classes.

Rationale for Closing the Volume

The decision to conclude this volume at the present juncture is motivated by several considerations:

- The collected results up to 2025 present a sufficiently complete and coherent picture of the current theoretical landscape.
- Continuous revisions in response to each new publication would compromise the structural integrity and readability of the monograph.
- This concluding collection of recent works serves as a strong foundation for future editions once the next wave of results stabilizes and new dominant patterns emerge.

Recent Articles (2024–2025)

Below we present a curated selection of publications from 2024 and 2025 that represent the most significant contemporary work on Jensen–Mercer inequalities and related generalizations.

{1} S. I. Butt, M. V.-Cortez, and H. Inam, Harmonic conformable refinements of Hermite-Hadamard Mercer inequalities by support line and related applications, *Mathematical and Computer Modelling of Dynamical Systems* **30** (1) (2024), 385–416.

{2} S. I. Butt, P. Agarwal and J. J. Nieto, New Hadamard-Mercer inequalities pertaining Atangana-Baleanu operator in Katugampola sense with applications, *Mediterranean J. Math.* **21** (1) (2024): 9.

{3} S. I. Butt and H. Inam, New fractional refinements of harmonic Hermite-Hadamard-Mercer type inequalities via support line, *Filomat* **38** (14) (2024), 5179–5207.

{4} S. I. Butt, M. Nadeem, J. Nasir, A. O. Akdemir, and M. S. Orujova, Jensen–Mercer variant of Hermite-Hadamard type inequalities via generalized fractional operator, *Filomat* **38** (29) (2024), 10463–10483.

{5} S. I. Bradanovic and N. Lovricevic, Generalized Jensen and Jensen–Mercer inequalities for strongly convex functions, *J. Inequal. Appl.* **2024** (1) (2024), 112.

{6} M. A. Khan, S. I. Bradanovic and H. A. Mahmoud, New improvements of the Jensen–Mercer inequality for strongly convex functions with applications, *Axioms* **13** (8) (2024), 553.

{7} M. Toseef, Z. Zhang, A. Mateen, H. Budak and A. Kashuri, Refinement of Jensen–Mercer and Hermite-Hadamard-Mercer inequalities for coordinated h-convex functions, *Ann.*

Math. Phys. **7** (223) (2024), 1–20.

{8} T. Hussain, L. Ciurdariu, and E. Grecu, A new inclusion on inequalities of the Hermite-Hadamard-Mercer type for three-times differentiable functions, *Math.* **12** (23) (2024), 3711.

{9} M. Kian and Z. P. Mazraj, Some generalizations of Mercer inequality and its operator extensions, *arXiv:2403.17557* (2024), 1–18.

{10} F. P. Mohebbi, M. Hassani, M. E. Omidvar, H. R. Moradi and S. Furuichi, Further Jensen–Mercer type inequalities for convex functions, *J. Math. Inequal.* **18** (1) (2024), 39–59.

{11} M. Toseef, Z. Zhang, D. Zhao and M. A. Ali, A new version of Jensen's Mercer inequality for interval-valued coordinated convex functions, *Research Square* **2024** (1) (2024), 1–25.

{12} M. J. S. Saïd and D. F. M. Torres, *Refinements of Jessen–Mercer inequality using time scale calculus*, *Dyn. Syst. Appl.*, **33** (2024), (1), 45–62.

{13} N. Azzouz, B. Benaissa, H. Budak and I. Demir, Hermite-Hadamard-Mercer type inequalities for fractional integrals via ψ-Hilfer operators, *Bound. Value Probl.* **2025** (12) (2025), 1–22.

{14} C. Yildiz, S. Erden, S. Kermausuor, D. Breaz and L.-I. Cotîrlă, New estimates on generalized Hermite-Hadamard-Mercer-type inequalities, *Bound. Value Probl.* **2025** (20) (2025), 1–18.

{15} L. Horvath, Integral Jensen–Mercer inequalities for signed measures with refinements, *Math.* **13** (3) (2025), 539.

{16} J. Khan, M. A. Khan and S. Sarwar, Improvements of Hermite-Hadamard-Mercer inequality using k-fractional integral, *Int. J. Optim. Theory Appl.* **15** (1) (2025), 1–12.

{17} R. Bibi and S. Ali, Refining Jensen–Mercer inequality and its applications in probability and statistics, *Math. Inequal. Appl.*, **2025** (2025), Art. 86.

{18} T. Hussain, L. Ciurdariu, M. Sittar and J. E. N. Valdes, Hermite-Hadamard-Mercer type inequalities for twice differentiable convex involving Riemann-Liouville fractional integrals, *Filomat* **39** (21) (2025), 7483–7497.

{19} Shaistan Alamgir, Linear Inequalities Via Interpolation Polynomial and Weighted Hermite-Hadamard Inequalities, University of Karachi, Karachi, 2024 (Unpublished Doctoral Thesis).

{20} G. A.-Gazić, J. Pečarić and M. Praljak, On generalization of Mercer's inequality involving averages of convex functions, *Applicable Analysis and Discrete Mathematics*, in press

Final Thoughts

The Jensen–Mercer inequality continues to expand in scope, depth, and applicability at an extraordinary pace. This concluding chapter captures the state of the field at a moment of intense and multifaceted development. We hope the present volume serves not only as a definitive reference for the current landscape but also as a catalyst for future discoveries across convex analysis, information theory, fractional calculus, operator theory, and beyond. The foundation laid here will undoubtedly support the next generation of innovations in this dynamically evolving field.

APPENDIX A

On Convexity and Related Results

A.1 On Convexity

In this section we list some of the basics properties of convex functions, P-convex functions and functions with nondecreasing increments. More details about these classes of functions can be found in monographs [204] and [224].

Definition A.1: Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be **convex** if for all $x, y \in I$ and all $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{A.1}$$

holds. If the inequality in ($A.1$) is reversed, then f is said to be **concave**.

Geometrically ($A.1$) means that the graph of f lies below its chords. Convex functions have the following properties:

1. Let I be an interval in $\mathbb{R}$, $p, q \in \mathbb{R}$. *For any* $x, y \in I$, $p, q \geq 0$, $p + q > 0$ ($A.1$) *is equivalent to*
$$f\left(\frac{px+qy}{p+q}\right) \leq \frac{pf(x)+qf(y)}{p+q}.$$
2. Let I be an interval in $\mathbb{R}$. *For any* $x_1, x_2, x_3 \in I$ *such that* $x_1 < x_2 < x_3$,
$$f(x_2) \leq \frac{x_2 - x_3}{x_1 - x_3} f(x_1) + \frac{x_1 - x_2}{x_1 - x_3} f(x_3) \tag{A.2}$$
holds.
3. Let I be an open interval in $\mathbb{R}$. *Function* $f : I \to \mathbb{R}$ *is convex iff it has at least one line of*

support at each $x_0 \in I$, *i.e., for all* $x \in I$

$$f(x) \geq f(x_0) + \lambda (x - x_0)$$

holds, *where* $\lambda \in \mathbb{R}$ *depends on* x_0 *and is given by* $\lambda = f'(x_0)$ *when* $f'(x_0)$ *exists, and* $\lambda \in \left[f'_-(x_0), f'_+(x_0)\right]$ when $f'_-(x_0) \neq f'_+(x_0)$.

4. *Function* $x \longmapsto f(x) - f(x_0) - \lambda(x - x_0)$, *representing the difference between convex function* f *and its line of support, is decreasing function for* $x < x_0$ *and increasing function for* $x > x_0$.

To define convexity of higher order we need the notion of divided difference .

Definition A.2: Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function. A ***n*-th order divided difference** of f at distinct points $x_0, \ldots, x_n \in I$ may be defined recursively by

$$\begin{aligned} [x_i] f &= f(x_i) \\ [x_0, \ldots, x_n] f &= \frac{[x_1, \ldots, x_n] f - [x_0, \ldots, x_{n-1}] f}{x_n - x_0}. \end{aligned}$$

The value $[x_0, \ldots, x_n] f$ is independent of the order of the points $x_0, \ldots, x_n$. This definition can extended by including the cases for more than one point coincide by applying the respective limits. For further details on the topic please see [136].

Definition A.3: A function $f : I \to \mathbb{R}$is said to be ***n*-convex** if

$$[x_0, \ldots, x_n] f \geq 0 \tag{A.3}$$

holds for all choices of $n+1$ distinct points $x_0, \ldots, x_n$ from I. If the inequality in (*A.3*) is reversed, then f is said to be ***n*-concave**.

It is easy to see that 2-convex (2-concave) functions are in fact convex (concave) functions. Some of the properties of n-convex functions are:

1. If a function $f : I \to \mathbb{R}$ is n-convex (n-concave), then there exists $f^{(n-2)}$ and it is convex (concave) function.
2. If $f^{(n)}$ exists, then f is n-convex (n-concave) iff $f^{(n)}(x) \geq 0$, for all $x \in I$ ($f^{(n)}(x) \leq 0$, for all $x \in I$).

Definition A.4: Let I, J be two intervals in $\mathbb{R}$ and $f : I \times J \to \mathbb{R}$ be a function. A **divided difference of order (m, n)** of f at $m+1$ distinct points $x_0, \ldots, x_m \in I$ and $n+1$ distinct points $y_0, \dots, y_n \in J$ is defined by

$$[x_0, \dots, x_m]\,[y_0, \dots, y_n]\, f = [x_0, \dots, x_m]\left([y_0, \dots, y_n]\, f\right)$$
$$= [y_0, \dots, y_n]\left([x_0, \dots, x_m]\, f\right).$$

Definition A.5: Let $m, n \in \mathbb{N}$ and let I, J be two intervals in $\mathbb{R}$. A function $f : I \times J \to \mathbb{R}$ is said to be **convex of order** $(\boldsymbol{m},\boldsymbol{n})$ if for all choices of $m+1$ distinct points $x_0, \dots, x_m$ from I and $n+1$ distinct points $y_0, \dots, y_n$ from J,

$$[x_0, \dots, x_m]\,[y_0, \dots, y_n]\, f \geq 0$$

holds.

A similar class of functions, which we define next, was considered in [213].

Definition A.6: A function $f : I \times J \to \mathbb{R}$ is said to be ***P*-convex of order *n*** if for all choices of $n+1$ distinct points $x_0, \dots, x_n \in I$ and $n+1$ distinct points $y_0, \dots, y_n \in J$

$$[x_0, \dots, x_i]\,[y_0, \dots, y_{k-i}]\, f \geq 0, \qquad i \in \{0, 1, \dots, n\}$$

holds, i.e., if f is convex of order $(i, n-i)$ for all $i \in \{0, 1, \dots, n\}$. We say that f is ***P*-concave of order *n*** if $-f$ is P-convex of order n. If a function f is P-convex of order 2, we simply say that f is ***P*-convex**.

In the same paper [213] several properties of P-convex functions of order n, related to the well-known properties of n-convex functions, were given:

1. A P-convex function of order n is not necessarily continuous on $I \times J$. Indeed, the function $f : I \times J = [a, b] \times [c, d] \to \mathbb{R}$ defined by
$$f(x, y) = \begin{cases} 0, & (x, y) \in I \times J \setminus (b, d) \\ 1, & (x, y) = (b, d) \end{cases},$$
is P-convex of order $n \in \mathbb{N}$, but it is not continuous.
2. If the n-th order partial derivatives of a function $f : I \times J \to \mathbb{R}$ exist, then f is P-convex of order n iff these partial derivatives are nonnegative.
3. If the $(k-1)$-th partial derivatives of a function $f : I \times J \to \mathbb{R}$ exist, then f is P-convex of order k iff these partial derivatives are nondecreasing in each argument.

Example A.1: *Functions* $f : [0, +\infty\rangle^2 \to \mathbb{R}$ *defined as*

$$f(x, y) = x^p y^q$$

are P-convex for $p, q \geq 1$, or $p, q \leq 0$. They are also convex for $p \geq 1, 1-p \leq q \leq 0$, or $1-q \leq p \leq 0, q \geq 1$, or $p, q \leq 0$.

A.1.1 Functions with Nondecreasing Increments

In 1963, H. D. Brunk defined an interesting class of multivariate real valued functions [46] (see also [140, 204]) known as functions with nondecreasing increments.

Definition A.7: A real valued function f on a k-dimensional rectangle $\mathbf{U} \subset \mathbb{R}^k$, where k is a fixed positive integer, is said to have nondecreasing increments if

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) \leq f(\mathbf{b}+\mathbf{h}) - f(\mathbf{b}) \textit{ whenever } \mathbf{0} \leq \mathbf{h} \in \mathbb{R}^k,\ \mathbf{a} \leq \mathbf{b},\ \mathbf{a}, \mathbf{b}+\mathbf{h} \in \mathbf{U}, \tag{A.4}$$

where partial order is defined by

$$(x_1, x_2, \ldots, x_k) \leq (y_1, y_2, \ldots, y_k) \Leftrightarrow x_1 \leq y_1, \ldots, x_k \leq y_k.$$

In the same paper [46], Brunk gave some examples and properties of the functions which we discuss below.

Examples of Functions with Nondecreasing Increments

(i) The simplest example of a function with nondecreasing increments is a constant function.

(ii) Lines of the form $\mathbf{x} = \mathbf{a}t + \mathbf{b}$, where $(0, \ldots, 0) \leq \mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^k$ whose direction cosines are nonnegative also belong to the family of functions with nondecreasing increments.

(iii) An important continuous function with nondecreasing increments is $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy$

1. Another useful continuous function with nondecreasing increments is $f : [0, \infty)^k \to \mathbb{R}$ defined by $f(x) = \prod_{i=1}^{k} x_i$.

(iv) An interesting and widely used example of such functions is the Cauchy functional equation

$$F(\mathbf{x}+\mathbf{y}) = F(\mathbf{x}) + F(\mathbf{y}).$$

Porperties of Functions with Nondecreasing Increments Functions with nondecreasing increments possess the following properties:

(i) A function with nondecreasing increments is not necessarily continuous.

(ii) If the first partial derivatives of a function $f : \mathbf{U} \to \mathbb{R}$ exists for $\mathbf{x} \in \mathbf{U}$, then $\mathbf{x}$ has nondecreasing increments if and only if each of these partial derivatives is nondecreasing in each arguments.

(iii) If the second partial derivatives of a function $f : \mathbf{U} \to \mathbb{R}$ exists for $\mathbf{x} \in \mathbf{U}$, then $\mathbf{x}$ has nondecreasing increments if and only if each of these partial derivatives is nonnegative.

(iv) If a function f with nondecreasing increments is continuous for $\mathbf{b} \leq \mathbf{x} \leq \mathbf{a}+\mathbf{b}$, where

$\mathbf{0} \leq \mathbf{a} \in \mathbb{R}^k$, then the function $f : [0,1] \to \mathbb{R}$ defined by $f(t) = f(t\mathbf{a} + \mathbf{b})$ is convex.

Remark A.1: We may note here that if $n = 2$ and if we only consider functions which have partial derivatives of the second order, then the class of P-convex functions and the class of functions with nondecreasing increments coincide.

We define here a special type of functions which belong to the class of functions with nondecreasing increments and themselves contain the class of convex functions. These functions are called Wright convex functions [204, p. 7].

Definition A.8: We say $f : I \to \mathbb{R}$ is a Wright convex function if $\forall\ a, b + h \in I$ with $a < b$ and $h > 0$, we have

$$f(a+h) - f(a) \leq f(b+h) - f(b)$$

Remark A.2: It is easily to see that in one dimension case functions with nondecreasing increments are Wright convex functions. Also, continuous Wright convex functions are convex functions. Thus, the class of convex functions is a proper subclass of the Wright convex functions.

A.1.2 Functions with Nondecreasing Increments of Order n

In [139] we can find the following expressions for functions with nondecreasing increments of order n.

If we write $\Delta_{\mathbf{h_1}} f(\mathbf{x}) = f(\mathbf{x} + \mathbf{h_1}) - f(\mathbf{x})$ and inductively,

$$\Delta_{\mathbf{h_1}} \Delta_{\mathbf{h_2}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{x}) = \Delta_{\mathbf{h_1}} (\Delta_{\mathbf{h_2}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{x})) \quad \text{for} \quad n \geq 2,$$

where $\mathbf{x}, \mathbf{x} + \mathbf{h_1} + \cdots + \mathbf{h_n} \in \mathbf{U}$, $\mathbf{h_i} \in \mathbb{R}^k_*$ for $i \in \{1, \ldots, n\}$, where $\mathbb{R}^k_*$ denotes set of nonnegative $\mathbb{R}^k$. Using this notation with $\mathbf{h} = \mathbf{h_1}$, $\mathbf{s} = \mathbf{h_2}$, $\mathbf{b} = \mathbf{a} + \mathbf{s}$, condition (A.4) becomes

$$\Delta_{\mathbf{h_1}} \Delta_{\mathbf{h_2}} f(\mathbf{a}) \geq 0.$$

The Definition A.7 can also to be extended to the following form.

Definition A.9: Let $f : \mathbf{U} \to \mathbb{R}$ is said to be a *function with nondecreasing increments of order n* if

$$\Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{x}) \geq 0$$

holds, whenever $\mathbf{x}, \mathbf{x} + \mathbf{h_1} + \cdots + \mathbf{h_n} \in \mathbf{U}$, $\mathbf{h_i} \in \mathbb{R}^k_*$ for $i \in \{1, \ldots, n\}$.

A.1.3 Monotocity in Means

Here we describe an interesting concept of monotonicity in means (see also [207]).

Definition A.10: A finite sequence $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \in \mathbf{U}^n$ is said to be *nondecreasing in means* with respect to weights $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n_+$ if the inequalities

$$\mathbf{X_1} \leq A_2(\mathbf{X}; \mathbf{w}) \leq \cdots \leq A_n(\mathbf{X}; \mathbf{w}) \tag{A.5}$$

hold, where

$$A_j(\mathbf{X}; \mathbf{w}) = \frac{1}{W_j} \sum_{i=1}^{j} w_i \mathbf{X_i}, \quad W_j = \sum_{i=1}^{j} w_i.$$

If the directions of inequalities are reversed in (A.5), then the sequence $(\mathbf{X_1}, \ldots, \mathbf{X_n})$ is said to be *nonincreasing in means.*

A.2 On Operator Convexity

We denote by $\mathcal{B}(H)$ the C^*-algebra of all bounded (i.e, continuous) linear operators on a Hilbert space H with operator norm. On its subspace of all bounded selfadjoint operators we introduce a partial order as follows [157]:

Definition A.11: A selfadjoint operator $A \in \mathcal{B}(H)$ is **positive semi-definite** (simply, **positive**) and we write $A \geq 0$ if $(Ax, x) \geq 0$ for each $x \in H$.
For selfadjoint operators $A, B \in \mathcal{B}(H)$ we write $B \geq A$ (or $A \leq B$) if $B - A \geq 0$.

We denote by $Sp(A)$ the spectrum of an operator A, i.e., the set $\{\lambda \in \mathbb{C} : A - \lambda I$ is not invertible in $\mathcal{B}(H)\}$. We denote by $C([m, M])$ the set of all real valued continuous functions on an interval $[m, M] \subset \mathbb{R}$.

Definition A.12: A function $f \in C([m, M])$ *is said to be* **operator monotone** if it preserves the operator order, i.e., if $A \leq B$ implies $f(A) \leq f(B)$, for all selfadjoint operators A and B from $\mathcal{B}(H)$ such that $Sp(A), Sp(B) \subseteq [m, M]$.

Definition A.13: A function $f \in C([m, M])$ *is said to be* **operator increasing** if f is operator monotone. A function $f \in C([m, M])$ *is said to be* **operator decreasing** *if* $-f$ is operator monotone.

Definition A.14: *A real valued continuous function f defined on an interval I is said to be* **operator convex** *if*

$$f((1 - \lambda) A + \lambda B) \leq (1 - \lambda) f(A) + \lambda f(B) \tag{A.6}$$

holds for all $\lambda \in [0, 1]$ and for all selfadjoint operators A and B from $\mathcal{B}(H)$ *with spectrum contained in I. A real valued continuous function f is said to be* **operator concave** *if the inequality in (A.6) is reversed.*

Example A.2: *A function $f : [0, +\infty\rangle \to \mathbb{R}$, $f(t) = t^p$ is operator monotone if $0 \leq p \leq 1$. A function $f : \langle 0, +\infty\rangle \to \mathbb{R}$, $f(t) = t^p$ is operator convex if $1 \leq p \leq 2$ or $-1 \leq p \leq 0$, and it is operator concave if $0 \leq p \leq 1$.*

Example A.3: *A function $f : \langle 0, +\infty\rangle \to \mathbb{R}$, $f(t) = \ln t$ is operator monotone and operator concave.*

Example A.4: *Exponential function is neither operator monotone nor operator convex.*

Remark A.3: For a *real continuous function f defined on an interval $I = [\alpha, +\infty\rangle$* and bounded from below the following is equivalent:

(i) f is operator monotone on I,
(ii) f is operator concave on I.

Proof of the assertion in Remark A.3 can be found in [100, p. 16]. The first statement in Example A.2 could be rewritten as the following theorem, which has been proved in some more general form by Löwner [161] in 1934.

Theorem A.1 ***(Löwner-Heinz inequality):*** *Let A and B be two positive semi-definite operators from $\mathcal{B}(H)$. If $A \geq B$, then $A^p \geq B^p$ for all $p \in [0,1]$.*

In [100, p. 220, 232, 250] we can found the following theorems related to the preservation of operator order.

Theorem A.2: *Let A, B be two selfadjoint operators from $\mathcal{B}(H)$ such that $Sp(A) \subseteq [m_1, M_1]$, and $Sp(B) \subseteq [m_2, M_2]$ for some scalars $M_j > m_j > 0, j \in \{1,2\}$. If $A \geq B$, then the following inequalities hold:*
(i) For all $p > 1$

$$K(m_1, M_1, p)\, A^p \geq B^p,$$
$$K(m_2, M_2, p)\, A^p \geq B^p.$$

(ii) For all $p < -1$

$$K(m_1, M_1, p)\, B^p \geq A^p,$$
$$K(m_2, M_2, p)\, B^p \geq A^p.$$

where the generalized Kantorovich constant $K(m, M, p)$ is defined by

$$K(m, M, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \right)^p$$

for all $p \in \mathbb{R}$, where $K(m, M, 0) = \lim\limits_{p \to 0} K(m, M, p) = 1$ and $K(m, M, 1) = \lim\limits_{p \to 1} K(m, M, p) = 1$.

Theorem A.3: *Let A, B be two selfadjoint operators from $\mathcal{B}(H)$ such that $Sp(B) \subseteq [m, M]$ for some scalars $M > m$. If $A \geq B$, then*

$$S\left(e^{M-m}\right) e^A \geq e^B$$

holds, where the Specht ratio $S(h)$ for $h > 0$ is defined by $S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \ln h}$ $(h \neq 1)$ and $S(1) = 1$.

At the end of this section we define normalized positive linear map.

Definition A.15: A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is said to be **linear** if it is additive and homogenous, i.e., if $\Phi(\alpha A + \beta B) = \alpha\Phi(A) + \beta\Phi(B)$ for each $\alpha, \beta \in \mathbb{C}$ and for all $A, B \in \mathcal{B}(H)$.
A linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is said to be **positive** if it preserves the operator order, i.e., if $A \geq 0$ implies $\Phi(A) \geq 0$ for each positive $A \in \mathcal{B}(H)$.
A linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is said to be **normalized** if it preserves the identity operator, i.e., $\Phi(1_H) = 1_K$.

More about examples and properties of positive linear mappings can be found in [100]. We denote by $\boldsymbol{P}[\mathcal{B}(H), \mathcal{B}(K)]$ the class of all positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$.

A.3 On Jensen and Companion Inequalities

More details about inequalities in this section can be found in monographs [204] and [183]. Throughout this section we assume that I and $[a, b]$ are intervals in $\mathbb{R}$. Jensen gave his famous inequality in 1905.

Theorem A.4*(Jensen's inequality)*: *Let I be an interval in $\mathbb{R}$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ $(n \geq 2)$, and $\boldsymbol{w} = (w_1, \ldots, w_n)$ be a positive n-tuple. If a function $f : I \to \mathbb{R}$ is convex, then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i) \tag{A.7}$$

holds, where $W_n = \sum_{i=1}^{n} w_i$. However, if f is concave on I, then the reversed inequality

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \geq \frac{1}{W_n}\sum_{i=1}^{n} W_i f(x_i) \tag{A.8}$$

holds.

If f is strictly convex (strictly concave), then (A.7) ((A.8)) is strict unless $x_i = c$ for all $i \in \{j : w_j > 0\}$.

Following results is also valid [117]:

Theorem A.5: *Let $f : (a, b) \to \mathbb{R}$ be a convex function on $(a, (a + b)/2]$ and $f(x) = -f(a + b - x)$ for every $x \in (a, b)$. If $x_i \in (a, b)$, $w_i > 0$, $(x_i + x_{n-i+1})/2 \in (a, (a + b)/2]$ and $\dfrac{w_i x_i + w_{n-i+1} x_{n-+1}}{w_i + w_{n-i+1}} \in (a, (a + b)/2]$ for $i \in \{1, 2, \ldots, n\}$, then (A.7) holds.*

The condition "$\boldsymbol{w}$ is positive $n-$tuple" in Jensen inequality can be replaced by "$\boldsymbol{w}$ is a non$-$negative $n-$tuple" and $W_n > 0$. It is reasonable to inquire whether the condition "$\boldsymbol{w}$ is a non$-$negative $n-$tuple" can be relaxed at the expense of restricting $\mathbf{x}$ more severely. Steffensen (see [232]) was the first to answer this query in Theorem A.6 (see the proof of corresponding Theorem in [204]).

Theorem A.6 ***(Jensen-Steffensen inequality)***: *If a function $f: I \to \mathbb{R}$ is convex (concave), $\boldsymbol{x} \in I^n$ is monotone n-tuple and $\boldsymbol{w}$ is real n-tuple such that*

$$0 \leq W_k \leq W_n, \quad k \in \{1, \ldots, n-1\}, \quad W_n > 0, \tag{A.9}$$

where $W_k = \sum_{i=1}^{k} w_i$, then inequality (A.7) ((A.8)) is valid.

Again, for strictly convex (strictly concave) function f (A.7) ((A.8)) remains strict unless $x_i = c$ for all $i \in \{j : w_j \neq 0\}$.

In 1979. Vasić and Pečarić obtained a simple consequence of (A.7) using the substitutions $w_1 \to W_n$, $x_1 \to \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i$, $w_i \to -w_i$, $x_i \to x_i$ for $2 \leq i \leq n$.

Theorem A.7 ***(Reverse Jensen's inequality)***: *Let $\boldsymbol{w}$ be a real n-tuple such that*

$$w_1 > 0,\ w_i \leq 0,\ i \in \{2, \ldots, n\},\ W_n > 0,$$

and let n-tuple $\boldsymbol{x} \in I^n$ satisfy $\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \in I$. If a function $f: I \to \mathbb{R}$ is convex , then

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \geq \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(x_i\right) \tag{A.10}$$

holds.

Converse Jensen's inequality was proved by Lah and Ribarič in 1973.

Theorem A.8 ***(Converse Jensen's Inequality)***: *Let I be an interval in $\mathbb{R}$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in [a, b]^n \subseteq I^n$ $(n \geq 2)$, and let $\boldsymbol{w} = (w_1, \ldots, w_n)$ be a positive n-tuple. If a function $f: I \to \mathbb{R}$ is convex on I, then*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(x_i\right) \leq \frac{M - \overline{x}}{b-a} f\left(a\right) + \frac{\overline{x} - a}{b-a} f\left(b\right)$$

holds, where $\overline{x} = \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i$. Equality holds in the above iff $x_i = m$ for $i \in I \subset \{1, \ldots, n\}$ and $x_i = M$ for $i \in \{1, \ldots, n\} \backslash I$.

One of the famous refinement of Jensen inequality is Hermite-Hadamard inequality. The Hermite-Hadamard inequality is given below [105]:

Theorem A.9 ***(*Hermite-Hadamard Inequality*)***: *Let $f: I \to \mathbb{R}$ be a convex function, where I is an interval and $a, b \in I$ such that $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{A.11}$$

If f is concave, then (A.11) holds in the reversed direction.

Another interesting inequality is the Petrović inequality (see e.g. [204, page 152]). It states that

Theorem A.10: **(Petrović inequality)** *If $f : [0, \infty) \to \mathbb{R}$ is a continuous convex function and $A_1, \cdots, A_n$ are positive operators such that $\sum_{i=1}^{n} A_i = MI$ for some scalar $M > 0$, then*

$$\sum_{i=1}^{n} f(A_i) \leq f\left(\sum_{i=1}^{n} A_i\right) + (n-1)f(0).$$

Now, we state integral versions of Jensen inequality as under:

Theorem A.11: *Let $f : [a, b] \to I$ be a continuous function. If the function $H : [a, b] \to \mathbb{R}$ is nondecreasing, bounded and $H(a) \neq H(b)$, then for every continuous convex function $\varphi : I \to \mathbb{R}$ the inequality*

$$\varphi\left(\frac{\int_a^b f(t)dH(t)}{\int_a^b dH(t)}\right) \leq \frac{\int_a^b \varphi(f(t))dH(t)}{\int_a^b dH(t)} \tag{A.12}$$

holds.

Inequality (A.12) can hold under different set of assumptions. For example, for a monotonic f, assumptions on H can be relaxed. The following Theorem gives Jensen-Steffensen's inequality.

Theorem A.12: *If $f : [a, b] \to J$ is continuous and monotonic (either nonincreasing or nondecreasing) and $H : [a, b] \to \mathbb{R}$ is either continuous or of bounded variation satisfying*

$$H(a) \leq H(t) \leq H(b) \quad \text{for all } t \in [a, b], \qquad H(b) > H(a),$$

then (A.12) holds.

If we replace the assumption of monotonicity of f over the whole interval $[a, b]$ in Theorem A.12 with monotonicity over subintervals, we obtain the following, Jensen-Boas inequality.

Theorem A.13: *If $H : [a, b] \to \mathbb{R}$ is continuous or of bounded variation satisfying*

$$H(a) \leq H(x_1) \leq H(y_1) \leq H(x_2) \leq \cdots \leq H(y_{k-1}) \leq H(x_k) \leq H(b)$$

for all $x_i \in (y_{i-1}, y_i)$ ($y_0 = a$, $y_k = b$), and $H(b) > H(a)$, and if f is continuous and monotonic (either nonincreasing or nondecreasing) in each of the k intervals (y_{i-1}, y_i), then inequality (A.12) holds.

If we replace the interval I with a convex set U from a real vector space V, and $x_1, \ldots, x_n$ with vectors in U, then inequality (A.7) remains valid for convex functions defined on U. Jensen's inequality can be also generalized for positive linear functionals.

A.4 On Jessen Inequality

A.4.1 Isotonic Linear Functional

[173] Let E be a nonempty set, $\mathfrak{A}$ be an algebra of subsets of E, and L be a subspace of the vector

space $\mathbb{R}^E$ over $\mathbb{R}$ which contains $\mathbf{1}$, that is L having the following properties for $f, g : E \to \mathbb{R}$:

L1 : $f, g \in L \Rightarrow (\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

L2 : $\mathbf{1} \in L$, i.e., if $f(t) = 1$ for all $t \in E$, then $f \in L$;

L3 : $f \in L$, $E_1 \in \mathfrak{A} \Rightarrow f.\chi_{E_1} \in L$,

where χ_{E_1} is the indicator function of E_1. It follows from L_2, L_3 that $\chi_{E_1} \in L$ for every $E_1 \in \mathfrak{A}$.

If L additionally has property

L4 : $(\forall f, g \in L)\,(\min\{f, g\} \in L \wedge \max\{f, g\} \in L)$,

then it is a *lattice*. Obviously, $\left(\mathbb{R}^E, \leq\right)$ (with standard ordering) is a lattice. It can also be easily verified that a subspace $X \subseteq \mathbb{R}^E$ is a lattice if and only if $x \in X$ implies $|x| \in X$. This is a simple consequence of the fact that for every $x \in X$ the functions $|x|$, x^- and x^+ can be defined by

$$|x|\,(t) = |x\,(t)|\,, \quad x^+\,(t) = \max\{0, x\,(t)\}\,, \quad x^-\,(t) = -\min\{0, x\,(t)\}\,, \qquad t \in E,$$

and $x^+ + x^- = |x|$, $x^+ - x^- = x$,

$$\min\{x, y\} = \frac{1}{2}\left(x + y - |x - y|\right), \quad \max\{x, y\} = \frac{1}{2}\left(x + y + |x - y|\right).$$

An isotonic linear functional $A : L \to \mathbb{R}$ is a functional satisfying the following properties:

A1 : $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f, g \in L, \alpha, \beta \in \mathbb{R}$;

A2 : $f \in L, f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$;

If A additionally satisfy the condition $A(\mathbf{1}) = 1$, then it is *normalized*.

It follows from L_3 that for every $E_1 \in \mathfrak{A}$ such that $A(\chi_{E_1}) > 0$, the functional A_1 defined for all $f \in L$ as $A_1(f) = \dfrac{A(f.\chi_{E_1})}{A(\chi_{E_1})}$ is an isotonic linear functional with $A(\mathbf{1}) = 1$. Furthermore, we observe that

$$A(\chi_{E_1}) + A(\chi_{E \setminus E_1}) = 1,$$

$$A(f) = A(f.\chi_{E_1}) + A(f.\chi_{E \setminus E_1}).$$

In 1931, Jessen [124] gave the following generalization of Jensen's inequality.

Theorem A.14 ***(Jessen's Inequality)*:** *Let L satisfy properties L1, L2 on a nonempty set E, and assume that φ is a continuous convex function on an interval $I \subset \mathbb{R}$. If A is a positive linear functional on L with $A(1) = 1$, then for all $f \in L$ such that $\varphi(f) \in L$ we have $A(f) \in I$ and*

$$\varphi(A(f)) \leq A(\varphi(f)).$$

Similarly as Jensen's inequality, Jessen's inequality also has its converse. Beesack and Pečarić proved it in [36] in 1985.

Theorem A.15 ***(Converse Jessen's Inequality)*:** *Let L satisfy properties L1 and L2 on a nonempty set E, and assume that φ is a convex function on an interval $I = [a, b]$ $(-\infty < a < b < \infty)$. If A is a positive linear functional on L with $A(1) = 1$, then for all $f \in L$ such that $\varphi(f) \in L$ $(a \leq f(t) \leq b$ for each $t \in E)$, the following inequality holds*

$$A(\varphi(f)) \leq \frac{b - A(f)}{b-a} f(a) + \frac{A(f) - a}{b-a} f(b).$$

Mond and Pečarić [186] in 1997 obtained the following operator variant of the classical Jensen's inequality and its converse also.

Theorem A.16: *Let $A_1, \ldots, A_k \in \mathcal{B}(H)$ be selfadjoint operators such that $mI \leq A_j \leq MI$ for some scalars $m < M$ and let $x_1, \ldots x_k \in H$ satisfy $\sum_{j=1}^{k} (x_j, x_j) = 1$. If $f \in C([m, M])$ is a convex function on $[m, M]$, then the following inequality holds*

$$f\left(\sum_{j=1}^{k} (A_j x_j, x_j)\right) \leq \sum_{j=1}^{k} (f(A_j) x_j, x_j).$$

Theorem A.17: *Let $A_1, \ldots, A_k \in \mathcal{B}(H)$ and $x_1, \ldots, x_k \in H$ be as in the previous theorem. If $f \in C([m, M])$ is a convex function on $[m, M]$, then the following inequality holds*

$$\sum_{j=1}^{k} (f(A_j) x_j, x_j) \leq \frac{M - \sum_{j=1}^{k} (A_j x_j, x_j)}{M-m} f(m) + \frac{\sum_{j=1}^{k} (A_j x_j, x_j) - m}{M-m} f(M).$$

Remark A.4: Analogous results for $k = 1$ can be found in [185].

The next theorem gives the Davis-Choi-Jensen inequality (see [100, p. 22]) for operator convex functions.

Theorem A.18: *If $\Phi \in \boldsymbol{P}[\mathcal{B}(H), \mathcal{B}(K)]$ is normalized positive linear map and $f \in C([m, M])$ is an operator convex function on an interval $[m, M]$, then*

$$f(\Phi(A)) \leq \Phi(f(A)) \tag{A.13}$$

holds for each selfadjoint operator A with spectrum contained in $[m, M]$.

Hansen, Pečarić and Perić [108] in 2005 proved the following generalization.

Theorem A.19: *If $\Phi_1, \ldots, \Phi_k \in \boldsymbol{P}[\mathcal{B}(H), \mathcal{B}(K)]$ are positive linear maps with $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$ and $f \in C([m, M])$ is operator convex function on $[m, M]$, then*

$$f\left(\sum_{j=1}^{k} \Phi_j(A_j)\right) \leq \sum_{j=1}^{k} \Phi_j(f(A_j)) \tag{A.14}$$

holds for all selfadjoint operators $A_1, \ldots, A_k \in \mathcal{B}(H)$ with spectrums contained in $[m, M]$.

Now we show that some inequalities which held true for convex functions also hold for functions with nondecreasing increments.

As we will see from the following two theorems, P-convex functions and functions with nondecreasing increments have a common property: the well known Jensen-Steffensen's inequality holds for them.

Theorem A.20: *[213, Theorem 6]Let $a \leq \xi_1 \leq \cdots \leq \xi_m \leq b$, $c \leq \eta_1 \leq \cdots \leq \eta_m \leq d$ and $w_1, \ldots, w_m$ be real numbers such that*

$$0 \leq W_j \leq W_m \quad (j \in \{1, \ldots, m-1\}), \quad W_m > 0. \tag{A.15}$$

where $W_j = \sum_{i=1}^{j} w_i$. Then for any P-convex function $\varphi : [a, b] \times [c, d] \to \mathbb{R}$,

$$\varphi\left(\frac{1}{W_m}\sum_{i=1}^{m} w_i\xi_i, \frac{1}{W_m}\sum_{i=1}^{m} w_i\eta_i\right) \leq \frac{1}{W_m}\sum_{i=1}^{m} w_i\varphi(\xi_i, \eta_i) \tag{A.16}$$

holds.

Remark A.5: Although it is not explicitly stated, it can be easily checked that inequality ($A.16$) also holds if $a \leq \xi_m \leq \cdots \leq \xi_1 \leq b$ *and* $c \leq \eta_m \leq \cdots \leq \eta_1 \leq d$, *i.e.,* it holds whenever n-tuples $\boldsymbol{\xi} \in [a, b]^m$ and $\boldsymbol{\eta} \in [c, d]^m$ are monotone in the same direction. But, neither ($A.16$) nor its reverse generally hold if they are monotone in opposite directions. Also, if function φ is P-concave, the inequality in ($A.16$) is reversed.

Theorem A.21 ***([209, Theorem 2]):*** *Let $f : K \to \mathbb{R}$, where $K \subset \mathbb{R}^k$ is a k-dimensional rectangle, be a continuous function with nondecreasing increments. Let $\boldsymbol{w}$ be a real m-tuple such that ($A.15$), and let $\boldsymbol{\xi}^{(i)} \in K$ $(i \in \{1, 2, \ldots, m\})$ be such that*

$$\boldsymbol{\xi}^{(1)} \leq \cdots \leq \boldsymbol{\xi}^{(m)} \quad \text{or} \quad \boldsymbol{\xi}^{(1)} \geq \cdots \geq \boldsymbol{\xi}^{(m)}.$$

Then

$$f\left(\frac{1}{W_m}\sum_{i=1}^{m} w_i\boldsymbol{\xi}^{(i)}\right) \leq \frac{1}{W_m}\sum_{i=1}^{m} w_i f(\boldsymbol{\xi}^{(i)}) \tag{A.17}$$

holds, where $W_m = \sum_{i=1}^{m} w_i$.

The Theorem A.21 gives the Jensen's inequality for functions with nondecreasing increments [208] as its direct consequence (see also [26]).

Theorem A.22: *Let $f : K \to \mathbb{R}$ be a continuous function with nondecreasing increments, let $\mathbf{w}$ be a nonnegative n-tuple such that $W_m > 0$ and let $\boldsymbol{\xi}^{(i)} \in \mathbf{U}$ where $i \in \{1, \ldots, m\}$ be such that $\boldsymbol{\xi}^{(1)} \leq \cdots \leq \boldsymbol{\xi}^{(m)}$ or $\boldsymbol{\xi}^{(1)} \geq \cdots \geq \boldsymbol{\xi}^{(m)}$. Then (A.17) holds.*

Remark A.6: Let f and $\boldsymbol{\xi}^{(i)} \in K$ where $i \in \{1, \ldots, n\}$ be as defined in Theorem A.21 and let $\mathbf{w}$ be a real m-tuple such that $w_1 > 0, \quad w_i \leq 0, \quad i \in \{2, \ldots, m\}, W_m > 0$ and $\frac{1}{W_m}\sum_{i=1}^{m} w_m\boldsymbol{\xi}^{(i)} \in$

U, then the inequality (A.17) holds in reverse direction.

We now state integral version of Jensen-Steffensen's inequaltiy for functions with nondecreasing increments [207].

Theorem A.23: *If* $\mathbf{f} : [a, b] \to \mathbf{J}$ *is a nondecreasing continuous function and* $H : [a, b] \to \mathbb{R}$ *is of bounded variation satisfying*

$$H(a) \leq H(t) \leq H(b) \quad \text{for all } t \in [a, b], \qquad H(b) > H(a). \tag{A.18}$$

If $\varphi : \mathbf{J} \to \mathbb{R}$ *is a continuous function with nondecreasing increments, then the following inequality holds*

$$\varphi\left(\frac{\int_a^b \mathbf{f}(t)dH(t)}{\int_a^b dH(t)}\right) \leq \frac{\int_a^b \varphi(\mathbf{f}(t))dH(t)}{\int_a^b dH(t)}, \tag{A.19}$$

where $\int_a^b \mathbf{f}dH = (\int_a^b f_1 dH, \ldots, \int_a^b f_k dH)$.

Remark A.7: If $\dfrac{\int_a^b \mathbf{f}(t)dH(t)}{\int_a^b dH(t)} \in \mathbf{U}$ and we have either $H(x) \leq H(a)$ or $H(x) \geq H(b)$, then the reverse inequality in ($A.19$) holds.

At this stage we prove Jensen-Boas inequality for functions with nondecreasing increments as follows.

Theorem A.24: *Let* $\mathbf{f} : [a, b] \to \mathbf{J}$ *be a continuous and monotonic (either nonincreasing or nondecreasing) map in each of the* k *intervals* (b_{i-1}, b_i). *Let* $H : [a, b] \to \mathbb{R}$ *is continuous or of bounded variation satisfying*

$$H(a) \leq H(a_1) \leq H(b_1) \leq H(a_2) \leq \cdots \leq H(b_{k-1}) \leq H(a_k) \leq H(b) \tag{A.20}$$

for all $a_i \in (b_{i-1}, b_i)$ $(b_0 = a,\ b_k = b)$, *and* $H(b) > H(a)$. *If* φ *is continuous function having nondecreasing increments in each of the* k *intervals* (b_{i-1}, b_i), *then the inequality* ($A.19$) *holds*

Remark A.8: If $\dfrac{\int_a^b \mathbf{\Upsilon}(t)dB(t)}{\int_a^b dB(t)} \in \mathbf{U}$ and $\forall x \in a_i$ such that $(b_{i-1}, b_i)(b_0 = a, b_l = b)$ and we have either $B(x) \geq B(b)$ or $B(x) \leq B(a)$, then the reverse inequality in ($A.19$) holds.

In 1981 Slater proved the following interesting companion inequality to the Jensen inequality (see [231]).

Theorem A.25: *Let the function* $f : (a, b) \to \mathbb{R}$ *be monotonic and convex,* $x_i \in (a, b)$ *and* $w_i \geq 0$ $i \in \{\ldots, n\}$ *with* $W_n > 0$. *If* $\sum_{i=1}^n w_i f'_+(x_i) \neq 0$, *then*

$$\frac{1}{W_n}\sum_{i=1}^n w_i f(x_i) \leq f\left(\frac{\sum_{i=1}^n w_i x_i f'_+(x_i)}{\sum_{i=1}^n w_i f'_+(x_i)}\right), \tag{A.21}$$

holds. When f *is strictly convex on* (a, b), *inequality (A.21) becomes equality iff* $x_i = c$ *for some* $c \in (a, b)$ *and for all* i *with* $w_i > 0$.

In [212] Pečarić noted that (A.21) remains true if we replace the condition of monotonicity of f by $\frac{\sum_{i=1}^{n} w_i x_i f'_+(x_i)}{\sum_{i=1}^{n} w_i f'_+(x_i)} \in (a,b)$, which is in fact more general and can hold for suitable points in (a,b) and not necessarily for monotone functions. In the same paper, one can find the multidimensional case of Slater inequality.

Γ: Let X be a real normed linear space and let X^* be the dual space of X, that is, the real space of all linear functionals $X^* : X \to \mathbb{R}$. Let C be an open convex subset of the real normed linear space X. If $f : C \to \mathbb{R}$ is a convex function, then for any fixed point $y \in C$, we can define the abstract subdifferential $\partial f(y)$ of f at y as follows:

$$\partial f(y) = \{a^*(y,\cdot) \in X^* : f(x) \geq f(y) + a^*(y, x-y), \forall x \in C$$

The set $\partial f(y)$ is non-empty for all $y \in C$ (see [61, p. 108 Theorem B]). Also, when f is strictly convex, the inequality

$$f(x) \geq f(y) + a^*(y, x-y), \forall x \in C$$

is strict unless $x = y$.

$$f(x) \geq f(y) + a^*(y; x-y), \quad \forall\, x, y \in C \tag{A.22}$$

is strict unless $x = y$.

For convenience we take $m = f'_+(y)$ and so (A.22) becomes

$$f(x) \geq f(y) + f'_+(y)(x-y), \quad \forall\, x, y \in (a,b).$$

The following theorem is proved by Matic and Pečarić in [165].

Theorem A.26: *Let $f : C \to \mathbb{R}$ be a convex function defined on an open convex subset C in the real normed linear space X. For the given vectors $x_i \in C$, $w_i \geq 0$ $i \in \{1, \ldots, n\}$ such that $W_n > 0$, let us denote*

$$\bar{x} = \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \text{ and } \bar{y} = \frac{1}{W_n} \sum_{i=1}^{n} w_i f(x_i).$$

If c, d are arbitrary vectors in C, then

$$f(c) + a^*(c; \bar{x} - c) \leq \bar{y} \leq f(d) + \frac{1}{W_n} \sum_{i=1}^{n} w_i a^*(x_i; x_i - d), \tag{A.23}$$

holds. Also, when f is strictly convex, we have equality in the left inequality in (A.23) iff $x_i = c$ holds for all indices i with $w_i > 0$, while equality holds in the right inequality in (A.23) iff $x_i = d$ holds for all indices i with $w_i > 0$.

Levinson in [158] proved that:

Theorem A.27: *If a real-valued function f defined on $[0,2a] \subset \mathbb{R}$ has a nonnegative third derivative, then*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \le \frac{1}{W_n}\sum_{i=1}^{n} w_i f(y_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i y_i\right) \tag{A.24}$$

holds for $0 < x_i < a$, $y_i = 2a - x_i$ and $w_i > 0$, $i \in \{1,\ldots,n\}$ such that $W_n = \sum_{i=1}^{n} w_i$.

Remark A.9: If $a = \frac{1}{2}$, $w_1 = \cdots = w_n = 1$ and $f(x) = \ln x$, then Levinson's inequality ($A.24$) becomes the renowned Fan's inequality

$$\frac{G_n}{G_n'} \le \frac{A_n}{A_n'},$$

where

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad A_n' = \frac{1}{n}\sum_{i=1}^{n}(1-x_i)$$

and

$$G_n = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \quad G_n' = \left(\prod_{i=1}^{n}(1-x_i)\right)^{1/n}.$$

In [206], Pečarić showed that instead of variables with sum equal to $2a$, we can use variables with constant difference and that result becomes a source of some further generalizations [204, p. 74,75]. In fact, he proved that if f is a real-valued 3-convex function on $[a,b]$ and x_i, y_i for $i \in \{1,\ldots,n\}$ are $2n$ points in $[a,b]$ such that $y_1 - x_1 = y_2 - x_2 = \cdots = y_n - x_n > 0$ and $w_i > 0$, $i \in \{1,\ldots,n\}$, then ($A.24$) is valid.

Now we define the notion of majorization as follows:

Definition A.16: (**Majorization**) Given two real row n-tuples $\mathbf{x} = (x_1,\ldots,\ x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$, $\mathbf{y}$ is said to majorize $\mathbf{x}$ or $\mathbf{x}$ is majorized by $\mathbf{y}$, symbolically $\boldsymbol{y} \succ \boldsymbol{x}$, if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$$

holds for $k \in \{1,\ldots,n-1\}$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where $x_{[1]} \ge \ldots \ge x_{[n]}$, and $y_{[1]} \ge \ldots \ge y_{[n]}$, are the entries of $\mathbf{x}$ and $\mathbf{y}$, respectively, in nonincreasing order.

Note that $\boldsymbol{y} \succ \boldsymbol{x}$ implies that, for a fixed sum, the components of $\boldsymbol{y}$ are more diverse than those of $\boldsymbol{x}$. For instance, $\boldsymbol{y} \succ \overline{\boldsymbol{y}}$ always holds for $\overline{\boldsymbol{y}} = (\overline{y},\ldots,\overline{y})$, where

$$\overline{y} = \frac{1}{m}\sum_{i=1}^{m} y_i.$$

This notion and notation of majorization was introduced by Hardy et al. in [110]. Now, we state the well-known majorization theorem from the same book as follows (see also [164]).

Theorem A.28: *(**Majorization Theorem**) Let* $\mathbf{x}$, $\mathbf{y}$ $\in \mathbb{R}^n$*. The inequality*

$$\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$$

holds for every continuous convex function $f : \mathbb{R} \to \mathbb{R}$ *iff* $\mathbf{x} \prec \mathbf{y}$.

In 1947, Fuchs gave a weighted generalization of the well-known majorization theorem for convex functions and two sequences monotonic in the same sense [99], (see also [204, p. 323], [164, p. 580]).

Theorem A.29: *Let* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *be two decreasing real* n*-tuples, let* $\mathbf{w} = (w_1, \ldots, w_n)$ *be a real* n*-tuple such that*

$$\sum_{i=1}^{k} w_i\, x_i \;\leq\; \sum_{i=1}^{k} w_i\, y_i \;\; for \;\; k \in \{1, \ldots, n-1\};$$

and

$$\sum_{i=1}^{n} w_i\, x_i \;=\; \sum_{i=1}^{n} w_i\, y_i;$$

then, for every continuous convex function $f : I \to \mathbb{R}$*, we have*

$$\sum_{i=1}^{m} w_i f(x_i) \leq \sum_{i=1}^{m} w_i f(y_i)\;\; .$$

Here we give a definition of integral version of Majorization, which is stated as follows.

Definition A.17: [204, p. 324] f is majorized by g ($f(t) \prec g(t)$) for $t \in [a, b]$ if they are nonincreasing in $[a, b]$ and

$$\begin{aligned}\int_a^x f(t)d(t) \;&\leq\; \int_a^x g(t)d(t), \quad a < x < b\\ \int_a^b f(t)d(t) \;&=\; \int_a^b g(t)d(t).\end{aligned}$$

The integral version of the majorization theorem is given in the book (see [204, P.325]).

Theorem A.30: f *is majorized by* g ($f(t) \prec g(t)$ *for* $t \in [a, b]$ *iff they are nonincreasing in* $[a, b]$ *and*

$$\int_a^b \varphi(f(t))d(t) \;\leq\; \int_a^b \varphi(g(t))d(t),$$

holds for every continuous convex function $\varphi : [a, b] \to \mathbb{R}$*, provided that the integrals exist.*

Now, we state other versions of majorization theorem for functions with nondecreasing increments from [208] (see also [47, p. 354]). Throughout this section $\mathbf{K} \subset \mathbb{R}^p$ and here partial ordering is defined in similar manner as defined in earlier in this section with definition of functions with nondecreasing increments.

Theorem A.31: *Let* $\mathbf{f}, \mathbf{g} : [a,b] \to \mathbf{K}$ *be nonincreasing continuous mappings and let* $H \in BV[a,b]$. *If*

$$\begin{aligned} \int_a^x \mathbf{f}(t)\, dH(t) \ &\leq \ \int_a^x \mathbf{g}(t)\, dH(t), \quad \forall\, x \in (a,b), \\ \textit{and} \quad \int_a^b \mathbf{f}(t)\, dH(t) \ &= \ \int_a^b \mathbf{g}(t)\, dH(t), \end{aligned}$$

then $\forall$ *continuous mapping* $\varphi : \mathbf{K} \to \mathbb{R}$ *with nondecreasing increments inequality (A.25) is valid.*

Theorem A.32: *Let* $\mathbf{f},\ \mathbf{g} : [a,b] \to \mathbf{K}$ *be nondecreasing continuous mappings and let* $H \in BV[a,b]$. *If*

$$\begin{aligned} \int_x^b \mathbf{f}(t)\, dH(t) \ &\leq \ \int_x^b \mathbf{g}(t)\, dH(t), \quad \forall \quad x \in (a,b), \\ \textit{and} \quad \int_a^b \mathbf{f}(t)\, dH(t) \ &= \ \int_a^b \mathbf{g}(t)\, dH(t), \end{aligned}$$

then $\forall$ *continuous mapping* $\varphi : \mathbf{K} \to \mathbb{R}$ *with nondecreasing increments we get*

$$\int_a^b \varphi(\mathbf{f}(t))\, dH(t) \leq \int_a^b \varphi(\mathbf{g}(t))\, dH(t). \tag{A.25}$$

The following Theorems are consequence of Theorems *A*.31 and *A*.32 respectively [208] (see also [204, p. 328]).

Theorem A.33: *Let* $f, g : [a,b] \to J$ *be two nonincreasing continuous functions and let* $H : [a,b] \to \mathbb{R}$ *be a function of bounded variation. If*

$$\begin{aligned} \int_a^x f(t)\, dH(t) \ &\leq \ \int_a^x g(t)\, dH(t), \quad \textit{for each} \quad x \in (a,b), \\ \textit{and} \quad \int_a^b f(t)\, dH(t) \ &= \ \int_a^b g(t)\, dH(t), \end{aligned}$$

hold, then for every continuous convex function $\varphi : J \to \mathbb{R}$ *the following inequality holds*

$$\int_a^b \varphi(f(t))\, dH(t) \leq \int_a^b \varphi(g(t))\, dH(t). \tag{A.26}$$

Theorem A.34: *If* $f, g : [a,b] \to J$ *are two nondecreasing continuous functions such that*

$$\begin{aligned} \int_x^b f(t)\, dH(t) \ &\leq \ \int_x^b g(t)\, dH(t), \quad \textit{for each} \quad x \in (a,b), \\ \textit{and} \quad \int_a^b f(t)\, dH(t) \ &= \ \int_a^b g(t)\, dH(t), \end{aligned}$$

then inequality (A.26) holds.

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Synopsis of the Volume

As a cornerstone of the **Pečarić PSR Series on Mathematical Inequalities,** this treatise offers an exhaustive analytical framework for the Jensen-Mercer and Jessen-Mercer inequalities. It bridges profound mathematical theory—spanning convex analysis, operator theory, and multivariate functions—with sophisticated computational methodologies, addressing complex paradigms in information theory, economics, and numerical analysis. This volume, featuring contributions from experts across Pakistan and Croatia, is curated to empower researchers and graduate students with advanced insights into refinements, generalizations, and diverse applications such as entropy measures and fractional integral operators.

Profiles of the Authors

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