NULL CONTROLLABILITY OF DEGENERATE AND NON-DEGENERATE SINGULAR PARABOLIC EQUATIONS : THEORETICAL AND NUMERICAL ASPECT

Hamed Ould Sidi

Urmcd research unit
Faculty of Legal, Economic and Social Sciences
University of Nouakchott al Aasriya, Mauritania

Author
Hamed Ould Sidi
Urmcd Research Unit
Faculty of Legal, Economic and Social Sciences
University of Nouakchott al Aasriya, Mauritania
hamedouldsidi@yahoo.fr

Editor
Muhammad Imran
Government College University,
Faisalabad, Pakistan
drmimranchaudhry@gcuf.edu.pk

PTOLEMY INSTITUTE OF SCIENTIFIC RESEARCH AND TECHNOLOGY (PISRT)
HTTPS://PISRT.ORG/

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Preface

The present work concerns the study of the controllability problem for the linear heat equation, posed in a bounded domain.

Nowadays, the problem of control occupies an important place in the general theory of partial differential equations, in particular because of its many physical applications (fluid mechanics, thermodynamics, propagation phenomena, engineering). Generically, it is a question of intervening on a given evolution system (E) in order to control its solution, i.e. to bring it from an initial (arbitrary) state to a prescribed final state . The system (E) is, depending on the case, hyperbolic (vibratory phenomena), parabolic (heat equation), or of a more complex type. We can also ask the control vector (function) to verify a constraint, such as, to minimize a certain functional. In recent years, the study of this (these) problem (s) has required the implementation of fairly complex theoretical and numerical tools.

Control theory took off at the end of the $70s^\circ$ with the H.U.M (Hilbert Uniqueness Method) method of J.L. Lions. The years 90° are marked by two strong points. First, by the arrival of microlocal techniques by C. Bardos, G. Lebeau and J. Rauch. Then, the proof and the use of global Carleman inequalities by Fursikov [17] (and also, for the heat equation in particular, by Lebeau and Robbiano [20]) for the trajectory controllability of second order parabolic equations.

A large number of related mathematical problems are equally relevant: stabilization of solutions, problem of uniqueness and unique extension.

1. Introduction

The field of control theory is attracting more and more researchers for its application in several areas of everyday life.

In mathematics, control refers to the theory that aims to understand how a command allows humans to act on a system they want to control. This definition naturally covers a large number of fields of application; an engineer may want to control a mechanical system by applying forces to it, an economist may want to act on a financial equilibrium by modifying a rate, a chemist may want to improve his process by regulating the temperature, etc..

It is interesting to note that, despite the diversity of concrete situations that can be understood in this way, “control theory” provides a framework common to all these universes. It is therefore remarkable that one achieves general results, which can be used in many fields.

In our study, we focus on the case of PDE control theory. Partial differential equations allow mathematicians to describe the behavior of a quantity that depends on several variables. For example, the temperature of the ocean depends on both where and when it is measured. To describe its evolution, the equations which occur naturally are called EDP because they involve variations compared to the various variables.

PDEs are omnipresent in physics: especially in fluid mechanics or in quantum mechanics. Let us quote as examples the Navier-Stokes equation to describe the motion of a liquid and the Schrodinger equation to describe a quantum particle. These two examples describe systems which at the same time have a so-called "free" evolution when one does not act on them, but which can also have a continuous evolution, when one tries to exert forces on them. This is where control comes in.

Take a fluid (typically water) and fill a basin with it. The movement of the water is then governed by an EDP. Suppose that we can exert a force at a place in the basin (thanks to the presence of dams, valves and propeller, etc.). In control theory, mathematicians seek to determine whether this action located at a specific spot in the basin may be sufficient to "control" all of the liquid. We can ask ourselves a multitude of questions about the scope of possible actions.

- If the pelvis is initially agitated by an external cause, can we guide it towards dead calm?

This is the question of returning to rest.

- If so, can this objective be achieved in as short a time as desired, or is there a minimum time to get there below which is impossible?
- And if we want to spend less energy, is there one action that is more economical than the others? This is the question of optimal cost, crucial in engineering or economics.
- Finally, let's talk about stabilization: is it possible to design a mechanism that regulates the water level at the surface of the basin despite external disturbances (rains, evaporation, etc.)?

The controllability of systems of parabolic partial differential equations has undergone an important development since the 90° in particular thanks to the contributions of [20] and [17]: the obtaining of so-called Carleman inequalities and their use in this framework have led to new controllability results called trajectory controllability.
This notion is relevant for dissipative systems such as heat equations or Navier-Stokes equations or related systems.

Indeed, for these equations one cannot expect to obtain exact controllability in the usual sense. The notion of approximate controllability, which has been extensively studied and which will be very useful in a first step here, is not really satisfactory for real applications. The exact control on the trajectories corresponds to real objectives. In our study, we focus on the case of the heat equation.

Let Ω be a regular and bounded open set of $\mathbb{R}^n$ et $\mathscr{O} \subset\subset \Omega$. Let $T > 0$.
We pose $\mathrm{Q} = \Omega \times (0,T)$ et $\Sigma = \partial\Omega \times (0,T)$. In order to explain the methodology that we follow, we first place ourselves in the simple case of the following equation

$$\begin{cases} \partial_t y - \Delta y &= v 1_{\mathscr{O}} \;\; in \;\; \Omega \times (0,T) \\ y &= 0 \quad sur \quad \partial\Omega \times (0,T) \\ y(x,t=0) &= y^0(x) \;\; dans \;\; \Omega \end{cases} \tag{1.0.1}$$

The function v is called control. This control is distributed here because it applies in a non-zero measurement subdomain of Ω. Controllability exact in $\mathrm{L}^2(\Omega)$ at time T of such an equation would be

$$\forall y^T, y^0 \in \mathrm{L}^2(\Omega), \exists v \in \mathrm{L}^2(\mathrm{Q}) \;\; such \;\; that \;\; y(T) = y^T. \tag{1.0.2}$$

It is well known that in the case of parabolic equations, in particular in the case of the heat equation that we have written, the exact controllability is wrong because of the regularizing character of the heat semigroup. The exact controllability is equivalent to the following observability inequality [28]

$$||q(T)||^2_{\mathrm{L}^2(\Omega)} \leq C \iint_{(0,T)\times\mathscr{O}} |q|^2 dxdt, \tag{1.0.3}$$

for the dual system

$$\begin{cases} -\partial_t q - \Delta q &= 0 \;\; dans \;\; Q = \Omega \times (0,T) \\ q &= 0 \quad sur \quad \partial\Omega \times (0,T) \\ q(x,t=T) &= q^T(x) \;\; dans \;\; \Omega \end{cases} \tag{1.0.4}$$

By considering an orthonormal basis of Laplacian eigenfunctions on Ω we indeed note that such an inequality is impossible.

A second notion often approached in the theory of control of partial differential equations is that of approximate controllability. We then want to demonstrate that

$$\forall \varepsilon > 0, \forall y^T, y^0 \in \mathrm{L}^2(\Omega), \exists v \in \mathrm{L}^2(\mathrm{Q}) \ \ such \ \ that \ \ ||y(T) - y^T||_{\mathrm{L}^2} \leq \varepsilon. \tag{1.0.5}$$

In the case of parabolic equations such as the heat equation that we have written the approximate controllability is true. As it corresponds to the density of the image of the heat semi-group, it is equivalent to the following uniqueness problem for the dual system [11]

$$q = 0 \ \ sur \ \ (0;T) \times \mathscr{O} \Rightarrow q = 0 \ \ sur \ \ \mathrm{Q}. \tag{1.0.6}$$

The notion that interests us here is that of zero controllability. It is expressed as follows:

$$\forall y^0 \in \mathrm{L}^2(\Omega), \exists v \in \mathrm{L}^2(\mathrm{Q}) \ \ such \ \ that \ \ y(T) = 0. \tag{1.0.7}$$

This notion is equivalent to that of trajectory controllability in the case of linear equations:

$$\forall y^0 \in \mathrm{L}^2(\Omega), \exists v \in \mathrm{L}^2(\mathrm{Q}) \ \ such \ \ that \ \ y(T) = z(T), \tag{1.0.8}$$

where $z(t)$ is a solution of (1.0.1) without control for an initial data $z^0 \in \mathrm{L}^2(\Omega)$. The controllability at zero is also equivalent to the following observability inequality for the dual system (1.0.3)

$$||q(0)||^2_{\mathrm{L}^2(\Omega)} \leq C_{abc} \iint_{(0,T)\times\mathscr{O}} |q|^2 dxdt, \tag{1.0.9}$$

(compare (1.0.3) with (1.0.9)). It is this last type of inequality that we will call in the following observability inequality.

In 1996, Fursikov and Imanuvilov showed how obtaining global Carleman inequalities for parabolic operators made it possible to prove observability inequalities of the type of (1.0.9) and to understand the dependence of C_{abc} depending on the terms of zero order and first order in variables space of the time-dependent parabolic operator [17]. This approach allowed the demonstration of null controllability results for semi-linear parabolic equations [29].

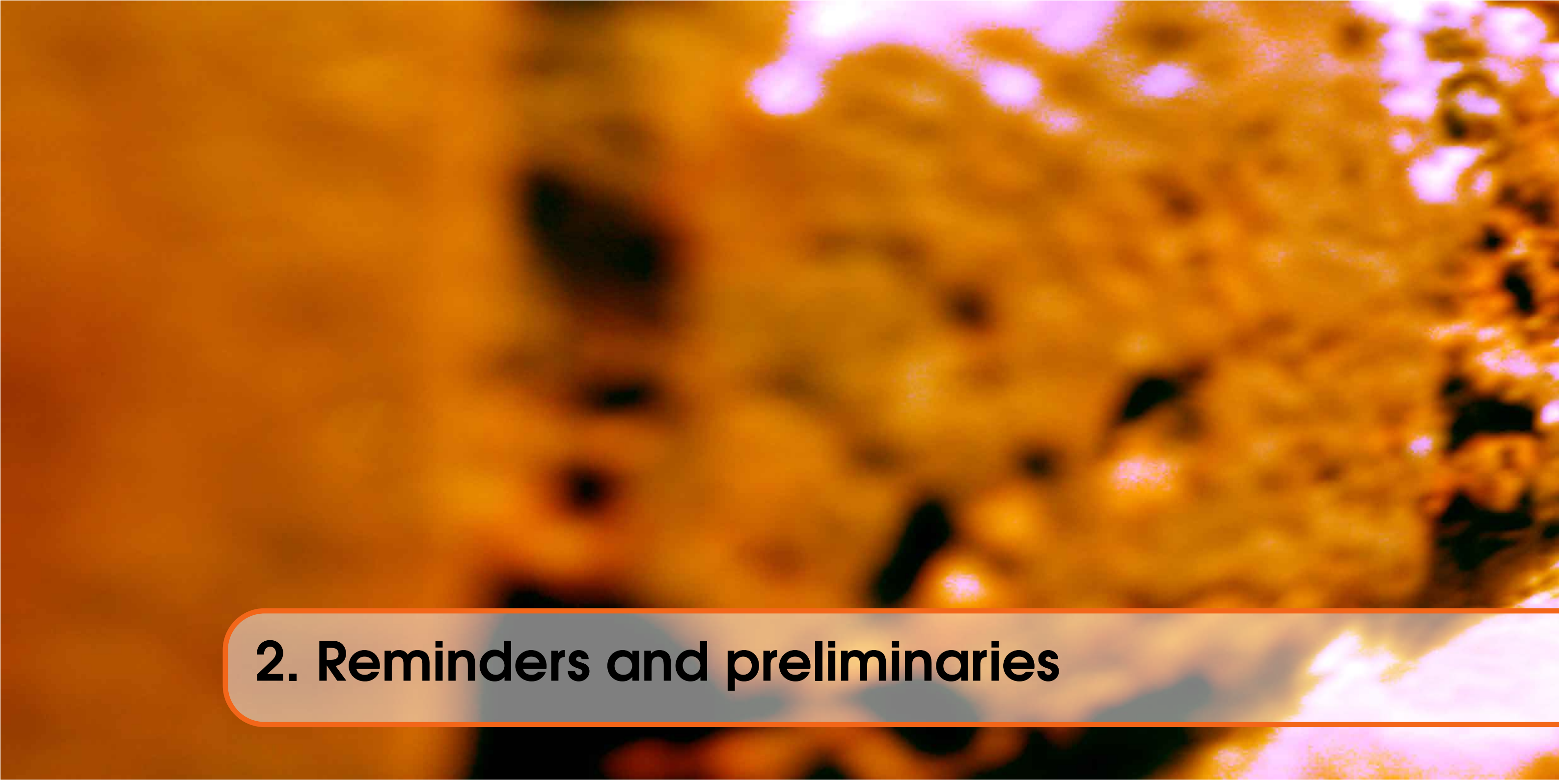
For the the numerical solution, we have considered an inverse source problem to study numerically the null controllability of a class of degenerate and singular parabolic equations. The solvability of the regularized inverse problem of determining the source term to obtain final temperature identically null, that was formulated as the minimizer of a least squares functional with the Tikhonov regularization, is studied. It is proved the Lipschitz continuity of the input-output operator $F : h \longrightarrow u$. Lipschitz continuity of the gradient functional was also proved, which implies the convergence of the descent method. Some numerical simulations are presented to validate the results of [14] .

1.1 Plan of the book

This document is made up of three chapters. Each chapter is made up of several sections in which we develop the different aspects of the chapter topic, and preceded by a detailed introduction.

After the introduction, we will find, in chapter II, the main results of the control for the linear heat equation.

The chapter III concerns the numerical study of the null controllability for the linear heat equation.



2. Reminders and preliminaries

The lemmas, as well as the mathematical tools that follow, will be used constantly in the chapters to come.

2.1 Functional spaces

In this chapter we group together the main properties and functional spaces that we will use. Most of the content of this paragraph is taken from the citeHB references.

In the following, Ω denotes an open bounded of $\mathbb{R}^n$ endowed with the Lebesgue measure dx, and of sufficiently regular border $\partial\Omega$. X and Y are two Banach spaces of respective norms $|.|_X$, $|.|_Y$.

2.1.1 $\mathrm{L}^p(\Omega)$ spaces - Sobolev spaces .

We denote by $\mathrm{L}^1(\Omega)$ the space (of classes of equivalences) of integrable functions in the sense of Lebesgue on Ω has values in $\mathbb{R}$. That is, as we usually do, we confuse two functions that coincide almost everywhere (p.p for short).
For $u \in \mathrm{L}^1(\Omega)$, we denote

$$\|u\|_{\mathrm{L}^1(\Omega)} = \int_\Omega |u(x)|dx.$$

When there is no confusion, we will write $\int_\Omega |u|$ au lieu de $\int_\Omega |u(x)|dx$.

Definition 2.1.1 Let $p \in \mathrm{R}$ with $1 \leq p < \infty$, we pose :

$$\mathrm{L}^p(\Omega) = \{u : \Omega \to \mathbb{R}; u \textit{ mesurable and } |u|^p \in \mathrm{L}^1(\Omega)\}$$

We note

$$\|u\|_{\mathrm{L}^p(\Omega)} = (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}} \tag{2.1.1}$$

Definition 2.1.2 We pose

$$\mathrm{L}^\infty(\Omega) = \{u : \Omega \to \mathbb{R}; \textit{mesurable and } \exists \textit{ a constant } C \textit{ such that } |u(x)| \leq C \textit{ p.p sur } \Omega.\}$$

on note

$$\|u\|_{\mathrm{L}^\infty(\Omega)} = \inf\{C; |u(x)| \leq C \ p.p \ on \ \Omega\}$$

R If $u \in \mathrm{L}^\infty$ we have

$$|u(x)| \leq \|u\|_{\mathrm{L}^\infty} \ p.p \ on \ \Omega$$

Notation: Let $1 \leq p \leq \infty$; the conjugate exponent of p i.e is denoted by p' $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 2.1.1 (Hölder inequality). Let $u \in \mathrm{L}^p$ et $v \in \mathrm{L}^{p'}$ with $1 \leq p \leq \infty$. Then $u.v \in \mathrm{L}^1$ et

$$\int |uv| \leq \|u\|_{\mathrm{L}^p} \|v\|_{\mathrm{L}^{p'}}$$

Theorem 2.1.2 L^p is a vector space and $\|.\|_{\mathrm{L}^p}$ is a standard for everything $1 \leq p \leq \infty$.

Theorem 2.1.3 (Fischer-Riesz) L^p is a Banach space for everything $1 \leq p \leq \infty$.

In the particular case $p = 2$, the relation

$$(u, v) = \int_\Omega u(x)v(x)dx \ \forall u, \ v \in \mathrm{L}^2(\Omega), \tag{2.1.2}$$

define a dot product in $\mathrm{L}^2(\Omega)$, whose associated norm is none other than the norm $\|.\|_{\mathrm{L}^2}$ defined in (2.1.1), and we have:

Proposition 2.1.4 Space $\mathrm{L}^2(\Omega)$ endowed with the dot product (2.1.2) is a Hilbert space.

Definition 2.1.3 $\mathrm{H}^1(\Omega)$ is the space of functions which belong to $\mathrm{L}^2(\Omega)$ and whose derivatives in the sense of distributions belong to $\mathrm{L}^2(\Omega)$

$$\mathrm{H}^1(\Omega) = \{u \in \mathrm{L}^2(\Omega); \frac{\partial u}{\partial x_i} \in \mathrm{L}^2(\Omega), 1 \leq i \leq n\}.$$

$\mathrm{H}^1(\Omega)$ is the Sobolev space of order 1.

We endow $\mathrm{H}^1(\Omega)$ with the dot product

$$(u, v)_{1,\Omega} = \int_\Omega (uv + \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}) dx = (u, v) + (\nabla u, \nabla v), \tag{2.1.3}$$

And we note:

$$\|u\|_{1,\Omega} = (u, u)_{1,\Omega}^{\frac{1}{2}} = (\int_\Omega (u^2 + \sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2) dx)^{\frac{1}{2}} = (\|u\|^2 + \|\nabla u\|^2)^{\frac{1}{2}}. \tag{2.1.4}$$

the corresponding standard.

Proposition 2.1.5 The space $\mathrm{H}^1(\Omega)$ is a Hilbert space for the dot product (2.1.3).

Let us denote by $\mathrm{D}(\Omega)$ the vector space of infinitely differentiable functions on Ω with compact support in Ω.

Definition 2.1.4 We define $H_0^1(\Omega)$ as being the adhesion of $D(\Omega)$ in $H^1(\Omega)$, i.e

$$H_0^1(\Omega) = \overline{D(\Omega)}$$

R Si Ω est borné, $D(\Omega)$ is not dense in $H^1(\Omega)$ so that $H_0^1(\Omega) \subset H^1(\Omega)$ with strict inclusion and $H_0^1(\Omega) \neq H^1(\Omega)$; However , if $\Omega = \mathbb{R}^n$, $D(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$, i.e $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.

Proposition 2.1.6 $H_0^1(\Omega)$ endowed with the norm induced by $H^1(\Omega)$ is a Hilbert space.

Theorem 2.1.7 (Trace theorem) Let Ω be a bounded open of class $\mathbb{C}^1$, there exists a continuous linear operator $\gamma_0 \in L(H^1(\Omega), L^2(\partial\Omega))$ such as

$$\gamma_0 u = u\mid_{\partial\Omega}, \forall u \in \mathbb{C}^1(\overline{\Omega}).$$

$L^2(\partial\Omega)$ is the space of (class of) real functions, integral square over $\partial\Omega$.

From the trace theorem, we can give the following characterization of the functions of $H_0^1(\Omega)$ which explains the important role played by the latter in the resolution of partial differential equations coupled with boundary conditions, i.e. when the value u is prescribed on the $\partial\Omega$ border.

Definition 2.1.5 The functions of $H_0^1(\Omega)$ are the functions of $H^1(\Omega)$ which vanish on the boundary $\Gamma = \partial\Omega$,

$$H_0^1(\Omega) = \{u/u \in H^1(\Omega); u = 0 \ on \ \Gamma\} = \ the \ core \ of \ \gamma_0.$$

R We denote the dual space of $H_0^1(\Omega)$ par $H^{-1}(\Omega)$.

2.1.2 The space $L^p(a,b;X)$

We give a brief introduction of integrability in the Bochner sense of functions defined over an interval, with vector value. For a complete and detailed study, we refer to J. Diestel and J. J. Uhl Jr. **[DU]**

Let X be a Banach space and $-\infty < a < b < +\infty$.

A function $f : [a,b] \to X$ is said to be simple if there exists $E_1, \ldots\ldots, E_m$ measurable sets of $[a,b]$ and $x_1, \ldots\ldots, x_m \in X$ such that:

$$f(t) = \sum_{i=1}^{m} \chi_{E_i}(t) x_i.$$

We will say that $f : [a,b] \to X$ is measurable if there is a sequence of simple functions f_k, $f_k : [a,b] \to X$, such that

$$f_k \to f \ pp \ on \ [a,b].$$

A function $f : [a,b] \to X$ measurable is said to be integral (in the Bochner sense) if there exists a sequence of simple functions f_k, $f_k : [a,b] \to X$, such as

$$\lim_k \int_a^b \|f - f_k\|_X = 0$$

In this case, $\int_a^b f(t)dt$ is defined by

$$\int_a^b f(t)dt = \lim_k \int_a^b f_k(t)dt,$$

where $\int_a^b f_k(t)dt$ is naturally defined.

Theorem 2.1.8 (Bochner) $f:[a,b]\to \mathrm{X}$ measurable is integrable if and only if $\|f\|_{\mathrm{X}}\in \mathrm{L}^1(a,b)$.

For $1\leq p\leq\infty$, we set

$$\mathrm{L}^p(a,b;\mathrm{X})=\{f:[a,b]\to \mathrm{X}\ integrable\ such\ that\ \|f\|_{\mathrm{X}}\in \mathrm{L}^p(a,b).\}$$

As in the scalar case, we do not distinguish between two almost everywhere equal functions. Equipped with the norm

$$\|f\|_{\mathrm{L}^p(a,b,\mathrm{X})}=\Big(\int_a^b \|f\|_{\mathrm{X}}^p\Big)^{\frac{1}{p}}\ \ if\ p<\infty$$

and

$$\|f\|_{\mathrm{L}^\infty(a,b,\mathrm{X})}=\inf\{C;\|f(t)\|_{\mathrm{X}}\leq C\ a.e\ on\ [a,b]\}\ \ if\ p=\infty,$$

$\mathrm{L}^p(a,b;\mathrm{X})$ is a Banach space. If X is Hilbert for the dot product $(.,.)_{\mathrm{X}}$, $\mathrm{L}^2(a,b;\mathrm{X})$ is a Hilbert space for the dot product:

$$(u,v)_{\mathrm{L}^2(a,b;\mathrm{X})}=\int_a^b (u(t),v(t))_{\mathrm{X}}dt.$$

Definition 2.1.6

- We call a vector field an application $v:\mathbb{R}^n\to\mathbb{R}^n$, which with $x=(x_1,.....,x_n)$ associates $v(x)=(v_1(x),......,v_n(x))$.
- For a function $u:\mathbb{R}^n\to\mathbb{R}$, its gradient is the vector field defined by

$$\nabla u(x)=(\frac{\partial u}{\partial x_1}(x),......,\frac{\partial u}{\partial x_n}(x)).$$

- For a vector field $v:\mathbb{R}^n\to\mathbb{R}^n$, we call divergence the function

$$div\ v(x)=\frac{\partial v_1}{\partial x_1}(x)+......+\frac{\partial v_n}{\partial x_n}(x).$$

- We call Laplacian of a function $u:\mathbb{R}^n\to\mathbb{R}$ the function

$$\Delta u(x)=div(\nabla u(x))=\frac{\partial^2 u}{\partial x_1^2}(x)+......+\frac{\partial^2 u}{\partial x_n^2}(x).$$

R Let ψ and ϕ be scalar fields and ϑ represents a vector field. We have the following relations:

- $\nabla(\psi\phi)=(\nabla\psi)\phi+(\nabla\phi)\psi$, this formula follows immediately from the product rule.
- $div(\psi\vartheta)=(\nabla\psi).\vartheta+(div(\vartheta))\psi$.
- $\Delta(\psi\phi)=\Delta(\psi)\phi+2\nabla(\psi)\nabla(\phi)+\psi\Delta(\phi)$.

Definition 2.1.7

- We call normal to the domain Ω a vector field $\nu(x)$ defined on the edge $\partial\Omega$ of Ω and such that at any point $x\in\partial\Omega$ where the edge is regular, $nu(x)$ is orthogonal to the edge and unitary ($|\nu(x)|=1$).
- We call external normal a normal which points towards the outside of the domain at any point.
- We call the normal derivative of a regular function u on the edge of a domain Ω the function defined on the regular points of $\partial\Omega$ by $\frac{\partial u}{\partial\nu}(x)=\nabla u(x).\nu(x)$ (dot product of

vector $\nabla u(x)$ with the vector $\nu(x)$).

2.1.3 Green and Stocks formulas

A first classical formula of integration by parts is

$$\int_{\Omega} \partial_i uv = -\int_{\Omega} u\partial_i v + \int_{\Gamma} uv\nu_i,\ u,v \in \mathrm{H}^1(\Omega)$$

From this formula can easily be deduced the following:

$$\int_{\Omega} \Delta uv = -\int_{\Omega} \nabla u.\nabla v + \int_{\Gamma} \partial_\nu uv,\ u \in \mathrm{H}^2(\Omega)\ et\ v \in \mathrm{H}^1(\Omega).\ Formula\ of\ Green$$

$$\int_{\Omega} div\Upsilon\Phi = -\int_{\Omega} \Upsilon.\nabla\Phi + \int_{\Gamma} \Upsilon.\nu\Phi, \quad StokesFormula$$

where Φ is a scalar function of $\mathrm{H}^1(\Omega)$ and Υ a vector-valued function of $\mathrm{H}^1(\Omega)$.

2.2 Study of a functional resulting from the control problem

First, we will study a functional that we will often encounter during this work, and for that, introduce a general formalism to describe its properties, and which will be specified later.

We denote by Ω is an open bounded of $\mathbb{R}^N$ and of border $\partial\Omega = \Gamma$ of class $\mathbb{C}^2$.
We consider a time interval $[0;T]$ with $T > 0$. We denote by Q the cylinder open with base Ω and height T:

$$Q = \Omega \times (0,T).$$

We denote by $|\theta|_p$ the norm L^p of a function θ with $1 < p < \infty$ and by p' the conjugate of p ($\frac{1}{p} + \frac{1}{p'} = 1$). We denote by (f,g) the integral $\int_Q f(x,t)g(x,t)dxdt$.

For $0 < t_1 < t_2$, we denote by $X^p(t_1,t_2)$ the following Banach space:

$$X^p(t_1,t_2) = L^P(t_1,t_2;W_0^{1,p}(\Omega)) \cap W^{1,p}(t_1,t_2;L^p(\Omega)), \tag{2.2.1}$$

provided with the natural norm

$$||.||_{X^p(t_1,t_2)} = ||.||_{L^P(t_1,t_2;W_0^{1,p}(\Omega))} + ||.||_{W^{1,p}(t_1,t_2;L^p(\Omega))}.$$

Let $a = a(x,t) \in L^\infty(Q)$. We recall that there exists (see [22] and [19]) a constant $C > 0$ (which depends on a, Ω and T) and C_{t_1,t_2} (which depends on a, Ω, t_1 and t_2) such that, for all $k \in L^p(Q)$ and $w^0 \in L^p(\Omega)$, the solution w of

$$\left\{ \begin{array}{rcl} \partial_t w - \Delta w + aw &=& k\ \ in\ \ \Omega \times (0,T) \\ w &=& 0 \quad sur \quad \partial\Omega \times (0,T) \\ w(x,t=0) &=& w^0(x)\ \ in\ \ \Omega \end{array} \right. \tag{2.2.2}$$

satisfied

$$\left\{ \begin{array}{l} ||w||_{L^\infty(0,T;L^p(\Omega))} \leq C(|w^0|_p + ||k||_{L^p(Q)}) \\ ||w||_{X^p(t_1,t_2)} \leq C_{t_1,t_2}(|w^0|_p + ||k||_{L^p(Q)}). \end{array} \right. \tag{2.2.3}$$

Moreover, if $w^0 = 0$, we have (see [19]) for all $k \in L^p(Q)$, the solution w of

$$\left\{ \begin{array}{rcl} \partial_t w - \Delta w &=& k\ \ in\ \ \Omega \times (0,T) \\ w &=& 0 \quad on \quad \partial\Omega \times (0,T) \\ w(x,t=0) &=& 0\ \ in\ \ \Omega \end{array} \right. \tag{2.2.4}$$

satisfied

$$\begin{cases} w & \in & X^p(0,T) \\ \text{the operator } k & \in & L^p(Q) \to w\ inX^p(0,T) \text{ is linear continuous.} \end{cases} \tag{2.2.5}$$

By (2.2.5) and using Gronwall's lemma, we find that for all $k \in L^p(Q)$, the solution of

$$\begin{cases} \partial_t w - \Delta w + aw & = & k \ in\ \Omega \times (0,T) \\ w & = & 0 \quad on \quad \partial\Omega \times (0,T) \\ w(x,t=0) & = & 0 \ in\ \Omega \end{cases} \tag{2.2.6}$$

satisfied

$$\begin{cases} w & \in & X^p(0,T) \\ ||w||_{X^p(0,T)} & \leq & C_a(||k||_{L^p(Q)})\ with\ C_a = O(1+|a|_\infty \exp(|a|_\infty)). \end{cases} \tag{2.2.7}$$

We consider a set q of the form: $q = \mathscr{O} \times (0,T)$ where $\mathscr{O} \subset \Omega$ is open nonempty of Ω
We introduce an operator of $L^{p'}(\Omega) \times L^\infty(Q)$ in $L^1(q)$ which satisfies the following assumptions:

H1

$$\forall a \in L^\infty(Q),\ \ L(.,a)\ is\ linear\ continuous\ of\ L^{p'}(\Omega)\ dans\ L^1(q)$$

and if φ is solution of

$$\begin{cases} \partial_t \varphi - \Delta\varphi + a\varphi & = & 0 \ in\ \Omega \times (0,T) \\ \varphi & = & 0 \quad on \quad \partial\Omega \times (0,T) \\ \varphi(x,t=0) & = & \varphi^0 \in L^{p'}(\Omega) \ in\ \Omega, \end{cases} \tag{2.2.8}$$

then

H2

$$L(\varphi^0,a) = 0\ \ almost\ everywhere\ in\ q \Rightarrow \varphi = 0\ \ almost\ everywhere\ in\ Q;$$

H3

$$\begin{cases} \varphi_n^0 \rightharpoonup \varphi^0\ weakly\ in\ L^{p'}(\Omega) \\ a_n \rightharpoonup a\ weakly - *\ in\ L^\infty(Q) \end{cases}$$

$$\Rightarrow L(\varphi_n^0, a_n) \rightharpoonup L(\varphi^0,a)\ weakly\ in\ L^1(q);$$

H4

$$\forall \varphi^0 \in L^{p'}(\Omega),\ \ L(\varphi^0,.)\ is\ compact\ of\ L^\infty(Q)\ in\ L^1(q).$$

We will often write $L_a(\varphi^0)$ instead of $L(\varphi^0,a)$.

R The hypothesis $H2$ is none other than the unique continuation property for the solutions of (2.2.8). We will see later that the operator L that we are going to consider satisfies this property thanks to a result of C.Fabre [10].

Let $\alpha > 0$ and $y^1 \in L^p(\Omega)$. For $\varphi^0 \in L^{p'}(\Omega)$ and the solution φ of (2.2.8) with $\varphi(T) = \varphi^0$, we introduce the functional

$$J(\varphi^0;a,y^1) = \frac{1}{2}\int_q |L_a(\varphi^0)(x,t)|^2 dxdt + \alpha|\varphi^0|_{p'} - \int_\Omega y^1\varphi^0 dx. \tag{2.2.9}$$

Proposition 2.2.1 [11] For all $\alpha > 0$, $y^1 \in L^p(\Omega)$ and $a \in L^\infty(Q)$, $J(.;A,y^1)$ is a continuous function over $L^{p'}$, strictly convex and satisfies

$$\liminf_{|\varphi^0|_{p'} \to \infty} \frac{J(\varphi^0;a,y^1)}{|\varphi^0|_{p'}} \geq \alpha. \tag{2.2.10}$$

The functional $J(.;A,y^1)$ reaches its minimum at a single point $\hat{\varphi}^0 \in L^{p'}(\Omega)$. Moreover,

$$\hat{\varphi}^0 = 0 \Leftrightarrow |y^1|_p \leq \alpha.$$

Proof. Since the norm $L^{p'}(\Omega)$ is strictly convex and L_a is linear, it is clear that $J(.;A,y^1)$ is strictly convex .
On the other hand, (2.2.3) and the continuity of L_a imply the continuity of $J(.;A,y^1)$. Now in order to show (2.2.10), we assume that there exists a sequence φ_n^0 in $L^{p'}(\Omega)$ such that:

$$|\varphi_n^0|_{p'} \to +\infty \tag{2.2.11}$$

and

$$\liminf_{n\to\infty} \frac{J(\varphi_n^0;a,y^1)}{|\varphi_n^0|_{p'}} \leq \alpha. \tag{2.2.12}$$

For $\tilde{\varphi}_n^0 = \frac{\varphi_n^0}{|\varphi_n^0|_{p'}}$ we denote by $\tilde{\varphi}_n$ the solution of (2.2.8) with $\tilde{\varphi}_n(T) = \tilde{\varphi}_n^0$.
Since $|\tilde{\varphi}_n^0|_{p'} = 1$, we can extract a subsequence (still noted $\tilde{\varphi}_n^0$ by abuse of language), which converges weakly in $L^{p'}(\Omega)$ towards an element $\tilde{\varphi}^0 \in L^{p'}(\Omega)$. According to (2.2.3), $(\tilde{\varphi}_n)_n$ weakly converges in $L^{p'}(Q)$ towards $\tilde{\varphi}$ solution of (2.2.8) with $\tilde{\varphi}(T) = \tilde{\varphi}^0$.
According to $(H1)$, $L_a(\tilde{\varphi}_n^0)$ converges weakly in $L^1(q)$ to $L_a(\tilde{\varphi}^0)$. However (2.2.11) and (2.2.12) imply that there exists a subsequence (always denoted by $(\tilde{\varphi}_n)_n$) such that

$$\int_q |L_a(\tilde{\varphi}_n^0)|^2 dxdt \to 0 \; if \; n \to +\infty. \tag{2.2.13}$$

Hence $L_a(\tilde{\varphi}^0) = 0$ in q.
From $(H2)$ we deduce that $\tilde{\varphi}^0 = 0$ in Q and

$$\tilde{\varphi}^0 = 0. \tag{2.2.14}$$

Then we find that

$$J(\varphi_n^0;a,y^1) \geq |\varphi_n^0|_{p'}(\alpha - \int_\Omega y^1 \tilde{\varphi}_n^0 dx), \tag{2.2.15}$$

and since $\tilde{\varphi}_n^0$ weakly converges to 0 in $L^{p'}(\Omega)$, it follows that

$$\liminf_{n\to\infty} \frac{J(\varphi_n^0;a,y^1)}{|\varphi_n^0|_{p'}} \geq \alpha,$$

which in contradiction with (2.2.12).
Hence (2.2.10).
Now, if $|y^1|_p \leq \alpha$, we have $J(\varphi^0;a,y^1) \geq 0$ for all $\varphi^0 \in L^{p'}(\Omega)$ and therefore $\hat{\varphi}^0 = 0$.
If $\hat{\varphi}^0 = 0$, then

$$\forall \varphi^0 \in L^{p'}(\Omega),\; \forall t > 0,\; \lim_{t\to 0^+} \frac{J(t\varphi^0;a,y^1)}{t} \geq 0.$$

and we easily find that $|y^1|_p \leq \alpha$. ■

In order to study the nonlinear case, we need to specify the dependence between the minimums with the potential. More precisely we have the

Proposition 2.2.2 [11]
If we denote by M the operator

$$\begin{array}{rcl} M : L^\infty(Q)\times L^p(\Omega) & \to & L^{p'}(\Omega) \\ (a,y^1) & \to & \hat{\varphi}^0 \end{array} \tag{2.2.16}$$

and if K is a compact subset of $L^p(\Omega)$ and B a bounded subset of $L^\infty(Q)$, then $M(B\times K)$ is a bounded subset of $L^{p'}(\Omega)$.

Proof. We will first show that (2.2.10) is uniform in $(a,y^1)\in B\times K$. We reason absurdly by following the same arguments as in the previous proof. Suppose there is a sequence of functions $(a_n)_n$ of $L^\infty(Q)$, $(y^1_n)_n$ from $L^p(\Omega)$ and $(\varphi^0_n)_n$ from $L^{p'}(\Omega)$ such that (we denote by φ_n the solution of (2.2.8) associated with the function a_n checking $\varphi_n(T)=\varphi^0_n$ and by $L_n(.)$ the function $L_{a_n}(.)$)

$$\exists a\in L^\infty(Q),\ \ a_n \rightharpoonup a\ \ weakly*\ in\, L^\infty(Q) \tag{2.2.17}$$

and

$$\exists y^1\in L^p(\Omega),\ \ y^1_n\to y^1\ \ strongly\ in\, L^p(\Omega), \tag{2.2.18}$$

such that (2.2.11) occurs and

$$\liminf_{n\to\infty}\frac{J(\varphi^0_n;a_n,y^1_n)}{|\varphi^0_n|_{p'}}<\alpha. \tag{2.2.19}$$

we designate by $\tilde{\varphi}^0_n=\frac{\varphi^0_n}{|\varphi^0_n|_{p'}}$ et $\tilde{\varphi}_n$ the solution of (2.2.8) associated with a_n and with the initial data $\tilde{\varphi}_n(T)=\tilde{\varphi}^0_n$.
As $|\tilde{\varphi}^0_n|_{p'}=1$, we can extract a subsequence, again denoted by $\tilde{\varphi}^0_n$, which weakly converges in $L^{p'}(\Omega)$ to an element $\tilde{\varphi}^0\in L^{p'}(\Omega)$. Now (From (H3)) $L_n(\tilde{\varphi}^0_n)$ weakly converges in $L^1(q)$ towards $L_a(\tilde{\varphi}^0)$. We denote by $\tilde{\varphi}$ the weak limit of $\tilde{\varphi}_n$ in $L^{p'}(Q)$. Using (2.2.3) and passing to the limit in the equation satisfied by φ_n, we can show that $\tilde{\varphi}$ is the solution of (2.2.8) associated with a and initial data $\tilde{\varphi}(T)=\tilde{\varphi}^0$. From (2.2.11) and (2.2.19), we deduce (after extracting a subsequence)

$$\int_q |L_n(\tilde{\varphi}^0_n)|dxdt\to 0\ if\ n\to+\infty. \tag{2.2.20}$$

Using (H3) and (H2), we get

$$\tilde{\varphi}=0. \tag{2.2.21}$$

Let us now show that if $J_n=\frac{J(\varphi^0_n;a_n,y^1_n)}{|\varphi^0_n|_{p'}}$, then $\liminf_{n\to\infty}J_n\geq\alpha$.
For this we notice that

$$J_n\geq(\alpha-\int_\Omega y^1_n\tilde{\varphi}^0_n dx), \tag{2.2.22}$$

and since $\tilde{\varphi}^0_n$ weakly converges to 0 in $L^{p'}(\Omega)$ and y^1_n strongly converges in $L^p(\Omega)$, then

$$\liminf_{n\to\infty}J_n\geq\alpha,$$

which contradicts (2.2.19).

Now, noticing that $J(0;a,y^1)=0$, it comes that

$$J(\hat{\varphi}^0;a,y^1)\leq 0,$$

and we can deduce that the image of M is bounded in $L^{p'}(\Omega)$.
which ends the demonstration.

■

3. Control of the linear heat equation

In this chapter we study the problem of internal controllability of the heat equation. The control is supposed to act on a subset of the domain where the solutions are defined.

3.1 Introduction

We consider the equation of inhomogeneous heat in $\Omega \times (0,T)$ with a control acting on $\mathscr{O} \times (0,T)$

$$\left\{ \begin{array}{rcl} \partial_t y - \Delta y + ay & = & v 1_{\mathscr{O}} \;\; in \;\; \Omega \times (0,T) \\ y & = & 0 \quad on \quad \partial\Omega \times (0,T) \\ y(x,t=0) & = & y^0 \;\; in \;\; \Omega \end{array} \right. \tag{3.1.1}$$

We aim to change the dynamics of the system by acting on the $\mathscr{O}$ subset of the domain Ω.

R Physical interpretation: The problem (3.1.1) is not only a model of heat propagation. For example, (3.1.1) is also known as the diffusion equation, and models the diffusion or migration of a concentration or density through the Ω domain.

The interest of the above heat equation analysis is based not only on the fact that it is a model for a large class of physical phenomena, but also one of the most important parabolic type partial differential equations.

R The following properties have very important consequences on control problems.

1. Irreversibility: With a datum at time $t = 0$ the solution is well defined for any positive time. It is even very regular (analytical compared to the space and time variables).
2. The propagation speed is infinite: More precisely for any $y(x,0) \geq 0$ with support in an arbitrarily small ball we have, for all $t > 0$ and all $x \in \Omega$, $y(x,t) > 0$.

Existence and Uniqueness

Proposition 3.1.1 Given $y^0 \in \mathrm{L}^2(\Omega)$ and $v \in \mathrm{L}^2(\mathrm{O} \times (0,T))$ there exists a unique solution y_v of (3.1.1) with

$$y_v \in \mathbb{C}([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega)).$$

Moreover, $\exists C = C(\Omega, T, ||a||_\infty) > 0$, such that

$$||y_v||_{\mathbb{C}([0,T];\mathrm{L}^2(\Omega))} + ||y_v||_{\mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega))} \leq C\big(||y^0||_{L^2(\Omega)} + ||v||_{L^2(Q)}\big).$$

Definition 3.1.1

1. We say that the equation (3.1.1) is exactly controllable at time T if,

$$\forall y^T, y^0 \in \mathrm{L}^2(\Omega), \exists v \in \mathrm{L}^2(\mathrm{Q}) \;\; such \;\; that \;\; y_v(T) = y^T.$$

2. The system (3.1.1) is said to be approximately controllable at time T if,

$$\forall \varepsilon > 0, \forall y^T, y^0 \in \mathrm{L}^2(\Omega), \exists v \in \mathrm{L}^2(\mathrm{Q}) \;\; such \;\; that \;\; ||y_v(T) - y^T||_{\mathrm{L}^2} \leq \varepsilon.$$

3. We say that the system (3.1.1) is controllable at zero if,

$$\forall y^0 \in \mathrm{L}^2(\Omega), \exists v \in \mathrm{L}^2(\mathrm{Q}) \;\; such \;\; that \;\; y_v(T) = 0.$$

We introduce the space of reachable states at time T starting from y^0 given by:

$$\mathrm{R}(T, y^0) = \{y_v(T),\ y_v \ solution \ of \ (3.1.1) \ with \ v \in \mathrm{L}^2(Q)\} \subseteq \mathrm{L}^2(\Omega).$$

R From the regularizing properties of the heat equation we know that if $\theta \subset \Omega \setminus \overline{\mathscr{O}}$, the solution y is of class $\mathbb{C}^\infty$ on $\theta \times]0,T[$. It is not possible to characterize the regularity class of $y(T)$ using classical spaces.
Therefore (3.1.1) is not exactly controllable.

Definition 3.1.2 We say that the system (3.1.1) is controllable at trajectories at time T if,

$$\forall y^0 \in \mathrm{L}^2(\Omega), \exists v \in \mathrm{L}^2(\mathrm{Q}) \;\; such \;\; that \;\; y_v(T) = z(T),$$

where $z(t)$ is a solution of (3.1.1) without control for an initial data $z^0 \in \mathrm{L}^2(\Omega)$.

R

1. Linearity: The system (3.1.1) which is considered here to be linear, the concepts of controllability at trajectories and of controllability at zero are equivalent.
2. The speed of propagation is infinite: In particular, we will see that the heat equation is controllable in an approximate way for any arbitrary time T and with a control in any subset of Ω.
3. The heat equation (3.1.1) has a strong regularizing effect on the initial data y^0. Note that the solution $y(x,t)$ is $\mathbb{C}^\infty$ in x for each $t > 0$, even if the initial data is discontinuous. It follows in particular that the heat equation is irreversible. In general the problem

$$\begin{cases} \partial_t y - \Delta y + ay &= v 1_{\mathscr{O}} \;\; in \;\; \Omega \times (0,T) \\ y &= 0 \quad on \quad \partial\Omega \times (0,T) \\ y(x, t=T) &= y^0 \;\; in \;\; \Omega \end{cases}$$

is not well put, and does not admit a solution.

3.2 Approximate controllability of the heat equation

There are several possible proofs for the property of approximate controllability. Here we present two of them. One is presented below and uses the Hahn-Banach theorem, the second is constructive and uses a variational technique.
We give it in the next subsection.

3.2.1 Topological approach for approximate controllability

Definition 3.2.1 The system (3.1.1) is approximately controllable at time T if, for any initial data $y^0 \in \mathrm{L}^2(\Omega)$, $\overline{\mathrm{R}(T,y^0)} = \mathrm{L}^2(\Omega)$.

Considering the linearity of the system that we have considered, it is easy to see that the following propositions are verified:

1. $\forall y^0 \in \mathrm{L}^2(\Omega)$, $\underline{\mathrm{R}(T,y^0)} = \mathrm{R}(T,0) + \underline{y(v=0,y^0)}(T)$.
2. $\forall y^0 \in \mathrm{L}^2(\Omega)$, $\overline{\mathrm{R}(T,y^0)} = \mathrm{L}^2(\Omega) \Leftrightarrow \overline{\mathrm{R}(T,0)} = \mathrm{L}^2(\Omega)$.
3. The system (3.1.1) is controllable exactly if and only if $\mathrm{R}(T,0) = \mathrm{L}^2(\Omega)$ and in an approximate way if and only if $\overline{\mathrm{R}(T,0)} = \mathrm{L}^2(\Omega)$.

We assume that $y^0 = 0$, i.e., we consider

$$\begin{cases} \partial_t y - \Delta y + ay &= v1_{\mathcal{O}} \ \ in \ \ \Omega\times(0,T) \\ y &= 0 \quad on \quad \partial\Omega\times(0,T) \\ y(x,t=0) &= 0 \ \ in \ \ \Omega \end{cases} \tag{3.2.1}$$

We introduce the linear operator

$$\mathscr{A}_T : v \in \mathrm{L}^2(Q) \to \mathscr{A}_T v = y_v(.,T) \in \mathrm{L}^2(\Omega)$$

with y_v the solution of (3.2.1) associated with v. According to the proposition 3.1.1, we have

$$\mathscr{A}_T \in \mathscr{L}(\mathrm{L}^2(Q), \mathrm{L}^2(\Omega)).$$

The system (3.2.1) is controllable in an approximate way to time $T \Leftrightarrow \mathrm{R}(\mathscr{A}_T)$ is dense in $\mathrm{L}^2(\Omega)$.

Or,

$$\overline{\mathrm{R}(\mathscr{A}_T)} = \ker(\mathscr{A}_T^*)^{\perp}.$$

Où $\mathscr{A}_T^*$ is the assistant operator of $\mathscr{A}_T$.

Objective: Show that

$$\ker(\mathscr{A}_T^*) = 0.$$

Let's fix $\varphi^0 \in \mathrm{L}^2(\Omega)$, and consider the following adjunct problem:

$$\begin{cases} -\partial_t\varphi - \Delta\varphi + a\varphi &= 0 \ \ in \ \ Q = \Omega\times(0,T) \\ \varphi &= 0 \quad on \quad \partial\Omega\times(0,T) \\ \varphi(x,t=T) &= \varphi^0(x) \ \ in \ \ \Omega \end{cases} \tag{3.2.2}$$

It is a retrograde problem and it is well posed.

Theorem 3.2.1 Given $a \in \mathrm{L}^\infty(Q)$, then $\forall\varphi^0 \in \mathrm{L}^2(\Omega)$, the system (3.2.2) admits a unique solution φ verifying:

$$\varphi \in \mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega)) \cap \mathbb{C}^0([0,T];\mathrm{L}^2(\Omega)), \ \ \partial_t\varphi \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$$

$$||\varphi||_{\mathbb{C}([0,T];\mathrm{L}^2(\Omega))} + ||\varphi||_{\mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega))} + ||\partial_t\varphi||_{\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))} \leq C||\varphi^0||_{\mathrm{L}^2(\Omega)}$$

for a positive constant C.

Proposition 3.2.2 Let's fix $y^0, \varphi^0 \in \mathrm{L}^2(\Omega)$ et $v \in \mathrm{L}^2(Q)$. Then

$$\int_\Omega y_v(x,T)\varphi^0 dx - \int_\Omega y^0(x)\varphi(x,0)dx = \int_0^T \int_{\mathscr{O}} v\varphi dxdt.$$

y_v and φ are respectively the solutions of (3.1.1) and (3.2.2) for y^0,v et φ^0.

Indeed,

We multiply the equation satisfied by φ by y and then the equation of y by φ. After integrations by parts and taking into account the conditions on board it comes:

$$\begin{aligned}
\int_0^T \int_{\mathscr{O}} v\varphi dxdt &= \iint_{\Omega\times(0,T)} (y_t - \Delta y + ay)\varphi dxdt \\
&= -\iint_{\Omega\times(0,T)} (\varphi_t + \Delta\varphi - a\varphi) y_v dxdt + \int_\Omega y_v \varphi dx|_0^T \\
&\quad + \int_0^T \int_{\partial\Omega} (-\frac{\partial y}{\partial n}\varphi + y\frac{\partial \varphi}{\partial n}) d\sigma dt. \\
&= \int_\Omega y_v(x,T)\varphi(T,x)dx - \int_\Omega y^0(x)\varphi(x,0)dx.
\end{aligned}$$

When $y^0 = 0$, the previous Proposition provides the equality

$$(\mathscr{A}_T v, \varphi^0)_{\mathrm{L}^2(\Omega)} = (v, \varphi 1_{\mathscr{O}}) \ \ \forall \varphi^0 \in \mathrm{L}^2(\Omega),$$

ie,

$$\mathscr{A}_T^* : \varphi^0 \in \mathrm{L}^2(\Omega) \to \varphi 1_{\mathscr{O}} \in \mathrm{L}^2(Q) \ \ and \ \ \mathscr{A}_T^* \in \mathscr{L}(\mathrm{L}^2(\Omega), \mathrm{L}^2(Q)).$$

It follows that the study of the approximate controllability for the problem (3.1.1) is equivalent to the following uniqueness problem for the adjoint system (3.2.2) :

Theorem 3.2.3 (Continuation unique) [10]
Suppose $a \in L^\infty(Q)$. Let φ solution of (3.2.2), checking $\varphi = 0$ on $\mathscr{O}\times(0,T)$, then $\varphi^0 = 0$ and therefore $\varphi = 0 \ \ on \ \ \Omega\times(0,T)$.

Theorem 3.2.4 Let $\mathscr{O}$ be a nonempty open such that $\mathscr{O} \subset \Omega$.
Then (3.1.1) is approximate controllable for all $T > 0$.

It is easy to see that the method used above has a character general and that, for a linear problem, the study of the controllability approach comes down to the study of a single continuation question for the assistant problem.

3.2.2 Variational approach for approximate controllability

In this section, we give a new proof for the result of the approximate controllability. The main ingredients that we will develop are variational in nature.
This proof has the advantage of being constructive and it allows to calculate the approximate controls explicitly.

We introduce,

$$\mathscr{U}_{ad}(y^1,\varepsilon) = \{v \in \mathrm{L}^2(Q) : ||y_v(T) - y^1||_{L^2(\Omega)} \le \varepsilon, \ \ y_v \ \ the \ \ solution \ \ of \ \ (3.2.1)\}$$

This set contain infinity of elements. In addition, it is convex, closed in $L^2(\Omega)$:

$$\mathscr{U}_{ad}(y^1,\varepsilon) = \mathscr{A}_T^{-1}(\overline{B}(y^1;\varepsilon)).$$

Question:

Given $y^1 \in L^2(\Omega)$ and $\varepsilon > 0$, can we give a constructive method which computes $v \in L^2(Q)$ than

$$||y_v(T) - y^1||_{L^2(\Omega)} \leq \varepsilon? \tag{3.2.3}$$

Answer: Yes

We then consider the following problem:

$$\begin{cases} minimize & \frac{1}{2}||v||^2_{L^2(Q)} = \frac{1}{2}\iint_Q |v(x,t|^2 dxdt \\ v \in & \mathscr{U}_{ad}(y^1,\varepsilon) \end{cases} \tag{3.2.4}$$

which admits a unique solution $\hat{v} \in \mathrm{L}^2(Q)$.

Objective: To build this minimum standard control.

We can write

$$\mathscr{U}_{ad}(y^1,\varepsilon) = \{v \in \mathrm{L}^2(Q) : \mathscr{A}_T(v) \in \overline{B}(y^1,\varepsilon)\}.$$

We introduce

$$F(v) = \frac{1}{2}\iint_Q |v(x,t|^2 dxdt \ \ et \ \ G(z) = \begin{cases} 0 & if \ \ z \in \overline{B}(y^1,\varepsilon) \\ +\infty & sinon \end{cases}$$

Note that $F : L^2(Q) \to \mathbb{R}$ et $G : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ are convex and proper functions.

We can then reformulate our problem (3.2.4) as follows:

$$\inf_{v \in L^2(Q)} [F(v) + G(\mathscr{A}_T(v))].$$

We will now use the Fenchel-Rockafellar duality theorem:

Let X, Y be two Hilbert spaces, $A \in \mathscr{L}(X,Y)$ a continuous linear operator, and $F : X \to \mathbb{R} \cup \{+\infty\}$ and $G : Y \to \mathbb{R} \cup \{+\infty\}$ two convex functions scb (semi continuous below) and clean.

We consider the following minimization problems:

$$(P) : \alpha = \inf_{x \in X}[F(x) + G(Ax)] \ et \ (P') : \alpha^* = \inf_{y \in Y}[F^*(A^*y) + G^*(-y)]$$

où F^* et G^* are the conjugate functions of F et G :

$$F^*(z) = \sup_{x \in X}[< z,x >_X - F(x)] \ et \ G^*(w) = \sup_{y \in Y}[< w,y >_Y - G(y)].$$

Suppose also that there is $x \in X$ et $y \in Y$ such as

$$F(x) + G(Ax) < \infty, \ \ F^*(A^*y) + G^*(-y) < \infty.$$

Then :

Theorem 3.2.5 [8] Under the previous assumptions :

1. Suppose $0 \in int(DomG - ADomF)$, then $\alpha + \alpha^* = 0$ et $\exists \hat{y} \in Y$ such that $\alpha^* = F^*(A^*\hat{y}) + G^*(-\hat{y})$.
2. Suppose $0 \in int(A^*DomG^* + DomF^*)$, then $\alpha + \alpha^* = 0$ et $\exists \hat{x} \in X$ such that $\alpha = F(\hat{x}) + G(A\hat{x})$.

3. Suppose that the problems (p) and (p ') admit a solution $\hat{x} \in X$ et $\hat{y} \in Y$ respectively. Then

$$\alpha + \alpha^* = 0 \Leftrightarrow A^*\hat{y} \in \partial F(\hat{x}) \text{ and } -\hat{y} \in \partial G(A\hat{x}).$$

Let's use the Fenchel-Rockafellar duality theorem:
The (3.2.4) problem is equivalent to the dual problem

$$\inf_{\varphi^0 \in L^2(\Omega)} [F^*(\mathscr{A}_T^*\varphi^0) + G^*(-\varphi^0)].$$

Où F^* et G^* are the conjugate functions of F et G :

$$F^*(v) = \sup_{w \in L^2(Q)} [(v,w)_{L^2(Q)} - F(w)] = \frac{1}{2}\iint_Q |v(x,t|^2 dxdt,$$

$$G^*(\varphi^0) = \sup_{\psi^0 \in L^2(\Omega)} [(\varphi^0, \psi^0)_{L^2(\Omega)} - G(\psi^0)] = (\varphi^0, y^1)_{L^2(\Omega)} + \varepsilon||\varphi^0||_{L^2(\Omega)}.$$

The dual problem is then the following

$$\begin{cases} \textit{minimize} & J_\varepsilon(\varphi^0, y^1) = \frac{1}{2}\iint_{\mathscr{O}\times(0,T)} |\varphi(x,t)|^2 dxdt + \varepsilon||\varphi^0||_{L^2(\Omega)} - (\varphi^0, y^1)_{L^2(\Omega)} \\ \varphi^0 \in & L^2(\Omega), \end{cases} \tag{3.2.5}$$

From the Fenchel-Rockafellar theorem (3.2.5), we deduce that

$$\min_{v \in \mathscr{U}_{ad}(y^1,\varepsilon)} \frac{1}{2}\iint_Q |v(x,t|^2 dxdt = -\inf_{\varphi^0 \in L^2(\Omega)} J_\varepsilon(\varphi^0, y^1).$$

The following lemma guarantees that the minimum of J_ε gives a control for our problem.

Lemma 1. *If $\hat{\varphi}^0$ is a minimum of J_ε in $\mathrm{L}^2(\Omega)$ and $\hat{\varphi}$ is the solution of* (3.2.2) *with the initial data $\hat{\varphi}^0$, then $\hat{v} = \hat{\varphi}1_{\mathscr{O}}$ is the solution of the problem* (3.2.4).

Proof. In the following, we simply denote J_ε by J.
assuming that J reaches its minimum in $\hat{\varphi}^0 \in \mathrm{L}^2(\Omega)$. So for everything $\psi^0 \in \mathrm{L}^2(\Omega)$ et $h \in \mathbb{R}$ we have

$$J(\hat{\varphi}^0) \leq J(\hat{\varphi}^0 + h\psi^0).$$

On the other hand,

$$\begin{aligned} J(\hat{\varphi}^0 + h\psi^0) &= \frac{1}{2}\int_0^T\int_{\mathscr{O}} |\hat{\varphi} + h\psi|^2 dxdt + \varepsilon||\hat{\varphi}^0 + h\psi^0||_{\mathrm{L}^2(\Omega)} \\ &\quad - \int_\Omega y^1(\hat{\varphi}^0 + h\psi^0)dx \\ &= \frac{1}{2}\int_0^T\int_{\mathscr{O}} |\hat{\varphi}|^2 dxdt + \frac{h^2}{2}\int_0^T\int_{\mathscr{O}} |\psi|^2 dxdt + h\int_0^T\int_{\mathscr{O}} \hat{\varphi}\psi dxdt \\ &\quad + \varepsilon||\hat{\varphi}^0 + h\psi^0||_{\mathrm{L}^2(\Omega)} - \int_\Omega y^1(\hat{\varphi}^0 + h\psi^0)dx. \end{aligned}$$

Thus

$$\begin{aligned} 0 \leq\ & \varepsilon[||\hat{\varphi}^0 + h\psi^0||_{\mathrm{L}^2(\Omega)} - ||\hat{\varphi}^0||_{\mathrm{L}^2(\Omega)}] + \frac{h^2}{2}\int_{(0,T)\times\mathscr{O}} \psi^2 dxdt \\ & + h[\int_0^T\int_{\mathscr{O}} \hat{\varphi}\psi dxdt - \int_\Omega y^1\psi^0 dx]. \end{aligned}$$

Since

$$||\hat{\varphi}^0 + h\psi^0||_{L^2(\Omega)} - ||\hat{\varphi}^0||_{L^2(\Omega)} \leq |h|||\psi^0||_{L^2(\Omega)}$$

we get

$$0 \leq \varepsilon|h|||\psi^0||_{L^2(\Omega)} + \frac{h^2}{2}\int_0^T\int_{\mathscr{O}} \psi^2 dxdt + h\int_0^T\int_{\mathscr{O}} \hat{\varphi}\psi dxdt - h\int_{\Omega} y^1\psi^0 dx$$

pour tout $h \in \mathbb{R}$ et $\psi^0 \in L^2(\Omega)$.

Dividing by $h > 0$ and passing to the limit $h \to 0$ we get

$$0 \leq \varepsilon||\psi^0||_{L^2(\Omega)} + \int_0^T\int_{\mathscr{O}} \hat{\varphi}\psi dxdt - \int_{\Omega} y^1\psi^0 dx.$$

The same calculation with $h < 0$ gives

$$\left|\int_0^T\int_{\mathscr{O}} \hat{\varphi}\psi dxdt - \int_{\Omega} y^1\psi^0 dx\right| \leq \varepsilon||\psi^0||_{L^2(\Omega)} \;\; \forall\psi^0 \in L^2(\Omega).$$

On the other hand, If we take control $v = \hat{\varphi}$ in (3.1.1), and by multiplying (3.1.1) by ψ solution of (3.2.2) and by integrating by parts we obtain that

$$\int_0^T\int_{\mathscr{O}} \hat{\varphi}\psi dxdt = \int_{\Omega} y(T)\psi^0 dx.$$

From the last two relationships, it follows that

$$\left|\int_{\Omega}(y(T) - y^1)\psi^0 dx\right| \leq \varepsilon||\psi^0||_{L^2(\Omega)} \;\; \forall\psi^0 \in L^2(\Omega).$$

which equals

$$||y(T) - y^1||_{L^2(\Omega)} \leq \varepsilon.$$

Which ends the proof of the lemma. ■

Let us now show that J reaches its minimum in $L^2(\Omega)$. But first of all we recall an essential theorem in calculus of variations, optimal control, etc.

Theorem 3.2.6 [5]
Let E be a reflexive Banach space, $A \subset E$ a closed, non-empty convex and $\varphi : A \to]-\infty, +\infty]$ a convex function, sci, $\varphi \neq +\infty$ such that

$$\lim_{x\ inA,\ ||x|| \to \infty} \varphi(x) = +\infty.$$

Then φ reaches its minimum on A, i.e. There exists $x_0 \in A$ such that

$$\varphi(x_0) = \min_A \varphi.$$

Theorem 3.2.7 [11] Let $y^1 \in \mathrm{L}^2(\Omega)$ the desired final state, $a \in L^\infty(Q)$ and $\varepsilon > 0$. Then

1. $J(.,y^1)$ is strictly convex, continues in $\mathrm{L}^2(\Omega)$.
2. $$\liminf_{||\varphi^0||_{\mathrm{L}^2(\Omega)} \to \infty} \frac{J(\varphi^0, y^1)}{||\varphi^0||_{\mathrm{L}^2(\Omega)}} \geq \varepsilon. \tag{3.2.6}$$
3. There exists $\hat{\varphi}^0 \in \mathrm{L}^2(\Omega)$ such that
$$J(\hat{\varphi}^0, y^1) = \min_{\varphi^0 \in \mathrm{L}^2(\Omega)} J(\varphi^0, y^1). \tag{3.2.7}$$

Proof. It is easy to see that J is strictly convex and continues in $\mathrm{L}^2(\Omega)$. According to the theorem 3.2.6, the existence of the minimum is assured if J is coercive, i.e.

$$J(\varphi^0) \to \infty \ \ when \ \ ||\varphi^0||_{\mathrm{L}^2(\Omega)} \to \infty. \tag{3.2.8}$$

In fact, we will prove that

$$\liminf_{||\varphi^0||_{\mathrm{L}^2(\Omega)} \to \infty} \frac{J(\varphi^0)}{||\varphi^0||_{\mathrm{L}^2(\Omega)}} \geq \varepsilon. \tag{3.2.9}$$

Obviously, (3.2.9) implies (3.2.8) and the proof of the lemma is complete. In order to prove (3.2.9) let $(\varphi_n^0) \subset \mathrm{L}^2(\Omega)$ be an initial data sequence for the adjoining system with $||\varphi_n^0||_{\mathrm{L}^2(\Omega)} \to \infty$. We normalize them

$$\tilde{\varphi}_n^0 = \frac{\varphi_n^0}{||\varphi_n^0||_{\mathrm{L}^2(\Omega)}},$$

so what $||\tilde{\varphi}_n^0||_{\mathrm{L}^2(\Omega)} = 1$.

On the other hand, either $\tilde{\varphi}_n$ the solution of (3.2.2) with the initial data $\tilde{\varphi}_n^0$. Then

$$\frac{J(\varphi_n^0)}{||\varphi_n^0||_{\mathrm{L}^2(\Omega)}} = \frac{1}{2}||\varphi_n^0||_{\mathrm{L}^2(\Omega)} \int_0^T \int_{\mathscr{O}} |\tilde{\varphi}_n|^2 dxdt + \varepsilon - \int_\Omega y^1 \tilde{\varphi}_n^0 dx.$$

A distinction is made between the following two cases:

1. $\liminf_{n\to\infty} \int_0^T \int_{\mathscr{O}} |\tilde{\varphi}_n|^2 > 0$. In this case we immediately obtain that
$$\liminf_{n\to\infty} \frac{J(\varphi_n^0)}{||\varphi_n^0||_{\mathrm{L}^2(\Omega)}} = \infty.$$
2. $\liminf_{n\to\infty} \int_0^T \int_{\mathscr{O}} |\tilde{\varphi}_n|^2 = 0$. In this case since $\tilde{\varphi}_n^0$ is bounded in $\mathrm{L}^2(\Omega)$, we can extract a subsequence (always denoted by $\tilde{\varphi}_n^0$) such that $\tilde{\varphi}_n^0 \rightharpoonup \psi^0$ weakly in $\mathrm{L}^2(\Omega)$ and $\tilde{\varphi}_n \rightharpoonup \psi$ weakly in $\mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{H}^{-1}(\Omega))$, where ψ is the solution of (3.2.1) with the initial data ψ^0 en $t = T$.
Moreover, according to the lower semi-continuity,
$$\int_0^T \int_{\mathscr{O}} \psi^2 dxdt \leq \liminf_{n\to\infty} \int_0^T \int_{\mathscr{O}} |\tilde{\varphi}_n|^2 dxdt = 0$$
and so $\psi = 0 \ \ in \ \ \mathscr{O} \times (0,T)$.
The result of single continuation implies that $\psi = 0 \ \ in \ \ \Omega \times (0,T)$ and hence $\psi^0 = 0$.

therefore, $\tilde{\varphi}_n^0 \rightharpoonup 0$ weakly in $\mathrm{L}^2(\Omega)$.
Hence $\int_\Omega y^1 \tilde{\varphi}_n^0 dx$ tends to 0.
Thus,

$$\liminf_{n\to\infty} \frac{J(\varphi_n^0)}{||\varphi_n^0||_{\mathrm{L}^2(\Omega)}} \geq \liminf_{n\to\infty} [\varepsilon - \int_\Omega y^1 \tilde{\varphi}_n^0 dx] = \varepsilon,$$

which give (3.2.6).

■

R This proposition and the lemma 1 give a second proof of the theorem 3.2.4. This approach not only guarantees the existence of control, but also provides a method to achieve control by minimizing a convex, continuous and coercive functional in $\mathrm{L}^2(\Omega)$.

In the proof of coercivity, the relevance of the term $\varepsilon||\varphi^0||_{\mathrm{L}^2(\Omega)}$is clear.
Indeed, the coercivity of J strongly depends on this term. It's not just for technical reasons.

The existence of a minimum of J with $\varepsilon = 0$ implies the existence of a check which verifies $y(T) = y^1$. But that's not true, unless y^1 is very regular in $\Omega \setminus \mathscr{O}$. Therefore, for a general $y^1 \in \mathrm{L}^2(\Omega)$, the term $\varepsilon||\varphi^0||_{\mathrm{L}^2(\Omega)}$ is required.

Note that the two tests are based on the unique continuation property which guarantees that if φ is a solution of (3.2.2), checking

$$\varphi = 0 \ \ on \ \ \mathscr{O} \times (0,T),$$

then

$$\varphi^0 = 0 \ \ and\ therefore\ \varphi = 0\ on\ \ \Omega \times (0,T).$$

3.3 Null controllability of the heat equation

Our next goal is to show null controllability or zero controllability for the problem (3.1.1)
Question: Given $y^0 \in L^2(\Omega)$, Ccan we find $v \in L^2(Q)$ such that the solution y_v of the system

$$\begin{cases} \partial_t y - \Delta y + ay &= v1_{\mathscr{O}} \ \ in \ \ \Omega \times (0,T) \\ y &= 0 \quad on \quad \partial\Omega \times (0,T) \\ y(x,t=0) &= y^0 \ \ in \ \ \Omega \end{cases}$$

check $y_v(T) = 0$ *in* Ω.

Theorem 3.3.1 The following conditions are equivalent:

1. There is a constant $C > 0$, such that $\forall y^0 \in L^2(\Omega)$, $\exists v \in L^2(Q)$, with

$$||v||_{L^2(Q)}^2 \leq C||y^0||_{L^2(\Omega)}^2 \tag{3.3.1}$$

such as the solution $y_v \in \mathbb{C}([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega))$ of the system (3.1.1) corresponding to y^0 and v satisfied $y_v(T) = 0$ *in* $L^2(\Omega)$.

2. There exists a constant $C > 0$, such that the following observability inequality

$$||\varphi(0)||_{\mathrm{L}^2(\Omega)}^2 \leq C \iint_{\mathscr{O}\times(0,T)} |\varphi(x,t)|^2 dxdt, \tag{3.3.2}$$

takes place, for any solution $\varphi \in \mathbb{C}([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega))$ of the adjoining system (3.2.2) with the initial data $\varphi^0 \in L^2(\Omega)$.

Proof. $1 \Rightarrow 2$: Take $y^0 \in L^2(\Omega)$ and consider a $\hat{v} \in L^2(Q)$ which satisfies (3.3.1) and $y_{\hat{v}}(T) = 0$ in Ω. ($y_{\hat{v}}$, solution of the system (3.1.1) associated with $\hat{v}$ with the initial data y^0.)

For $\varphi^0 \in L^2(\Omega)$, let φ be the solution of the assistant system (3.2.2) with the initial data φ^0 . From the proposition 3.2.2 we deduce

$$\begin{aligned}(\varphi(0), y^0) = \int_\Omega \varphi(x,0)y^0(x)dx &= -\iint_{\mathscr{O}\times(0,T)} \hat{v}(x,t)\varphi(x,t)dxdt \\ &\leq ||\varphi 1_{\mathscr{O}}||_{L^2(Q)}||\hat{v}||_{L^2(Q)} \\ &\leq \sqrt{C}||\varphi 1_{\mathscr{O}}||_{L^2(Q)}||y^0||_{L^2(\Omega)}, \quad \forall y^0 \in L^2(\Omega).\end{aligned}$$

From this last inequality we get the observability inequality (3.3.2) for the aide system.

$2 \Rightarrow 1$: We divide the proof into two stages. First, we construct a series of controls $v_\varepsilon \in L^2(Q)$ with $\varepsilon > 0$ which provide the approximate controllability of (3.1.1). Second, we go to the limit as ε approaches zero and we conclude. Step 1: Let $y^0 \in L^2(\Omega)$ and $\varepsilon > 0$. We introduce the functional J_ε defined by

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\iint_{\mathscr{O}\times(0,T)} |\varphi|^2 dxdt + \varepsilon||\varphi^0||_{L^2(\Omega)} + (\varphi(0), y^0)_{L^2(\Omega)}, \ \forall \varphi^0 \in L^2(\Omega).$$

Here φ is the solution of the system (3.2.2) with the initial data φ^0.

It is not difficult to verify that J_ε is strictly convex, continuous and coercive in $L^2(\Omega)$. It therefore admits a unique minimum φ_ε^0, whose associated solution is denoted by φ_ε. Let us now introduce the control $v_\varepsilon = \varphi_\varepsilon 1_{\mathscr{O}}$ and denote by y_ε the solution of the system (3.1.1) associated with v_ε.

Since J_ε reaches its minimum in φ_ε^0, then for each $\varphi^0\ inL^2(\Omega)$, the function

$$\begin{aligned} g: h \mapsto J_\varepsilon(\varphi_\varepsilon^0 + h\varphi^0) &= \frac{1}{2}\iint_{\mathscr{O}\times(0,T)} (\varphi_\varepsilon^2 + 2h\varphi_\varepsilon\varphi + h^2\varphi^2)dxdt \\ &+\varepsilon\sqrt{\int_\Omega \left((\varphi_\varepsilon^0)^2 + 2h\varphi_\varepsilon^0\varphi^0 + h^2(\varphi^0)^2\right)dxdt} \\ &+\int_\Omega (y^0\varphi_\varepsilon(0) + hy^0\varphi(0))dx\end{aligned}$$

reaches its minimum in 0.

Hence $g'(0) = 0$.

i.e.

$$\iint_{\mathscr{O}\times(0,T)} \varphi_\varepsilon\varphi dxdt + \varepsilon\left(\frac{\varphi_\varepsilon^0}{||\varphi_\varepsilon^0||_{L^2(\Omega)}}, \varphi^0\right) + \int_\Omega y^0\varphi(0)dx = 0, \ \forall \varphi^0 \in L^2(\Omega). \tag{3.3.3}$$

For $\varphi^0 = \varphi_\varepsilon^0$, we get

$$\iint_{\mathscr{O}\times(0,T)} |\varphi_\varepsilon|^2 dxdt + \varepsilon||\varphi_\varepsilon^0||_{L^2(\Omega)} + \int_\Omega \varphi_\varepsilon(0)y^0 dx = 0.$$

Or,

$$v_\varepsilon = \varphi_\varepsilon 1_{\mathscr{O}}.$$

Hence

$$\begin{aligned}||v_\varepsilon||^2_{L^2(Q)} = \iint_{\mathscr{O}\times(0,T)} |\varphi_\varepsilon|^2 dxdt &\leq -\int_\Omega \varphi_\varepsilon(0) y^0 dx \\ &\leq ||y^0||_{L^2(\Omega)} ||\varphi_\varepsilon(0)||_{L^{2(\Omega)}} \\ &\leq \frac{C}{2}||y^0||^2_{L^2(\Omega)} + \frac{1}{2C}||\varphi_\varepsilon(0)||^2_{L^{2(\Omega)}} \\ &\leq \frac{C}{2}||y^0||^2_{L^2(\Omega)} + \frac{1}{2}\iint_{\mathscr{O}\times(O,T)} |\varphi_\varepsilon|^2 dxdt.\end{aligned}$$

From where finally

$$||v_\varepsilon||^2_{L^2(Q)} \leq C||y^0||^2_{L^2(\Omega)}, \tag{3.3.4}$$

where C is the observability constant in (3.3.2).

On the other hand, we have According to the proposition 3.2.2 :

$$\iint_{\mathscr{O}\times(0,T)} \varphi_\varepsilon \varphi dxdt = (y_\varepsilon(T), \varphi^0)_{L^2(\Omega)} - (y^0, \varphi(0))_{L^2(\Omega)}.$$

This equation combined with (3.3.3), results in

$$\begin{aligned}(y_\varepsilon(T), \varphi^0)_{L^2(\Omega)} &= -\varepsilon\left(\frac{\varphi^0_\varepsilon}{||\varphi^0_\varepsilon||_{L^2(\Omega)}}, \varphi^0\right) \\ &\leq \varepsilon||\varphi^0||_{L^2(\Omega)}.\end{aligned}$$

Hence

$$||y_\varepsilon(T)||_{L^2}(\Omega) \leq \varepsilon. \tag{3.3.5}$$

Step 2: Since the sequence v_ε is bounded in $L^2(Q)$, we can extract a subsequence, again denoted v_ε by abuse of language, which converges weakly in $L^2(Q)$ to an element v. From there, we deduce from the classical results on the heat equation, that $y_\varepsilon = y_{v_\varepsilon}$ converges to $y = y_v$ in $L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$. In particular, this gives weak convergence for $\{y_\varepsilon(t)\}(t \in [0,T])$ in $L^2(\Omega)$. In particular, one can pass to the limit under the boundary conditions, and one obtains:

$$y(T) = 0.$$

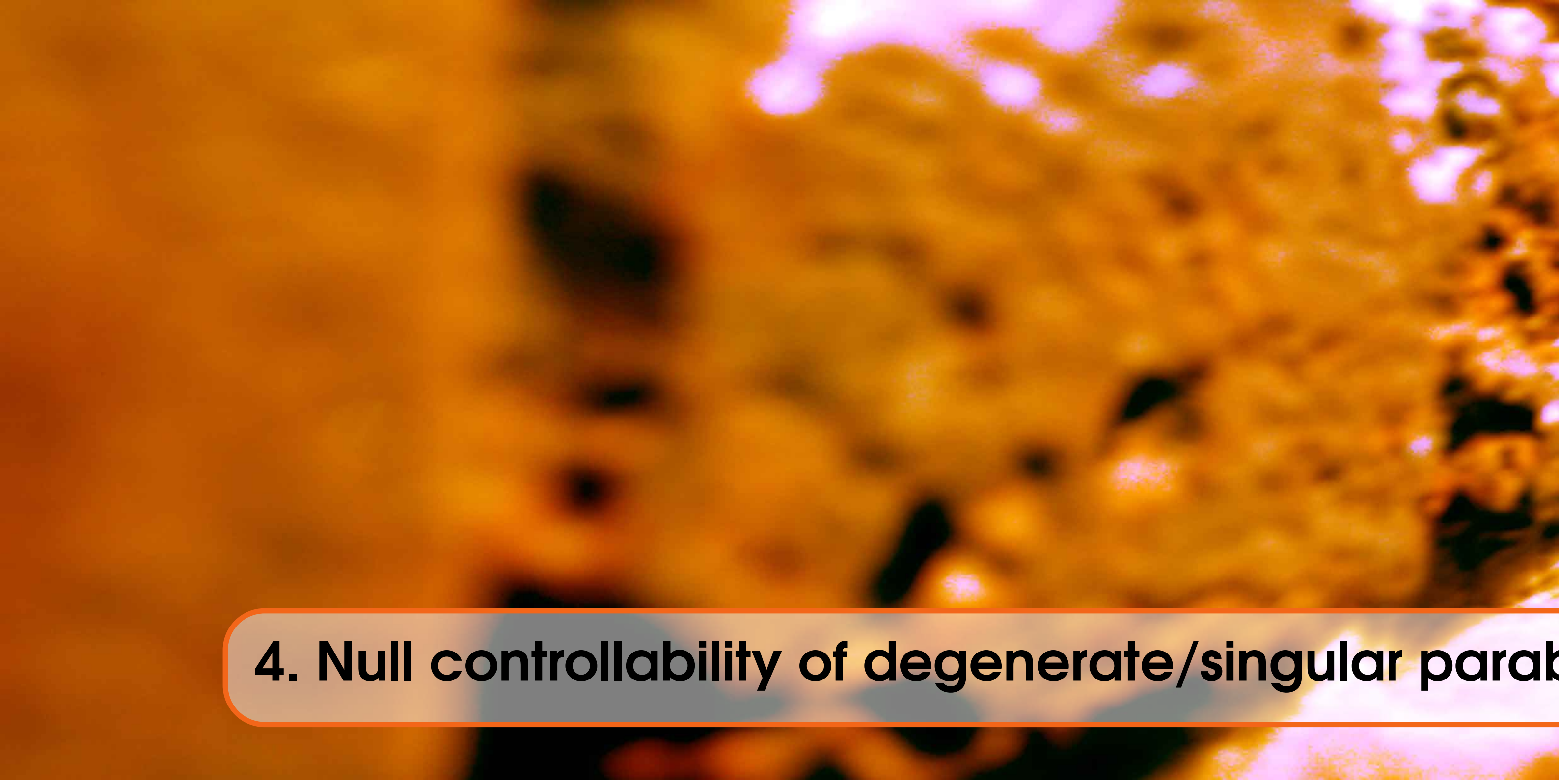
■

Proposition 3.3.2 Suppose that $y^0 \in L^2(\Omega)$ and $a \in L^\infty(Q)$. Then there exists a control $v \in L^2(\mathscr{O}\times(0,T))$ such that the solution y of associated (3.1.1) satisfies

$$y(.,T) = 0 \;\; in\; \Omega.$$

Moreover

$$||v||^2_{L^2(\mathscr{O}\times(0,T))} \leq \exp\{C(1 + \frac{1}{T} + T||a||_\infty + ||a||_\infty^{\frac{2}{3}}\}||y^0||^2_{L^2(\Omega)} \tag{3.3.6}$$



4. Null controllability of degenerate/singular parab

In the recent years, an increasing interest has been devoted to the study of the controllability for parabolic equations and has become an active research area. After the pioneering works [7, 18, 23, 24, 25]. Indeed many problems coming from physics, biology and economics are described by degenerate/singular parabolic equations, whose linear prototype is

$$u_t - (au_x)_x - \frac{\lambda}{b(x)}u = h(t,x), \quad (t,x) \in (0,T) \times (0,1), \tag{4.0.1}$$

More recently, several works were done in the controllability of purely ($\lambda = 0$) degenerate equations in divergence or in non divergence form with boundary degeneracy, see [**Alabau**, 12, 21]. The results on Carleman estimates for purely degenerate problems with an interior degenerate point are obtained in [13], for a regular degeneracy, and in [15], for a globally non smooth degeneracy. The case of parabolic operators with singular lower order terms. is treated in [2, 3]. And the parabolic problem with singular potential is considered, in [9, 27].

Furthermore, in numerical aspects, very few results are known regarding study of the null controllability in degenerate/singular parabolic equations, even though this class of operators occurs in interesting theoretical and applied problems. As far as we know, [1] is the unique published work on this subject; it concerns the numerical study of null controllability of the heat distribution in a degenerate/singular parabolic equation with degeneracy and singularity at the boundary of the domain.

In particular, our results complements the ones of [14] and [16], In fact we validate numerically the results obtained by [14], we consider the numerical reconstruction of the source term $h(t,x)$ to obtain $u(T,.) = 0$. To this end, we adopt the classical Tikhonov regularization to reformulate the inverse problem into a related optimization problem, for which we develop an iterative thresholding algorithm by using the corresponding adjoint system.

Consider the following problem

$$\begin{cases} u_t - (au_x)_x - \dfrac{\lambda}{b(x)}u = h(t,x), & (t,x) \in Q, \\ u(0) = u(1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{4.0.2}$$

where $u_0 \in L^2(0,1)$, $T > 0$ fixed and $Q := (0,T) \times (0,1)$. Moreover, we assume that the constant λ satisfy suitable assumptions described below and the functions a and b degenerate at the same interior point x_0 of the spatial domain $(0,1)$ (for the precise assumptions we refer to section 4.1).

Let us recall that in inverse source problems, the source term has to satisfy some condition otherwise uniqueness may be false, see [26]. Let $C_0 > 0$ be given and for $t_0 \in (0,T)$ given, let $T' := \frac{T+t_0}{2}$. In [6, 18], the authors make the assumption that source terms h satisfy the condition

$$|h_t(t,x)| \le C_0|h(T',x)|, \text{ for almost all } (t,x) \in Q. \tag{4.0.3}$$

Therefore they define the set $\mathscr{S}(C_0)$ of admissible source terms as

$$\mathscr{S}(C_0) := \{h \in H^1(0,T;L^2(0,1)) : h \text{ satisfies (4.0.3)}\}.$$

The rest of this article is organized as follows. In Section 4.1, we recall the well-posedness of the problem (4.0.2). Then Section 4.1, we study numerically the null controllability of 4.0.2. To this end, we reformulate our inverse source problem as a minimization problem with the Tikhonov regularization and provide several numerical examples.

Throughout the paper, C denotes a generic positive constant, which may vary from line to line.

4.1 Well-posedness

The ways in which a and b degenerate at x_0 can be quite different, and for this reason, following [16], to establish our results, we give the following definitions and assumptions:

Hypothesis 1. *Double weakly degenerate case (WWD) There exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ in $[0,1] \setminus \{x_0\}$, $a, b \in C^1([0,1] \setminus \{x_0\})$ and there exists $\alpha, \beta \in (0,1)$ such that $(x - x_0)a' \le \alpha a$ and $(x - x_0)b' \le \beta b$ a.e. in $[0,1]$.*

Hypothesis 2. *Weakly strongly degenerate case (WSD) There exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ in $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\})$, $b \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, $\exists\, \alpha \in (0,1), \beta \in [1,2)$ such that $(x - x_0)a' \le \alpha a$ and $(x - x_0)b' \le \beta b$ a.e. in $[0,1]$.*

Hypothesis 3. *Strongly weakly degenerate case (SWD) There exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ in $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, $b \in C^1([0,1] \setminus \{x_0\})$, $\exists \alpha \in [1,2)$, $\beta \in (0,1)$ such that $(x - x_0)a' \le \alpha a$ and $(x - x_0)b' \le \beta b$ a.e. in $[0,1]$.*

Hypothesis 4. *Double strongly degenerate case (SSD). There exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ in $[0,1] \setminus \{x_0\}$, $a, b \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, there exists $\alpha, \beta \in [1,2)$ such that $(x - x_0)a' \le \alpha a$ and $(x - x_0)b' \le \beta b$ a.e. in $[0,1]$.*

Typical examples for the previous degeneracies and singularities are $a(x) =| x - x_0 |^\alpha$ and $b(x) =| x - x_0 |^\beta$.

For the well–posedness of the problem (4.0.2), as in [16], we consider different classes of weighted Hilbert spaces, which are suitable to study the four different situations given above, namely the (WWD), (WSD), (SWD) and (SSD) cases. Thus, we consider the Hilbert spaces

$$H_a^1(0,1) := \left\{u \in W_0^{1,1}(0,1) \,:\, \sqrt{a}u_x \in L^2(0,1)\right\}$$

and

$$H^1_{a,b}(0,1) := \left\{ u \in H^1_a(0,1) : \frac{u}{\sqrt{b}} \in L^2(0,1) \right\}$$

endowed with the inner products

$$\langle u,v\rangle_{H^1_a} := \int_0^1 au'v'\,dx + \int_0^1 uv\,dx,$$

and

$$\langle u,v\rangle_{H^1_{a,b}} := \int_0^1 au'v'\,dx + \int_0^1 uv\,dx + \int_0^1 \frac{uv}{b}\,dx,$$

respectively.

In order to deal with the singularity of b we need the following inequality proved in [16, Proposition 2.14].

Lemma 2. *If one among Hypotheses 1-3 holds with $\alpha+\beta \le 2$, then there exists a constant $C>0$ such that for all $u \in H^1_{a,b}(0,1)$ we have*

$$\int_0^1 \frac{u^2}{b(x)}\,dx \le C\int_0^1 a(x)|u'|^2\,dx. \tag{4.1.1}$$

In order to study well-posedness of problem (4.0.2) and in view of Lemma 2 , we consider the space

$$\mathscr{H} := H^1_{a,b}(0,1),$$

where the Hardy-Poincaré inequality (4.1.1) holds.

We underline that, from Lemma 2, the standard norm $\|.\|^2_{\mathscr{H}}$ is equivalent to

$$\|.\|^2_{\sim} := \int_0^1 a(u')^2 dx.$$

From now on, we make the following assumptions on a, b and λ:

Hypothesis 5. *1. One among the Hypothesis 1, 2 or 3 holds true with $\alpha+\beta \le 2$ and we assume that*

$$\lambda \in \left(0, \frac{1}{C^\star}\right). \tag{4.1.2}$$

2. Hypotheses 1, 2, 3 or 4 hold with $\lambda < 0$.

Using the lemma 2, the next inequality is proved in [16, Proposition 2.18], which is crucial not only to obtain the well-posedness of the problem (4.0.2), but also to prove that the inverse problem posed as weak solution minimization problem has a solution.

Proposition 4.1.1 Assume Hypothesis 5. Then there exist a positive constant $\Lambda \in (0,1]$ such that for all $u \in \mathscr{H}$, there holds

$$\int_0^1 a(u')^2\,dx - \lambda\int_0^1 \frac{u^2}{b}\,dx \ge \Lambda\int_0^1 a(u')^2\,dx. \tag{4.1.3}$$

Now, let us go back to problem (4.0.2), recalling the following definition:

Definition 4.1.1 Let $u_0 \in L^2(0,1)$ and $h \in L^2(Q)$. A function u is said to be a (weak) solution of (4.0.2) if

$$u \in C([0,T];L^2(0,1)) \cap L^2(0,T;\mathcal{H})$$

and satisfies the following differential equation

$$\begin{aligned}
&\int_0^1 u(T,x)\varphi(T,x)\,dx - \int_0^1 u_0(x)\varphi(0,x)\,dx - \iint_Q u(t,x)\varphi_t(t,x)\,dx\,dt \\
&= -\iint_Q a(x)u_x(t,x)\varphi_x(t,x)\,dx\,dt + \lambda \iint_Q \frac{u(t,x)\varphi(t,x)}{b}\,dx\,dt \\
&\quad + \iint_Q h(t,x)\varphi(t,x)\,dx\,dt
\end{aligned}$$

for all $\varphi \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;\mathcal{H})$.

Finally, we introduce the Hilbert space

$$H^2_{a,b}(0,1) := \{u \in H^1_a(0,1) : au' \in H^1(0,1)\,\text{and}\, Au \in L^2(0,1)\},$$

where

$$Au := (au')' + \frac{\lambda}{b}u,$$

with domain

$$D(A) := H^2_{a,b}(0,1).$$

R Observe that if $u \in D(A)$, then $\frac{u}{\sqrt{b}} \in L^2(0,1)$, so that $u \in \mathcal{H}$ and inequality (2) holds.

Hence, the next result holds thanks to the theory of semigroups.

Proposition 4.1.2 The following assertions hold.

(i) The operator $(A, D(A))$, is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(0,1)$. Moreover, the semigroup is analytic.

(ii) For all $u_0 \in D(A)$ and $h \in H^1(0,T;L^2(0,1))$, the problem (4.0.2) admits a unique strict solution belonging to the class

$$u \in C\big([0,T];D(A)\big) \cap C^1([0,T];L^2(0,1))$$

and there exists a positive constant C such that

$$\begin{aligned}
&\sup_{t\in[0,T]} \left(\|u(t)\|^2_{H^1_a(0,1)}\right) + \int_0^T \left(\|u_t\|^2_{L^2(0,1)} + \|(au_x)_x\|^2_{L^2(0,1)}\right) dt \\
&\qquad \le C\left(\|u_0\|^2_{H^1_a(0,1)} + \|h\|^2_{L^2(Q)}\right).
\end{aligned} \tag{4.1.4}$$

If moreover $u_0 \in L^2(0,1)$, then for all $\varepsilon \in (0,T)$ there holds

$$u \in C\big([\varepsilon,T];D(A)\big) \cap C^1([\varepsilon,T];L^2(0,1)).$$

(iii) For all $u_0 \in L^2(0,1)$ and for all $h \in L^2(0,T;L^2(0,1))$, problem (4.0.2) has a unique weak solution $u \in C\big([0,T];L^2(0,1)\big) \cap L^2(0,T;\mathscr{H})$ such that

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(0,1)} + \int_0^T \|u(t)\|^2_{\mathscr{H}} dt \le C_T \left(\|u_0\|^2_{L^2(0,1)} + \|h\|^2_{L^2(Q)}\right), \tag{4.1.5}$$

for some positive constant C_T. Further, for all $\varepsilon \in (0,T)$ there holds

$$u \in L^2\big(\varepsilon,T;D(A)\big) \cap H^1\big(\varepsilon,T;L^2(0,1)\big).$$

If moreover $h \in H^1(0,T;L^2(0,1))$ and $\varepsilon \in (0,T)$, we have

$$u \in H^1\big([\varepsilon,T];D(A)\big) \cap H^2([\varepsilon,T];L^2(0,1))$$

.

Proof. The proof of statement (i) can be found in [16], whereas statements (ii) and (iii) are a consequence of (i) and [4, Proposition 3.3 and Proposition 3.8]. ■

4.2 The null controllability

In this section, we develop a numerical approach to study the null controllability of 4.0.2, with $a(x) = \mid x - x_0 \mid^\alpha$ and $b(x) = \mid x - x_0 \mid^\beta$. α and β two real constants and $x_0 \in (0,1)$.

4.2.1 Theory

We recall the following result which shows theoretically the null controllability of problem 4.0.2

Theorem 4.2.1 — (14). Assume Hypotheses 5 Then, given $u_0 \in L^2(0,1)$, there exists $h \in L^2(Q)$ such that the solution u of 4.0.2 satisfies

$$u(T,x) = 0 \text{ for every } x \in [0,1]$$

Moreover,

$$\int_Q h^2 dxdt \le C \int_0^1 {u_0}^2(x)dx$$

for some positive constant C.

4.2.2 Numerical approach

In this subsection we study null controllability from the numerical viewpoint. To this end, let us define our inverse problem which we use in computations.

Inverse Source Problem (ISP). Let u be the solution to (4.0.2). Determine the source term $h(t,x)$ to obtain $u(T,.) = 0$.

Numerically, we treat Problem (ISP) by interpreting its solution as a minimizer of the following least squares functional with the Tikhonov regularization

$$\min_{h\in\mathscr{U}} J(h), \quad J(h) = \frac{1}{2}\|u(T,.)\|^2_{L^2(0,1)} + \frac{\varepsilon}{2}\|h\|^2_{L^2(Q)}, \tag{4.2.1}$$

where $\varepsilon > 0$ stands for the regularization parameter and $\mathscr{U}$ is the set of admissible unknown sources defined in the following way

$$\mathscr{U} := \{h \in H^1(0,T;L^2(0,1)) : \|h\|_{H^1(0,T;L^2(0,1))} \leq r, r > 0\}. \tag{4.2.2}$$

Evidently, the set $\mathscr{U}$ is a bounded, closed, and convex subset of $H^1(0,T;L^2(0,1))$.

In this section, we focus on the determination of an unknown source term h to obtain $u(T,.) = 0$ in a one dimensional parabolic equation which is not only degenerate but also singular at the same interior point, this inverse problem can be formulate as a minimization problem of the functional J. To show that the minimization problem and the direct problem are well-posed, we prove that the solution's behavior changes continuously with the source term, for this we prove the Lipschitz continuity of the input-output operator $F : h \longrightarrow u$, where u is the weak solution of (4.0.2) with term source h. And, we prove the differentiability of the functional J, which gives the existence of the gradient of J, that is computed using the adjoint state method. Finally, to show the convergence of the descent method, we prove that the gradient of J is Lipschitz continuous, this gives that $\lim\limits_{k\to\infty} \| \nabla J(h^k) \|_{L^2(\Omega)} = 0$ and $\big(J(h^k)\big)_k$ is a monotone decreasing sequence , where $(h^k)_k$ is the sequence of iterations obtained by the Landweber iteration algorithm $h^{k+1} = h^k - t_k \nabla J(h^k)$ and t_k is chosen by the inaccurate linear search by the Armijo-Goldstein Rule. Also we provide several numerical examples to validate the work of [14].

We are now going to show the existence of minimizers to the problem (4.2.1). To do so, we need the following lemma.

Lemma 3. *Assume Hypothesis 5. Let u be the weak solution of* (4.0.2) *corresponding to a given source term h. Then, the input-output operator $F : H^1(0,T;L^2(0,1)) \to C([0,T];L^2(0,1)) \cap L^2(0,T;\mathscr{H})$ defined as $F(h) := u$ is Lipschitz continuous.*

Proof. First, take $u_0 \in D(A)$. Then, let the source term h be perturbed by a small amount δh such that $h + \delta h \in \mathscr{U}$. Consider $\delta u = u^\delta - u$, where u^δ is the weak solution of (4.0.2) with source term $h^\delta := h + \delta h$. Then $\delta u \in C^1\big([0,T];L^2(0,1)\big) \cap C\big(0,T;D(A)\big)$ satisfies the following sensitivity problem:

$$\begin{cases} \partial_t \delta u - \partial_x \big(a \partial_x \delta u\big) - \dfrac{\lambda}{b} \delta u = \delta h(t,x), & (t,x) \in Q, \\ \delta u(0) = \delta u(1) = 0, & t \in (0,T), \\ \delta u(0,x) = 0, & x \in (0,1). \end{cases} \tag{4.2.3}$$

Let $v(t,x)$ be a smooth function. From equation (4.2.3) and by the Gauss Green identity [16, Lemma 2.21], we have

$$\int_0^1 \partial_t \delta u v \, dx + \int_0^1 \Big(a \partial_x \delta u v_x - \frac{\lambda}{b} \delta u v\Big) \, dx = \int_0^1 \delta h v \, dx.$$

We take δu as a mutual test function for v to deduce

$$\frac{1}{2} \frac{d}{dt} \int_0^1 (\delta u)^2 \, dx + \int_0^1 \Big(a (\partial_x \delta u)^2 - \frac{\lambda}{b} (\delta u)^2\Big) \, dx = \int_0^1 \delta h \delta u \, dx.$$

Then, using Lemma 4.1.1, by the Cauchy-Schwarz inequality we obtain

$$\frac{1}{2} \frac{d}{dt} \|\delta u(t)\|^2_{L^2(0,1)} + \Lambda \int_0^1 a (\partial_x \delta u(t))^2 \, dx \leq \frac{1}{2} \|\delta u(t)\|^2_{L^2(0,1)} + \frac{1}{2} \|\delta h(t)\|^2_{L^2(0,1)}, \tag{4.2.4}$$

for every $t \leq T$, from which

$$\frac{d}{dt}\|\delta u(t)\|^2_{L^2(0,1)} \leq \|\delta u(t)\|^2_{L^2(0,1)} + \|\delta h(t)\|^2_{L^2(0,1)}.$$

Applying Gronwall's inequality, we obtain

$$\begin{aligned}\|\delta u(t)\|^2_{L^2(0,1)} &\leq e^T\left(\|\delta u(0)\|^2_{L^2(0,1)} + \|\delta h\|^2_{L^2(Q)}\right)\\ &= e^T\|\delta h\|^2_{L^2(Q)},\end{aligned} \tag{4.2.5}$$

for every $t \leq T$. From (4.2.4) and (4.2.5), we immediately get

$$\int_0^T \|\sqrt{a}\delta u_x(t)\|^2_{L^2(0,1)}dt \leq C_T\|\delta h\|^2_{L^2(Q)}, \tag{4.2.6}$$

for every $t \leq T$ and some universal constant $C_T > 0$. Thus, by (4.2.5) and (4.2.6), we obtain

$$\sup_{t\in[0,T]} \|\delta u(t)\|^2_{L^2(0,1)} + \int_0^T \|\delta u(t)\|^2_{\mathscr{H}}dt \leq C_T\|\delta h\|^2_{L^2(Q)},$$

from which it follows that

$$\|\delta u\|^2_{C([0,T];L^2(0,1))} + \|\delta u\|^2_{L^2(0,T;\mathscr{H})} \leq C\|\delta h\|^2_{H^1(0,T;L^2(0,1))},$$

if $u_0 \in D(A)$. Since $D(A)$ is dense in $L^2(0,1)$, the same inequality holds if $u_0 \in L^2(0,1)$. This completes the proof Lemma 3. ■

An immediate consequence of Lemma 3 is the following result

Proposition 4.2.2 Assume Hypothesis 5. Then, the functional J is continuous on $\mathscr{U}$ and there exists a minimizer $h^\star \in \mathscr{U}$ of $J(h)$, i.e.

$$J(h^\star) = \min_{h\in\mathscr{U}} J(h).$$

Proposition 4.2.3 . Let u the weak solution of (4.0.2) with source term h. The input-output operator $F : H^1(0,T;L^2(0,1)) \to C([0,T];L^2(0,1)) \cap L^2(0,T;\mathscr{H})$, $F(h) = u$ is G-derivable.

The most important issue in numerical solutions of inverse problems is the Lipschitz continuity of the gradient, which ensures the convergence of the method of descent, for that we have the follows result

Proposition 4.2.4 Let h and δh, such that $h+\delta h \in \mathscr{U}$, than ∇J is Lipschitz continuous

$$\| \nabla J(h+\delta h) - \nabla J(h) \|_{L^2(Q)} \leqslant L_1 \| \delta h \|_{H^1(0,T;L^2(0,1))} \tag{4.2.7}$$

with the Lipschitz constant $L_1 > 0$.

Proof of Proposition 4.2.3. Let δh be a small variation such that $h+\delta h \in \mathscr{U}$, we define the function

$$F'(h) : \delta h \in \mathscr{U} \to \delta u, \tag{4.2.8}$$

where δu is the solution of the variational problem

$$\begin{gathered}\int_\Omega \partial_t(\delta u)v\,dx + \int_\Omega (a(x)\partial_x(\delta u)\partial_x v - \frac{\lambda}{b(x)}\delta u v)\,dx = \int_\Omega \delta h v dx \quad \forall v \in H^1_0(\Omega)\\ \delta u(0,t) = \delta u(1,t) = 0 \quad \forall t \in]0,T[\\ \delta u(x,0) = 0 \quad \forall x \in \Omega.\end{gathered} \tag{4.2.9}$$

We set

$$\phi(h) = F(h+\delta h) - F(h) - F'(h)\delta h. \tag{4.2.10}$$

We want to show that

$$\phi(h) = o(\delta h). \tag{4.2.11}$$

We easily verify that the function ϕ is the solution of variational problem

$$\begin{gathered}\int_\Omega \partial_t \phi v\, dx + \int_\Omega \Big(a(x)\partial_x\phi\partial_x v - \frac{\lambda}{b(x)}\phi v\Big)\, dx = \int_\Omega (\delta h - (\delta h)^2) v dx \quad \forall v \in H_0^1(\Omega)\\ \phi(0,t) = \phi(1,t) = 0 \quad \forall t \in]0,T[\\ \phi(x,0) = 0 \quad \forall x \in \Omega.\end{gathered} \tag{4.2.12}$$

In the same way as that used in the proof of continuity, we deduce

$$\|\phi\|^2_{C([0,T];L^2(0,1))} + \|\phi\|^2_{L^2(0,T;\mathscr{H})} \leq C\|\delta h - (\delta h)^2\|^2_{H^1(0,T;L^2(0,1))},$$

Hence, the function $F(h) = u$ is G-derivable and we deduce the existence of the gradient of the functional J. ■

Before starting the demonstration of Proposition 4.2.4, we compute the gradient of J using the adjoint state method.

We define the Gâteaux derivative of u at h in the direction $f \in L^2(\Omega\times]0,T[)$, by

$$\hat{u} = \lim_{s\to 0} \frac{u(h+sf) - u(h)}{s}, \tag{4.2.13}$$

$u(h+sf)$ is the weak solution of (4.0.2) with source term $h+sf$, and $u(h)$ is the weak solution of (4.0.2) with source term h.

We compute the Gâteaux (directional) derivative of (4.0.2) at h in some direction $f \in L^2(\Omega\times]0,T[)$, and we get the so-called tangent linear model:

$$\begin{gathered}\partial_t \hat{u} - A\hat{u} = f\\ \hat{u}(0,t) = \hat{u}(1,t) = 0 \quad \forall t \in]0,T[\\ \hat{u}(x,0) = 0 \quad \forall x \in \Omega.\end{gathered} \tag{4.2.14}$$

We introduce the adjoint variable P, and we integrate,

$$\int_0^1\int_0^T \partial_t \hat{u} P\, dt\, dx - \int_0^1\int_0^T A\hat{u}P\, dx = \int_0^1\int_0^T fP\, dt\, dx, \tag{4.2.15}$$

$$\int_0^1 \Big([\hat{u}P]_0^T - \int_0^T \hat{u}\partial_t P\, dt\Big)\, dx - \int_0^T \langle A\hat{u}, P\rangle_{L^2(\Omega)} dt = \langle f, P\rangle_{L^2(\Omega\times]0,T[)}, \tag{4.2.16}$$

$$\begin{aligned}&\int_0^1 [\hat{u}(T)P(T) - \hat{u}(0)P(0)]dx - \int_0^T \langle \hat{u}, \partial_t P\rangle_{L^2(\Omega)} dt - \int_0^T \langle A\hat{u}, P\rangle_{L^2(\Omega)} dt\\ &= \langle f, P\rangle_{L^2(\Omega\times]0,T[)}.\end{aligned} \tag{4.2.17}$$

Let us take $P(x=0) = P(x=1) = 0$, then we may write $\langle \hat{u}, AP\rangle_{L^2(\Omega)} = \langle A\hat{u}, P\rangle_{L^2(\Omega)}$. With $P(T) = 0$ we may now rewrite (4.2.17) as

$$\int_0^T \langle \hat{u}, \partial_t P + AP\rangle_{L^2(\Omega)} dt = -\langle f, P\rangle_{L^2(\Omega\times]0,T[)}$$

this gives

$$\begin{aligned} &\int_0^T \langle \hat{u}, \partial_t P + AP \rangle_{L^2(\Omega)} dt = -\langle f, P \rangle_{L^2(\Omega \times]0,T[)} \\ &\quad P(x=0) = P(x=1) = 0, \quad P(T) = 0. \end{aligned} \tag{4.2.18}$$

The discretization in time of (4.2.18), using the Rectangular integration method, gives

$$\begin{aligned} &\sum_{j=0}^{M+1} \langle \hat{u}(t_j), \partial_t P(t_j) + AP(t_j) \rangle_{L^2(\Omega)} \Delta t = \langle -P, f \rangle_{L^2(\Omega \times]0,T[)} \\ &\qquad P(x=0) = P(x=1) = 0, \quad P(T) = 0. \end{aligned} \tag{4.2.19}$$

With

$$t_j = j\Delta t, \quad j \in \{0, 1, 2, \dots, M+1\},$$

where Δt is the step in time and $T = (M+1)\Delta t$.

The Gâteaux derivative of J at h in the direction $f \in L^2(\Omega)$ is given by

$$\hat{J}(f) = \lim_{s \to 0} \frac{J(h+sf) - J(h)}{s}.$$

After some computations, we arrive at

$$\hat{J}(f) = \langle u(T), \hat{u}(T) \rangle_{L^2(\Omega)} + \langle \varepsilon h, f \rangle_{L^2(\Omega \times]0,T[)}. \tag{4.2.20}$$

The adjoint model is

$$\begin{aligned} \partial_t P(T) + AP(T) = \frac{1}{\Delta t} u(T), \quad \partial_t P(t_j) + AP(t_j) = 0 \quad \forall t_j \neq T \\ P(x=0) = P(x=1) = 0 \quad \forall t_j \in]0;T[\\ P(T) = 0. \end{aligned} \tag{4.2.21}$$

From equations (4.2.19), (4.2.20) and (4.2.21), the gradient of J is given by

$$\frac{\partial J}{\partial h} = -P + \varepsilon h. \tag{4.2.22}$$

Problem (4.2.21) is retrograde, we make the change of variable $t \longleftrightarrow T - t$.

Proof of Proposition 4.2.4. We have $\nabla J(h) = -P_1 + \varepsilon h$ with P_1 is the solution of the adjoint model(with change of variable $t_j \longleftrightarrow T - t_j$)

$$\begin{cases} \begin{cases} \partial_t P_1(0) - AP_1(0) = \dfrac{1}{\Delta t} u_1(T) \\ \partial_t P_1(t_j) - AP_1(t_j) = 0 \qquad \forall t_j \neq 0 \end{cases} \\ P_1(x,t) = 0 \quad \forall x \in \partial\Omega, \quad \forall t \in]0;T[\\ P_1(x,0) = 0. \end{cases}$$

where u_1 is the weak solution of (4.0.2) with source term h, and $\nabla J(h + \delta h) = -P_2(T) + \varepsilon(h + \delta h)$ with P_2 is the solution of the adjoint model (with change of variable $t_j \longleftrightarrow T - t_j$)

$$\begin{cases} \begin{cases} \partial_t P_2(0) - AP_2(0) = \dfrac{1}{\Delta t} u_2(T) \\ \partial_t P_2(t_j) - AP_2(t_j) = 0 \qquad \forall t_j \neq 0 \end{cases} \\ P_2(x,t) = 0 \; \forall x \in \partial\Omega, \quad \forall t \in]0;T[\\ P_2(x,0) = 0. \end{cases}$$

where u_2 is the weak solution of (4.0.2) with source term $h+\delta h$.

Let $\delta P = P_1 - P_2$, we easily verify that δP is the solution of the variational problem

Hence, δP is weak solution of (4.0.2) with $h = (u_2(T)u_1(T))\mathbb{1}_0$. We apply the estimate in proposition 4.1.2, we obtain and there exists a positive constant C such that

$$\sup_{t\in[0,T]} \|\delta P(t)\|^2_{L^2(0,1)} + \int_0^T \|\delta P(t)\|^2_{\mathscr{H}} dt \le C\|(u_2(T)-u_1(T))\mathbb{1}_0\|^2_{L^2(Q)}, \tag{4.2.23}$$

then

$$\| \delta P \|^2_{L^2(0,T;\mathscr{H})} \leqslant C \left(\| (u_2(T)-u_1(T))\mathbb{1}_0 \|^2_{L^2(Q)} \right), \tag{4.2.24}$$

and

$$\| \delta P \|^2_{C([0,T];L^2(0,1))} \leqslant C \left(\| (u_2(T)-u_1(T))\mathbb{1}_0 \|^2_{L^2(Q)} \right), \tag{4.2.25}$$

the constant C depending only on Ω and T.

we showed above the Lipschitz continuity of the input-output operator

$$\begin{aligned} F : H^1(0,T;L^2(0,1)) &\longrightarrow C([0,T];L^2(0,1)) \cap L^2(0,T;\mathscr{H}) \\ h &\longmapsto u \end{aligned} \tag{4.2.26}$$

from where

$$\| (u_2(T)-u_1(T))\mathbb{1}_0 \|^2_{L^2(Q)} \leqslant C \| \delta h \|^2_{H^1(0,T;L^2(0,1))} \tag{4.2.27}$$

therefore

$$\| \delta P \|^2_{L^2(0,T;\mathscr{H})} \leqslant C \| \delta h \|^2_{H^1(0,T;L^2(0,1))}, \tag{4.2.28}$$

and

$$\| \delta P \|^2_{C([0,T];L^2(0,1))} \leqslant C \| \delta h \|^2_{H^1(0,T;L^2(0,1))}, \tag{4.2.29}$$

We have

$$\begin{aligned} \| \nabla J(u_0+\delta u_0) - \nabla J(u_0) \|_{L^2(Q)} &= \| \delta P + \varepsilon \delta h \|_{L^2(Q)} \\ &\leqslant \| \delta P \|_{L^2(Q)} + \| \varepsilon \delta h \|_{L^2(Q)} . \end{aligned} \tag{4.2.30}$$

therefore

$$\| \nabla J(h+\delta h) - \nabla J(h) \|_{L^2(Q)} \leqslant (\sqrt{C}+\varepsilon) \| \delta h \|_{H^1(0,T;L^2(0,1))} . \tag{4.2.31}$$

This completes the proof of the theorem. ■

4.2.3 Algorithm and simulations

In this subsection, a numerical algorithm on the basis of the conjugate gradient method is designed to treat the inverse problem and some numerical experiments are also performed.

The main steps for descent method at each iteration are:

- Calculate u^k solution of (4.0.2) with source term h^k
- Calculate P^k solution of the adjoint problem
- Calculate the descent direction $d_k = -\nabla J(h^k)$
- Find $t_k = \underset{t>0}{argmin}\, J(h^k + td_k)$
- Update the variable $h^{k+1} = h^k + t_k d_k$.

The algorithm ends when $\left|\nabla J(h)\right| < \mu$, where μ is a given small precision.

The value t_k is chosen by the inaccurate linear search by the Armijo-Goldstein Rule as follows:

Let $\alpha_i, \beta \in [0,1[$ and $\alpha > 0$

if $J(h^k + \alpha_i d_k) \leq J(h^k) + \beta \alpha_i d_k^T d_k$, $t_k = \alpha_i$ and stop.

if not, $\alpha_i = \alpha \alpha_i$.

Now, we are going to reconstruct the solution of problem 4.0.2 in all three cases : the purely singular case $\alpha = 0$, purely degenerate case $\lambda = 0$, and degenerate-singular case.

For the simulations, in all the tests below we take $x_0 = 0.5$, $u_0(x) = \dfrac{x(x-1)}{T}$, step in space $N = 100$ and step in time $M = 100$.

In the figures below, u_0 is drawn red and the rebuilt function u in blue.

1. The purely singular case $\alpha = 0$, with $\beta < 2$ and $\lambda < 0$ (example $\beta = \dfrac{1}{2}$ and $\lambda = -1$

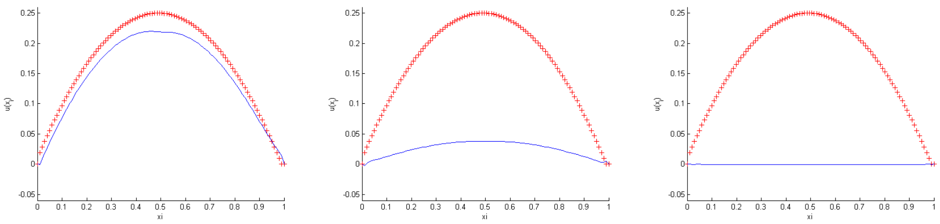


Figure 01. Temperature at $t = t_1$ (left), at $t = t_{15}$ (in the midst). Final temperature showing that $u(T) \simeq 0$ (right).

2. The purely degenerate case $\lambda = 0$, with $0 < \alpha < 2$ (example $\alpha = \dfrac{1}{2}$

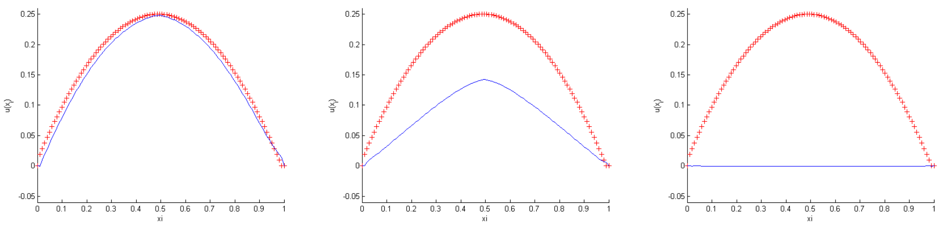


Figure 02. Temperature at $t = t_1$ (left), at $t = t_{15}$ (in the midst). Final temperature showing that $u(T) \simeq 0$ (right).

3. The degenerate and singular case
Double weakly degenerate case (WWD) in this case we have $\alpha, \beta \in (0,1)$, for our tests we take for example $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\lambda = -1$.

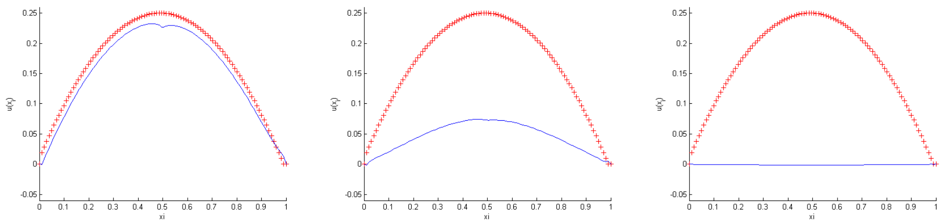


Figure 03. Temperature at $t = t_1$ (left), at $t = t_{15}$ (in the midst). Final temperature showing that $u(T) \simeq 0$ (right).

Weakly strongly degenerate case (WSD) in this case we have $\alpha \in (0,1)$, $\beta \in [1,2)$, for our tests we take for example $\alpha = \frac{1}{4}$, $\beta = \frac{4}{3}$, $\lambda = -1$.

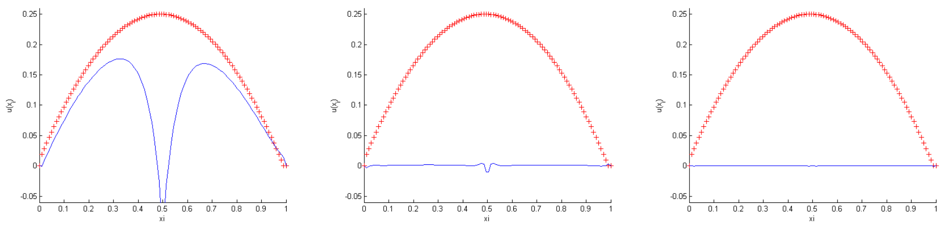


Figure 04. Temperature at $t = t_1$ (left), at $t = t_{15}$ (in the midst). Final temperature showing that $u(T) \simeq 0$ (right).

For case $\alpha + \beta = 2$ example $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $\lambda = -1$.

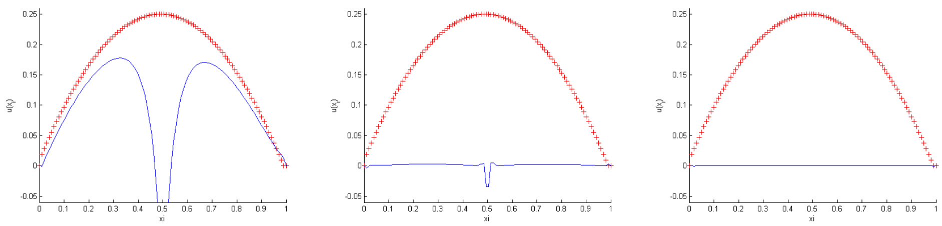


Figure 05. Temperature at $t = t_1$ (left), at $t = t_{15}$ (in the midst). Final temperature showing that $u(T) \simeq 0$ (right).

Strongly weakly degenerate case (SWD) in this case we have $\alpha \in [1,2)$, $\beta \in (0,1)$, for our tests we take for example $\alpha = \frac{4}{3}$, $\beta = \frac{1}{2}$, $\lambda = -1$.

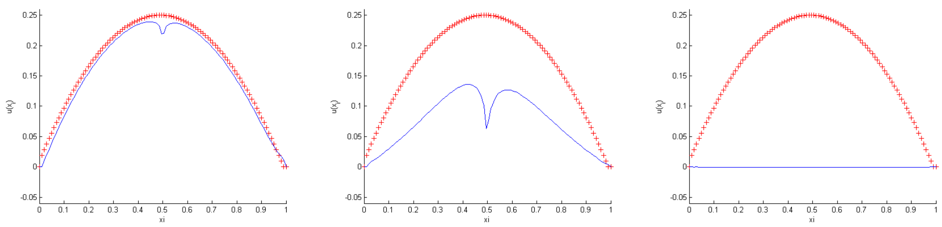


Figure 06. Temperature at $t = t_1$ (left), at $t = t_{15}$ (in the midst). Final temperature showing that $u(T) \simeq 0$ (right).

Double strongly degenerate case (SSD) in this case we have $\alpha, \beta \in [1,2)$, for our tests we take for example $\alpha = \frac{5}{4}$, $\beta = \frac{5}{4}$, $\lambda = -1$.

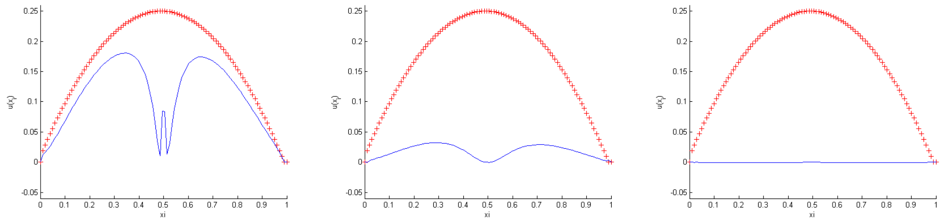


Figure 07. Temperature at $t = t_1$ (left), at $t = t_{15}$ (in the midst). Final temperature showing that $u(T) \simeq 0$ (right).

In all case (purely singular case $\alpha = 0$, purely degenerate case $\lambda = 0$), (Fig. 01- Fig 02) show that our problem 4.0.2 is null controllable, witch valid the results obtained by [**Alabau**, 9, 12, 13, 21, 27].

In degenerate and singular case, (Fig. 03- Fig 07) show that we can rebuild the source term h to obtain $u(T,.) = 0$. This valid numerically the results of [14].

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Dr. Hamed Ould Sidi is presently working at University of Nouakchott Al-Asriya, Nouakchott. He is Doctorate form Hassan First University, Settat, Marroco in Applied Mathematics. He has published four research article in internation journals. He presetented his research in four internation confrences. His main research interests has been on Partial Differential Equations and Control Theory.

This work summarizes the different mathematical methods allowing the approximate zero control of the linear heat equation. In the frist chapter we give a general introduction to the control theory. In the 2nd chapter we group together the main properties and functional spaces that we will use. For the third chapter we study the problem of internal controllability of the heat equation. The control is supposed to act on a subset of the domain where the solutions are defined. To study the practical part we end with the fourth chapter which consists in studying the null controllability for a degenerate singular unidimensional problem.

Author
Hamed Ould Sidi

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