BOUNDEDNESS OF LITTLEWOOD–PALEY OPERATORS WITH VARIABLE KERNEL ON WEIGHTED HERZ SPACES WITH VARIABLE EXPONENT

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Abstract. Let $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ be a homogeneous function of degree zero. In this article, we obtain some boundedness of the parameterized Littlewood–Paley operators with variable kernels on weighted Herz spaces with variable exponent.

Index Terms: Parameterized Littlewood-Paley operators; variable kernel; weighted Herz spaces; Muckenhoupt; variable exponents.

1. Introduction

The boundedness of Littlewood-Paley operators on function spaces are one of the very important tools, not only in harmonic analysis, but also in potential theory and in partial differential equations (see [1, 2, 3, 4, 5, 6], for details).

In 2004, Ding, Lin and Shao [7] investigated the $L^2$-boundedness for a class of Marcinkiewicz integral operators with variable kernels $\mu_\Omega$ and $\mu_\Omega,s$, related to the Littlewood-Paley function $\mu_{\Omega,\lambda}^*$ and the area integral $g_{\lambda}^*$. In 2006, the authors [8] proved the $L^p$-boundedness of the Littlewood-Paley operators with variable kernels. In 2009, Xue and Ding [9] established the weighted estimate for Littlewood-Paley operators and their commutators.

In 1960, Hörmander [10] introduced the parameterized Littlewood-Paley operators for the first time. Now, let us recall the definitions of the parameterized Lusin area integral and Littlewood-Paley $g_{\lambda}^*$ function.

Let $S^{n-1}(n \geq 2)$ be the unit sphere in $\mathbb{R}^n$ with normalized Lebesgue measure $d\sigma(x')$. Take $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})(r \geq 1)$ to be a homogeneous function

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Parameterized Littlewood-Paley operators with variable kernels

of degree zero and

\[
\int_{\mathbb{R}^{n-1}} \Omega(x, z') d\sigma(z') = 0, \quad \text{for all } x \in \mathbb{R}^n,
\]

(1)

where \( \Omega \) satisfies the following conditions:

1. For any \( x, z \in \mathbb{R}^n \) and any \( \lambda > 0 \), we have \( \Omega(x, \lambda z) = \Omega(x, z) \);
2. \( \|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})} := \sup_{r \geq 0, y \in \mathbb{R}^n} \left( \int_{\mathbb{R}^{n-1}} |\Omega(rz' + y, z')|^r d\sigma(z') \right)^{\frac{1}{r}} < \infty \).

The parameterized Littlewood-Paley operators \( \mu_{\Omega, s}^\rho \) and \( \mu_{\Omega, \lambda}^{\ast, \rho} \) with variable kernels, which are related to the Lusin area integral and the Littlewood-Paley \( g^\ast \lambda \) function are defined by

\[
\mu_{\Omega, s}^\rho(f)(x) = \left( \int \int_{\Gamma(x)} \left| \frac{1}{t^\rho} \int_{|y-z| \leq t} \frac{\Omega(y, y-z)}{|y-z|^{n-\rho}} f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}}
\]

and

\[
\mu_{\Omega, \lambda}^{\ast, \rho}(f)(x) = \left( \int_{\mathbb{R}^{n+1}} \left( \frac{t}{t + |x-y|} \right)^{\lambda n} \left| \frac{1}{t^\rho} \int_{|y-z| \leq t} \frac{\Omega(y, y-z)}{|y-z|^{n-\rho}} f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}},
\]

where \( \Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1} : |x - y| < t \text{ and } \lambda > 1\} \).

In 2013, Wei and Tao [11] investigated the boundedness of parameterized Littlewood-Paley operators on weighted weak Hardy spaces. Lin and Xuan [12] established the boundedness for commutators of parameterized Littlewood-Paley operators and area integrals on weighted Lebesgue spaces \( L^p(w) \).

The theory of the variable exponent function spaces has been rapidly developed after the work [13], where Kováčik and Rákosník have clarified fundamental properties of Lebesgue spaces with variable exponent. After that, many researchers have been interested in the theory of the variable exponent spaces (see [14, 15, 16, 17, 18, 19, 20]).

The generalization of the Muckenhoupt weights with variable exponent \( A_{p(\cdot)} \) has been considered in [21, 22, 23, 24]. The equivalence between the Muckenhoupt condition and the boundedness of the Hardy-Littlewood maximal operator on weighted Lebesgue spaces with variable exponent were discussed in [21, 22]. After that, Cruz-Uribe and Wang [25] proved the boundedness of some classical operators on weighted Lebesgue spaces with variable exponent \( L^{p(\cdot)}(w) \).

Recently, Izuki and Noi [26] introduced the weighted Herz spaces with variable exponent, and also studied the boundedness of fractional integrals on those spaces.

In this paper, we establish the boundedness of parameterized Littlewood-Paley operators with variable kernels on weighted Herz spaces with variable exponent.
Definition 1.1. Let \( p(\cdot) : E \to [1, \infty) \) be a measurable function. The variable exponent Lebesgue space is defined as

\[
L^{p(\cdot)}(E) = \left\{ f \text{ measurable} : \int_E \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx < \infty \text{ for some constant } \eta > 0 \right\}.
\]

The space \( L^{p(\cdot)}_{\text{loc}}(E) \) is defined as

\[
L^{p(\cdot)}_{\text{loc}}(E) = \left\{ f \text{ measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset E \right\}.
\]

The Lebesgue spaces \( L^{p(\cdot)}(E) \) is a Banach spaces with the norm defined as

\[
\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \eta > 0 : \int_E \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx \leq 1 \right\}.
\]

We denote \( p_- = \text{ess inf}\{p(x) : x \in E\} \), \( p_+ = \text{ess sup}\{p(x) : x \in E\} \), then \( P(E) \) consists of all \( p(\cdot) \) satisfying \( p_- > 1 \) and \( p_+ < \infty \).

Definition 1.2. [27] Let \( p(\cdot) : \mathbb{R}^n \to [1, \infty) \). A measurable function \( p(\cdot) \) is said to be globally log-Hölder continuous if it satisfies

1. \( |p(x) - p(y)| \leq \frac{1}{\log(1/|x-y|)}, \ x, y \in \mathbb{R}^n, |x-y| \leq 1/2; \)
2. \( |p(x) - p_{\infty}| \leq \frac{1}{\log(e+|x|)}, \ x \in \mathbb{R}^n, \)

for some \( p_{\infty} \geq 1 \). The set of \( p(\cdot) \) satisfying conditions (1) and (2) is denoted by \( LH(\mathbb{R}^n) \).

We know that, if \( p(\cdot) \in LH(\mathbb{R}^n) \cap P(\mathbb{R}^n) \), the Hardy-Littlewood maximal operator \( M \) is bounded on \( L^{p(\cdot)}(\mathbb{R}^n) \) (see[28]).

Definition 1.3. [29] Suppose that \( p(\cdot) \in P(\mathbb{R}^n) \) and \( w \) is a weight function. The weighted Lebesgue spaces with variable exponent \( L^{p(\cdot)}(w) \) is the set of all complex-valued measurable function \( f \) such that \( f^{1/p(\cdot)} \in L^{p(\cdot)}(\mathbb{R}^n) \). The space \( L^{p(\cdot)}(w) \) is a Banach space equipped with the norm

\[
\|f\|_{L^{p(\cdot)}(w)} = \|f^{1/p(\cdot)}\|_{L^{p(\cdot)}}.
\]

\( p'(\cdot) \) is the conjugate of \( p(\cdot) \) such that \( \frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1 \). Next, we introduce the classical Muckenhoupt \( A_p \) weight.

Definition 1.4. [30] Let \( 1 < p < \infty \), then \( w \in A_p \) for every cube \( Q \),

\[
\left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} \, dx \right)^{p-1} \leq C < \infty.
\]

We say that \( w \in A_1 \) if it satisfies \( Mw(x) \leq w(x) \) for all \( x \in \mathbb{R}^n \). The set \( A_1 \) consists of all Muckenhoupt \( A_1 \) weights.
**Definition 1.5.** [21, 25] Given \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and a weight \( w \), then \( w \in A_{p(\cdot)} \) if

\[
\sup_{B:\text{ball}} |B|^{-1} \|w^{1/p(\cdot)} \chi_B\|_{L^{p(\cdot)}(w)} \|w^{-1/p(\cdot)} \chi_B\|_{L^{p'(\cdot)}(w)} < \infty.
\]

**Definition 1.6.** [25] Given \( p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and \( 1/p_1(x) - 1/p_2(x) = \mu/n \) such that \( 0 < \mu < n \). Then \( w \in A_{p_1(\cdot),p_2(\cdot)} \) if

\[
\|w \chi_B\|_{L^{p_2(\cdot)}(w)} \|w^{-1} \chi_B\|_{L^{p_1'(\cdot)}(w)} \leq |B|^n\frac{n+\mu}{n}
\]

holds for all balls \( B \in \mathbb{R}^n \).

**Definition 1.7.** [25] Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and \( w \) be a weight. We say that \( (p(\cdot),w) \) is an \( M \)-pair if the maximal operator \( M \) is bounded on \( L^{p(\cdot)}(w) \) and \( L^{p'(\cdot)}(w^{-1}) \).

Now, we need to give the definition of weighted Herz space with variable exponent. For all \( k \in \mathbb{Z} \), we denote \( B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \} \), \( C_k = B_k \setminus B_{k-1} \), \( \chi_k = \chi_{C_k} \).

**Definition 1.8.** [26] Suppose that \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), \( 0 < q < \infty \), \( \alpha \in \mathbb{R} \). The homogeneous weighted Herz space with variable exponent \( \dot{K}^\alpha_{p(\cdot)}(w) \) is the collection of \( f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\},w) \) such that

\[
\|f\|_{\dot{K}^\alpha_{p(\cdot)}(w)} := \left( \sum_{k=-\infty}^{\infty} 2^{\alpha q k} \|f \chi_k\|_{L^{p(\cdot)}(w)}^q \right)^{1/q} < \infty.
\]

It is easy to see that if \( w = 1 \), then \( \dot{K}^\alpha_{p(\cdot)}(w) = \dot{K}^\alpha_{p(\cdot)}(\mathbb{R}^n) \) is the Herz space with variable exponent [17]. If \( w = 1 \) and \( p(\cdot) = p \), then \( K^\alpha_{p(\cdot)}(w) = K^\alpha_{p(\cdot)}(\mathbb{R}^n) \) is the classical Herz space introduced in [31]. If \( p(\cdot) = p \), then \( \dot{K}^\alpha_{p(\cdot)}(w) = \dot{K}^\alpha_{p(\cdot)}(w) \) is the weighted Herz space [32].

**Definition 1.9.** We say a kernel function \( \Omega(x,z) \) satisfies the \( L^r \)-Dini condition \( (r \geq 1) \), if

\[
\int_0^1 \frac{\omega_r(\delta)}{\delta} (1 + |\log \delta|^r) d\delta < \infty,
\]

where \( \omega_r(\delta) \) denotes the integral modulus of continuity of order \( r \) of \( \Omega \) defined by

\[
\omega_r(\delta) = \sup_{x \in \mathbb{R}^n, |\rho| < \delta} \left( \int_{S_1} \left| \Omega(x,\rho z') - \Omega(x,z') \right|^r d\sigma(z') \right)^{1/r},
\]

where \( \rho \) is the rotation in \( \mathbb{R}^n \), \( \|\rho\| = \sup_{z' \in S^{n-1}} \|\rho z' - z'\| \).
2. Preliminaries and notations

In order to prove our main theorems, we need the following Lemmas.

**Lemma 2.1.** [3] Suppose that $X \subset M$ is a Banach function space.

1. (The generalized Hölder inequality) For all $f \in X$ and $g \in X'$, we have
   \[
   \int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \|f\|_X \|g\|_X.
   \]

2. For all $f \in X$, we have
   \[
   \sup \left\{ \left| \int_{\mathbb{R}^n} f(x)g(x) \, dx \right| : \|g\|_{X'} \leq 1 \right\} = \|f\|_X.
   \]

In particular, space $(X')' = X$.

As an application of the generalized Hölder inequality above, we have the following Lemma.

**Lemma 2.2.** Let $X$ be a Banach function space, we have

\[
1 \leq \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'}
\]

hold for all balls $B$.

**Lemma 2.3.** [24] Let $X$ be a Banach function space. If the Hardy-Littlewood maximal operator $M$ is weakly bounded on $X$, that is

\[
\|\chi_{\{Mf > \lambda\}}\|_X \leq \lambda^{-1} \|f\|_X,
\]

holds for all $f \in X$ and $\lambda > 0$, then we get

\[
\sup_{B: \text{Ball}} \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'} < \infty.
\]

**Remark 2.1.** [29] The weighted Banach function space $X(\mathbb{R}^n, W)$ is a Banach function space equipped the norm $\|f\|_{X(\mathbb{R}^n, W)} := \|fW\|_X$. The associated space of $X(\mathbb{R}^n, W)$ is a Banach function space and equals $X'(\mathbb{R}^n, W^{-1})$.

**Remark 2.2.** If $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, by comparing the definition of the weighted Banach function space with weighted variable Lebesgue space, we have

1. If $X = L^{p_1(\cdot)}(\mathbb{R}^n)$ and $W = w$, then we obtain
   \[
   L^{p_1(\cdot)}(\mathbb{R}^n, w) = L^{p_1(\cdot)}(w^{p_1(\cdot)}).
   \]

2. If $X = L^{p_1(\cdot)}(\mathbb{R}^n)$ and $W = w^{-1}$. Using Lemma 2.3, we obtain
   \[
   L^{p_1(\cdot)}(\mathbb{R}^n, w^{-1}) = L^{p_1(\cdot)}(w^{-p_1(\cdot)}) = (L^{p_1(\cdot)}(w^{p_1(\cdot)}))'.
   \]
Lemma 2.4. [33] Suppose that $X$ is a Banach space. Let $M$ be bounded on the associated space $X'$. Then there exists a constant $0 < \delta < 1$ such that

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \leq \left(\frac{|E|}{|B|}\right)^\delta$$

holds for all balls $B$ and all measurable sets $E \subset B$.

Lemma 2.5. [26] Suppose that $p_1(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w^{p_1(\cdot)} \in A_1$. Let $M$ be a bounded on $L^{p_1(\cdot)}(w^{p_1(\cdot)})$ and $L^{p'_1(\cdot)}(w^{-p'_1(\cdot)})$, then there exist constants $\delta_1, \delta_2 \in (0, 1)$ such that

$$\frac{\|\chi_S\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}}{\|\chi_B\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}} \leq C \left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_S\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_B\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}} \leq C \left(\frac{|S|}{|B|}\right)^{\delta_2},$$

hold for all balls $B$ and all measurable sets $S \subset B$.

Lemma 2.6. [34] Suppose that $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^n-1)$ satisfies equation (1) and definition (1.9), $\lambda > 2, 2p - n > 0, 1 < p < \infty$. Then for all $f \in L^p(w)$ there exists $C > 0$ independent of $f$ such that

$$\|\mu_{\Omega, \lambda}^\mu f\|_{L^n(w)} \leq C \|f\|_{L^n(w)}$$

and

$$\|\mu_{\Omega, \lambda}^\mu f\|_{L^n(w)} \leq C \|f\|_{L^n(w)}.$$

Lemma 2.7. [25] Assume that for $p_0, 1 < p_0 < \infty$ and every $w_0 \in A_{p_0},$

$$\int_{\mathbb{R}^n} f(x)p_0w_0(x)dx \leq \int_{\mathbb{R}^n} g(x)p_0w_0(x)dx, \quad (f, g) \in \mathcal{F}.$$

Then for any $M$-pair $(p(\cdot), w),$

$$\|f\|_{L^{p(\cdot)}(w)} \leq C\|g\|_{L^{p(\cdot)}(w)}, \quad (f, g) \in \mathcal{F},$$

Lemma 2.7 holds for $p_0 = 1$ and the maximal operator is bounded on $L^{p(\cdot)}(w^{-1})$. We know the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$ (see [33]).

Combining Lemma 2.6 with Lemma 2.7, we obtain the following conclusion.

Corollary 2.8. Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n), \Omega \in L^\infty(\mathbb{R}^n) \times L^{r}(S^n-1)(r \geq 1)$ and $w \in A_{p(\cdot)}$. Then the parameterized Littlewood-Paley operators $\mu_{\Omega, \lambda}^\mu$ and $\mu_{\Omega, \lambda}^\nu$ with variable kernels are bounded on $L^{p(\cdot)}(w)$.

3. Main Theorems and their proofs

In this section, we will prove the boundedness of the parameterized Littlewood-Paley operators with variable kernels on variable weighted Herz spaces.
First, we consider $L \theta$.

**Theorem 3.2.** Let $P(\cdot) \in LH(\mathbb{R}^\alpha) \cap \mathcal{P}(\mathbb{R}^\alpha)$, $0 < q_1 \leq q_2 < \infty$, $\lambda > 2$, $a + b - n > 0$ and $\Omega \in L^\infty(\mathbb{R}^\alpha) \times L^2(S^{n-1})$ satisfies (1.1) and (1.2). If $w(P(\cdot)) \in A_1$ and $\lambda \delta_1 < \alpha < n \delta_2$, where $\delta_1, \delta_2$ are the constants in Lemma 2.5, then the operator $\mu_{\Omega,s}$ is bounded from $K_{\Omega,s}^{(\alpha,q_2)}(w(P(\cdot)))$ to $K_{\Omega,s}^{(\alpha,q_1)}(w(P(\cdot)))$.

**Theorem 3.2.** Let $P(\cdot) \in LH(\mathbb{R}^\alpha) \cap \mathcal{P}(\mathbb{R}^\alpha)$, $0 < q_1 \leq q_2 < \infty$, $\lambda > 2$, $a + b - n > 0$ and $\Omega \in L^\infty(\mathbb{R}^\alpha) \times L^2(S^{n-1})$ satisfies (1.1) and (1.2). If $w(P(\cdot)) \in A_1$ and $\lambda \delta_1 < \alpha < n \delta_2$, where $\delta_1, \delta_2$ are the constants in Lemma 2.5, then the operator $\mu_{\Omega,s}$ is bounded from $K_{\Omega,s}^{(\alpha,q_2)}(w(P(\cdot)))$ to $K_{\Omega,s}^{(\alpha,q_1)}(w(P(\cdot)))$.

**Remark 3.1.** As it is well known that, $\mu_{\Omega,s}^\lambda f(x) \leq 2^\lambda \mu_{\Omega,s}^\lambda f(x)$ (see [6], p.89). Therefore, we give only the proof of Theorem 3.2.

**Proof of Theorem 3.2**

Let $f \in K_{\Omega,s}^{(\alpha,q_1)}(w(P(\cdot)))$. By the Jensen inequality, we have

$$
\left| \mu_{\Omega,s}^\lambda f \right|_{K_{\Omega,s}^{(\alpha,q_2)}(w(P(\cdot)))} \leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_2 k} \left( \mu_{\Omega,s}^\lambda f \right)_{K_{\Omega,s}^{(\alpha,q_1)}(w(P(\cdot)))}^{q_2} \leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left( \mu_{\Omega,s}^\lambda f \right)_{K_{\Omega,s}^{(\alpha,q_1)}(w(P(\cdot)))}^{q_1}
$$

Denote $f_j = f \chi_j$ for each $j \in \mathbb{Z}$, then $f = \sum_{j=-\infty}^{\infty} f_j$, so we have

$$
\left| \mu_{\Omega,s}^\lambda f \right|_{K_{\Omega,s}^{(\alpha,q_2)}(w(P(\cdot)))} \leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left( \mu_{\Omega,s}^\lambda f \right)_{K_{\Omega,s}^{(\alpha,q_1)}(w(P(\cdot)))}^{q_1}
$$

First, we consider $L_2$. Using Lemma 2.1 and $-2 \leq k - j \leq 2$, it is easy to get

$$
L_2 = \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left( \sum_{j=-\infty}^{k+2} \left( \mu_{\Omega,s}^\lambda f_j \right)_{K_{\Omega,s}^{(\alpha,q_1)}(w(P(\cdot)))} \right)^{q_1}
$$

$$
= \sum_{k=-\infty}^{\infty} \left( \sum_{j=k-2}^{k+2} 2^{\alpha (k-j)} \left( \mu_{\Omega,s}^\lambda f_j \right)_{K_{\Omega,s}^{(\alpha,q_1)}(w(P(\cdot)))} \right)^{q_1}
$$
Now we need to consider $\mu_{\Omega, \lambda}^s f_j$. Applying the Minkowski inequality, we conclude that

$$
|\mu_{\Omega, \lambda}^s f_j(x)| = \left( \int_0^\infty \int_{S^n} \left( \frac{t}{t + |x - y|}\right)^{\lambda n} \frac{1}{\Omega(y, y - z)} \left| \int_{|y - z|^{-1}}^t f_j(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}
$$

$$
\leq \int_{S^n} f_j(z) \left( \int_0^\infty \int_{|y - z|^{-1}}^t \left( \frac{t}{t + |x - y|}\right)^{\lambda n} \frac{1}{\Omega(y, y - z)} \left| \int_{|y - z|^{-1}}^t f_j(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} dz
$$

$$
\leq \int_{S^n} f_j(z) \left( \int_0^\infty \int_{|y - z|^{-1}}^t \left( \frac{t}{t + |x - y|}\right)^{\lambda n} \frac{1}{\Omega(y, y - z)} \left| \int_{|y - z|^{-1}}^t f_j(z) dz \right|^2 \frac{dydt}{t^{2p+n+1}} \right)^{\frac{1}{2}} dz
$$

$$
+ \int_{S^n} f_j(z) \left( \int_0^\infty \int_{|y - z|^{-1}}^t \left( \frac{t}{t + |x - y|}\right)^{\lambda n} \frac{1}{\Omega(y, y - z)} \left| \int_{|y - z|^{-1}}^t f_j(z) dz \right|^2 \frac{dydt}{t^{2p+n+1}} \right)^{\frac{1}{2}} dz.
$$

For $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ and $2\rho - n > 0$, the following inequality holds

$$
\int_{|y - z|^{-1} \leq t} \frac{\Omega(y, y - z)}{|y - z|^{2n-2\rho}} dy \leq \int_{\mathbb{R}^n} \int_0^t \frac{\Omega(sy' + z, y')}{s^{2(n-2\rho)}} s^{-n-1} ds d\sigma(y')
$$

$$
\leq \|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})}^2 t^{2\rho - n}.
$$

Since $|x - z| \leq |x - y| + |y - z| \leq |x - y| + t$. For $\lambda > 2$, taking $0 < \delta < (\lambda - 2)n$, we have

$$
\int_0^{|x - z|} \int_0^{|y - z|^{-1}} \left( \frac{t}{t + |x - y|}\right)^{\lambda n} \frac{1}{|y - z|^{2n-2\rho}} \frac{dydt}{t^{2p+n+1}} \leq \int_0^{|x - z|} \int_0^{|y - z|^{-1}} \left( \frac{t}{t + |x - y|}\right)^{\lambda n-2n-\delta} \frac{dydt}{|y - z|^{2n+\delta}} \leq \frac{1}{|x - z|^{2n+\delta}} \int_0^{|x - z|} \frac{dydt}{t^{2p-n-\delta+1}} \leq \frac{\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})}^2}{|x - z|^{2n+\delta}} \int_0^{|x - z|} t^{\delta-1} dt \leq C|x - z|^{-2n}.
$$

If we take $1 < \lambda_1 < 2$, then $\lambda_1 n - n > 0$ and $\lambda_1 n - 2n < 0$, so we have

$$
\int_{|x - z|}^\infty \int_{|y - z|^{-1}}^t \left( \frac{t}{t + |x - y|}\right)^{\lambda n} \frac{1}{|y - z|^{2n-2\rho}} \frac{dydt}{t^{2p+n+1}} \leq \int_{|x - z|}^\infty \int_{|y - z|^{-1}}^t |x - z|^{-\lambda_1 n} \frac{1}{|y - z|^{2n-2\rho}} \frac{dydt}{t^{2p-\lambda_1 n + n+1}}.
$$
Now we have two cases: $1 < q$ and $0 < q \leq 1$. Noting that for $x \in A_k$, $z \in A_j$ and $j \leq k - 2$, then $|x - z| \sim |x|$. By the virtue of the generalized Hölder inequality, we have

$$\mu_{1,\lambda}^p(f_j)(x) \leq C 2^{-kn} \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_j \|_{L^{p_1}(w^{p_1}(\cdot))}. $$

Applying Lemma 2.3 and Lemma 2.5, we take $\| . \|_{L^{p_1}(w^{p_1}(\cdot))}$ for each side, we have

$$\| \mu_{1,\lambda}^p(f_j)(x) \chi_k \|_{L^{p_1}(w^{p_1}(\cdot))} \leq C 2^{-kn} \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_k \|_{L^{p_1}(w^{p_1}(\cdot))},$$

$$\leq C 2^{-kn} \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_k \|_{L^{p_1}(w^{p_1}(\cdot))},$$

$$\leq C \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_k \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_k \|_{L^{p_1}(w^{p_1}(\cdot))},$$

$$\leq C \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_k \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_k \|_{L^{p_1}(w^{p_1}(\cdot))},$$

$$\leq 2^{(j-k)n2} \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))}.$$

Thus, we have

$$L_1 \leq C \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left( \sum_{j=-\infty}^{k-2} 2^{(j-k)n2} \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \right)^{q_1},$$

$$\leq C \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} 2^{j \alpha n2} \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \right)^{q_1}.$$

Now we have two cases: $1 < q_1 < \infty$ and $0 < q_1 \leq 1$. When $1 < q_1 < \infty$, by using the Hölder inequality, we have

$$L_1 \leq C \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} 2^{j \alpha n2} \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \right)^{q_1}.$$
we have

Applying Lemma 2.3 and Lemma 2.5, we can take

\[
\|f_j\|_{L^p(\mathbb{R}^n)}^{q_1}
\]

Finally, we estimate \(L_3\). Noting that for \(x \in A_k, y \in A_j\) and \(j \geq k + 2\), then \(|y - x| \sim |y|\). By the virtue of the generalized Hölder inequality, we have

\[
m_{\frac{2-jm}{2-jm}j}(f_j(x)) \leq C 2^{-jm} \|f_j\|_{L^p(\mathbb{R}^n)}(\omega_{p_1}) \|\mathcal{X}_j\|_{L^p(\mathbb{R}^n)}(\omega_{p_1})^{jm}.
\]

Applying Lemma 2.3 and Lemma 2.5, we can take \(\|\mathcal{X}_{\omega_{p_1}}\|_{L^p(\mathbb{R}^n)}\) for each side, we have

\[
\|\mu_{\frac{2-jm}{2-jm}j}^{\omega_{p_1}}(f_j(x))\mathcal{X}_j\|_{L^p(\mathbb{R}^n)}(\omega_{p_1})
\]

Finally, we estimate \(L_3\). Noting that for \(x \in A_k, y \in A_j\) and \(j \geq k + 2\), then \(|y - x| \sim |y|\). By the virtue of the generalized Hölder inequality, we have

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\]
Thus, we have

$$L_3 \leq C \sum_{k=-\infty}^{\infty} 2^{\alpha q_k} \left( \sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_1} \| f_j \|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1}$$

$$\leq C \sum_{k=-\infty}^{\infty} \left( \sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha+n\delta_1)} f_j^{q_1} \right) \frac{q_1}{q_1} \right)^{q_1} \sum_{k=-\infty}^{\infty} 2^{(k-j)(\alpha+n\delta_1)q_1/2} \| f_j \|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}^{q_1}$$

$$\leq C \sum_{j=-\infty}^{\infty} 2^{j\alpha q_1} \| f_j \|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}^{q_1} \sum_{k=-\infty}^{j-2} 2^{(k-j)(\alpha+n\delta_1)q_1/2}$$

$$\leq C \sum_{j=-\infty}^{\infty} \| f_j \|_{K^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}^{q_1}.$$

When $0 < q_1 < 1$, applying the Jensen inequality, we obtain

$$L_3 \leq C \sum_{j=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} 2^{j\alpha q_1} \| f_j \|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \sum_{k=-\infty}^{j-2} 2^{(k-j)(\alpha+n\delta_2)q_1}$$

This completes the proof of Theorem 3.2.
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