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On the oscillation of fractional differential equations via ψ-Hilfer fractional derivative

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Abstract: In this paper, we study the oscillatory theory for fractional differential equations (FDEs) via ψ-Hilfer fractional derivative. Sufficient conditions are established for the oscillation of solutions FDEs.

Keywords: Caputo derivative, ψ-Hilfer fractional derivative, oscillation, Riemann-Liouville operator.

1. Introduction

Over the decades, the fractional calculus has been building a great history and consolidating itself in several scientific areas such as: mathematics, physics and engineering, among others. The emergence of new fractional integrals and derivatives, makes the wide number of definitions becomes increasingly larger and clears its numerous applications. Recently, the existence of solutions of initial and boundary value problems for differential equations involving Hilfer fractional derivative has a considerable attention [1–8].

Very recently, Almeida [9] introduced a new fractional derivative named by ψ-fractional derivative with respect to another function, which extended the classical fractional derivative and also studied some properties like semigroup law, Taylor's Theorem and so on. Thereafter, Sousa and Oliveira [10,11] initially studied a Cauchy problem for fractional ordinary differential equation with ψ-Hilfer operator with respect to another function, in order to unify the wide number of fractional derivatives in a single fractional operator and consequently, open a window for new applications and established a new Gronwall inequality to derive a prior bound of a solution. The authors studied the Leibniz type rule: ψ-Hilfer fractional operator in [12].

The oscillation theory as a part of the qualitative theory of differential equations has been developed rapidly in the last decades and there has been a great deal of work on the oscillatory behavior of integer order differential equations. However, there are only very few papers dealing with the oscillation of FDEs; see [13–15]. The study of oscillation and other qualitative properties of fractional dynamical systems such as stability, existence, and uniqueness of solutions is necessary to analyze the systems under consideration [16,17].

Motivated by [18] and the aforementioned papers, we study the oscillatory theory for ψ-Hilfer fractional type FDEs of the form

$$D_{a^+}^{\alpha,\beta;\psi}x(t) + f_1(t,x) = w(t) + f_2(t,x), \tag{1}$$

$$I_{a^+}^{1-\gamma;\psi}x(t) = b_1, \tag{2}$$

where $D_{a^+}^{\alpha,\beta;\psi}$ denotes the ψ-Hilfer fractional derivative of order $0 < \alpha < 1$ type $0 \leq \beta \leq 1$, $I_{a^+}^{1-\gamma;\psi}$ is the ψ-Riemann-Liouville fractional integral with $\gamma = \alpha + \beta(1-\alpha)$ and $b_1 > 0$.

We assume in this paper that the functions f_1, f_2 and w are continuous. The solution representation of (1)-(2) can be written as

$$x(t) = \begin{cases} \frac{b_1(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} \\ +\frac{1}{\Gamma(\alpha)}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\left[w(s)+f_2(s,x(s))-f_1(s,x(s))\right]ds \end{cases}. \tag{3}$$

We only take those solutions which are continuous and continuable to (a,∞), and are not identically zero on any half-line (b,∞) for some $b \geq a$. The term "solution"' henceforth applies to such solutions of equations (1) or (3). A solution is said to be oscillatory if it has arbitrarily large zeros on $(0,\infty)$; otherwise, it is called nonoscillatory.

2. Main results

We will make use of the conditions:

$$xf_i(t,x) > 0 \quad (i=1,2), \quad x \neq 0, t \leq a \tag{4}$$

and

$$|f_1(t,x)| \geq p_1(t)\,|x|^v, \quad |f_2(t,x)| \leq p_2(t)\,|x|^u, \quad x \neq 0, t \geq a, \tag{5}$$

where $p_1, p_2 \in C\left([a,\infty], \mathbb{R}^+\right)$ and $u, v > 0$ are real numbers.

We will use the following lemma [[19], Lemma 1]

Lemma 1. *For $\mathscr{X} \geq 0$ and $\mathscr{Y} > 0$, we have*

$$\mathscr{X}^\lambda + (\lambda-1)\mathscr{Y}^\lambda - \lambda\mathscr{X}\mathscr{Y}^{\lambda-1} \geq 0, \quad \lambda > 1 \tag{6}$$

and

$$\mathscr{X}^\lambda + (1-\lambda)\mathscr{Y}^\lambda - \lambda\mathscr{X}\mathscr{Y}^{\lambda-1} \leq 0, \quad \lambda < 1, \tag{7}$$

where equality holds if and only if $\mathscr{X} = \mathscr{Y}$.

Now we may give our first theorem when $f_2 = 0$.

Theorem 2. *Let $f_2 = 0$ and condition (4) hold. If*

$$\lim_{t\to\infty} \inf(\psi(t))^{1-\gamma}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}w(s)ds = -\infty, \tag{8}$$

and

$$\lim_{t\to\infty} \sup(\psi(t))^{1-\gamma}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}w(s)ds = \infty. \tag{9}$$

Proof. Let $x(t)$ be a non-oscillatory solution of equations (1)-(2) with $f_2 = 0$. Suppose that $T > a$ is large enough so that $x(t) > 0$ for $t \leq T$.

Let $F(t) = w(t) + f_2(t,x(t)) - f_1(t,x(t))$, then we see from (3) that

$$\begin{aligned} x(t) \leq & \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}|b_1| + \frac{1}{\Gamma(\alpha)}\int_a^T \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}|F(s)|\,ds \\ & + \frac{1}{\Gamma(\alpha)}\int_a^T \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}w(s)ds, \quad t \geq T, \end{aligned} \tag{10}$$

and hence

$$\Gamma(\alpha)(\psi(t))^{1-\gamma}x(t) \leq c(T) + (\psi(t))^{1-\gamma}\int_T^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}w(s)ds, \tag{11}$$

where,

$$c(T) = \frac{1}{\Gamma(\gamma)}\left(\frac{\psi(T)}{\psi(T)-\psi(a)}\right)^{1-\gamma}|b_1| + \int_a^T \left(\frac{\psi(T)}{\psi(T)-\psi(s)}\right)^{1-\alpha}|F(s)|\,ds. \tag{12}$$

Note that the improper integral on the right is convergent. Applying the limit inferior of both sides of inequality (11) as $t \to \infty$, we obtain a contradiction to condition (8). In case $x(t)$ is eventually negative, a similar argument leads to a contradiction with (9). □

Next we have the following results.

Theorem 3. *Let conditions* (1)-(2) *and* (2) *hold with* $v > 1$ *and* $u = 1$. *If*

$$\lim_{t\to\infty}\inf(\psi(t))^{1-\gamma}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left[w(s)+\mathscr{H}_v(s)\right]ds = -\infty \tag{13}$$

and

$$\lim_{t\to\infty}\sup(\psi(t))^{1-\gamma}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left[w(s)+\mathscr{H}_v(s)\right]ds = \infty \tag{14}$$

where

$$\mathscr{H}_v(s) = (v-1)v^{\frac{v}{(1-v)}}p_1^{\frac{1}{(1-v)}}(s)p_2^{\frac{1}{(v-1)}}(s),$$

then every solution of equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equations (3), say, $x(t) > 0$ for $r \geq T > a$. Using (5) in equation (3) with $u = 1$ and $v > 1$ and $t \geq T$, we find

$$\begin{aligned}\Gamma(\gamma)(\psi(t))^{1-\gamma}x(t) \leq c(T) + (\psi(t))^{1-\gamma}\Bigg[&\int_t^T \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}w(s)ds \\ &+ \int_t^T \psi^{'}(t)(\psi(t)-\psi(s))^{\alpha-1}\left[p_2(s)x(s)-p_1(s)x^v(s)\right]ds\Bigg].\end{aligned} \tag{15}$$

We apply (6) in Lemma 1 with

$$\lambda = v, \mathscr{X} = p_1^{\frac{1}{v}}x \quad \text{and} \quad \mathscr{Y} = \left(p_2 p_1^{\frac{-1}{v}}/v\right)^{\frac{1}{(v-1)}}$$

to obtain

$$p_2(t)x(t) - p_1(t)x^v(t) \leq (v-1)v^{\frac{v}{(1-v)}}p_1^{\frac{1}{(1-v)}}(t)p_2^{\frac{v}{(v-1)}}(t). \tag{16}$$

Using (16) in (15), we have

$$\Gamma(\gamma)(\psi(t))^{1-\gamma}x(t) \leq c(T) + (\psi(t))^{1-\gamma}\int_T^t \psi^{'}(t)\left[\psi(t)-\psi(s)\right]^{\alpha-1}\left[w(s)+\mathscr{H}_v(s)\right]ds, \quad t \geq T.$$

□

The rest of the proof is the similar as in that of Theorem 2.

Theorem 4. *Let condition* (4) *and* (5) *hold with* $v = 1$ *and* $u < 1$. *If*

$$\lim_{t\to\infty} \inf(\psi(t))^{1-\gamma} \int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1} \left[w(s)+\mathscr{H}_u(s)\right] ds = -\infty \tag{17}$$

and

$$\lim_{t\to\infty} \sup(\psi(t))^{1-\gamma} \int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1} \left[w(s)+\mathscr{H}_u(s)\right] ds = \infty, \tag{18}$$

where,

$$\mathscr{H}_u(s) = (1-u)u^{\frac{u}{(u-1)}} p_1^{\frac{u}{(u-1)}}(s) p_2^{\frac{1}{(1-u)}}(s),$$

then every solution of equations of (1)-(2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equations (3), say $x(t) > 0$ for $t \geq a > 1$. Using condition (5) in (3), with $v = 1$ and $u < 1$, we obtain

$$\begin{aligned}\Gamma(\alpha)(\psi(t))^{1-\gamma}x(t) \leq c(T) + (\psi(t))^{1-\alpha} \Bigg[&\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} w(s)ds \\ &+ \int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} \left[p_2(s)x^u(s)-p_1(s)x(s)\right] ds \Bigg]. \end{aligned} \tag{19}$$

Now we use (7) in Lemma 1 with

$$\lambda = u,\ \mathscr{X} = p_2^{\frac{1}{u}} x \quad \text{and} \quad \mathscr{Y} = \left(p_1 p_2^{\frac{-1}{u}}/u\right)^{\frac{1}{(u-1)}}$$

to get

$$p_2(t)x^u(t) - p_1(t)x(t) \leq (1-u)u^{\frac{u}{(1-u)}} p_1^{\frac{u}{(u-1)}}(t) p_2^{\frac{1}{(1-u)}}(t). \tag{20}$$

Using (20) in (19) then yields

$$\Gamma(\alpha)(\psi(t))^{1-\gamma}x(t) \leq c(T) + (\psi(t))^{1-\gamma} \int_t^T \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} \left[w(s)+\mathscr{H}_u(s)\right] ds, \quad t \geq T.$$

The rest of the proof is the similar as in that of Theorem 2. □

Theorem 5. *Let condition* (4) *and* (5) *hold with* $v > 1$ *and* $u < 1$. *If*

$$\lim_{t\to\infty} \inf(\psi(t))^{1-\gamma} \int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} \left[w(s)+\mathscr{H}_{v,u}(s)\right] ds = -\infty, \tag{21}$$

and

$$\lim_{t\to\infty} \sup(\psi(t))^{1-\gamma} \int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} \left[w(s)+\mathscr{H}_{v,u}(s)\right] ds = \infty, \tag{22}$$

where

$$\mathscr{H}_{v,u}(s) = (v-1)v^{\frac{v}{(1-v)}} \epsilon^{\frac{v}{(v-1)}}(s) p_1^{\frac{1}{(1-v)}}(s) + (1-u)u^{\frac{u}{(1-u)}} \epsilon^{\frac{u}{(u-1)}}(s) p_2^{\frac{1}{(1-u)}}(s)$$

with $\epsilon \in C\left([a,\infty],\mathbb{R}^+\right)$, *then every solution of equation* (1)-(2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1)-(2), say $x(t) > 0$ for $t \geq T > a$. Using (5) in equation (3) one can easily write that

$$\begin{aligned}\Gamma(\alpha)(\psi(t))^{1-\gamma}x(t) \geq c(T) + (\psi(t))^{1-\gamma}\int_T^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} w(s)ds \\ + (\psi(t))^{1-\gamma}\int_T^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left(\epsilon(s)x(s) - p_1(s)x^v(s)\right)ds \\ + (\psi(t))^{1-\gamma}\int_T^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left(p_2(s)x^u(s) - \epsilon(s)x(s)\right)ds,\ t \geq T. \qquad (23)\end{aligned}$$

We may bound the term $(\epsilon x - p_1 x^v)$ and $(p_2 x^u - \epsilon x)$ by using inequalities (16) (with $p_2 = \epsilon$) respectively; to get

$$\Gamma(\alpha)(\psi(t))^{1-\gamma}x(t) \leq c(T) + (\psi(t))^{1-\alpha}\int_T^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\left[w(s) + \mathscr{H}_{v,u}(s)\right]ds, \quad t \geq T.$$

The rest of the proof is the similar as in that of Theorem 2. □

Remark 1. The result obtained from (1) are with different nonlinearities and one can observe that the forcing term w is unbounded, and its oscillatory character is inherited by the solutions.

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