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Analytical solution of isotropic rectangular plates resting on Winkler and Pasternak foundations using Laplace transform and variation of iteration method

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Abstract: Dynamic analysis of isotropic thin rectangular plate resting on two-parameter elastic foundations is investigated. The governing system is converted to system of nonlinear ordinary differential equation using Galerkin method of separation. The Ordinary differential equation is analyzed using hybrid method of Laplace transform and Variation of iteration Method. The accuracies of the analytical solutions obtained are verified with existing literature and confirmed in good agreement. Thereafter, the analytical solutions are used for parametric studies. From the results, it is observed that, increase in elastic foundation parameters increases the natural frequency. Increase in aspect ratios increases the natural frequency. It is expected that the present study will add value to the existing knowledge in the field of vibration.

Keywords: Analytical solution, deflection, Laplace variation of iteration method, natural frequency, Winkler and Pasternak.

1. Introduction

Research into vibration analysis of thin isotropic rectangular plate resting on elastic nonlinear foundation is vast gaining significant awareness among researchers due to its wide applications and important in the field of engineering. Geotechnics engineers need to understand the behaviour of plates when embedded in soil for their design, structural engineers requires same information for the design of the structural foundations likewise highway engineers rely on the information for the highway pavement design. In the design of elastic soil foundation, the adoption of two-parameter foundations gives better results than the use of Winkler foundation alone, which is associated with limitation of shear interaction among the spring elements. In the study of dynamic behavior of plates, Jain et al. [1] worked on free vibration of rectangular plate. In another work, natural frequency of rectangular plate was determined by Bhat [2] using Rayleigh method. Few years later, Balkaya [3] investigated the dynamic response of rectangular plate using differential transform method (DTM). Thereafter, Gupta et al. [4] analyzed forced vibration of rectangular plate with varying thickness. In a further study, some other researchers [5–10] studied buckling and vibration of plates and beams.

Several authors already applied different method of solutions for analysis of thin rectangular plate. However, in numerical analysis [9–23], it is very important to carry out convergence and stability study which increases the computational time and cost otherwise the solution will diverge. Furthermore, exact method [24–26] are having the limitation of handling nonlinear problem due to the complex mathematics involved. These limitations had led to the introduction of semi-analytical methods. Ozturk and Coskun [27] used Homotopy perturbation method (HPM) in the study of plate dynamic behaviour. However, despite the effectiveness, there is setback of finding embedded parameters. In another study, Galerkin method of solution was adopted by Njoku [28] for vibration analysis of thin isotropic rectangular plate. The method suffers the limitation of extension of the series solution to provide precise result. In a later work, Pirbodaghi et al. [29] utilized Homotopy analysis method (HAM) for investigation of vibration analysis of beam. HAM suffers from limitation of assumption of solution for the expression. Variation of iteration method (VIM), was first proposed by He [30–36], has been applied to investigate many nonlinear partial differential equation. The approach uses

Lagrange multiplier to find the analytical solution with very fast convergence. This present study adopts the use of exact method to handle the linear part of the system governing equation and solving the rest of equation with very effective method of VIM. The advantage of this method over other hybrid method calls for its application in this research.

Despite the effectiveness of the method and high prediction of results, the author realized that, with several researches on dynamic analysis of plate, Laplace transform and VIM has not been used to determine analytical solution of thin rectangular isotropic plate resting on two-parameter foundations. Therefore, the present study is on determination of analytical solution of free vibration of thin isotropic rectangular plate resting on nonlinear foundation. The analytical solution obtained is used for investigation of the controlling parameters.

2. Problem formulation and mathematical analysis

Considering homogenous rectangular plate of uniform thickness resting on Winkler and Pasternak foundations as shown in Figure 1. The two opposite edge \( y = 0 \) and \( y = b \) are regarded as simply supported.

![Figure 1. Rectangular plate resting on two-parameter foundations](image)

Figure 1. Rectangular plate resting on two-parameter foundations

![Figure 2. Geometry of plate with boundary conditions](image)

Figure 2. Geometry of plate with boundary conditions

The domain are \( 0 \leq x \leq a, 0 \leq y \leq b \) where \( a \) and \( b \) represents the length and breadth of the rectangular plate as shown in Figure 2. The following assumptions are made for the development of the governing equation [37]:

1. Normal stresses in the direction transverse to the plate are considered small.
2. Thickness of plate is smaller compared to the other dimensions.
3. Plate is of constant thickness.
4. Normal to the undeformed middle surface remains straight and unstretched in length and still normal to the deformed middle surface.

The governing equation for thin isotropic rectangular plate as reported by Leissa [38] is;

\[
D \left( \frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y, t)}{\partial x^4} \right) + \frac{\rho h}{2} \frac{\partial^2 w(x, y, t)}{\partial t^2} + K_w w(x, y, t) + K_p w^3(x, y, t) = 0, \tag{1}
\]

where, \( w(x, y, t) \) represents the transverse deflection, \( D \) is the flexural rigidity \( \frac{E h^3}{12(1-\nu^2)} \), represents modulus of elasticity \( h \), represents the plate thickness, \( \nu \) represents the Poisson ratio of plate material, \( \rho \) represents the...
mass density of the plate, \( \omega \) represents the radial frequency \((\text{rad/s})\), \( k_w \), and \( k_p \) are Winkler foundation and Pasternak foundation parameter respectively.

Using the following dimensionless variables:

\[
W = \frac{w}{w_{\text{max}}}, \quad X = \frac{x}{a}, \quad Y = \frac{y}{b}.
\] (2)

According to Kantorovich type approximation, the free vibration of Equation (1) can be written as:

\[
w(x, y, t) = w(x, y)e^{i\omega t},
\] (3)

\[
\Omega^2 = \frac{a^4ph}{D}\omega^2, \quad k_w = \frac{a^4k_w}{D}, \quad k_p = \frac{a^4k_p w_{\text{max}}^2}{D},
\] (4)

\[
\frac{\partial^4 W(x, y)}{\partial X^4} + 2\lambda^2 \frac{W(x, y)}{\partial X^2\partial Y^2} + \lambda^4 \frac{\partial^4 W(x, y)}{\partial Y^4} - \Omega^2 W(x, y) + k_w W(x, y) + k_p W^3(x, y) = 0.
\] (5)

Assuming the two opposite edges of Figure 1, \( Y = 0 \) and \( Y = 1 \) to be simply supported, deflection function can be represented as follows:

\[
W = W(X)\sin(m\pi Y).
\] (6)

Substituting the derivative of Equation (6) into governing differential equation

\[
\frac{d^4 W(X)}{dX^4} - 2\lambda^2 m^2 \pi^2 \frac{d^2 W(x)}{dX^2} - (\Omega^2 - k_w - \lambda^4 m^4 \pi^4) W(X) + k_p W^3(X) = 0,
\] (7)

where \( \lambda \left( \frac{a}{b} \right) \) represents the aspect ratio, \( m \) is an integer, \( \Omega \) is the frequency parameter, \( a \) represents side length along \( x \)-axis.

### 2.1. Boundary conditions

Three boundary conditions are considered at \( X = 0 \) and \( X = l \) namely, Simply supported and clamped edge (SC), Simply supported and simply supported edge (SS) and Simply supported and free edge conditions (SF).

- **Clamped edge**: \( W = \frac{dW}{dX} = 0 \),

- **Simply supported**: \( W = \frac{d^2 W}{dX^2} - \nu(\lambda^2 m^2 \pi^2) W = 0 \),

- **Free edge**: \( \frac{d^2 W}{dX^2} - \nu(\lambda^2 m^2 \pi^2) W = 0, \frac{d^3 W}{dX^3} - (2 - \nu)(\lambda^2 m^2 \pi^2) \frac{dW}{dX} = 0 \),

### 3. Method of Solution: Laplace transform and variation iteration method

#### 3.1. Basic ideal of Laplace transform

If \( f(t) \) is a function of a variable \( t \). \( \mathcal{L}F(t) \) and is defined by the integral:

\[
\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t) dt.
\] (11)

Some of the properties used in this study include:

\[
\mathcal{L}\{1\} = \frac{1}{s}(s \geq 0),
\] (12)
3.2. Laplace and variation iteration method

Assuming the following nonlinear differential equation:

$$Lw(x) + Nw(x) = f(x),$$  
(17)

$L$ represents the linear operator, $N$ is nonlinear operator, $f$ is the source or analytical function. Variatlon iteration method uses the correction function for Equation (17) as:

$$w_{n+1}(x) = w_n(x) + \int_0^x \lambda(\xi) [Lw_n(\xi) + N\tilde{w}_n(\xi) - f(\xi)] d\xi, \quad n = 0, 1, 2, \ldots,$$  
(18)

where $\lambda$ is general Lagrange multiplier identified through variational theory. The subscript $n$ represents the $n$th term and $w_n$ is a constrained variation ($\delta \tilde{w}_n = 0$).

Laplace transform of both sides of Equation (12) gives:

$$\mathcal{L}\{w_{n+1}(x)\} = \mathcal{L}\{w_n(x)\} + \mathcal{L}\left[\int_0^x \lambda(x - \xi) [Lw_n(\xi) + N\tilde{w}_n(\xi) - f(\xi)] d\xi\right], \quad n = 0, 1, 2, \ldots$$  
(19)

Apply convolution to Equation (19), we get:

$$\mathcal{L}\{w_{n+1}(x)\} = \mathcal{L}\{w_n(x)\} + \mathcal{L}\left[\lambda(x) \mathcal{L}\{w_n(x)\} + \mathcal{L}\{N\tilde{w}_n(x) - f(x)\}\right]$$  

$$= \mathcal{L}\{w_n(x)\} + \mathcal{L}\left[\lambda(x) \mathcal{L}\{Lw_n(x) + N\tilde{w}_n(x) - f(x)\}\right].$$  
(20)

Optimal value of $\lambda(x - \xi)$ is obtained taking variation with respect to $\tilde{w}_n(x)$ given as:

$$\frac{\delta}{\delta \tilde{w}_n} \mathcal{L}\{w_{n+1}(x)\} = \frac{\delta}{\delta \tilde{w}_n} \mathcal{L}\{w_n(x)\} + \frac{\delta}{\delta \tilde{w}_n} \mathcal{L}\{\lambda(x)\} \mathcal{L}\{Lw_n(x) + N\tilde{w}_n(x) - f(x)\}.$$  
(21)

Applying variation with respect to $\tilde{w}_n(x)$ gives:

$$\mathcal{L}\{\delta \tilde{w}_{n+1}\} = \mathcal{L}\{\delta \tilde{w}_n\} + \delta \mathcal{L}\{\lambda\} \mathcal{L}\{w_n\}.$$  
(22)

Assume $L$ is linear differential operator with constant coefficients as:

$$L(w) = a_0 w^1 + a_1 w^2 + a_2 w^3 + \cdots + a_{n-2} w^{n-2} + a_{n-1} w^{n-1} + a_n w^n$$  
(23)

where $a_i$’s are constants. The coefficient contains non-constant terms of the form $x^k$. The Laplace transform of initial operator term is given as:
\[ \mathcal{L}[a_n w^n] = a_n s^n \mathcal{L}[w] - a_n \sum_{k=1}^{n} s^{k-1} w^{n-k}(0). \] (24)

The variation with respect to \( w \) is given as:

\[ \mathcal{L}[\delta w_{n+1}] = \mathcal{L}[\delta w_n] + \mathcal{L}[\lambda] \left[ \sum_{k=0}^{n} a_k s^k \right] \mathcal{L}[\delta w_n] = \left[ 1 + \mathcal{L}[\lambda] \left[ \sum_{k=0}^{n} a_k s^k \right] \right] \mathcal{L}[\delta w_n]. \] (25)

Extremum condition \( w_{n+1} \) needs that \( \delta w_{n+1} \). Meaning the right hand side of Equation (25) should be set to zero. Hence stationary condition is:

\[ \mathcal{L}[\lambda] = -\frac{1}{\sum_{k=0}^{n} a_k s^k}. \] (26)

3.3. Application of LVIM to the governing equation

Following the basic principle of LVIM, the governing equation is now analyzed as:

\[ \mathcal{L}[w_{n+1}(x)] = \mathcal{L}[w_n(x)] + \mathcal{L}[\lambda] \mathcal{L}\left[ \frac{d^4 W_n(X)}{dX^4} - 2\lambda^2 m^2 \pi^2 \frac{d^2 W_n(x)}{dx^2} \right. \\
\left. - (\Omega^2 - k_w - \lambda^2 m^4 \pi^4) W_n(x) - k_p W_n^3(x) \right] \\
= \mathcal{L}[w_n(x)] + \mathcal{L}[\lambda] \left( \lambda^4 m^4 \pi^4 - 2\pi^2 s^2 \lambda^2 m^2 + s^2 - \Omega^2 + k_w \right) \mathcal{L}[w_n(x)] \\
- w''''(0) - s w'''(0) - s^2 w''(0) - s^3 w'(0) + 2m^2 \lambda^2 \pi^2 w_n(0) + sw_n(0) - k_p \mathcal{L}[W^3(x)] \] (27)

Taking variation with respect to \( w_n(x) \) on both sides of Equation (27), we get

\[ \frac{\delta}{\delta w_n} \mathcal{L}[w_{n+1}(x)] = \frac{\delta}{\delta w_n} \mathcal{L}[w_n(x)] + \frac{\delta}{\delta w_n} \mathcal{L}[\lambda(x)] \left( \lambda^4 m^4 \pi^4 - 2\pi^2 s^2 \lambda^2 m^2 + s^2 - \Omega^2 + k_w \right) \mathcal{L}[w_n(x)] \\
- w''''(0) - s w'''(0) - s^2 w''(0) - s^3 w'(0) + 2m^2 \lambda^2 \pi^2 w_n(0) + sw_n(0) - k_p \mathcal{L}[W^3(x)] \] (28)

Simplifying Equation (28) gives,

\[ \mathcal{L}[\delta w_{n+1}] = \mathcal{L}[\delta w_n] + \mathcal{L}[\lambda] \left( \lambda^4 m^4 \pi^4 - 2\pi^2 s^2 \lambda^2 m^2 + s^2 - \Omega^2 + k_w \right) \mathcal{L}[\delta w_n] \]

\[ = \mathcal{L}[\delta w_n] \left( 1 + \mathcal{L}[\lambda] \left( \lambda^4 m^4 \pi^4 - 2\pi^2 s^2 \lambda^2 m^2 + s^2 - \Omega^2 + k_w \right) \right) \] (29)

Extremum condition \( w_{n+1} \) needs that \( \delta w_{n+1} = 0 \). Meaning the right hand side of Equation (29) should be set to zero.

\[ 1 + \mathcal{L}[\lambda] (\pi^4 - 2\pi^2 s^2 + s^4) = 0, \quad \mathcal{L}[\lambda] = -\frac{1}{(\pi^4 - 2\pi^2 s^2 + s^4)}. \] (30)

For simplicity we adopt,

\[ \mathcal{L}[\lambda] = -\frac{1}{s^4}. \] (31)

Substituting Equation (31) into Equation (27) results;
\[
\mathcal{L}[w_{n+1}(x)] = \mathcal{L}[w_n(x)] - \mathcal{L} \left[ \int_0^x \lambda(x - \zeta) \left( \frac{d^4 W_n(\zeta)}{d\zeta^4} - 2m^2 \pi^2 \lambda^2 \frac{d^2 W_n(\zeta)}{d\zeta^2} - (\Omega^2 - m^4 \pi^4 \lambda^4 - k_w) W_n(\zeta) - k_p W_n^3(\zeta) \right) d\zeta \right]
\]

\[
= \mathcal{L}[w_n(x)] - \mathcal{L} \left[ \frac{x^3}{6} \mathcal{L} \left[ \frac{d^4 W_n(x)}{dX^4} - 2m^2 \pi^2 \lambda^2 \frac{d^2 W_n(x)}{dX^2} - (\Omega^2 - m^4 \pi^4 \lambda^4 - k_w) W_n(x) - k_p W_n^3(x) \right] \right]
\]

Assuming
\[
\Phi_0 = \begin{cases} 
  w(0), & \text{if } L = \frac{d}{dx} \\
  w(0) + x w'(0), & \text{if } L = \frac{d^2}{dx^2} \\
  w(0) + x w'(0) + \frac{s^2}{12} w''(0), & \text{if } L = \frac{d^3}{dx^3} \\
  w(0) + x w'(0) + \frac{s^2}{2} w''(0) + \frac{s^3}{3!} w'''(0), & \text{if } L = \frac{d^4}{dx^4},
\end{cases}
\]

(33)

Applying condition 9 at \( x = 0 \) on Equation (34) gives
\[
w_0 = w'(0)x + \frac{1}{2!} w''(0) x^2 + \frac{1}{3!} w'''(0) x^3.
\]

(34)

Then
\[
\mathcal{L}[w_1] = \mathcal{L}[w_0] - \mathcal{L} \left[ \frac{x^3}{6} \mathcal{L} \left[ \frac{d^4 W_0(x)}{dX^4} - 2m^2 \pi^2 \lambda^2 \frac{d^2 W_0(x)}{dX^2} - (\Omega^2 - m^4 \pi^4 \lambda^4 - k_w) W_0(x) - k_p W_0^3(x) \right] \right]
\]

(36)

Inverse Laplace gives the first iteration:
\[
w_1 = -259459200 \pi^4 \beta \lambda^4 m^4 x^2 + \beta^3 k_p x^{10} - 10897286400 \pi^4 \alpha \lambda^4 m^4 + 630 \alpha \beta^2 k_p x^8
\]
\[
+ 98280 \alpha^2 k_p x^6 + 3603600 \alpha^3 k_p x^4 + 259459200 \Omega^2 \beta x^2
\]
\[
- 259459200 \beta k w x^2 + 10897286400 \Omega^2 \alpha - 10897286400 \alpha k_w \right] \frac{x^3}{130767436800} + \frac{1}{6} x (\beta x^2 + 6 \alpha),
\]

(38)

(38)

(39)

(40)
Inverse Laplace gives the second iteration:

\[
 w_2 = \left[ -142449935721085171226298826358784000000\pi^{12}\beta^3k_p\lambda^{12}m^{12}x^{22} + 450332877744754421964800000\pi^8\beta^5k_p^2\lambda^8m^8x^{30} - 29999564268605370602585328311599104000000\pi^{12}\alpha\beta^2k_p\lambda^{12}m^{12}x^{20} - 770228582800\pi^4\beta^7k_p\lambda^4m^4x^{38} + \ldots \right] x^5 + \frac{1}{3}x(\beta x^2 + 6\alpha) + \left[ -259459200\pi^4\beta\lambda^4m^4x^2 + 10897286400\pi^4\alpha\lambda^4m^4 + 630\alpha\beta^2k_p\lambda^8x^6 - 3603600\alpha^3k_p\lambda^4 + 259459200\Omega^2\beta x^2 - 259459200\beta k_w x^2 + 10897286400\Omega^2\alpha - 10897286400\alpha m \right] \frac{x^5}{1307674368000}, \tag{41}
\]

The same approach is continued till frequency parameter \(\Omega\) obtained converges. Substituting boundary condition at \(x = 1\) to find the unknowns introduced results into simultaneous equation.

<table>
<thead>
<tr>
<th>Table 1. Parameters for validation of the model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pasternak foundation</td>
</tr>
<tr>
<td>(K_w)</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

Table 1 contains parameters for validation of the approach to ascertain the correctness of the results.

\[
 \psi_{11}(\Omega) w_0 + \psi_{12}(\Omega) w_2 \\
 \psi_{21}(\Omega) w_0 + \psi_{22}(\Omega) w_2. \tag{42}
\]

The polynomials are represented as \(\psi_{11}, \psi_{12}, \psi_{21}\) and \(\psi_{22}\). Equation (42) can be written in matrix form as:

\[
 \begin{bmatrix} \psi_{11}(\Omega) & \psi_{12}(\Omega) \\ \psi_{21}(\Omega) & \psi_{22}(\Omega) \end{bmatrix} \begin{bmatrix} w_0 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{43}
\]

The following Characteristic determinant is obtained applying the non-trivial condition

\[
 \begin{bmatrix} \psi_{11}(\Omega) & \psi_{12}(\Omega) \\ \psi_{21}(\Omega) & \psi_{22}(\Omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{44}
\]

Solving Equation (44) gives the natural frequencies. Substitute the result obtained into Equation (43), we get

\[
 \begin{bmatrix} 245431 \\ -181919 \\ -3775 \end{bmatrix} \begin{bmatrix} 32120 \\ 125273 \\ 9460 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{45}
\]

Setting \(\alpha = 1\) and find \(\beta\)

\[
 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ -3.641035489 \end{bmatrix}. \tag{46}
\]
Same procedure is repeated for other modes.

\[
w(x) = \left[ 4.27 \times 10^{58} \pi^2 x^4 - 2.36 \times 10^{60} x^6 - 3.34 \times 10^{62} x^4 - 4.58 \times 10^{62} x^2 + 5.28 \times 10^{63} \right] x^5
\]

\[
-\frac{1}{3} \left[ -47399 \frac{13018}{x^2 - 6} \right] x - \left[ \frac{3.86 \times 10^8 x^2 - 4.45 \times 10^9}{1.30 \times 10^{10}} \right] x^5
\]

The following convergence criterion may be used

\[
\frac{|\Omega_j^{(i)} - \Omega_j^{(i-1)}|}{\Omega_j^{(i)}} \leq \varepsilon, \quad j = 1, 2, 3, \ldots, n
\]

(48)

where \(\varepsilon\) is the tolerance parameter taken to be 0.0001 for this study, \(\Omega_j\) represents the Eigenvalue.

The iteration converges at third iteration for first mode frequency parameter.

4. Results and discussion

The solution of Laplace and Variation iteration method is presented here. Table 2 shows the comparison of present results to that of previously published work. It is realized from the Table 2 that, good agreements is achieved with that of the past results. The fundamental modal shape of the thin rectangular plate are shown in Figures 3, 4, 5 and it is observed that the shape obeys classical plate theory. Also, Table 3 shows different deflection values of transverse displacement for the first three mode frequency parameters of SC, SS and SF boundary condition considered. Table 4 shows the convergence study, it is observed that the fundamental natural frequency converges at the third iteration while higher modes are obtained by increasing the number of iterations. This phenomenon is peculiar to vibration problem.

### Table 2. Showing validation of results

<table>
<thead>
<tr>
<th>Edge Condition/Dimensionless</th>
<th>Simply-supported (SS)</th>
<th>Simply supported -Clamped (SC)</th>
<th>Simply-supported -Free (SF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural frequency (\Omega_1)</td>
<td>Bhat et al. [2] Present</td>
<td>Leissa [38] Present</td>
<td>Leissa [38] Present</td>
</tr>
<tr>
<td></td>
<td>19.7392</td>
<td>19.7434</td>
<td>23.6463</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Leissa [38]</td>
<td>23.6486</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>11.7195</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>11.7606</td>
</tr>
</tbody>
</table>

![Figure 3. Fundamental mode shape of simply supported condition at both edges](image)
It is also observed that, the presence of elastic foundation and aspect ratio has no significant changes on the mode shape of the rectangular plate. Since dimensionless analysis is carried out, the results are valid for all thin plates. Table 4 shows that the value of frequency parameters $\Omega$ decreases in the order of $SC \geq SS \geq SF$.

4.1. Effect of foundation parameter on natural frequency

Table 5 illustrates the impact of foundation parameter on natural frequency. It is clear from the Figures 6, 7, and 8 that the foundation parameter has impact on natural frequency, increasing values of the foundation parameter increases the natural frequency. This satisfies the principle of classical vibration. Stiffness increment results to natural frequency increment. This also corroborated with finding reported in [38]. The effect of increase in natural frequency is much significant in higher values of the elastic foundation.

4.2. Effect of variation of aspect ratio on natural frequency

The influence of aspect ratio on natural frequency are shown in Table 6 and Figures 9, 10, 11 respectively. It is shown that, the natural frequency increase with increases in aspect ratio. This is because, the plate becomes more stiff as the aspect ratio increases resulting in the natural frequency increases.
Table 3. Results of different deflection values

<table>
<thead>
<tr>
<th>Transverse displacement</th>
<th>SF</th>
<th>SS</th>
<th>SC</th>
<th>SF</th>
<th>SS</th>
<th>SC</th>
<th>SF</th>
<th>SS</th>
<th>SC</th>
</tr>
</thead>
<tbody>
<tr>
<td>w[0]</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>w[0.05]</td>
<td>0.0500</td>
<td>0.0498</td>
<td>0.0497</td>
<td>0.0496</td>
<td>0.0492</td>
<td>0.0490</td>
<td>0.0489</td>
<td>0.0482</td>
<td>0.0479</td>
</tr>
<tr>
<td>w[0.10]</td>
<td>0.0998</td>
<td>0.0984</td>
<td>0.0978</td>
<td>0.0970</td>
<td>0.0935</td>
<td>0.0921</td>
<td>0.0916</td>
<td>0.0858</td>
<td>0.0837</td>
</tr>
<tr>
<td>w[0.15]</td>
<td>0.1492</td>
<td>0.1445</td>
<td>0.1425</td>
<td>0.1400</td>
<td>0.1288</td>
<td>0.1240</td>
<td>0.1224</td>
<td>0.1048</td>
<td>0.0983</td>
</tr>
<tr>
<td>w[0.20]</td>
<td>0.1981</td>
<td>0.1871</td>
<td>0.1825</td>
<td>0.1767</td>
<td>0.1514</td>
<td>0.1410</td>
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Table 4. Showing convergence study of the results

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<tr>
<th>Edge Condition</th>
<th>Iteration</th>
<th>(SS)</th>
<th>(SC)</th>
<th>(SF)</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td>Bhat et al. [39]</td>
<td>Present</td>
<td>Leissa [38]</td>
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<tr>
<td>Ω1</td>
<td>N3</td>
<td>19.7392</td>
<td>19.7434</td>
<td>23.6463</td>
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<tr>
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<td>N4</td>
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<td>23.6463</td>
</tr>
<tr>
<td>Ω2</td>
<td>49.3481</td>
<td>49.3271</td>
<td>58.6465</td>
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</tr>
<tr>
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<td>58.6240</td>
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Table 5. Variation elastic foundation coefficient on natural frequency

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<th>kw=120</th>
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<tr>
<td>SC</td>
<td>Ω1</td>
<td>23.7518</td>
<td>23.9613</td>
<td>24.5974</td>
<td>26.0604</td>
<td>27.5526</td>
<td>28.445534</td>
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<tr>
<td>Ω2</td>
<td>58.6889</td>
<td>58.7741</td>
<td>60.3274</td>
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<td></td>
<td></td>
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<tr>
<td>Ω2</td>
<td>27.8462</td>
<td>28.0252</td>
<td>28.5546</td>
<td>31.1514</td>
<td>31.94393</td>
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</table>
Figure 6. Variation of elastic foundation parameter on SF edge condition

Figure 7. Variation of elastic foundation parameter on SS-edge condition

Figure 8. Variation of elastic foundation parameter on SC-edge condition
Figure 9. Variation of Aspect ratio on SF edge condition

Figure 10. Variation of Aspect ratio on SS edge condition

Figure 11. Variation of Aspect ratio on SC edge condition
Table 6. Variation of aspect ratio on natural frequency

<table>
<thead>
<tr>
<th>Edge Condition</th>
<th>Natural frequency</th>
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<th>λ = 0.7</th>
<th>λ = 1.0</th>
<th>λ = 1.5</th>
<th>λ = 2.5</th>
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5. Conclusion

In this study, the dynamic analysis of isotropic rectangular plates resting on Winkler and Pasternak foundations is analyzed. The governing equation is transform to nonlinear ordinary differential equation using Galerkin method of separation. The nonlinear ordinary differential equations have been solved using Laplace transform and variation of iteration method. The accuracies of the obtained analytical solutions were ascertained with the results obtained by earlier researcher. The obtained analytical solutions were used to examine the effects of foundation parameter, aspect ratio. The rate of convergence is increased with the introduction of exact method for analyzing the linear part of the governing equation while the remaining part are treated with variation of iteration method, practical applications of the study are base plate of tower, steel hinged steel column structures and culvert covers. From the parametric studies, the following observations were established:

1. Increase in elastic foundation parameter increases the natural frequency.
2. Increase in aspect ratio increases the natural frequency.
3. Increasing the combine elastic foundation parameters increases the natural frequency.
4. Accurate higher mode frequency can be obtained with increase in number of iterations.
5. SF boundary condition has the least value of frequency parameter followed by SS edge condition.
6. The effect of increase in natural frequency is much significant in higher value of the elastic foundation.

Abbreviations

<table>
<thead>
<tr>
<th>Abbreviations</th>
<th>Nomenclature</th>
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<tbody>
<tr>
<td>a</td>
<td>Length of the plate</td>
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<tr>
<td>b</td>
<td>Width of the plate</td>
</tr>
<tr>
<td>C</td>
<td>Clamped edge plate</td>
</tr>
<tr>
<td>E</td>
<td>Young’s modulus</td>
</tr>
<tr>
<td>F</td>
<td>Free edge support</td>
</tr>
<tr>
<td>S</td>
<td>Simply supported edge</td>
</tr>
<tr>
<td>d/dx</td>
<td>Differential operator</td>
</tr>
<tr>
<td>w</td>
<td>Dynamic deflection</td>
</tr>
<tr>
<td>X</td>
<td>space coordinate along the length of thin plate Symbol</td>
</tr>
<tr>
<td>h</td>
<td>plate thickness</td>
</tr>
<tr>
<td>ρ</td>
<td>Mass density</td>
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<tr>
<td>D</td>
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References


