Intuitionistic fuzzy subgroups with respect to norms $(T, S)$

Rasul Rasuli

Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.; rasulirasul@yahoo.com

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Abstract: The purpose of this paper is introduce the notion of intuitionistic fuzzy subgroups with respect to norms $(t$-norm $T$ and $s$-norm $S$). Also we introduce intersection and normality of them and investigate some properties of them. Finally, we provide some results of them under group homomorphisms.

Keywords: Group theory, theory of fuzzy sets, intuitionistic fuzzy groups, norms, homomorphisms, intersection.

1. Introduction

Group theory, in modern algebra, the study of groups, which are systems consisting of a set of elements and a binary operation that can be applied to two elements of the set, which together satisfy certain axioms. Groups are vital to modern algebra; their basic structure can be found in many mathematical phenomena. Groups can be found in geometry, representing phenomena such as symmetry and certain types of transformations. Group theory has applications in physics, chemistry, and computer science, and even puzzles like Rubik’s Cube can be represented using group theory.

In 1965, Zadeh [1] introduced the notion of fuzzy sets. In 1971, Rosenfeld [2] introduced fuzzy sets in the realm of group theory and formulated the concepts of fuzzy subgroups of a group. An increasing number of properties from classical group theory have been generalized. Many authors have worked on fuzzy group theory [3–5]. Especially, some authors considered the fuzzy subgroups with respect to a $t$-norm and gave some results [5–7]. The concept of intuitionistic fuzzy set was introduced by Atanassov [8], as a generalization of the notion of fuzzy set. The theory of intuitionistic fuzzy set is expected to play an important role in modern mathematics in general as it represents a generalization of fuzzy set. After the concept of intuitionistic fuzzy set was introduced, several papers have been published by mathematicians to extend the classical mathematical concepts and fuzzy mathematical concepts to the case of intuitionistic fuzzy mathematics. In the case of intuitionistic fuzzy mathematics, there were some attempts to establish a significant and rational definition of intuitionistic fuzzy group. Zhan and Tan [9] defined intuitionistic fuzzy subgroup as a generalization of Rosenfeld’s fuzzy subgroup.

By starting with a given classical group they define intuitionistic fuzzy subgroup using the classical binary operation defined over the given classical group. The author by using norms, investigated some properties of fuzzy algebraic structures [10]. In this paper, by using norms $(t$-norm $T$ and $s$-norm $S$) we define intuitionistic fuzzy subgroups of group $G$ as $IFGN(G)$ and normality of $G$ as $NIFGN(G)$. Also we investigate algebraic structures and some related properties of them and prove that if $A, B \in IFGN(G)$ and $A, B \in NIFGN(G)$, then $A \cap B \in IFGN(G)$ and $A \cap B \in NIFGN(G)$. Next we define normality between $A, B \in IFGN(G)$ as $A \triangleright B$ and give characterizations about them. Later we investigate them under group homomorphism $\varphi : G \rightarrow H$ such that if $A \in IFGN(G)$ and $B \in IFGN(H)$, then $\varphi(A) \in IFGN(H)$ and $\varphi^{-1}(B) \in IFGN(G)$. Also if $A \in NIFGN(G)$ and $B \in NIFGN(H)$, then $\varphi(A) \in NIFGN(H)$ and $\varphi^{-1}(B) \in NIFGN(G)$. Finally, if $A, B \in IFGN(G)$ and $A \triangleright B$, then $\varphi(A) \triangleright \varphi(B)$ and if $A, B \in IFGN(H)$ and $A \triangleright B$, then $\varphi^{-1}(A) \triangleright \varphi^{-1}(B)$.

2. preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel. For details we refer to [6,8,11–16].

Definition 1. A group is a non-empty set $G$ on which there is a binary operation $(a, b) \rightarrow ab$ such that
(1) if \(a\) and \(b\) belong to \(G\) then \(ab\) is also in \(G\) (closure),
(2) \(a(bc) = (ab)c\) for all \(a, b, c \in G\) (associativity),
(3) there is an element \(e_G \in G\) such that \(ae_G = e_Ga = a\) for all \(a \in G\) (identity),
(4) if \(a \in G\), then there is an element \(a^{-1} \in G\) such that \(aa^{-1} = a^{-1}a = e_G\) (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group \(G\) is called
abelian if the binary operation is commutative, i.e., \(ab = ba\) for all \(a, b \in G\).

**Remark 1.** There are two standard notations for the binary group operation: either the additive notation, that
is \((a, b) \rightarrow a + b\) in which case the identity is denoted by 0, or the multiplicative notation, that is \((a, b) \rightarrow ab\)
for which the identity is denoted by \(e\).

**Proposition 1.** Let \(G\) be a group. Let \(H\) be a non-empty subset of \(G\). The following are equivalent:

1. \(H\) is a subgroup of \(G\).
2. \(x, y \in H\) implies \(xy^{-1} \in H\) for all \(x, y\).

**Definition 2.** Let \(H\) be subgroup of group \(G\). Then we say that \(H\) is normal subgroup of \(G\) if for all \(g \in G\) and
\(h \in H\), we have that \(g^{-1}hg \in H\).

**Definition 3.** Let \(G\) and \(H\) be any two groups and \(f : G \rightarrow H\) be a function. Then \(f\) is called a homomorphism
if \(f(xy) = f(x)f(y)\) for all \(x, y \in G\).

**Definition 4.** Let \(G\) be an arbitrary group with a multiplicative binary operation and identity \(e\). A fuzzy subset
of \(G\), we mean a function from \(G\) into \([0, 1]\).

**Definition 5.** For sets \(X, Y\) and \(Z\), \(f = (f_1, f_2) : X \rightarrow Y \times Z\) is called a complex mapping if \(f_1 : X \rightarrow Y\) and
\(f_2 : X \rightarrow Z\) are mappings.

**Definition 6.** Let \(X\) be a nonempty set. A complex mapping \(A = (\mu_A, \nu_A) : X \rightarrow [0, 1] \times [0, 1]\) is called
an intuitionistic fuzzy set (in short, \(IFS\)) in \(X\) if \(\mu_A + \nu_A \leq 1\) where the mappings \(\mu_A : X \rightarrow [0, 1]\) and
\(\nu_A : X \rightarrow [0, 1]\) denote the degree of membership (namely \(\mu_A(x)\)) and the degree of non-membership (namely
\(\nu_A(x)\)) for each \(x \in X\) to \(A\), respectively. In particular \(0\) and \(1\) denote the intuitionistic fuzzy empty set and
intuitionistic fuzzy whole set in \(X\) defined by \(0(x) = (0, 1)\) and \(1(x) = (1, 0)\), respectively.

We will denote the set of all \(IFS\)s in \(X\) as \(IFS(X)\).

**Definition 7.** Let \(X\) be a nonempty set and let \(A = (\mu_A, \nu_A)\) and \(B = (\mu_B, \nu_B)\) be \(IFS\)s in \(X\). Then

1. \(A \subseteq B\) iff \(\mu_A \leq \mu_B\) and \(\nu_A \geq \nu_B\).
2. \(A = B\) iff \(A \subseteq B\) and \(B \subseteq A\).

**Definition 8.** A \(t\)-norm \(T\) is a function \(T : [0, 1] \times [0, 1] \rightarrow [0, 1]\) having the following four properties:

1. \(T(x, 1) = x\) (neutral element),
2. \(T(x, y) \leq T(x, z)\) if \(y \leq z\) (monotonicity),
3. \(T(x, y) = T(y, x)\) (commutativity),
4. \(T(x, T(y, z)) = T(T(x, y), z)\) (associativity),
for all \(x, y, z \in [0, 1]\).

It is clear that if \(x_1 \geq x_2\) and \(y_1 \geq y_2\), then \(T(x_1, y_1) \geq T(x_2, y_2)\).

**Example 1.**
1. Standard intersection \(T\)-norm \(T_m(x, y) = \min\{x, y\}\).
2. Bounded sum \(T\)-norm \(T_p(x, y) = \max\{0, x + y - 1\}\).
3. Algebraic product \(T\)-norm \(T_p(x, y) = xy\).
(4) Drastic $T$-norm

\[
T_D(x, y) = \begin{cases} 
  y & \text{if } x = 1 \\
  x & \text{if } y = 1 \\
  0 & \text{otherwise.}
\end{cases}
\]

(5) Nilpotent minimum $T$-norm

\[
T_{nm}(x, y) = \begin{cases} 
  \min\{x, y\} & \text{if } x + y > 1 \\
  0 & \text{otherwise.}
\end{cases}
\]

(6) Hamacher product $T$-norm

\[
T_H(x, y) = \begin{cases} 
  0 & \text{if } x = y = 0 \\
  \frac{xy}{x+y-xy} & \text{otherwise.}
\end{cases}
\]

The drastic $t$-norm is the pointwise smallest $t$-norm and the minimum is the pointwise largest $t$-norm:

\[
T_D(x, y) \leq T(x, y) \leq T_{\min}(x, y) \quad \text{for all } x, y \in [0, 1].
\]

Recall that $t$-norm $T$ will be idempotent if for all $x \in [0, 1]$ we have $T(x, x) = x$.

\textbf{Lemma 1.} Let $T$ be a $t$-norm. Then

\[
T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),
\]

for all $x, y, w, z \in [0, 1]$.

\textbf{Definition 9.} An $s$-norm $S$ is a function $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$ having the following four properties:

1. $S(x, 0) = x$,
2. $S(x, y) \leq S(x, z)$ if $y \leq z$,
3. $S(x, y) = S(y, x)$,
4. $S(x, S(y, z)) = S(S(x, y), z)$,

for all $x, y, z \in [0, 1]$.

We say that $S$ is idempotent if for all $x \in [0, 1], S(x, x) = x$.

\textbf{Example 2.} The basic $S$-norms are $S_m(x, y) = \max\{x, y\}$, $S_b(x, y) = \min\{1, x + y\}$ and $S_p(x, y) = x + y - xy$ for all $x, y \in [0, 1]$, where $S_m$ is standard union, $S_b$ is bounded sum and $S_p$ is algebraic sum.

\textbf{Lemma 2.} Let $S$ be a $s$-norm. Then $S(S(x, y), S(w, z)) = S(S(x, w), S(y, z))$, for all $x, y, w, z \in [0, 1]$.

\textbf{Definition 10.} Let $A = (\mu_A, \nu_A) \in \text{IFS}(X)$ and $B = (\mu_B, \nu_B) \in \text{IFS}(X)$. Define intesection $A$ and $B$ as

\[
A \cap B = (\mu_A, \nu_A) \cap (\mu_B, \nu_B) = (\mu_{A \cap B}, \nu_{A \cap B})
\]

such that $\mu_{A \cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A \cap B}(x) = S(\nu_A(x), \nu_B(y))$ for all $x \in X$.

\textbf{Definition 11.} Let $\varphi$ be a function from set $X$ into set $Y$ such that $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in $X$ and $Y$ respectively. For all $x \in X$ and $y \in Y$, we define

\[
\varphi(A)(y) = (\varphi(\mu_A)(y), \varphi(\nu_A)(y))
\]

\[
= \begin{cases} 
  (\sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\}, \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\}) & \text{if } \varphi^{-1}(y) \neq \emptyset; \\
  (0, 1) & \text{if } \varphi^{-1}(y) = \emptyset.
\end{cases}
\]

Also $\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x))).$
3. Intuitionistic fuzzy subgroups with respect to norms (t-norm T and s-norm S)

Definition 12. Let G be a group. An \( A = (\mu_A, v_A) \in IFS(G) \) is said to be intuitionistic fuzzy subgroup with respect to norms (t-norm T and s-norm S) (in short, IFGN(G)) of G if

\[
\begin{align*}
(1) & \quad A(xy) \supseteq T(\mu_A(x), \mu_A(y)), S(v_A(x), v_A(y)), \\
(2) & \quad A(x^{-1}) \supseteq A(x),
\end{align*}
\]

for all \( x, y \in G \).

Remark 2. Conditions (1) and (2) of Definition 12 are equivalent to following conditions:

\[
\begin{align*}
(1) & \quad \mu_A(xy) \geq T(\mu_A(x), \mu_A(y)), \\
(2) & \quad \mu_A(x^{-1}) \geq \mu_A(x), \\
(3) & \quad v_A(xy) \leq S(v_A(x), v_A(y)), \\
(4) & \quad v_A(x^{-1}) \leq v_A(x),
\end{align*}
\]

for all \( x, y \in G \).

Example 3. Let \((\mathbb{Z}, +)\) be a group of integers. For all \( x \in G \) we define a fuzzy subset \( \mu_A \) and \( v_A \) of \( G \) as

\[
\mu_A(x) = \begin{cases} 0.65 & \text{if } x \in \{0, \pm 2, \pm 4, \ldots\}; \\
0.35 & \text{if } x \in \{\pm 1, \pm 3, \ldots\}. \end{cases}
\]

\[
v_A(x) = \begin{cases} 0.20 & \text{if } x \in \{0, \pm 2, \pm 4, \ldots\}; \\
0.80 & \text{if } x \in \{\pm 1, \pm 3, \ldots\}. \end{cases}
\]

Let \( T(x, y) = T_p(x, y) = xy \) and \( S(x, y) = S_p(x, y) = x + y - xy \) for all \( x, y \in [0, 1] \), then \( A = (\mu_A, v_A) \in IFGN(G) \).

Lemma 3. Let \( A = (\mu_A, v_A) \in IFS(G) \) such that \( G \) is finite group and \( T \) and \( S \) be idempotent. If \( A = (\mu_A, v_A) \) satisfies condition (1) of Definition 12, then \( A = (\mu_A, v_A) \in IFGN(G) \).

Proof. As \( G \) is finite so we have an \( x \in G \) such that \( x \neq e \) and \( x \) has finite order, say \( n > 1 \) then \( x^n = e \) and \( x^{-1} = x^{n-1} \). Now by using condition (1) of Definition 12 repeatedly, we get that

\[
\mu_A(x^{-1}) = \mu(x^{n-1}) = \mu_A(x^{n-2}x) \geq T(\mu_A(x^{n-1}), \mu_A(x)) \geq T(\mu_A(x), \mu_A(x),...,\mu_A(x)) = \mu_A(x)
\]

and

\[
v_A(x^{-1}) = v_A(x^{n-1}) = v_A(x^{n-2}x) \leq (v_A(x^{n-1}), v_A(x)) \leq S(v_A(x), v_A(x),...,v_A(x)) = v_A(x).
\]

Thus

\[
A(x^{-1}) = (\mu_A(x^{-1}), v_A(x^{-1})) \supseteq (\mu_A(x), v_A(x)) = A(x).
\]

Hence \( A = (\mu_A, v_A) \in IFGN(G) \). \( \square \)

Proposition 2. Let \( A = (\mu_A, v_A) \in IFGN(G) \) and \( T \) and \( S \) be idempotent. Then for all \( x \in G \), and \( n \geq 1 \),

\[
(1) \quad A(e) \supseteq A(x);
(2) \quad A(x^n) \supseteq A(x);
(3) \quad A(x) = A(x^{-1}).
\]

Proof. Let \( x \in G \) and \( n \geq 1 \).

\[
(1) \quad \mu_A(e) = \mu_A(xx^{-1}) \geq T(\mu_A(x), \mu_A(x^{-1})) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x)
\]
and

\[ v_A(x) = v_A(xx^{-1}) \leq S(v_A(x), v_A(x^{-1})) \leq S(v_A(x), v_A(x)) = v_A(x). \]

Hence

\[ A(e) = (\mu_A(e), v_A(e)) \supseteq (\mu_A(x), v_A(x)) = A(x). \]

(2)

\[ \mu_A(x^n) = \mu_A(x)^n \geq T(\mu_A(x), \mu_A(x), \ldots, \mu_A(x)) = \mu_A(x) \]

and

\[ v_A(x^n) = v_A(x^n) \leq S(v_A(x), v_A(x), \ldots, v_A(x)) = v_A(x). \]

Hence

\[ A(x^n) = (\mu_A(x^n), v_A(x^n)) \supseteq (\mu_A(x), v_A(x)) = A(x). \]

(3) As

\[ \mu_A(x) = \mu_A((x^{-1})^{-1}) \geq \mu_A(x^{-1}) \geq \mu_A(x) \]

and

\[ v_A(x) = v_A((x^{-1})^{-1}) \leq v_A(x^{-1}) \leq v_A(x) = v_A(x^{-1}). \]

So \( \mu_A(x) = \mu_A(x^{-1}) \) and \( v_A(x) = v_A(x^{-1}) \), therefore

\[ A(x) = (\mu_A(x), v_A(x)) = (\mu_A(x^{-1}), v_A(x^{-1})) = A(x^{-1}). \]

\( \square \)

**Proposition 3.** Let \( A = (\mu_A, v_A) \in IFGN(G) \) and \( T \) and \( S \) be idempotent. Then \( A(xy) = A(y) \) if and only if \( A(x) = A(e) \) for all \( x, y \in G \).

**Proof.** Let \( A(xy) = A(y) \) for all \( x, y \in G \). If we let \( y = e \), then we get that \( A(x) = A(e) \).

Conversely, suppose that \( A(x) = A(e) \) so from Proposition 2(1), we get that \( A(x) \supseteq A(y) \) and \( A(x) \supseteq A(xy) \) which mean that \( \mu_A(x) \supseteq \mu_A(y) \) and \( \mu_A(x) \supseteq \mu_A(xy) \) and \( v_A(x) \leq v_A(y) \) and \( v_A(x) \leq v_A(xy) \). Then

\[ \mu_A(xy) \geq T(\mu_A(x), \mu_A(y)) \]
\[ \geq T(\mu_A(y), \mu_A(y)) \]
\[ = \mu_A(y) \]
\[ = \mu_A(x^{-1}xy) \]
\[ \geq T(\mu_A(x), \mu_A(xy)) \]
\[ \geq T(\mu_A(xy), \mu_A(xy)) = \mu_A(xy). \]

So

\[ \mu_A(xy) = \mu_A(y). \] (1)

Also

\[ v_A(xy) \leq S(v_A(x), v_A(y)) \]
\[ \leq S(v_A(y), v_A(y)) \]
\[ = v_A(y) \]
\[ = v_A(x^{-1}xy) \]
\[ \leq S(v_A(x), v_A(xy)) \]
\[ \leq S(v_A(xy), v_A(xy)) \]
\[ = v_A(xy). \]
such that $A(\nu) = (\mu_A(\nu), v_A(\nu)) = (\mu_A(y), v_A(y)) = A(y)$.

\[\Box\]

**Definition 13.** Let $A = (\mu_A, v_A) \in IFS(G)$ and $B = (\mu_B, v_B) \in IFS(G)$. We define the composition of $A$ and $B$ as $A \circ B \in IFS(G)$ such that for all $x \in G$, we have

\[(A \circ B)(x) = (\mu_A(B(y)), v_A(B(y))) = (\mu_A(v_B(x)), v_A(v_B(x)))\]

such that

\[
\mu_{A \circ B}(x) = \begin{cases} 
\sup_{y = z} T(\mu_A(y), \mu_A(z)) & \text{if } x = yz; \\
0 & \text{if } x \neq yz,
\end{cases}
\]

and

\[
v_{A \circ B}(x) = \begin{cases} 
\inf_{y = z} S(v_A(y), v_A(z)) & \text{if } x = yz; \\
0 & \text{if } x \neq yz.
\end{cases}
\]

**Proposition 4.** Let $A^{-1} = (\mu_A^{-1}, v_A^{-1}) \in IFS(G)$ be the inverse of $A = (\mu_A, v_A) \in IFS(G)$ such that for all $x \in G$

\[A^{-1}(x) = (\mu_A^{-1}(x), v_A^{-1}(x)) = (\mu_A(x^{-1}), v_A(x^{-1})) = A(x^{-1}).\]

Let $T$ and $S$ be idempotent then $A \in IFGN(G)$ if and only if $A$ satisfies the following conditions:

1. $A \circ A \subseteq A$;
2. $A^{-1} = A$.

**Proof.** Let $x, y, z \in G$ such that $x = yz$. If $A = (\mu_A, v_A) \in IFGN(G)$, then

\[
\mu_A(x) = \mu_A(yz) \geq T(\mu_A(y), \mu_A(z)) = \mu_{A \circ A}(x)
\]

and

\[
v_A(x) = v_A(yz) \leq S(v_A(y), v_A(z)) = v_{A \circ A}(x).
\]

Which yield

\[(A \circ A)(x) = (\mu_{A \circ A}(x), v_{A \circ A}(x)) \subseteq (\mu_A(x), v_A(x)) = A(x).
\]

Then $A \circ A \subseteq A$. Also $A^{-1} = A$ comes from Proposition 2(3). Conversely, let $A \circ A \subseteq A$ and $A^{-1} = A$. As $A \circ A \subseteq A$, so

\[
\mu_A(yz) = \mu_A(x) \geq \mu_{A \circ A}(x) = \sup_{y = z} T(\mu_A(y), \mu_A(z)) \geq T(\mu_A(y), \mu_A(z))
\]

and

\[
v_A(yz) = v_A(x) \leq v_{A \circ A}(x) = \inf_{y = z} S(v_A(y), v_A(z)) \leq S(v_A(y), v_A(z)).
\]

Which mean that

\[A(yz) = (\mu_A(yz), v_A(yz)) \geq (T(\mu_A(y), \mu_A(z)), S(v_A(y), v_A(z))).
\]

As $A^{-1} = A$, so

\[(x) = (\mu_A(x), v_A(x)) = (\mu_A^{-1}(x), v_A^{-1}(x)) = A^{-1}(x).
\]

Therefore from (3) and (4) we get that $A \in IFGN(G)$. \[\Box\]
Corollary 1. Let \( A = (\mu_A, \nu_A) \in IFGN(G) \) and \( B = (\mu_B, \nu_B) \in IFGN(G) \) and \( G \) be commutative group. Then \( A \circ B \in IFGN(G) \) if and only if \( A \circ B = B \circ A \).

Proof. If \( A, B, A \circ B \in IFGN(G) \), then from Proposition 4 we get that \( A^{-1} = A, B^{-1} = B \) and \((B \circ A)^{-1} = B \circ A\). Now \( A \circ B = A^{-1} \circ B^{-1} = (B \circ A)^{-1} = B \circ A\).

Conversely, since \( A \circ B = B \circ A \) we have

\[
(A \circ B)^{-1} = (B \circ A)^{-1} = A^{-1} \circ B^{-1} = A \circ B.
\]

Also

\[
(A \circ B) \circ (A \circ B) = A \circ (B \circ A) \circ B = A \circ (A \circ B) \circ B = (A \circ A) \circ (B \circ B) \subseteq B \circ B.
\]

Now Proposition 4 gives us that \( A \circ B \in IFGN(G) \). \( \square \)

Proposition 5. Let \( A = (\mu_A, \nu_A) \in IFGN(G) \) and \( B = (\mu_B, \nu_B) \in IFGN(G) \). Then \( A \cap B = (\mu_{A \cap B}, \nu_{A \cap B}) \in IFGN(G) \).

Proof. Let \( x, y \in G \). Then

\[
\mu_{A \cap B}(xy) = T(\mu_A(xy), \mu_B(xy)) \\
\geq T(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y))) \\
= T(T(\mu_A(x), \mu_B(x)), T(\mu_A(y), \mu_B(y))) \\
= T(\mu_{A \cap B}(x), \mu_{A \cap B}(y)).
\]

And

\[
v_{A \cap B}(xy) = S(v_A(xy), v_B(xy)) \\
\leq S(S(v_A(x), v_A(y)), S(v_B(x), v_B(y))) \\
= S(S(v_A(x), v_B(x)), S(v_A(y), v_B(y))) \\
= S(v_{A \cap B}(x), v_{A \cap B}(y)).
\]

Which mean that

\[
(A \cap B)(xy) = (\mu_{A \cap B}(xy), v_{A \cap B}(xy)) \supseteq (T(\mu_{A \cap B}(x), \mu_{A \cap B}(y)), S(v_{A \cap B}(x), v_{A \cap B}(y))).
\]

Also

\[
\mu_{A \cap B}(x^{-1}) = T(\mu_A(x^{-1}), \mu_B(x^{-1})) \geq T(\mu_A(x), \mu_B(x)) = \mu_{A \cap B}(x)
\]

and

\[
v_{A \cap B}(x^{-1}) = S(v_A(x^{-1}), v_B(x^{-1})) \leq S(v_A(x), v_B(x)) = v_{A \cap B}(x).
\]

So

\[
(A \cap B)(x^{-1}) = (\mu_{A \cap B}(x^{-1}), v_{A \cap B}(x^{-1})) \supseteq (\mu_{A \cap B}(x), v_{A \cap B}(x)) = (A \cap B)(x).
\]

Thus \( A \cap B = (\mu_{A \cap B}, v_{A \cap B}) \in IFGN(G) \). \( \square \)

Corollary 2. Let \( I_n = \{1, 2, ..., n\} \). If \( \{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i \in I_n\} \subseteq IFGN(G) \). Then \( A = \cap_{i \in I_n} A_i \in IFGN(G) \).

Definition 14. We say that \( A = (\mu_A, \nu_A) \in IFGN(G) \) is normal if for all \( x, y \in G \), \( A(xy^{-1}) = A(y) \). Also we denote by \( NIFGN(G) \) the set of all normal intuitionistic fuzzy groups with respect to norms (\( t \)-norm \( T \) and \( s \)-norm \( S \)).

Proposition 6. Let \( A = (\mu_A, \nu_A) \in NIFGN(G) \) and \( B = (\mu_B, \nu_B) \in NIFGN(G) \). Then \( A \cap B = (\mu_{A \cap B}, v_{A \cap B}) \in NIFGN(G) \).
Corollary 3. As Proposition 5 we have that \( A \cap B = (\mu_{A \cap B}, v_{A \cap B}) \in \text{IFGN}(G) \). Let \( x, y, \in G \), then

\[
\mu_{A \cap B}(xyx^{-1}) = T(\mu_A(xy), \mu_B(xy^{-1})) = T(\mu_A(y), \mu_B(y)) = \mu_{A \cap B}(y)
\]

and

\[
v_{A \cap B}(xyx^{-1}) = S(v_A(xy), v_B(xy^{-1})) = S(v_A(y), v_B(y)) = v_{A \cap B}(y).
\]

Thus

\[
(A \cap B)(xyx^{-1}) = (\mu_{A \cap B}(xyx^{-1}), v_{A \cap B}(xyx^{-1})) = (\mu_{A \cap B}(y), v_{A \cap B}(y)) = (A \cap B)(y).
\]

Therefore \( A \cap B = (\mu_{A \cap B}, v_{A \cap B}) \in \text{NIFGN}(G) \).

Corollary 3. Let \( I_n = \{1, 2, ..., n\} \). If \( \{A_i = (\mu_{A_i}, v_{A_i}) \mid i \in I_n\} \subseteq \text{NIFGN}(G) \), then \( A = \cap_{i \in I_n} A_i \in \text{NIFGN}(G) \).

Definition 15. Let \( A = (\mu_A, v_A) \in \text{IFGN}(G) \) and \( B = (\mu_B, v_B) \in \text{IFGN}(G) \) such that \( A \subseteq B \). Then \( A \) is called normal of \( B \), written \( A \triangleright B \), if for all \( x, y, \in G \) we have

\[
A(xyx^{-1}) = (\mu_A(xy^{-1}), v_A(xy^{-1})) \supseteq (T(\mu_A(y), \mu_B(x)), S(v_A(y), v_B(x))).
\]

Proposition 7. (1) Let \( G_1 \) and \( G_2 \) are subgroups of \( G \). Then \( G_1 \) is a normal subgroup of \( G_2 \) if and only if \( 1_{G_1} \triangleright 1_{G_2} \).

(2) If \( T \) and \( S \) be idempotent, then every intuitionistic fuzzy subgroup with respect to norms (\( t \)-norm \( T \) and \( s \)-norm \( S \)) is normal fuzzy subgroup of itself.

Proof. (1) Let \( x \in G_2 \) and \( y \in G_1 \) then \( 1_{G_1}(x) = 1 \) and \( 1_{G_1}(y) = 1 \). If \( G_1 \triangleright G_2 \), then \( xyx^{-1} \in G_1 \) and so \( 1_{G_1}(xyx^{-1}) = 1 \). As \( 1_G = (1_G, 1_G) \in \text{IFGN}(G) \), so

\[
1_{G_1}(xyx^{-1}) = 1 \geq 1 = T(1, 1) = T(1_{G_1}(y), 1_{G_2}(x))
\]

and

\[
1_{G_1}(xyx^{-1}) = 1 \leq 1 = S(1, 1) = S(1_{G_1}(y), 1_{G_2}(x)).
\]

Then

\[
1_{G_1}(xyx^{-1}) = (1_{G_1}(xyx^{-1}), 1_{G_1}(xyx^{-1})) \supseteq (T(1_{G_1}(y), 1_{G_2}(x)), S(1_{G_1}(y), 1_{G_2}(x)));
\]

Hence \( 1_{G_1} \triangleright 1_{G_2} \).

(2) Let \( A = (\mu_A, v_A) \in \text{IFGN}(G) \) and \( x, y \in G \) then

\[
\mu_A(xy^{-1}) \supseteq T(\mu_A(xy), \mu_A(x^{-1}))
\]

\[
\supseteq T(T(\mu_A(x), \mu_A(y)), \mu_A(x))
\]

\[
= T(T(\mu_A(y), \mu_A(x)), \mu_A(x))
\]

\[
= T(\mu_A(y), T(\mu_A(x), \mu_A(x)))
\]

\[
= T(\mu_A(y), \mu_A(x)).
\]

And

\[
v_A(xy^{-1}) \leq S(v_A(xy), v_A(x^{-1}))
\]

\[
\leq S(S(v_A(x), v_A(y)), v_A(x))
\]

\[
= S(S(v_A(y), v_A(x)), v_A(x))
\]

\[
= S(v_A(y), S(v_A(x), v_A(x)))
\]

\[
= S(v_A(y), v_A(x)).
\]

Thus

\[
A(xyx^{-1}) = (\mu_A(xyx^{-1}), v_A(xyx^{-1})) \supseteq (T(\mu_A(y), \mu_A(x)), S(v_A(y), v_B(x))).
\]
Hence \( A = (\mu_A, v_A) \gg A = (\mu_A, v_A) \). 

\[ \square \]

**Proposition 8.** Let \( A = (\mu_A, v_A) \in NIFGN(G) \) and \( B = (\mu_B, v_B) \in IFGN(G) \) such that \( T \) and \( S \) be idempotent. Then \( A \cap B \gg B \).

**Proof.** Using Proposition 5, we get that \( A \cap B \in IFGN(G) \). Let \( x, y \in G \), then

\[
\mu_{A \cap B}(xyx^{-1}) = T(\mu_A(xy^{-1}), \mu_B(xyx^{-1})) \\
= T(\mu_A(y), \mu_B(xy^{-1})) \\
\geq T(\mu_A(y), T(\mu_B(xy), \mu_B(x^{-1}))) \\
\geq T(\mu_A(y), T(\mu_B(x), \mu_B(y)), \mu_B(x))) \\
= T(\mu_A(y), v_B(y), T(\mu_B(x), \mu_B(x))) \\
= T(\mu_A(y), T(\mu_B(y), \mu_B(x))) \\
= T(\mu_A(y), \mu_B(y), \mu_B(x)) \\
= T(\mu_{A \cap B}(y), \mu_B(x)).
\]

And

\[
v_{A \cap B}(xyx^{-1}) = S(v_A(xy^{-1}), v_B(xyx^{-1})) \\
= S(v_A(y), v_B(xy^{-1})) \\
\leq S(v_A(y), S(v_B(xy), v_B(x))) \\
\leq S(v_A(y), S(S(v_B(x), v_B(y)), v_B(x))) \\
= S(v_A(y), S(v_B(y), S(v_B(x), v_B(x)))) \\
= S(v_A(y), S(v_B(y), v_B(x))) \\
= S(S(v_A(y), v_B(y)), v_B(x)) \\
= S(v_{A \cap B}(y), v_B(x)).
\]

Thus

\[
(A \cap B)(xyx^{-1}) = (\mu_{A \cap B}(xyx^{-1}), v_{A \cap B}(xyx^{-1})) \geq (T(\mu_{A \cap B}(y), \mu_B(x)), S(v_{A \cap B}(y), v_B(x))).
\]

Which means that \( A \cap B \gg B \). 

\[ \square \]

**Proposition 9.** Let \( A = (\mu_A, v_A) \in IFGN(G) \) and \( B = (\mu_B, v_B) \in IFGN(G) \) and \( C = (\mu_C, v_C) \in IFGN(G) \) such that \( T \) and \( S \) be idempotent. If \( A \gg C \) and \( B \gg C \), then \( A \cap B \gg C \).

**Proof.** By Proposition 5 we will have that \( A \cap B \in IFGN(G) \). Let \( x, y \in G \), then

\[
\mu_{A \cap B}(xyx^{-1}) = T(\mu_A(xy^{-1}), \mu_B(xyx^{-1})) \\
\geq T(T(\mu_A(y), \mu_C(x)), T(\mu_B(y), \mu_C(x))) \\
= T(T(\mu_A(y), \mu_B(y)), T(\mu_C(x), \mu_C(x))) \\
= T(T(\mu_A(y), \mu_B(y)), \mu_C(x)) = T(\mu_{A \cap B}(y), \mu_C(x)).
\]

And

\[
v_{A \cap B}(xyx^{-1}) = S(v_A(xy^{-1}), v_B(xyx^{-1})) \\
\leq S(S(v_A(y), v_C(x)), S(v_B(y), v_C(x))) \\
= S(S(v_A(y), v_B(y)), S(v_C(x), v_C(x))) \\
= S(S(v_A(y), v_B(y)), v_C(x)) = S(v_{A \cap B}(y), v_C(x)).
\]
Therefore
\[(A \cap B)(xyx^{-1}) = (\mu_{A\cap B}(xyx^{-1}), \nu_{A\cap B}(xyx^{-1})) \supseteq (T(\mu_{A\cap B}(y), \mu_C(x)), S(\nu_{A\cap B}(y), \nu_C(x))).\]

Hence \(A \cap B \triangleright \triangleleft C. \)

**Corollary 4.** Let \(I_n = \{1, 2, \ldots, n\}\) and \(\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i \in I_n\} \subseteq \text{IFGN}(G)\) such that \(\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i \in I_n\} \triangleright B = (\mu_B, \nu_B).\) Then \(A = \cap_{i \in I_n} A_i \triangleright B = (\mu_B, \nu_B).\)

**4. Homomorphisms of IFGN(G)**

**Proposition 10.** Let \(A = (\mu_A, \nu_A) \subseteq \text{IFGN}(G)\) and \(H\) be a group. Suppose that \(\varphi : G \to H\) is a homomorphism. Then \(\varphi(A) \subseteq \text{IFGN}(H).\)

**Proof.** Let \(u, v \in H\) and \(x, y \in G\) such that \(u = \varphi(x)\) and \(v = \varphi(y)\) and \(\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)).\) Now

\[
\varphi(\mu_A)(uv) = \sup \{\mu_A(xy) \mid u = \varphi(x), v = \varphi(y)\}
\]

\[
\geq \sup \{T(\mu_A(x), \mu_A(y)) \mid u = \varphi(x), v = \varphi(y)\}
\]

\[
= T(\sup \{\mu_A(x) \mid u = f(x)\}, \sup \{\mu_A(y) \mid v = \varphi(y)\})
\]

and

\[
\varphi(\nu_A)(uv) = \inf \{\nu_A(xy) \mid u = \varphi(x), v = \varphi(y)\}
\]

\[
\leq \inf \{S(\nu_A(x), \nu_A(y)) \mid u = \varphi(x), v = \varphi(y)\}
\]

\[
= S(\inf \{\nu_A(x) \mid u = f(x)\}, \inf \{\nu_A(y) \mid v = \varphi(y)\})
\]

Which mean that

\[
\varphi(A)(uv) \supseteq (T(\varphi(\mu_A)(u), \varphi(\mu_A)(v)), S(\varphi(\nu_A)(u), \varphi(\nu_A)(v))).
\]

Also

\[
\varphi(\mu_A)(u^{-1}) = \sup \{\mu_A(x^{-1}) \mid u^{-1} = \varphi(x^{-1})\}
\]

\[
= \sup \{\mu_A(x^{-1}) \mid u^{-1} = \varphi^{-1}(x)\}
\]

\[
\geq \sup \{\mu_A(x) \mid u = \varphi(x)\}
\]

\[
= \varphi(\mu_A)(u)
\]

and

\[
\varphi(\nu_A)(u^{-1}) = \inf \{\nu_A(x^{-1}) \mid u^{-1} = \varphi(x^{-1})\}
\]

\[
= \inf \{\nu_A(x^{-1}) \mid u^{-1} = \varphi^{-1}(x)\}
\]

\[
\leq \inf \{\nu_A(x) \mid u = \varphi(x)\}
\]

\[
= \varphi(\nu_A)(u).
\]

Thus

\[
\varphi(A)(u^{-1}) = (\varphi(\mu_A)(u^{-1}), \varphi(\nu_A)(u^{-1})) \supseteq (\varphi(\mu_A)(u), \varphi(\nu_A)(u)) = \varphi(A)(u).
\]

Therefore \(\varphi(A) \subseteq \text{IFGN}(H).\)

**Proposition 11.** Let \(H\) be a group and \(B = (\mu_B, \nu_B) \subseteq \text{IFGN}(H).\) Suppose that \(\varphi : G \to H\) is a homomorphism. Then \(\varphi^{-1}(B) \subseteq \text{IFGN}(G).\)
Proof. Let \( x, y \in G \) and \( \varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(v_B)) = (\mu_B(\varphi), v_B(\varphi)) \). Now

\[
\varphi^{-1}(\mu_B)(xy) = \mu_B(\varphi(xy)) = \mu_B(\varphi(x)\varphi(y)) \geq T(\mu_B(\varphi(x)), \mu_B(\varphi(y))) = T(\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\mu_B)(y))
\]

and

\[
\varphi^{-1}(v_B)(xy) = v_B(\varphi(xy)) = v_B(\varphi(x)\varphi(y)) \leq S(v_B(\varphi(x)), v_B(\varphi(y))) = S(\varphi^{-1}(v_B)(x), \varphi^{-1}(v_B)(y)).
\]

So

\[
\varphi^{-1}(B)(xy) \supseteq (T(\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\mu_B)(y)), S(\varphi^{-1}(v_B)(x), \varphi^{-1}(v_B)(y))).
\]

Also

\[
\varphi^{-1}(\mu_B)(x^{-1}) = \mu_B(\varphi(x^{-1})) = \mu_B(\varphi^{-1}(x)) \geq \mu_B(\varphi(x)) = \varphi^{-1}(\mu_B)(x)
\]

and

\[
\varphi^{-1}(v_B)(x^{-1}) = v_B(\varphi(x^{-1})) = v_B(\varphi^{-1}(x)) \leq v_B(\varphi(x)) = \varphi^{-1}(v_B)(x).
\]

Thus

\[
\varphi^{-1}(B)(x^{-1}) = (\varphi^{-1}(\mu_B)(x^{-1}), \varphi^{-1}(v_B)(x^{-1})) \supseteq (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(v_B)(x)) = \varphi^{-1}(B)(x).
\]

Hence \( \varphi^{-1}(B) \in \text{IFGN}(G) \). \( \square \)

Proposition 12. Let \( A = (\mu_A, v_A) \in \text{IFGN}(G) \) and \( H \) be a group. Suppose that \( \varphi : G \to H \) is a homomorphism. Then \( \varphi(A) \in \text{IFGN}(H) \).

Proof. As Proposition 10 we have that \( \varphi(A) \in \text{IFGN}(H) \). Let \( x, y \in H \) such that \( \varphi(u) = x \) and \( \varphi(w) = y \) with \( u, w \in G \). Then

\[
\varphi(\mu_A(xy^{-1})) = \sup \{ \mu_A(w) \mid w \in G, \varphi(w) = xy^{-1} \}
\]

\[
= \sup \{ \mu_A(w) \mid w \in G, \varphi(w) = \varphi(u)\varphi(w)\varphi(u^{-1}) \}
\]

\[
= \sup \{ \mu_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1}) \}
\]

\[
= \sup \{ \mu_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1}) = y \}
\]

\[
= \sup \{ \mu_A(w) \mid w \in G, \varphi(w) = y \}
\]

\[
= \varphi(\mu_A(y))
\]

and

\[
\varphi(v_A(xy^{-1})) = \inf \{ v_A(w) \mid w \in G, \varphi(w) = xy^{-1} \}
\]

\[
= \inf \{ v_A(w) \mid w \in G, \varphi(w) = \varphi(u)\varphi(w)\varphi(u^{-1}) \}
\]

\[
= \inf \{ v_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1}) \}
\]

\[
= \inf \{ v_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1}) = y \}
\]

\[
= \inf \{ v_A(w) \mid w \in G, \varphi(w) = y \}
\]

\[
= \varphi(v_A(y)).
\]

Which yield

\[
\varphi(A)(xy^{-1}) = (\varphi(\mu_A(xy^{-1})), \varphi(v_A(xy^{-1}))) = (\varphi(\mu_A(y)), \varphi(v_A(y))) = \varphi(A)(y).
\]

Thus \( \varphi(A) \in \text{IFGN}(H) \). \( \square \)

Proposition 13. Let \( H \) be a group and \( B = (\mu_B, v_B) \in \text{IFGN}(H) \). Suppose that \( \varphi : G \to H \) is a homomorphism. Then \( \varphi^{-1}(B) \in \text{IFGN}(G) \).
Proof. By Proposition 11 we get that $\varphi^{-1}(B) \in IFGN(G)$. Let $x, y \in G$, then
\[
\varphi^{-1}(\mu_B)(xy^{-1}) = \mu_B(\varphi(xy^{-1})) \\
= \mu_B(\varphi(x)\varphi(y)\varphi(x^{-1})) \\
= \mu_B(\varphi(x)\varphi(y)\varphi^{-1}(x)) \\
= \mu_B(\varphi(y)) \\
= \varphi^{-1}(\mu_B)(y)
\]
and
\[
\varphi^{-1}(v_B)(xy^{-1}) = v_B(\varphi(xy^{-1})) \\
= v_B(\varphi(x)\varphi(y)\varphi(x^{-1})) \\
= v_B(\varphi(x)\varphi(y)\varphi^{-1}(x)) \\
= v_B(\varphi(y)) \\
= \varphi^{-1}(v_B)(y).
\]

Then
\[
\varphi^{-1}(B)(xy^{-1}) = (\varphi^{-1}(\mu_B)(xy^{-1}), \varphi^{-1}(v_B)(xy^{-1})) = (\varphi^{-1}(\mu_B)(y), \varphi^{-1}(v_B)(y)) = \varphi^{-1}(B)(y).
\]
Thus $\varphi^{-1}(B) \in NIFGN(G)$. □

Proposition 14. Let $A = (\mu_A, v_A) \in IFGN(G)$ and $B = (\mu_B, v_B) \in IFGN(G)$ such that $A \triangleright B$. If $\varphi : G \to H$ is a homomorphism, then $\varphi(A) \triangleright \varphi(B)$.

Proof. Using Proposition 10 we will have that $\varphi(A) \in IFGN(H)$ and $\varphi(B) \in IFGN(H)$. Let $x, y \in H$ and $u, v \in G$, then
\[
\varphi(\mu_A)(xy^{-1}) = \sup\{\mu_A(z) \mid z \in G, \varphi(z) = xy^{-1}\} \\
= \sup\{\mu_A(wh^{-1}) \mid u, v \in G, \varphi(u) = x, \varphi(v) = y\} \\
\geq \sup\{T(\mu_A(v), \mu_B(u)) \mid \varphi(u) = x, \varphi(v) = y\} \\
= T(\sup\{\mu_A(v) \mid y = \varphi(v)\}, \sup\{\mu_B(u) \mid x = \varphi(u)\}) \\
= T(\varphi(\mu_A)(y), \varphi(\mu_B)(x))
\]
and
\[
\varphi(v_A)(xy^{-1}) = \inf\{v_A(z) \mid z \in G, \varphi(z) = xy^{-1}\} \\
= \inf\{v_A(wh^{-1}) \mid u, v \in G, \varphi(u) = x, \varphi(v) = y\} \\
\leq \inf\{S(v_A(v), v_B(u)) \mid \varphi(u) = x, \varphi(v) = y\} \\
= S(\inf\{v_A(v) \mid y = \varphi(v)\}, \inf\{v_B(u) \mid x = \varphi(u)\}) \\
= S(\varphi(v_A)(y), \varphi(v_B)(x)).
\]

Then
\[
\varphi(A)(xy^{-1}) = (\varphi(\mu_A)(xy^{-1}), \varphi(v_A)(xy^{-1})) \supseteq (T(\varphi(\mu_A)(y), \varphi(\mu_B)(x)), S(\varphi(v_A)(y), \varphi(v_B)(x))).
\]
Thus $\varphi(A) \triangleright \varphi(B)$. □

Proposition 15. Let $A = (\mu_A, v_A) \in IFGN(H)$ and $B = (\mu_B, v_B) \in IFGN(H)$ such that $A \triangleright B$. If $\varphi : G \to H$ is a homomorphism, then $\varphi^{-1}(A) \triangleright \varphi^{-1}(B)$. 

Proof. As Proposition 11 we will have that $\varphi^{-1}(A) \in IFGN(G)$ and $\varphi^{-1}(B) \in IFGN(G)$. Let $x, y \in G$, then

$$
\varphi^{-1}(\mu_A)(xyx^{-1}) = \mu_A(\varphi(xy)x^{-1})
= \mu_A(\varphi(x)\varphi(y)\varphi(x^{-1}))
= \mu_A(\varphi(x)\varphi(y)\varphi^{-1}(x))
\geq T(\mu_A(\varphi(y)), \mu_B(\varphi(x)))
= T(\varphi^{-1}(\mu_A)(y), \varphi^{-1}(\mu_B)(x))
$$

and

$$
\varphi^{-1}(\nu_A)(xyx^{-1}) = \nu_A(\varphi(xy)x^{-1})
= \nu_A(\varphi(x)\varphi(y)\varphi(x^{-1}))
= \nu_A(\varphi(x)\varphi(y)\varphi^{-1}(x))
\leq S(\nu_A(\varphi(y)), \nu_B(\varphi(x)))
= S(\varphi^{-1}(\nu_A)(y), \varphi^{-1}(\nu_B)(x)).
$$

Then

$$
\varphi^{-1}(A)(xyx^{-1}) = (\varphi^{-1}(\mu_A)(xyx^{-1}), \varphi^{-1}(\nu_A)(xyx^{-1}))
\supseteq (T(\varphi^{-1}(\mu_A)(y), \varphi^{-1}(\mu_B)(x)), S(\varphi^{-1}(\nu_A)(y), \varphi^{-1}(\nu_B)(x))).
$$

Thus $\varphi^{-1}(A) \triangleright \varphi^{-1}(B)$. □

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