Reverse Hermite-Hadamard’s inequalities using $\psi$-fractional integral

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Abstract: Our purpose in this paper is to use $\psi$–Riemann-Liouville fractional integral operator which is the fractional integral of any function with respect to another increasing function to establish some new fractional integral inequalities of Hermite-Hadamard, involving concave functions. Using the concave functions, we establish some new fractional integral inequalities related to the Hermite-Hadamard type inequalities via $\psi$–Riemann-Liouville fractional integral operator.

Keywords: Fractional inequalities, $\psi$-Riemann-Liouville fractional integral, $\psi$-Riemann-Liouville derivative.

1. Introduction

The classical calculus of derivatives and integrals which involves integer orders is extended with fractional orders that belong to the real numbers. In last few decades, the fractional calculus theory receives more attention due to its significant applications in several scopes such as physics, fluid dynamics, computer networking, image processing, biology, signal processing, control theory and other scopes. Because of the importance of fractional calculus, many researchers have shown their intense interest. One of the prevalent approaches among researchers is the use of fractional derivatives and integral operators. As a consequence, several distinct kinds of fractional integrals and derivatives operators have been realized, such as the Liouville, Riemann-Liouville, Katugampola, Weyl types, Hadamard and some other types which can be found in Kilbas et al., [1].

Hilfer [2] in (2000), through his contribution to improve the fractional calculus, established a new fractional derivative operator for any real order $\delta$, which gives the Caputo derivative and the Riemann-Liouville fractional operator. The primary concept and properties and more information of $\psi$-Riemann–Liouville fractional derivative and integral can be found in [1]. In (2017), Almeida [3], introduced $\psi$-Caputo fractional derivative and investigated its significant properties. Recently, in 2018, Sousa and Oliveira [4], introduced a generalization of many existing fractional derivative operators called $\psi$-Hilfer derivative.

The mathematical inequalities play a very reliable role in classical integral and differential equations as well as in the past few years, many of useful mathematical inequalities have been originated by many authors, see [5–8]. One of the most significant integral inequalities is that discovered by Hermite [9] and Hadamard [10] for convex function $f$ as follows

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{a-b} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.$$ (1)

If $f$ is a concave function then both inequalities in (1) are held in a reversed direction. For some historical of Hermite-Hadamard inequalities [11] and the references therein. In the last few decades, these inequalities have been received a considerable attention by many authors and several articles have appeared in the literature, see [12–14]. In 2010, Dahmani [15], studied the Hermite-Hadamard type inequalities for concave functions by means of Riemann–Liouville fractional integral. Sarikaya et al., in 2013 [16], gave the Hermite-Hadamard type inequalities for convex function using Riemann–Liouville fractional integral. In 2014, Set et al., established Hermite-Hadamard type inequalities for s-convex functions in the second

Very recently, in 2020, Chudziak and Oldak introduced notion of a co-ordinated $(F, G)$-convex function defined on an interval in $\mathbb{R}^2$.

The main objective of this paper is to establish some new fractional integral Hermite-Hadamard inequalities for concave functions. In Section 4, we give some notations, definitions, results and preliminary facts which are used throughout this paper. In Section 3, we present the reverse Hermite-Hadamard’s inequalities for concave functions. In Section 4, we give some other related results of Hermite-Hadamard type inequalities which involving $\psi –$Riemann-Liouville fractional integral operator.

2. Basic definitions and tools

This section is dedicated for some basic definitions and properties of fractional integrals used to obtain and discuss our new results. We also outline some basic results related to this work.

Let $\delta > 0$, $m \in \mathbb{N}$, with $Y = [a, b]$ $(-\infty \leq a < t < b \leq \infty)$, be a finite or infinite interval. Assume that $f$ be an integrable function defined on $Y$ and $\psi : Y \to \mathbb{R}$ be an increasing function for all $t \in Y$, which belong to $C^1 (Y, \mathbb{R})$ with condition that $\psi'(t)$ must be nonzero along the interval $Y$. The $\psi –$Riemann-Liouville fractional derivative of order $\delta$ of a function $f$ are defined by [1,24]:

\[
D^\delta_{a} \psi f(t) = \left( \frac{1}{\psi'(t)} \right) \frac{d}{dt} \mathcal{I}^m_{a} \psi f(t) = \frac{1}{\Gamma(m-\delta)} \left( \frac{1}{\psi'(t)} \right) \frac{d}{dt} \int_{a}^{t} \psi' (\xi) [\psi(t) - \psi(\xi)]^{m-\delta-1} f(\xi) \, d\xi,
\]

and

\[
D^\delta_{b} \psi f(t) = \left( -\frac{1}{\psi'(t)} \right) \frac{d}{dt} \mathcal{I}^m_{b} \psi f(t) = \frac{1}{\Gamma(m-\delta)} \left( -\frac{1}{\psi'(t)} \right) \frac{d}{dt} \int_{t}^{b} \psi' (\xi) [\psi(\xi) - \psi(t)]^{m-\delta-1} f(\xi) \, d\xi.
\]

**Definition 1.** Let $\delta > 0$ and $f$ be an integrable function defined on $Y$ and $\psi(t) \in C^1 (Y, \mathbb{R})$ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in Y$. The left and right $\psi –$Riemann-Liouville fractional integral of order $\delta$ with respect to the function $\psi$ of a function $f$ are respectively defined by [1,24]:

\[
\mathcal{I}^\delta_{a} \psi f(t) = \frac{1}{\Gamma(\delta)} \int_{a}^{t} \psi'(\xi) [\psi(t) - \psi(\xi)]^{\delta-1} f(\xi) \, d\xi,
\]

and

\[
\mathcal{I}^\delta_{b} \psi f(t) = \frac{1}{\Gamma(\delta)} \int_{t}^{b} \psi'(\xi) [\psi(\xi) - \psi(t)]^{\delta-1} f(\xi) \, d\xi.
\]

**Lemma 1.** [1] Let $\delta > 0$ and $\mu > 0$. If $f(t) = |\psi(t) - \psi(\xi)|^{\mu-1}$, then

\[
\mathcal{I}^\delta_{a} \psi f(t) = \frac{\Gamma(\mu)}{\Gamma(\delta + \mu)} [\psi(t) - \psi(\xi)]^{\delta + \mu - 1}.
\]
Definition 2. The function \( f : (\Lambda \subseteq \mathbb{R}) \rightarrow \mathbb{R} \) is said to be concave function if the following inequality holds
\[
f (\lambda x + (1 - \lambda) y) \geq \lambda f (x) + (1 - \lambda) f (y),
\]
for all \( x, y \in \Lambda \) and \( \lambda \in [0, 1] \). We say that \( f \) is convex if the inequality (6) is reversed.

Theorem 3. [16] Let \( f : [a, b] \rightarrow \mathbb{R} \) be a convex positive function on \([a, b]\) with \( 0 \leq a < b \), then for all \( \delta > 0 \), the following inequality holds:
\[
\frac{f (a) + f (b)}{2} \leq \frac{\Gamma (\delta + 1)}{2 (a + b)^{\delta}} \left[ \mathcal{I}_{a}^{\delta} f (b) + \mathcal{I}_{b}^{\delta} f (a) \right] \leq f \left( \frac{a + b}{2} \right).
\]

Lemma 2. [15] Let \( h : [a, b] \rightarrow \mathbb{R} \) be a concave function. Then the following inequality holds:
\[
h (a) + h (b) \leq h (a + b - t) + h (t) \leq 2h \left( \frac{a + b}{2} \right).
\]

Theorem 4. [15] Let \( f \) and \( g \) be two positive functions on \([0, \infty)\). If \( f \) and \( g \) are a concave functions on \([0, \infty)\), then for all \( p > 1, q > 1 \) and \( \delta > 0 \), the following inequality holds:
\[
2^{-p-q} \left[ f (0) + f (\infty) \right]^{\frac{q}{p}} \left[ g (0) + g (\infty) \right]^{\frac{p}{q}} \left( \mathcal{I}^{\delta} x^{\delta-1} \right)^{2} \leq \mathcal{I}^{\delta} \left[ x^{\delta-1} f^{p} (x) \right] \mathcal{I}^{\delta} \left[ x^{\delta-1} g^{q} (x) \right].
\]

Theorem 5. [15] Let \( f \) and \( g \) be two positive functions on \([0, \infty)\). If \( f \) and \( g \) are concave functions on \([0, \infty)\), then for all \( p > 1, q > 1 \) and \( \delta > 0, \sigma > 0 \) the following inequality holds:
\[
2^{-p-q} \left[ f (0) + f (\infty) \right]^{\frac{q}{p}} \left[ g (0) + g (\infty) \right]^{\frac{p}{q}} \left( \mathcal{I}^{\delta} x^{\delta-1} \right)^{2} \leq \mathcal{I}^{\sigma} \left[ x^{\sigma-1} f^{p} (x) \right] \mathcal{I}^{\delta} \left[ x^{\delta-1} g^{q} (x) \right] + \mathcal{I}^{\delta} \left[ x^{\delta-1} g^{q} (x) \right].
\]

3. The reverse Hermite-Hadamard’s inequalities for fractional integral

Now, we give the reverse Hermite-Hadamard’s inequalities involving concave functions for \( \psi \)-Riemann-Liouville fractional integral operators.

Theorem 6. Let \( \psi : [a, b] \rightarrow \Lambda \subseteq \mathbb{R} \) with \( 0 \leq a < b \) be an increasing and bijective function having a continuous derivative \( \psi' (x) \neq 0 \ \forall \ x \in [a, b] \), \( \psi (0) = 0 \), \( \psi (1) = 1 \) and \( f : \Lambda \rightarrow \mathbb{R} \) be an increasing and differentiable function on \( \Lambda \) such that \( \psi \circ f : [a, b] \rightarrow \mathbb{R} \) be an integrable mapping on \([a, b]\) a continuous functions on \( \Lambda \), then the following inequality holds:
\[
f \left( \frac{\psi (a) + \psi (b)}{2} \right) \geq \frac{\Gamma (\delta + 1)}{2 [\psi (a) + \psi (b)]^{\delta}} \left[ \mathcal{I}_{a}^{\delta} (\psi \circ f) (b) + \mathcal{I}_{b}^{\delta} (\psi \circ f) (a) \right] \geq \frac{(f \circ \psi) (a) + (f \circ \psi) (b)}{2}.
\]

Proof. For any \( x, y \in [a, b] \), using the concavity of \( f \) and \( \psi \), we have
\[
(f \circ \psi) [\lambda x + (1 - \lambda) y] = f [\psi (\lambda x + (1 - \lambda) y)] \geq f [\lambda \psi (x) + (1 - \lambda) \psi (y)] \geq \lambda f [\psi (x)] + (1 - \lambda) f [\psi (y)] = \lambda (f \circ \psi) (x) + (1 - \lambda) (f \circ \psi) (y).
\]

Putting \( \lambda = \frac{1}{2} \) and using (9), we can write
\[
f \left( \frac{\psi (x) + \psi (y)}{2} \right) \geq \frac{(f \circ \psi) (x) + (f \circ \psi) (y)}{2}.
\]
Let

\[ \psi (x) = \psi (t) \psi(a) + [1 - \psi (t)] \psi (b), \]  

(11)

and

\[ \psi (y) = [1 - \psi (t)] \psi(a) + \psi (t) \psi (b), \]  

(12)

where \( x, y \) are variables containing \( t \). By substituting (11) and (12) in (10), we get

\[ 2f \left( \frac{\psi(a) + \psi(b)}{2} \right) \geq f [\psi (t) \psi(a) + [1 - \psi (t)] \psi (b)] + f [[1 - \psi (t)] \psi(a) + \psi (t) \psi (b)]. \]  

(13)

Now, multiplying both sides of (13) by \( \psi' (t) \psi^{\delta-1} (t) \), then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain

\[ \frac{2}{\delta} f \left( \frac{\psi(a) + \psi(b)}{2} \right) \geq \int_0^1 \psi' (t) \psi^{\delta-1} (t) f [\psi (t) \psi(a) + [1 - \psi (t)] \psi (b)] \, dt 
+ \int_0^1 \psi' (t) \psi^{\delta-1} (t) f [[1 - \psi (t)] \psi(a) + \psi (t) \psi (b)] \, dt. \]  

(14)

From (11), we have

\[ \frac{d}{dt} \psi (x) = \frac{d}{dt} [\psi (t) \psi (a) + [1 - \psi (t)] \psi (b)] \Rightarrow \frac{\psi' (x) \, dx}{\psi (a) - \psi (b)} = \psi' (t) \, dt, \]

and

\[ \psi (t) = \frac{\psi (x) - \psi (b)}{\psi (a) - \psi (b)}. \]

So, we have

\[ I_1 = \int_a^b \left( \frac{\psi (x) - \psi (b)}{\psi (a) - \psi (b)} \right)^{\delta-1} f (\psi (x)) \, \frac{\psi' (x) \, dx}{\psi (a) - \psi (b)} 
= \int_a^b \psi' (x) \left( \frac{\psi (b) - \psi (x)}{\psi (b) - \psi (a)} \right)^{\delta-1} (f \circ \psi) (x) \, \frac{dx}{\psi (b) - \psi (a)} 
= \frac{\Gamma (\delta)}{[\psi (b) - \psi (a)]^\delta} \int_a^b \psi' (x) \, (f \circ \psi) (x) \, dx. \]  

(15)

Also from (12), we have

\[ \frac{d}{dt} \psi (y) = \frac{d}{dt} [[1 - \psi (t)] \psi (a) + \psi (t) \psi (b)] \Rightarrow \frac{\psi' (y) \, dy}{\psi (b) - \psi (a)} = \psi' (t) \, dt, \]

and

\[ \psi (t) = \frac{\psi (y) - \psi (a)}{\psi (b) - \psi (a)}. \]

So, we have

\[ I_2 = \int_a^b \left( \frac{\psi (y) - \psi (a)}{\psi (b) - \psi (a)} \right)^{\delta-1} f (\psi (y)) \, \frac{\psi' (y) \, dy}{[\psi (b) - \psi (a)]} 
= \frac{1}{[\psi (b) - \psi (a)]^\delta} \int_a^b \psi' (y) \, (f \circ \psi) (y) \, dy 
= \frac{\Gamma (\delta)}{[\psi (b) - \psi (a)]^\delta} \int_{\psi (a)}^{\psi (b)} (f \circ \psi) (a). \]  

(16)

Using (14), (15) and (16), we get
Lemma 3. Let \( f \left( \frac{\psi(a) + \psi(b)}{2} \right) \geq \frac{\Gamma(\delta + 1)}{2 |\psi(b) - \psi(a)|^\delta} \left[ \mathcal{I}_a^{\delta \psi} (f \circ \psi) (b) + \mathcal{I}_b^{\delta \psi} (f \circ \psi) (a) \right] \),

which is the first inequality in (8). To prove the second inequality and using the concavity of \( f \) and \( \psi \), we can write for \( \lambda \in [0,1] \)

\[
\psi(t) \psi(a) + [1 - \psi(t)] \psi(b) \geq \psi(t) (f \circ \psi)(a) + [1 - \psi(t)] (f \circ \psi)(b),
\]

and

\[
\psi(t) \psi(a) + \psi(t) \psi(b) \geq [1 - \psi(t)] (f \circ \psi)(a) + \psi(t) (f \circ \psi)(b).
\]

Adding (18) and (19), we obtain

\[
f \left( \psi(t) \psi(a) + [1 - \psi(t)] \psi(b) \right) + f \left( [1 - \psi(t)] \psi(a) + \psi(t) \psi(b) \right) \geq (f \circ \psi)(b) + (f \circ \psi)(a).
\]

Now, multiplying both sides of (20) by \( \psi'(t) \psi^{\delta - 1}(t) \), then integrating the resulting inequality with respect to \( t \) over \([0,1]\), we obtain

\[
\frac{\Gamma(\delta)}{|\psi(b) - \psi(a)|^\delta} \left[ \mathcal{I}_a^{\delta \psi} (f \circ \psi) (b) + \mathcal{I}_b^{\delta \psi} (f \circ \psi) (a) \right] \geq \frac{(f \circ \psi)(b) + (f \circ \psi)(a)}{\delta}.
\]

Hence, by combining the inequalities (17) and (20), we get the desired inequality (8). \( \square \)

**Remark 1.** (i) If we put \( \psi(x) = x, \forall x \in [a,b] \), for \( f \) a convex function, then both inequalities (8) reversed and Theorem 6 reduce to Theorem 3 obtained by Sarikaya et al., in [16] for classical Riemann-Liouville fractional integral. (ii) Applying inequalities (8) for \( \psi(x) = x, \forall x \in [a,b] \) and \( \delta = 1 \), for \( f \) a convex function, we obtain the classical Hermite-Hadamard inequalities (1).

4. Hermite-Hadamard type inequalities for fractional integral

In this section, we generalize some Hermite-Hadamard type inequalities involving concave functions introduced by Dahmani [15] using the Riemann-Liouville fractional integral with respect to other monotone and bijective function. In present part, we use only the left-sided fractional integrals (4). Moreover, we consider \( a = 0 \) to obtain and discuss our results. We first prove the following lemma:

**Lemma 3.** Let \( \psi : [0,\infty) \rightarrow \Lambda \) be an increasing and bijective function having a continuous derivative \( \psi'(t) \neq 0 \) \( \forall \ t \in [0,\infty) \), \( \psi(0) = 0 \), \( \psi(1) = 1 \) and \( h : \Lambda \rightarrow \mathbb{R} \) be an increasing and differentiable function on \( \Lambda^0 \) such that \( (h \circ \psi) : [0,\infty) \rightarrow \mathbb{R} \) be an integrable mapping on \( [0,\infty) \). If \( h \) is a concave functions on \( \Lambda \), then we have

\[
(h \circ \psi)(c) + (h \circ \psi)(d) \leq h[(\psi(c) + \psi(d) - \psi(t)) + (h \circ \psi)(t)] \leq 2h \left( \frac{\psi(c) + \psi(d)}{2} \right).
\]

**Proof.** Since \( h \) be a concave function on \( \Lambda \), so for any \( c,d \in [0,\infty) \), we can write

\[
h \left( \frac{\psi(c) + \psi(d)}{2} \right) = h \left( \psi(c) + \psi(d) + \psi(t) - \psi(t) \right) \geq h \left( \psi(c) + \psi(d) - \psi(t) \right) + h \left( \psi(t) \right) / 2.
\]

If we choose \( \psi(t) = \lambda \psi(c) + [1 - \lambda] \psi(d) \), then we have

\[
\frac{1}{2} \left[ h \left( \psi(c) + \psi(d) - \lambda \psi(c) - [1 - \lambda] \psi(d) \right) + h \left( \lambda \psi(c) + [1 - \lambda] \psi(d) \right) \right]
\]

\[
= \frac{1}{2} \left[ h \left( \lambda \psi(d) + [1 - \lambda] \psi(c) \right) + h \left( \lambda \psi(c) + [1 - \lambda] \psi(d) \right) \right].
\]
Proof. Since
\[
\frac{1}{2} \left[ h \left( \lambda \psi (d) + [1 - \lambda] \psi (c) \right) + h \left( \lambda \psi (c) + [1 - \lambda] \psi (d) \right) \right] \geq \frac{1}{2} \left[ (h \circ \psi) (c) + (h \circ \psi) (d) \right].
\]
(24)

By (23) and (24), we get
\[
(h \circ \psi) (c) + (h \circ \psi) (d) \leq h [\psi (c) + \psi (d) - \psi (t)] + (h \circ \psi) (t) \leq 2h \left( \frac{\psi (c) + \psi (d)}{2} \right),
\]
which is the required inequality (22). □

**Theorem 7.** Let \( \psi : [0, \infty) \rightarrow \Lambda \) be an increasing positive and bijective function having a continuous derivative \( \psi' (x) \neq 0 \ \forall \ x \in [0, \infty) \), \( \psi (0) = 0 \), \( \psi (1) = 1 \) and \( f, g : \Lambda \rightarrow \mathbb{R} \) be an increasing and differentiable functions on \( \Lambda \). If \( f \) and \( g \) are a concave functions on \( \Lambda \). Then for all \( p > 1 \), \( q > 1 \) and \( \delta > 0 \), the following inequality holds:
\[
2^{-p-q} [f (0) + (f \circ \psi) (x)]^p [g (0) + (g \circ \psi) (x)]^q \left( \mathcal{I}^\delta \psi \psi^{\delta-1} (x) \right)^2 \leq \mathcal{I}^\delta \psi \left[ \psi^{\delta-1} (x) (f \circ \psi)^p (x) \right] \mathcal{I}^\delta \psi \left[ \psi^{\delta-1} (x) (g \circ \psi)^q (x) \right].
\]
(25)

Proof. Since \( f^p \) and \( g^q \) are a concave functions on \( \Lambda \), so by Lemma (3), for any \( x, y > 0 \), we have
\[
f^p (0) + (f \circ \psi)^p (x) \leq f^p [\psi (x) - \psi (y)] + (f \circ \psi)^p (y) \leq 2 (f \circ \psi)^p \left( \frac{x}{2} \right),
\]
(26)
and
\[
g^q (0) + (g \circ \psi)^q (x) \leq g^q [\psi (x) - \psi (y)] + (g \circ \psi)^q (y) \leq 2 (g \circ \psi)^q \left( \frac{x}{2} \right).
\]
(27)

Multiplying both sides of (26) and (27) by \( \frac{\psi^{\delta-1} (x)}{\Gamma (\delta)} [\psi (x) - \psi (y)]^{\delta-1} \psi^{\delta-1} (y) \), \( y \in (0, x) \) and integrating the resulting inequalities with respect to \( y \) over \((0, x)\), we obtain
\[
\frac{f^p (0) + (f \circ \psi)^p (x)}{\Gamma (\delta)} \int_0^x \psi' (y) [\psi (x) - \psi (y)]^{\delta-1} \psi^{\delta-1} (y) \ dy \leq \frac{1}{\Gamma (\delta)} \int_0^x \psi' (y) [\psi (x) - \psi (y)]^{\delta-1} \psi^{\delta-1} (y) \left( f \circ \psi ight)^p (y) \ dy + \frac{1}{\Gamma (\delta)} \int_0^x \psi' (y) [\psi (x) - \psi (y)]^{\delta-1} \psi^{\delta-1} (y) \left( f \circ \psi ight)^p \left( \frac{x}{2} \right) \ dy \leq 2 \frac{f^p (0) + (f \circ \psi)^p (x)}{\Gamma (\delta)} \frac{x}{2} \int_0^x \psi' (y) [\psi (x) - \psi (y)]^{\delta-1} \psi^{\delta-1} (y) \ dy,
\]
(28)
and
\[
\frac{g^q (0) + (g \circ \psi)^q (x)}{\Gamma (\delta)} \int_0^x \psi' (y) [\psi (x) - \psi (y)]^{\delta-1} \psi^{\delta-1} (y) \ dy \leq \frac{1}{\Gamma (\delta)} \int_0^x \psi' (y) [\psi (x) - \psi (y)]^{\delta-1} \psi^{\delta-1} (y) \left( g \circ \psi ight)^q (y) \ dy + \frac{1}{\Gamma (\delta)} \int_0^x \psi' (y) [\psi (x) - \psi (y)]^{\delta-1} \psi^{\delta-1} (y) \left( g \circ \psi ight)^q \left( \frac{x}{2} \right) \ dy \leq 2 \frac{g^q (0) + (g \circ \psi)^q (x)}{\Gamma (\delta)} \frac{x}{2} \int_0^x \psi' (y) [\psi (x) - \psi (y)]^{\delta-1} \psi^{\delta-1} (y) \ dy.
\]
(29)

Using the change of variable \( \psi (u) = \psi (x) - \psi (y) \), where \( u \in [0, \infty) \) is a variable containing \( y \), we have
\[
\frac{d}{dy} [\psi (u)] = \frac{d}{dy} [\psi (x) - \psi (y)] \implies \psi' (u) \ du = -\psi' (y) \ dy.
\]
Then, we can write
\[
\frac{1}{\Gamma(\delta)} \int_0^x \psi'(y) \left(\psi(x) - \psi(y)\right)^{\delta-1} \psi^{\delta-1}(y) f^p \left(\psi(x) - \psi(y)\right) \, dy
\]
\[
= \frac{1}{\Gamma(\delta)} \int_0^x \psi'(u) \left(\psi(x) - \psi(u)\right)^{\delta-1} \psi^{\delta-1}(u) (f \circ \psi)^p (u) \, du = \mathcal{I}^{\delta \psi} \left[\psi^{\delta-1}(x) (f \circ \psi)^p (x)\right],
\]  
(30)

and
\[
\frac{1}{\Gamma(\delta)} \int_0^x \psi'(y) \left(\psi(x) - \psi(y)\right)^{\delta-1} \psi^{\delta-1}(y) g^q \left(\psi(x) - \psi(y)\right) \, dy
\]
\[
= \frac{1}{\Gamma(\delta)} \int_0^x \psi'(u) \left(\psi(x) - \psi(u)\right)^{\delta-1} \psi^{\delta-1}(u) (g \circ \psi)^q (u) \, du = \mathcal{I}^{\delta \psi} \left[\psi^{\delta-1}(x) (g \circ \psi)^q (x)\right].
\]  
(31)

Now, by using (28) and (30), we get
\[
[f^p(0) + (f \circ \psi)^p(x)] \mathcal{I}^{\delta \psi} \psi^{\delta-1}(x) \leq 2 \mathcal{I}^{\delta \psi} \left[\psi^{\delta-1}(x) (f \circ \psi)^p (x)\right] \leq 2 (f \circ \psi)^p \left(\frac{x}{2}\right) \mathcal{I}^{\delta \psi} \psi^{\delta-1}(x),
\]  
(32)

and using (29) and (31), we get
\[
[g^q(0) + (g \circ \psi)^q(x)] \mathcal{I}^{\delta \psi} \psi^{\delta-1}(x) \leq 2 \mathcal{I}^{\delta \psi} \left[\psi^{\delta-1}(x) (g \circ \psi)^q (x)\right] \leq 2 (g \circ \psi)^q \left(\frac{x}{2}\right) \mathcal{I}^{\delta \psi} \psi^{\delta-1}(x).
\]  
(33)

The inequalities (32) and (33) yields
\[
\left[f^p(0) + (f \circ \psi)^p(x)\right] \left[g^q(0) + (g \circ \psi)^q(x)\right] \left(\mathcal{I}^{\delta \psi} \psi^{\delta-1}(x)\right)^2 \leq 4 \mathcal{I}^{\delta \psi} \left[\psi^{\delta-1}(x) (f \circ \psi)^p (x)\right] \mathcal{I}^{\delta \psi} \left[\psi^{\delta-1}(x) (g \circ \psi)^q (x)\right].
\]  
(34)

On the other hand, we have \( f \) and \( g \) are positive functions and \( \psi \) is increasing function on \([0, \infty)\). Then for any \( x > 0, p \geq 1, q \geq 1 \), we can write
\[
\left[f^p(0) + (f \circ \psi)^p(x)\right]^{\frac{1}{q}} \geq f(0) + (f \circ \psi)(x),
\]  
(35)

and
\[
\left[g^q(0) + (g \circ \psi)^q(x)\right]^{\frac{1}{q}} \geq g(0) + (g \circ \psi)(x).
\]  
(36)

Multiplying both sides of (35) and (36) by \( \frac{\psi'(y)}{\Gamma(\delta)} |\psi(x) - \psi(y)|^{\delta-1} \psi^{\delta-1}(y), y \in (0, x) \), then integrating the resulting inequalities with respect to \( y \) over \((0, x)\), we get
\[
\frac{f^p(0) + (f \circ \psi)^p(x)}{2} \mathcal{I}^{\delta \psi} \psi^{\delta-1}(x) \geq 2^{-p} \left[f(0) + (f \circ \psi)(x)\right]^p \mathcal{I}^{\delta \psi} \psi^{\delta-1}(x),
\]  
(37)

and
\[
\frac{g^q(0) + (g \circ \psi)^q(x)}{2} \mathcal{I}^{\delta \psi} \psi^{\delta-1}(x) \geq 2^{-q} \left[g(0) + (g \circ \psi)(x)\right]^q \mathcal{I}^{\delta \psi} \psi^{\delta-1}(x).
\]  
(38)

The inequalities (37) and (38) yields
\[
\frac{1}{4} \left[f^p(0) + (f \circ \psi)^p(x)\right] \left[g^q(0) + (g \circ \psi)^q(x)\right] \left(\mathcal{I}^{\delta \psi} \psi^{\delta-1}(x)\right)^2 \geq 2^{-p-q} [f(0) + (f \circ \psi)(x)]^p [g(0) + (g \circ \psi)(x)]^q \left(\mathcal{I}^{\delta \psi} \psi^{\delta-1}(x)\right)^2.
\]  
(39)

Combining the inequalities (34) and (39), we obtain the desired inequality (25).

\[\square\]

**Remark 2.** (i) If we put \( \psi(x) = x \) for all \( x \in [0, \infty) \), then Lemma 3 reduce to Lemma 2 and Theorem 7 reduce to Theorem 4 obtained by Dahmani in [15].
(ii) Applying Theorem 7 for \( \psi(x) = x \) for all \( x \in [0, \infty) \), \( \delta = 1 \), we obtain Theorem 5 obtained by Set et al., in [25].

No, we give the following version of Theorem 7 with two parameters for \( \psi \)-Riemann-Liouville fractional integral operator.

**Theorem 8.** Let \( \psi \colon [0, \infty) \to \Lambda \) be an increasing positive and bijective function having a continuous derivative \( \psi'(x) \neq 0 \ \forall \ x \in [0, \infty) \), \( \psi(0) = 0 \), \( \psi(1) = 1 \) and \( f, g \colon \Lambda \to \mathbb{R} \) be an increasing and differentiable functions on \( \Lambda^{\circ} \) such that \( (f \circ \psi), (g \circ \psi) : [0, \infty) \to \mathbb{R} \) are two integrable mappings on \( [0, \infty) \). If \( f \) and \( g \) are a concave functions on \( \Lambda \). Then for all \( p > 1 \), \( q > 1 \) and \( \delta > 0 \), \( \sigma > 0 \), the following inequality holds:

\[
2^{2-p-q} [f(0) + (f \circ \psi)(x)]^p [g(0) + (g \circ \psi)(x)]^q \left( T^{\delta \sigma p} \psi^{\sigma - 1}(x) \right)^2 \\
\leq \frac{\Gamma(\sigma)}{\Gamma(\delta)} T^{\delta \sigma p} \left[ \psi^{\sigma - 1}(x) (f \circ \psi)^p(x) \right] + T^{\delta \sigma p} \left[ \psi^{\sigma - 1}(x) (f \circ \psi)^p(x) \right] \\
\times \frac{\Gamma(\sigma)}{\Gamma(\delta)} T^{\delta \sigma p} \left[ \psi^{\sigma - 1}(x) (g \circ \psi)^q(x) \right] + T^{\delta \sigma p} \left[ \psi^{\sigma - 1}(x) (g \circ \psi)^q(x) \right].
\]

**Proof.** By using Lemma 3 and as \( f^p \) and \( g^q \) are concave functions on \( \Lambda \), then we have for any \( x, y > 0 \)

\[
f^p(0) + (f \circ \psi)^p(x) \leq f^p(\psi(x) - \psi(y)) + (f \circ \psi)^p(y) \leq 2 (f \circ \psi)^p \left( \frac{x}{2} \right),
\]

and

\[
g^q(0) + (g \circ \psi)^q(x) \leq g^q(\psi(x) - \psi(y)) + (g \circ \psi)^q(y) \leq 2 (g \circ \psi)^q \left( \frac{x}{2} \right).
\]

Now, multiplying both sides of (41) and (42) by \( \frac{\psi'(y)}{\psi'(x)} [\psi(x) - \psi(y)]^\delta \psi^{\sigma - 1}(y) \), \( y \in (0, x) \), then integrating the resulting inequalities with respect to \( y \) over \( (0, x) \), we obtain

\[
\frac{f^p(0) + (f \circ \psi)^p(x)}{\Gamma(\delta)} \int_0^x \psi'(y) [\psi(x) - \psi(y)]^\delta \psi^{\sigma - 1}(y) \, dy \\
\leq \frac{1}{\Gamma(\delta)} \int_0^x \psi'(y) [\psi(x) - \psi(y)]^\delta \psi^{\sigma - 1}(y) f^p(\psi(x) - \psi(y)) \, dy \\
+ \frac{1}{\Gamma(\delta)} \int_0^x \psi'(y) [\psi(x) - \psi(y)]^\delta \psi^{\sigma - 1}(y) (f \circ \psi)^p(y) \, dy \\
\leq \frac{2 (f \circ \psi)^p \left( \frac{x}{2} \right)}{\Gamma(\delta)} \int_0^x \psi'(y) [\psi(x) - \psi(y)]^\delta \psi^{\sigma - 1}(y) \, dy,
\]

and

\[
\frac{g^q(0) + (g \circ \psi)^q(x)}{\Gamma(\delta)} \int_0^x \psi'(y) [\psi(x) - \psi(y)]^\delta \psi^{\sigma - 1}(y) \, dy \\
\leq \frac{1}{\Gamma(\delta)} \int_0^x \psi'(y) [\psi(x) - \psi(y)]^\delta \psi^{\sigma - 1}(y) g^q(\psi(x) - \psi(y)) \, dy \\
+ \frac{1}{\Gamma(\delta)} \int_0^x \psi'(y) [\psi(x) - \psi(y)]^\delta \psi^{\sigma - 1}(y) (g \circ \psi)^q(y) \, dy \\
\leq \frac{2 (g \circ \psi)^q \left( \frac{x}{2} \right)}{\Gamma(\delta)} \int_0^x \psi'(y) [\psi(x) - \psi(y)]^\delta \psi^{\sigma - 1}(y) \, dy.
\]

Using the change of variable \( \psi(u) = \psi(x) - \psi(y) \), where \( u \in [0, \infty) \) is a variable containing \( y \), we have

\[
\frac{1}{\Gamma(\delta)} \int_0^x \psi'(y) [\psi(x) - \psi(y)]^\delta \psi^{\sigma - 1}(y) f^p(\psi(x) - \psi(y)) \, dy \\
= \frac{\Gamma(\sigma)}{\Gamma(\sigma) \Gamma(\delta)} \int_0^x \psi'(u) [\psi(x) - \psi(u)]^\delta \psi^{\sigma - 1}(u) f^p(u) \, du = \frac{\Gamma(\sigma)}{\Gamma(\delta)} T^{\delta \sigma p} \left[ \psi^{\sigma - 1}(x) (f \circ \psi)^p(x) \right],
\]

(45)
and
\[
\frac{1}{\Gamma(\delta)} \int_0^x \psi'(y) [\psi(x) - \psi(y)]^{\delta-1} \psi^{\delta-1}(y) \Delta y [\psi(x) - \psi(y)] dy
= \frac{\Gamma(\sigma)}{\Gamma(\sigma) \Gamma(\delta)} \int_0^x \psi'(u) [\psi(x) - \psi(u)]^{\sigma-1} \psi^{\sigma-1}(u) (g \circ \psi)^q(u) du
= \frac{\Gamma(\sigma)}{\Gamma(\delta)} T_{\psi}^\varphi \left[ \psi^{\delta-1}(x) (g \circ \psi)^q(x) \right].
\]
(46)

By using (43) and (45), we obtain
\[
[f^p(0) + (f \circ \psi)^p(x)] T_{\psi}^\varphi \psi^{\sigma-1}(x) \leq \frac{\Gamma(\sigma)}{\Gamma(\delta)} T_{\psi}^\varphi \left[ \psi^{\delta-1}(x) (f \circ \psi)^p(x) \right] + T_{\psi}^\varphi \left[ \psi^{\sigma-1}(x) (g \circ \psi)^q(x) \right]
\leq 2 (f \circ \psi)^p \left( \frac{x}{2} \right) T_{\psi}^\varphi \psi^{\sigma-1}(x),
\]
(47)
and using (44) and (46), we get
\[
[g^q(0) + (g \circ \psi)^q(x)] T_{\psi}^\varphi \psi^{\sigma-1}(x) \leq \frac{\Gamma(\sigma)}{\Gamma(\delta)} T_{\psi}^\varphi \left[ \psi^{\delta-1}(x) (g \circ \psi)^q(x) \right] + T_{\psi}^\varphi \left[ \psi^{\sigma-1}(x) (g \circ \psi)^q(x) \right]
\leq 2 (g \circ \psi)^q \left( \frac{x}{2} \right) T_{\psi}^\varphi \psi^{\sigma-1}(x).
\]
(48)
The inequalities (47) and (48) imply that
\[
\left[ f^p(0) + (f \circ \psi)^p(x) \right] [g^q(0) + (g \circ \psi)^q(x)] \left( T_{\psi}^\varphi \psi^{\sigma-1}(x) \right)^2
\leq \frac{\Gamma(\sigma)}{\Gamma(\delta)} T_{\psi}^\varphi \left[ \psi^{\delta-1}(x) (f \circ \psi)^p(x) \right] + T_{\psi}^\varphi \left[ \psi^{\sigma-1}(x) (g \circ \psi)^q(x) \right]
\times \frac{\Gamma(\sigma)}{\Gamma(\delta)} T_{\psi}^\varphi \left[ \psi^{\delta-1}(x) (g \circ \psi)^q(x) \right] + T_{\psi}^\varphi \left[ \psi^{\sigma-1}(x) (g \circ \psi)^q(x) \right].
\]
(49)

Similarly as before, we have \( f \) and \( g \) are positive functions and \( \psi \) is increasing function on \([0, \infty)\). Then for any \( x > 0, p \geq 1, q \geq 1 \), we can write
\[
\frac{f^p(0) + (f \circ \psi)^p(x)}{2} T_{\psi}^\varphi \psi^{\sigma-1}(x) \geq 2^{-p} [f(0) + (f \circ \psi)(x)]^p T_{\psi}^\varphi \psi^{\sigma-1}(x),
\]
(50)
and
\[
\frac{g^q(0) + (g \circ \psi)^q(x)}{2} T_{\psi}^\varphi \psi^{\sigma-1}(x) \geq 2^{-q} [g(0) + (g \circ \psi)(x)]^q T_{\psi}^\varphi \psi^{\sigma-1}(x).
\]
(51)
The inequalities (50) and (51) imply that
\[
\frac{1}{4} \left[ f^p(0) + (f \circ \psi)^p(x) \right] [g^q(0) + (g \circ \psi)^q(x)] \left( T_{\psi}^\varphi \psi^{\sigma-1}(x) \right)^2
\geq 2^{-p-q} [f(0) + (f \circ \psi)(x)]^p [g(0) + (g \circ \psi)(x)]^q \left( T_{\psi}^\varphi \psi^{\sigma-1}(x) \right)^2.
\]
(52)

Combining the inequalities (49) and (52), we obtain the desired inequality (40).

\textbf{Remark 3.} If we put \( \psi(x) = x \) for all \( x \in [0, \infty) \), then Theorem 7 reduce to Theorem 5 obtained by Dahmani in [15].

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\textbf{References}


