Čebyšev inequalities for co-ordinated QC-convex and (s, QC)-convex

1. Introduction

In 1882, Čebyšev [1] gave the following inequality

\[ |T(f, g)| \leq \frac{1}{12} (b - a)^2 \|f'\|_\infty \|g'\|_\infty, \]  

(1)

where \( f, g : [a, b] \rightarrow \mathbb{R} \) are absolutely continuous function, whose first derivatives \( f' \) and \( g' \) are bounded and

\[ T(f, g) = \frac{1}{b - a} \int_a^b f(x) g(x) \, dx - \left( \frac{1}{b - a} \int_a^b f(x) \, dx \right) \left( \frac{1}{b - a} \int_a^b g(x) \, dx \right), \]  

(2)

and \( \| \cdot \|_\infty \) denotes the norm in \( L_\infty [a, b] \) defined as \( \|f\|_\infty = \text{ess sup}_{t \in [a, b]} |f(t)| \).

During the past few years, many researchers have given considerable attention to the inequality (1). Various generalizations, extensions and variants have been appeared in the literature [2–6].

Recently, Guezane-Lakoud and Aissaoui [2] gave the analogue of the functional (2) for functions of two variables and established the following Čebyšev type inequalities for functions whose mixed derivatives are co-ordinated quasi-convex and \( a \)-quasi-convex and \( s \)-quasi-convex functions.

**Abstract:** In this paper, we establish some new Čebyšev type inequalities for functions whose modulus of the mixed derivatives are co-ordinated quasi-convex and \( a \)-quasi-convex and \( s \)-quasi-convex functions.

**Keywords:** Čebyšev inequalities, quasi-convexity, \((s, QC)\)-convexity, \((a, QC)\)-convexity.
2. Preliminaries

Throughout this paper, we denote by Δ, the bidimensional interval in \([0, \infty)^2\), \(\Delta : = [a, b] \times [c, d]\) with \(a < b\) and \(c < d\), \(k : = (b - a)(d - c)\) and \(\frac{\partial^2 f}{\partial \lambda \partial \omega}\) by \(f_{\lambda \omega}\).

Definition 1. [7] A function \(f : \Delta \rightarrow \mathbb{R}\) is said to be convex on the co-ordinates on \(\Delta\) if

\[
f(\lambda x + (1 - \lambda)t, \omega y + (1 - \omega)v) \leq \lambda f(x, y) + \omega(1 - \lambda)f(t, y) + (1 - \lambda)(1 - \omega)f(t, v)
\]

holds for all \(\lambda, \omega \in [0, 1]\) and \((x, y), (t, y), (t, v) \in \Delta\).

Definition 2. [8] A function \(f : \Delta \rightarrow \mathbb{R}\) is said to be quasi-convex on the co-ordinates on \(\Delta\) if

\[
f(\lambda x + (1 - \lambda)t, \omega y + (1 - \omega)v) \leq \max\{f(x, y) + f(t, y) + f(t, v)\}
\]

holds for all \(\lambda, \omega \in [0, 1]\) and \((x, y), (t, y), (t, v) \in \Delta\).

Definition 3. [9] For some \(\alpha \in (0, 1]\), a function \(f : \Delta \rightarrow \mathbb{R}\) is said to be \((\alpha, QC)\)-convex on the co-ordinates on \(\Delta\), if

\[
f(\lambda x + (1 - \lambda)t, \omega y + (1 - \omega)v) \leq \lambda^\alpha \max\{f(x, y) + f(t, y)\} + (1 - \lambda^\alpha)\max\{f(t, y) + f(t, v)\}
\]

holds for all \(\lambda, \omega \in [0, 1]\) and \((x, y), (t, y), (t, v) \in \Delta\).

Definition 4. [10] For some \(s \in [-1, 1]\), a function \(f : \Delta \rightarrow [0, \infty)\) is said to be \((s, QC)\)-convex on the co-ordinates on \(\Delta\), if

\[
f(\lambda x + (1 - \lambda)t, \omega y + (1 - \omega)v) \leq \lambda^s \max\{f(x, y) + f(t, y)\} + (1 - \lambda^s)\max\{f(t, y) + f(t, v)\}
\]

holds for all \(\lambda \in (0, 1)\), \(\omega \in [0, 1]\) and \((x, y), (t, y), (t, v) \in \Delta\).

Lemma 1. [11] Let \(f : \Delta \rightarrow \mathbb{R}\) be a partial differentiable mapping on \(\Delta\) in \(\mathbb{R}^2\). If \(f_{\lambda \omega} \in L_1(\Delta)\) then for any \((x, y) \in \Delta\), we have the equality;

\[
f(x, y) = \frac{1}{b - a} \int_a^b f(t, y) dt + \frac{1}{d - c} \int_c^d f(x, v) dv - \frac{1}{k} \int_a^b \int_c^d f(t, v) dt dv + \frac{1}{k} \int_a^b \int_0^{1 - t} (x - t) (y - v)
\]

\[
\times \left( \int_0^{1 - t} f_{\lambda \omega} (\lambda x + (1 - \lambda)t, \omega y - (1 - \omega)v) \lambda d\lambda \right) dv dt.
\]

3. Main result

Theorem 1. Let \(f, g : \Delta \rightarrow \mathbb{R}\) be partially differentiable functions such that their second derivatives \(f_{\lambda \omega}\) and \(g_{\lambda \omega}\) are integrable on \(\Delta\). If \(f_{\lambda \omega}^\prime\) and \(g_{\lambda \omega}\) are co-ordinated quasi-convex on \(\Delta\), then

\[
|T(f, g)| \leq \frac{49}{360} MNk^2,
\]

where \(T(f, g)\) is defined as in (5), \(M = \max_{x, t \in [a, b], y, v \in [c, d]} [\|f_{\lambda \omega}(x, y)\| + \|f_{\lambda \omega}(x, v)\| + \|f_{\lambda \omega}(t, y)\| + \|f_{\lambda \omega}(t, v)\|]\), and \(N = \max_{x, t \in [a, b], y, v \in [c, d]} [\|g_{\lambda \omega}(x, y)\| + \|g_{\lambda \omega}(x, v)\| + \|g_{\lambda \omega}(t, y)\| + \|g_{\lambda \omega}(t, v)\|]\), and \(k = (b - a)(d - c)\).
Proof. From Lemma 1, we have
\[
 f(x, y) - \frac{b}{\pi^2} \int_{a}^{b} f(t, y) dt - \frac{1}{\pi^2} \int_{c}^{d} f(x, \nu) d\nu + \frac{b}{\pi^2} \int_{a}^{b} \int_{c}^{d} f(t, \nu) d\nu dt
\]
\[
= \frac{b}{\pi^2} \int_{a}^{b} \int_{c}^{d} (x-t)(y-\nu) \left( \int_{0}^{1} f_{\lambda}(\lambda x + (1-\lambda)t, wy - (1-\nu)v) d\lambda \right) d\nu dt, \tag{8}
\]
and
\[
 g(x, y) - \frac{b}{\pi^2} \int_{a}^{b} g(t, y) dt - \frac{1}{\pi^2} \int_{c}^{d} g(x, \nu) d\nu + \frac{b}{\pi^2} \int_{a}^{b} \int_{c}^{d} g(t, \nu) d\nu dt
\]
\[
= \frac{b}{\pi^2} \int_{a}^{b} \int_{c}^{d} (x-t)(y-\nu) \left( \int_{0}^{1} g_{\lambda}(\lambda x + (1-\lambda)t, wy - (1-\nu)v) d\lambda \right) d\nu dt. \tag{9}
\]
Multiplying (8) by (9), and then integrating the resulting equality with respect to \(x\) and \(y\) over \(\Delta\), using modulus and Fubini's Theorem, and multiplying the result by \(\frac{1}{k}\), we get
\[
|T(f, g)| \leq \frac{1}{k} \int_{a}^{b} \int_{c}^{d} \int_{a}^{b} \int_{c}^{d} |x-t||y-\nu| \times \left( \int_{0}^{1} f_{\lambda}(\lambda x + (1-\lambda)t, wy - (1-\nu)v) d\lambda \right) d\nu dt d\lambda dydx.
\]
\[
\times \left[ \int_{0}^{1} g_{\lambda}(\lambda x + (1-\lambda)t, wy - (1-\nu)v) d\lambda \right] d\nu \int_{a}^{b} \int_{c}^{d} |x-t||y-\nu| d\nu dt \int_{a}^{b} \int_{c}^{d} |x-t||y-\nu| dydx. \tag{10}
\]
Since \(|f_{\lambda}|\) and \(|g_{\lambda}|\) are co-ordinated quasi-convex, we deduce
\[
|T(f, g)| \leq \frac{1}{k} MN \int_{a}^{b} \int_{c}^{d} \left( \int_{a}^{b} \int_{c}^{d} |x-t||y-\nu| d\nu dt \right)^2 dydx = \frac{49}{3600} k^2 MN, \tag{11}
\]
where we have used the fact that
\[
\int_{a}^{b} \int_{c}^{d} \left( \int_{a}^{b} \int_{c}^{d} |x-t||y-\nu| d\nu dt \right)^2 dydx = \frac{49}{3600} k^5. \tag{12}
\]
The proof is completed. \(\square\)

Theorem 2. Under the assumptions of Theorem 1, we have
\[
|T(f, g)| \leq \frac{1}{k^2} \int_{a}^{b} \int_{c}^{d} \left[ M |g(x, y)| + N |f(x, y)| \right] \left[ (x-a)^2 + (b-x)^2 \right] \times \left[ (y-c)^2 + (d-y)^2 \right] dydx, \tag{13}
\]
where \(T(f, g)\) is defined as in (5), \(M, N, \) and \(k\) are as in Theorem 1.

Proof. From Lemma 1, (8) and (9) are valid. Let \(G(x, y) = \frac{1}{k} g(x, y)\) and \(F(x, y) = \frac{1}{k^2} f(x, y)\). Multiplying \(G(x, y)\) by \(F(x, y)\), then integrating the resultant equalities with respect to \(x\) and \(y\) over \(\Delta\), and by using the modulus, we get
\[ |T(f, g)| \leq \frac{1}{2\pi} \left[ \int_a^b \int_c^d |g(x, y)| \left( \int_a^b \int_c^d |x-t| |y-v| \right)^{\frac{1}{2}} dt \right] \]
\multiplytext{\times} \left( \int_0^1 |f_{\lambda w} (\lambda x + (1-\lambda)t, wy - (1-w)v)| \, d\lambda \right) \, dydx \times \left( \int_0^1 |g_{\lambda w} (\lambda x + (1-\lambda)t, wy - (1-w)v)| \, d\lambda \right) \, dydx.

Since \( |f_{\lambda w}| \) and \( |g_{\lambda w}| \) are co-ordinated quasi-convex, (14) implies
\[ |T(f, g)| \leq \frac{1}{2\pi} \left[ \int_a^b \int_c^d |g(x, y)| \left( \int_a^b \int_c^d |x-t| |y-v| \, d\lambda \right) \, dydx \right] \]
\multiplytext{+} \left( \int_0^1 |f(x, y)| \left( \int_a^b \int_c^d |x-t| |y-v| \, d\lambda \right) \, dydx \right) \]
\multiplytext{=} \frac{1}{2\pi} \left( \int_a^b \int_c^d (M |g(x, y)| + N |f(x, y)|) \left( \int_a^b \int_c^d |x-t| |y-v| \, d\lambda \right) \, dydx \right).

By a simple computation, we easily obtain
\[ \int_a^b \int_c^d |x-t| |y-v| \, d\lambda \, dydx = \frac{1}{4} \left[ (x-a)^2 + (b-x)^2 \right] \left[ (y-c)^2 + (d-y)^2 \right]. \]

Substituting (16) in (15), we get the desired result. \( \square \)

**Theorem 3.** Let \( f, g : \Delta \rightarrow \mathbb{R} \) be partially differentiable functions, such that their second derivatives \( f_{\lambda w} \) and \( g_{\lambda w} \) are integrable on \( \Delta \). If \( |f_{\lambda w}| \) and \( |g_{\lambda w}| \) are co-ordinated \( \alpha \)-quasi-convex on \( \Delta \), for some \( \alpha \in (0, 1] \), then
\[ |T(f, g)| \leq \frac{49}{3600} MNk^2, \]
where \( T(f, g) \) is defined as in (5), \( M, N, \) and \( k \) are as in Theorem 1.

**Proof.** Clearly the inequalities (8)-(10) are valid, using the co-ordinated \( \alpha \)-quasi-convexity of \( |f_{\lambda w}| \) and \( |g_{\lambda w}| \), (10) gives
\[ |T(f, g)| \leq \frac{1}{2\pi} \left[ \int_a^b \int_c^d \int_a^b \int_c^d |x-t| |y-v| \int_0^1 |\lambda^a \max \{|f_{\lambda w}(x, y)| + |f_{\lambda w}(x, v)|\}| \right] \]
\multiplytext{+} \left[ \int_a^b \int_c^d \int_a^b \int_c^d |\lambda^a \max \{|g_{\lambda w}(x, y)| + |g_{\lambda w}(x, v)|\}| \right] \, dydx \]
\multiplytext{=} \frac{1}{2\pi} \left[ \int_a^b \int_c^d \int_a^b \int_c^d |\lambda^a \max \{|f_{\lambda w}(x, y)| + |f_{\lambda w}(x, v)|\}| \right] \, dydx.
Theorem 4. Under the assumptions of Theorem 3, we have

\[
|T(f,g)| \leq \frac{1}{18}\left[\int_a^b \int_c^d (M|g(x,y)| + N|f(x,y)|) \times \left((a-x)^2 + (b-x)^2\right) \left((c-y)^2 + (d-y)^2\right) dydx, \right.
\]

(19)

where \(T(f,g)\) is defined as in (5) and \(M, N, \) and \(k\) are as in Theorem 3.

Proof. By the same argument given in Theorem 2, we easily obtain the inequality (14), using the \(\alpha\)-quasi-convexity on the co-ordinates of \(|f_{1w}|\) and \(|g_{1w}|\), we get

\[
|T(f,g)| \leq \frac{1}{18}\left[\int_a^b \int_c^d |g(x,y)| \int_c^d |x-t| |y-v| \times \left(M \int_0^1 \lambda^\alpha dwd\lambda + M \int_0^1 (1 - \lambda^\alpha) dwd\lambda\right) dtdy \right.
\]

\[
+ \int_a^b \int_c^d |f(x,y)| \int_c^d |x-t| |y-v| \times \left(N \int_0^1 \lambda^\alpha dwd\lambda + N \int_0^1 (1 - \lambda^\alpha) dwd\lambda\right) dtdy.
\]

(20)

Substituting (16) in (20), we get the desired result. \(\square\)

Theorem 5. Let \(f, g : \Delta \to \mathbb{R}\) be partially differentiable functions such that their second derivatives \(f_{1w}\) and \(g_{1w}\) are integrable on \(\Delta\), and let \(s \in (-1, 1)\) fixed. If \(|f_{1w}|\) and \(|g_{1w}|\) are co-ordinated s-quasi-convex on \(\Delta\), then

\[
|T(f,g)| \leq \frac{49}{900(s+1)^2} MNk^2,
\]

(21)

where \(T(f,g)\) is defined as in (5) and \(M, N, \) and \(k\) are as in Theorem 1.

Proof. Clearly inequalities (8)-(10) are satisfied. Using second definition of the co-ordinated s-quasi-convex of \(|f_{1w}|\) and \(|g_{1w}|\), (10) gives;
\[ |T(f,g)| \leq \frac{1}{k^2} \int_{a}^{b} \int_{c}^{d} \left[ \int_{a}^{b} \left| x-t \right| |y-v| \int_{0}^{1} \left[ \lambda^s \max \left\{ |f_{2w}(x,y)| + |f_{2w}(x,v)| \right\} \right] d\lambda d\nu \right] dy dx \\
\quad + \left( 1 - \lambda \right)^s \max \left\{ |f_{2w}(t,y)| + |f_{2w}(t,v)| \right\} d\lambda d\nu \right] dy dx \\
\quad \times \left[ \int_{a}^{b} \int_{c}^{d} \left| x-t \right| |y-v| \int_{0}^{1} \left[ \lambda^s \max \left\{ |g_{2w}(x,y)| + |g_{2w}(x,v)| \right\} \right] d\lambda d\nu \right] dy dx \\
= \frac{1}{k^2} \int_{a}^{b} \int_{c}^{d} \left[ \int_{a}^{b} \int_{c}^{d} \left| x-t \right| |y-v| \left[ \max \left\{ |f_{2w}(x,y)| + |f_{2w}(x,v)| \right\} \int_{0}^{1} \lambda^s d\lambda \right] \\
\quad + \left( 1 - \lambda \right)^s \max \left\{ |f_{2w}(t,y)| + |f_{2w}(t,v)| \right\} \int_{0}^{1} \lambda^s d\lambda \right] d\lambda d\nu \right] dy dx \\
\quad \times \left[ \int_{a}^{b} \int_{c}^{d} \left| x-t \right| |y-v| \left[ \max \left\{ |g_{2w}(x,y)| + |g_{2w}(x,v)| \right\} \int_{0}^{1} \lambda^s d\lambda \right] \\
\quad + \left( 1 - \lambda \right)^s \max \left\{ |g_{2w}(t,y)| + |g_{2w}(t,v)| \right\} \int_{0}^{1} \lambda^s d\lambda \right] d\lambda d\nu \right] dy dx \\
\quad \leq \frac{M \left( \frac{1}{k^2} \right)^2}{k^2} \int_{a}^{b} \int_{c}^{d} \left[ \int_{a}^{b} \int_{c}^{d} \left| x-t \right| |y-v| \left( \frac{M}{k^2} + \frac{M}{k^2} \right) \right] \\
\quad \times \left[ \int_{a}^{b} \int_{c}^{d} \left| x-t \right| |y-v| \left( \frac{N}{k^2} + \frac{N}{k^2} \right) \right] dy dx \\
\quad = \frac{4MN}{(s+1)^2 k^2} \int_{a}^{b} \int_{c}^{d} \left( \int_{a}^{b} \int_{c}^{d} \left| x-t \right| |y-v| \right)^2 dy dx. \tag{22} \]

Substituting (12) in (22), we get the desired result. \( \square \)

**Theorem 6.** Under the assumptions of Theorem 5, we have

\[ |T(f,g)| \leq \frac{1}{4(s+1)^2} \int_{a}^{b} \int_{c}^{d} \left( M |g(x,y)| + N |f(x,y)| \right) \times \left( (x-a)^2 + (b-x)^2 \right) \left( (y-c)^2 + (d-y)^2 \right) dy dx, \tag{23} \]

where \( T(f,g) \) is defined as in (5) and \( M, N, \) and \( k \) are as in Theorem 1.

**Proof.** By the same argument given in Theorem 2, we easily obtain the inequality (14), using the second definition of \( s \)-quasi-convexity on the co-ordinates of \( |f_{2w}| \) and \( |g_{2w}| \), we get

\[ |T(f,g)| \leq \frac{1}{2s} \int_{a}^{b} \int_{c}^{d} |g(x,y)| \times \int_{a}^{b} \int_{c}^{d} |x-t| |y-v| \times \left( M \int_{0}^{1} \lambda^s d\lambda + N \int_{0}^{1} \lambda^s d\lambda \right) dy dx \\
\quad + \int_{a}^{b} \int_{c}^{d} |f(x,y)| \times \int_{a}^{b} \int_{c}^{d} |x-t| |y-v| \times \left( N \int_{0}^{1} \lambda^s d\lambda + N \int_{0}^{1} \lambda^s d\lambda \right) dy dx \\
\quad = \frac{1}{(s+1)^2} \int_{a}^{b} \int_{c}^{d} \left( M |g(x,y)| + N |f(x,y)| \right) \int_{a}^{b} \int_{c}^{d} |x-t| |y-v| dy dx. \tag{24} \]
Substituting (16) in (24), we get the desired result. □

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Conflicts of Interest: “The authors declare no conflict of interest.”

References


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