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Exploring effective iterative methods for nonlinear equations by using variational iteration technique

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Abstract: The purpose of this paper is to introduce and evaluate novel iterative methods for approximating solutions to nonlinear equations, which leverage the power of the variational iteration technique. Specifically, we present a comprehensive analysis of the proposed methods and demonstrate their effectiveness through various examples. Moreover, we provide a comparative analysis with other existing methods and conclude that the newly developed methods offer a competitive alternative. Our results highlight the potential of this approach in generating a diverse set of iterative methods for solving nonlinear equations. Therefore, this study contributes to the ongoing efforts to improve the efficiency and accuracy of nonlinear equation solving techniques.

Keywords: Nonlinear equations; Iterative method; Convergence; Newton's method; Taylor series; Examples

1. Introduction

Finding roots of nonlinear equations efficiently has widespread applications in numerical analysis. Due to such importance and significant applications in various branches of science, several methods are being developed for solving $f(x) = 0$, using different techniques such as Taylor series, quadrature formulas, homotopy perturbation method, Adomian decomposition and variational iteration technique [1–23]. Newton method for a single nonlinear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, 3 \cdots .$$

This is an important and basic method [21], which converges quadratically. To improve the local order of convergence, many modified methods have been proposed, see [2,3] and [9–14].

In this paper, we use a novel technique suggested by He [6] for the development of iterative schemes for linear and nonlinear problems. We implement He's variational iteration technique to suggest and analyze some new iterative methods for solving the nonlinear equations. We would like to mention that the variational iteration technique was developed by He [6] and has been used to solve a wide class of problems arising in various branches of pure and applied sciences. The variational iteration technique is very reliable and efficient technique. See also Noor and Mohyud-Din [15] and the references therein. Essentially using the idea and technique of He [6], Noor and Shah [16] has suggested and analyzed some iterative methods for solving the nonlinear equations. Now we have used this technique to gain higher order convergent iterative methods. An approximation technique to remove the higher derivative of the function is also introduced. We show that the new methods include only first derivative of the functions and these are free from higher order derivatives. Several examples are given to illustrate the efficiency and performance of these new methods and their comparison with other iterative methods. These new methods can be considered as alternative to the existing higher order methods.

2. Construction of iterative methods

In this section, we use a special relation for the implementation of He's variational iteration technique [6]. We develop the main recurrence relation which generates efficient iterative schemes for the approximate solution of nonlinear equation. Consider the nonlinear equation of the type

$$f(x) = 0. \tag{1}$$

We assume that p is a simple root and γ is an initial guess sufficiently close to p. We consider the approximate solution x_n of (1) such that $f(x_n) \neq 0$.

Let $g(x_n)$ be any arbitrary function and λ be a parameter which is usually called the Lagrange's multiplier and can be identified by the optimality condition. Consider the following iterative relation

$$x_{n+1} = \phi(x_n) + \lambda\,[f(\phi(x_n))g(\phi(x_n))], \tag{2}$$

where $\phi(x_n)$ is the arbitrary auxiliary function of order $p \geq 1$. Relation (2) is a generalized relation. We note that, if $\phi(x_n) = I$ and $p = 1$, then (2) reduces to the following iterative relation

$$x_{n+1} = x_n + \lambda\,[f(x_n)g(x_n)], \tag{3}$$

which was considered and analyzed by He [6]. See also Noor [11,13]. Thus we conclude that our scheme (2) includes the He's scheme as a special case. In this paper, our aim is to analyze the relation (2) for obtaining higher order methods and for this, we will study the arbitrary auxiliary function for $p = 2$, and the generated methods will be of fourth order. Using the optimality criteria, we can get the value of λ from (2) as

$$\lambda = -\frac{\phi'(x_n)}{[g'(\phi(x_n))f(\phi(x_n)) + g(\phi(x_n))f'(\phi(x_n))]}. \tag{4}$$

From (2) and (4), we have

$$x_{n+1} = \phi(x_n) - \frac{f(\phi(x_n))g(\phi(x_n))}{[g'(\phi(x_n))f(\phi(x_n)) + g(\phi(x_n))f'(\phi(x_n))]}. \tag{5}$$

Let us consider

$$\phi(x_n) = y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{6}$$

Using (5) in (6), we obtain the following iterative relation for solving the nonlinear equations as:

$$x_{n+1} = y_n - \frac{f(y_n)g(y_n)}{[f'(y_n)g(y_n) + f(y_n)g'(y_n)]}, \tag{7}$$

where $g(y_n)$ is the auxiliary function. We observe that, If p is the root of $f(x)$, then for $x = p$, we have $f(p) = 0$ and

$$\frac{g'(y)}{g(y)} \approx \frac{g'(p)}{g(p)}. \tag{8}$$

Also we have

$$\frac{g'(x)}{g(x)} = \frac{g'(p)}{g(p)}. \tag{9}$$

Combining (8) and (9), and replacing in (7), we obtain the following iterative scheme

$$x_{n+1} = y_n - \frac{f(y_n)g(x_n)}{[f'(y_n)g(x_n) + f(y_n)g'(x_n)]}. \tag{10}$$

From the above scheme, for different values of the auxiliary function $g(x_n)$, one can obtain several iterative methods of fourth order convergence for solving nonlinear equations. Here our aim is to improve the efficiency of the above iterative scheme by removing $f'(y_n)$ from the main scheme. Using the Taylor series technique, we have

$$f(y_n) \simeq f(x_n) + (y_n - x_n)f'(x_n) + \frac{(y_n - x_n)^2}{2} f''(x_n) = \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^2}. \quad (11)$$

Let us approximate

$$f''(x_n) \simeq \frac{[f'(y_n) - f'(x_n)]}{y_n - x_n}. \quad (12)$$

Using (12) in (11) and simplifying, we obtain

$$f'(y_n) \approx \frac{f'(x_n)}{f(x_n)} [f(x_n) - 2f(y_n)]. \quad (13)$$

Using (13) in (10), we get the relation

$$x_{n+1} = y_n - \frac{f(x_n)f(y_n)g(x_n)}{f'(x_n)[f(x_n) - 2f(y_n)]g(x_n) + f(x_n)f(y_n)g'(x_n)}. \quad (14)$$

This is the main iterative scheme for generating 4th order convergent methods. We will use some special values of $g(x_n)$ and get the iterative methods as:
I. Let $g(x_n) = e^{-\alpha x_n}$. Then from (14), we obtain the following iterative method for solving the nonlinear Eq. (1).

Algorithm 1. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{f(x_n)f(y_n)}{f'(x_n)[f(x_n) - 2f(y_n)] - \alpha f(y_n)f(x_n)}. n = 0,\ 1,\ 2,\ \cdots.$$

If $\alpha = 0$, then Algorithm 1 reduces to the well known Ostrowski method [17].
II. Let $g(x_n) = e^{-\alpha f(x_n)}$. Then from (14), we have the following iterative scheme for solving the nonlinear Eq. (1).

Algorithm 2. For a given x_0 , find the approximate solution x_{n+1} by the iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{f(x_n)f(y_n)}{f'(x_n)\left([f(x_n) - 2f(y_n)] - \alpha f(y_n)f(x_n)\right)},$$

If $\alpha = 0$, then Algorithm 2 reduces to the well known Ostrowski method [17].
III. Let $g(x_n) = e^{\frac{\alpha}{f'(x_n)}}$. Then $g'(x_n) = e^{\frac{\alpha}{f'(x_n)}} \left(-\frac{\alpha f''(x_n)}{[f'(x_n)]^2}\right)$.
Now from (14), we get after combining with (11), the following iterative method for solving the nonlinear Eq. (1).

Algorithm 3. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{f(y_n)[f(x_n)]^2}{f'(x_n)f(x_n)[f(x_n) - 2f(y_n)] - 2\alpha[f(y_n)]^2}.$$

If $\alpha = 0$, then Algorithm 3 reduces to the well known Ostrowski method [17].
IV. Let $g(x_n) = e^{-\frac{\alpha f(x_n)}{f'(x_n)}}$. Then from (15), we have the following iterative scheme after combining with (11) for solving the nonlinear Eq. (1).

Algorithm 4. For a given x_0 , find the approximate solution x_{n+1} by the iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{f(x_n)f(y_n)}{[f(x_n) - 2f(y_n)]\,[f'(x_n) - \alpha f(y_n)]}.$$

If $\alpha = 0$, then Algorithm 4 reduces to the well known Ostrowski method [17].

3. Convergence analysis

In this section, we consider the convergence criteria of the main iterative scheme (14) developed in §2.

Theorem 1. *Assume that the function $f : D \subset R \rightarrow R$ for an open interval D has a simple root $p \in D$. Let $f(x)$ be smooth sufficiently in some neighborhood of the root and then (14) has fourth-order convergence.*

Proof. Let p be a simple root of$f(x)$. Since f is sufficiently differential, then expanding $f(x)$ and $f'(x)$ in Taylor's series about p, we get

$$f(x_n) = f'(p)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)\right], \tag{15}$$

and

$$f'(x_n) = f'(p)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^7)\right]. \tag{16}$$

Where $c_k = \frac{1}{k!}\frac{f^{(k)}(p)}{f'(p)}, k = 2, 3, \cdots$. and $e_n = x_n - p$.

From (15) and (16), we get

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} =& e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4{c_2}^3 - 3c_4)e_n^4 + (8c_2^4 - 20c_3c_2^2 \\ &+ 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 + (13c_2c_5 - 28c_2^2c_4 - 5c_6 - 16c_2^5 + 52c_2^3c_3 \\ &+ 17c_3c_4 - 33c_2c_3^2)e_n^6 + O(e_n^7).\end{aligned} \tag{17}$$

Using (17), we have

$$\begin{aligned}y_n =& p + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 - (8c_2^4 - 20c_3c_2^2 + 6c_3^2 \\ &+ 10c_2c_4 \ - 4c_5)e_n^5 + (13c_2c_5 - 28c_2^2c_4 - 5c_6 - 16c_2^5 + 52c_2^3c_3 \ + 17c_3c_4 \\ &- 33c_2c_3^2)e_n^6 + O(e_n^7).\end{aligned} \tag{18}$$

From (18), we obtain

$$\begin{aligned}f(y_n) =& f'(p)[c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 5c_2^3 - 3c_4)e_n^4 - (12c_2^4 - 24c_3c_2^2 \\ &+ 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 + O(e_n^7)].\end{aligned} \tag{19}$$

Now from (15) and (19), we get

$$f(x_n) - 2f(y_n) = f'(p)\left[e_n - c_2e_n^2 + (4c_2^2 - 3c_3)e_n^3 + (14c_2c_3 - 10c_2^3 \ -5c_4)e_n^4 + O(e_n^5)\right], \tag{20}$$

and

$$f(x)f(y)g(x) = [f'(p)]^2\left[g(p)c_2e_n^3 + \left\{g'(p)c_2 - g(p)c_2^2 + 2g(p)c_3\right\}e_n^4 + O(e_n^5)\right]. \tag{21}$$

From (15), (20) and (21), we obtain

$$f'(x_n)[f(x_n)-2f(y_n)]g(x_n)+f(x_n)f(y_n)g'(x_n)=[f'(p)]^2\Big[g(p)e_n+(g(p)c_2+g'(p))e_n^2+\Big(\frac{1}{2}g''(p)$$
$$+2g(p)c_2^2\Big)e_n^3+\Big(-\frac{1}{2}g''(p)c_2-2g'(p)c_3+\frac{1}{6}g'''(p)\ +3g'(p)c_2^2-2g(p)c_2^3-g(p)c_4+5g(p)c_2c_3\Big)e_n^4+O(e_n^5)\Big]. \tag{22}$$

Now with the help of (21) and (22), we get

$$\frac{f(x)f(y)g(x)}{f'(x_n)[f(x_n)-2f(y_n)]g(x_n)+f(x_n)f(y_n)g'(x_n)}$$
$$=c_2e_n^2+(2c_3-2c_2^2)e_n^3+\left(\frac{g'(p)}{g(p)}c_2^2+3c_2^3+c_4-6c_2c_3\right)e_n^4+O(e_n^5). \tag{23}$$

Now from (18) and (23), we get

$$x_{n+1}=p+\left(\frac{g'(p)}{g(p)}c_2^2+3c_2^3+c_4-6c_2c_3\right)e_n^4+O(e_n^5). \tag{24}$$

Finally, the error equation is

$$e_{n+1}=\left(\frac{g'(p)}{g(p)}c_2^2+3c_2^3+c_4-6c_2c_3\right)e_n^4+O(e_n^5). \tag{25}$$

Thus we conclude that (14) has fourth-order convergence and subsequently all the methods derived from (14) has also fourth order convergence. □

4. Numerical results

We now present some examples to illustrate the efficiency of the new developed two-step iterative methods (see Tables 1–6). We compare the Newton method (NM), Turab's method (TM), Ostrowski method (OM) Algorithm 1 (NR1), Algorithm 2 (NR2), Algorithm 3 (NR3) and Algorithm 4 (NR4) which are introduced here in this paper. We also note that these methods do not require the computation of second derivative to carry out the iterations. All computations are done using the MAPLE using 60 digits floating point arithmetics (Digits: =60).

We will use $\varepsilon=10^{-32}$.The following stopping criteria are used for computer programs.

(i) $|x_{n+1}-x_n|<\varepsilon$. (ii) $|f(x_{n+1})|<\varepsilon$.

The computational order of convergence p approximated for all the examples in Tables 1–6, (see [20]) by means of

$$\rho\approx\frac{\ln(|x_{n+1}-x_n|/|x_n-x_{n-1}|)}{\ln(|x_n-x_{n-1}|/|x_{n-1}-x_{n-2}|)},$$

along with the total number of functional evaluations (TNFE) as required for the iterations. We consider the following nonlinear equations as test problems which are same as Noor and Noor [16].

$$\begin{aligned}
f_1(x)&=\sin^2x-x^2+1,\\
f_2(x)&=x^2-e^x-3x+2,\\
f_3(x)&=(x-1)^3-1,\\
f_4(x)&=x^3-10,\\
f_5(x)&=xe^{x^2}-\sin^2x+3\cos x+5,\\
f_6(x)&=e^{x^2+7x-30}-1.
\end{aligned}$$

Table 1. ($f_1,\ x_0 = 1$)

Methods	IT	TNFE	x_n	$\lvert f(x_n)\rvert$	δ	ρ
NM	7	14	1.404491648215341226	1.04e-50	7.33e-26	2.00003
TM	4	16	1.404491648215341226	0.00e-01	7.33e-26	4.29576
OM	4	12	1.404491648215341226	0.00e-01	5.64e-28	4.24367
Algorithm 1	4	12	1.404491648215341226	0.00e-01	2.14e-27	4.27401
Algorithm 2	4	12	1.404491648215341226	0.00e-01	4.20e-18	3.97200
Algorithm 3	4	12	1.404491648215341226	0.00e-01	3.48e-17	4.29898
Algorithm 4	4	12	1.404491648215341226	0.00e-01	2.00e-16	4.52494
NM	7	14	1.404491648215341226	1.04e-50	7.33e-26	2.00003
TM	4	16	1.404491648215341226	0.00e-01	7.33e-26	4.29576
OM	4	12	1.404491648215341226	0.00e-01	5.64e-28	4.24367
Algorithm 1	4	12	1.404491648215341226	0.00e-01	9.57e-23	4.25801
Algorithm 2	4	12	1.404491648215341226	0.00e-01	1.80e-43	3.87150
Algorithm 3	4	12	1.404491648215341226	0.00e-01	2.33e-22	4.34817
Algorithm 4	4	12	1.404491648215341226	0.00e-01	2.83e-24	4.09265

Table 2. ($f_2,\ x_0 = 2$)

Methods	IT	TNFE	x_n	$\lvert f(x_n)\rvert$	δ	ρ
NM	6	12	0.2575302854398607	2.93e-55	9.10e-28	2.00050
TM	4	16	0.2575302854398607	1.00e-59	7.74e-56	3.86670
OM	4	12	0.2575302854398607	0.00e-01	2.70e-23	4.15500
Algorithm 1	4	12	0.2575302854398607	2.00e-59	4.23e-24	4.29911
Algorithm 2	4	12	0.2575302854398607	1.00e-59	2.26e-16	4.10939
Algorithm 3	4	12	0.2575302854398607	0.00e-01	2.60e-25	4.49624
Algorithm 4	4	12	0.2575302854398607	0.00e-01	1.35e-40	3.92927
NM	6	12	0.2575302854398607	2.93e-55	9.10e-28	2.00050
TM	4	16	0.2575302854398607	1.00e-59	7.74e-56	3.86670
OM	4	12	0.2575302854398607	0.00e-01	2.70e-23	4.55500
Algorithm 1	4	12	0.2575302854398607	0.00e-01	3.37e-40	3.85293
Algorithm 2	4	12	0.2575302854398607	0.00e-01	5.62e-18	4.91925
Algorithm 3	4	12	0.2575302854398607	1.00e-59	2.83e-24	4.52741
Algorithm 4	4	12	0.2575302854398607	1.00e-59	5.76e-27	3.96131

Table 3. ($f_3,\ x_0 = 3.5$)

Methods	IT	TNFE	x_n	$\lvert f(x_n)\rvert$	δ	ρ
NM	8	16	2.0000000000000000	2.06e-42	8.28e-22	2.00025
TM	5	20	2.0000000000000000	0.00e-01	6.86e-43	3.86708
OM	5	15	2.0000000000000000	0.00e-01	2.21e-49	3.90897
Algorithm 1	5	15	2.0000000000000000	0.00e-01	1.00e-40	4.28367
Algorithm 2	6	18	2.0000000000000000	0.00e-01	1.40e-41	3.95934
Algorithm 3	4	12	2.0000000000000000	0.00e-01	4.30e-17	3.96235
Algorithm 4	4	12	2.0000000000000000	0.00e-01	1.43e-21	3.84029
NM	8	16	2.0000000000000000	2.06e-42	8.28e-22	2.00025
TM	5	20	2.0000000000000000	0.00e-01	6.86e-43	3.86708
OM	5	15	2.0000000000000000	0.00e-01	2.21e-49	3.90897
Algorithm 1	4	12	2.0000000000000000	0.00e-01	5.48e-24	3.85377
Algorithm 2	6	18	2.0000000000000000	0.00e-01	8.02e-57	3.97842
Algorithm 3	5	15	2.0000000000000000	0.00e-01	2.43e-53	3.98698
Algorithm 4	4	12	2.0000000000000000	0.00e-01	9.19e-17	3.83639

Table 4. (f_4, $x_0 = 1.5$)

Methods	IT	TNFE	x_n	$\|f(x_n)\|$	δ	ρ
NM	7	14	2.15443469003188	2.06e-54	5.64e-28	2.00003
TM	4	16	2.15443469003188	1.00e-58	5.64e-28	4.21798
OM	4	12	2.15443469003188	8.00e-59	3.73e-32	4.18546
Algorithm 1	4	12	2.15443469003188	8.00e-59	2.72e-20	5.14348
Algorithm 2	5	15	2.15443469003188	1.00e-58	2.82e-34	3.92450
Algorithm 3	4	12	2.15443469003188	1.00e-58	1.04e-50	3.92059
Algorithm 4	4	12	2.15443469003188	1.00e-58	8.17e-17	4.33019
NM	7	14	2.15443469003188	2.06e-54	5.64e-28	2.00003
TM	4	16	2.15443469003188	1.00e-58	5.64e-28	4.21798
OM	4	12	2.15443469003188	8.00e-59	3.73e-32	4.18546
Algorithm 1	4	12	2.15443469003188	8.00e-59	2.04e-32	4.46637
Algorithm 2	5	15	2.15443469003188	8.00e-59	8.44e-46	3.97295
Algorithm 3	4	12	2.15443469003188	8.00e-59	1.79e-36	4.04813
Algorithm 4	4	12	2.15443469003188	8.00e-59	3.63e-20	4.41414

Table 5. (f_5, $x_0 = -2$)

Methods	IT	TNFE	x_n	$\|f(x_n)\|$	δ	ρ
NM	9	18	-1.207647827130918	2.27e-40	2.73e-21	2.000851
TM	5	20	-1.207647827130918	1.10e-58	2.73e-21	4.004843
OM	5	15	-1.207647827130918	8.00e-59	3.71e-43	4.136952
Algorithm 1	5	15	-1.207647827130918	8.00e-59	3.06e-24	4.012770
Algorithm 2	6	18	-1.207647827130918	1.10e-58	2.00e-22	4.02808 0
Algorithm 3	5	15	-1.207647827130918	8.00e-59	1.08e-48	4.520711
Algorithm 4	5	15	-1.207647827130918	8.00e-59	3.29e-48	3.975131
NM	9	18	-1.207647827130918	2.27e-40	2.73e-21	2.00085
TM	5	20	-1.207647827130918	1.10e-58	2.73e-21	4.00484
OM	5	15	-1.207647827130918	8.00e-59	3.71e-43	4.13695
Algorithm 1	5	15	-1.207647827130918	8.00e-59	1.36e-30	4.03378
Algorithm 2	7	21	-1.207647827130918	8.00e-59	4.13e-30	4.05656
Algorithm 3	5	15	-1.207647827130918	8.00e-59	5.78e-45	4.24036
Algorithm 4	5	15	-1.207647827130918	8.00e-59	4.57e-51	3.98835

Table 6. (f_6, $x_0 = 3.5$)

Methods	IT	TNFE	x_n	$\|f(x_n)\|$	δ	ρ
NM	13	26	3.00000000000000	1.52e-47	4.21e-25	2.00023
TM	7	28	3.00000000000000	0.00e-01	4.21e-25	3.83827
OM	6	18	3.00000000000000	0.00e-01	6.93e-17	3.97578
Algorithm 1	6	18	3.00000000000000	0.00e-01	5.25e-23	4.03116
Algorithm 2	6	18	3.00000000000000	0.00e-01	5.14e-19	4.04841
Algorithm 3	6	18	3.00000000000000	0.00e-01	7.08e-18	4.15183
Algorithm 4	6	18	3.00000000000000	0.00e-01	5.14e-19	4.12008
NM	13	26	3.00000000000000	1.52e-47	4.21e-25	2.00023
TM	7	28	3.00000000000000	0.00e-01	4.21e-25	3.83827
OM	6	18	3.00000000000000	0.00e-01	6.93e-17	3.97578
Algorithm 1	6	18	3.00000000000000	0.00e-01	1.54e-19	3.98918
Algorithm 2	10	18	3.00000000000000	2.00e-58	1.13e-30	4.04841
Algorithm 3	6	18	3.00000000000000	2.00e-58	2.47e-17	4.05576
Algorithm 4	6	18	3.00000000000000	2.00e-58	4.96e-16	3.92802

5. Conclusion

The focus of this study is to present new fourth-order convergent methods for solving nonlinear equations. Notably, all of these methods are designed to be free from the need for second derivatives. To evaluate their effectiveness, we compare these new methods with the standard Newton method, and our findings indicate that the proposed methods generally exhibit superior performance. Furthermore, when assessing the efficiency of these methods, we apply the definition of the efficiency index, as outlined in [4]. This allows us to provide a more comprehensive analysis of the strengths and limitations of these newly proposed methods for solving nonlinear equations. Overall, our results suggest that these new methods have the potential to provide valuable contributions to the field of numerical mathematics and to enable more accurate and efficient solutions with the efficiency index [5] as $p^{\frac{1}{m}}$,where p is the order of the method and m is the number of functional evaluations per iteration required by the method, we have that all of the methods obtained have the efficiency index equal to $4^{\frac{1}{3}} \approx 1.5874$, which is better than the one of Newton's method $2^{\frac{1}{2}} \approx 1.4142$.

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