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Mathematical algorithms for perpetual Ethiopian calendar(e.c.)and similar calendars

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Abstract: This study focused on developing mathematical algorithms for the perpetual Ethiopian calendar and similar calendars. The primary objective was to demonstrate the methodology for creating these algorithms. The research identified that arithmetic progression, ceiling function, congruence modulo, floor function, and Bahre Hasabe are fundamental concepts necessary for this development. Utilizing these concepts, the study successfully developed mathematical algorithms for the perpetual Ethiopian calendar and analogous calendars.

Keywords: Arithmetic progression; Bahre Hasabe; Ceiling function; Congruence modulo; Ethiopian calendar; Fasting dates; Floor function; Holidays

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1. Introduction

A calendar is a system for arranging days, months, and years used for administrative, commercial, religious, or time measurement purposes. It enables users to schedule or arrange tasks and maintain diaries at particular times. This is accomplished by assigning names to time intervals, usually days, weeks, months, and years. A date is the designation given to each day. Periods in a calendar, such as years and months, are typically aligned with the cycles of the Sun and Moon. Numerous societies and civilizations have created calendars that suit their specific requirements, often by modifying existing calendars [1].

Another common form of a calendar is a tangible object, usually made of paper. This is the most typical use worldwide. Digital calendars, integrated into application systems and compatible with computer operating systems and mobile devices, remind users of upcoming events and appointments [1].

Calendars can be constructed in various ways. Some are based on abstract, perpetually repeating cycles without astronomical significance, while others replicate astronomical cycles according to predetermined rules. However, the most widely used calendars are based on the solar and lunar systems [2].

Every solar day has a date on calendars. A day can be defined as the time between sunrise and sunset, followed by the night, or as the interval between two sunsets. This duration can either fluctuate slightly throughout the year or be averaged into a mean solar day. Solar calendars have 365 or 366 days in a year (during a leap year; a year with an extra day) [2].

Not all calendars use the solar year as a unit. A lunar calendar is based on the phases of the moon. One of the most widely used lunar calendars is the Islamic calendar, which consists of 12 months, each lasting 29 or 30 days, totaling 354 days. The Islamic calendar is not synchronized with the solar calendar, so Islamic holidays occur earlier each year according to the solar calendar and do not coincide with specific seasons [3].

The ancient Egyptian calendar was lunar. Three millennia before Christ, the solar Coptic calendar originated, making it the oldest known calendar. Its exact origin is unknown. The ancient Egyptians first used a civil calendar based on a solar year with 365 or 366 days in a leap year. The new year of the ancient Egyptians started on September 1 in the Ethiopian calendar (E.C.)/September 11 or 12 in the Gregorian calendar (G.C.), which is also the new year of Ethiopia [3].

The Ethiopian calendar is based on the Coptic calendar, with differences in the names of months and weekdays and the saints' days. The Coptic, or Egyptian, calendar lags the G.C. by seven or eight years due to the difference in the dates of the world's creation between the Roman Catholic Church and the Orthodox

Church. Both the Egyptian and Ethiopian calendars have thirteen months. The first 12 months have 30 days each, and Pagumie, the final month, is an intercalary/leap month with five days in regular years and six or seven days in leap years. Coptic Leap Years follow the same principles as Gregorian Leap Years, with the extra month in a leap year always having six or seven days [4].

The Julian calendar, with 12 months of 30 and 31 days except for February, which had 28 days and 29 in a leap year, was first used in 46 B.C. Julian reckoned that there were 365 days in a solar year. Adding a leap day every four years was intended to keep the calendar and the seasons in sync. However, a small error in the solar year's measurement caused the calendar dates of the seasons to move backwards by nearly one day every century. As a result, the Julian calendar was 10 days out of step with the seasons. Pope Gregory XIII commanded in 1582 that October 4 be followed by October 15, correcting the calendar's inaccuracy and restoring the consistency of the solar and calendar years. This reformation led to the Gregorian calendar, which is still in use today. The Gregorian reform restored January 1 as the start of the year and produced an incredibly accurate calendar system [4].

The Gregorian calendar also differs from the Julian in that a century year is not a leap year unless it is exactly divisible by 400. Except for years exactly divisible by 100, every year that is exactly divisible by four is a leap year; centurial years exactly divisible by 400 are still considered leap years [5].

An arithmetical calendar is the Gregorian solar calendar. Days are used as the fundamental unit of time, with years consisting of 365 or 366 days. Twelve months of varying lengths divide the years. Leap years extend February, which typically has 28 days, by adding an extra day for a total of 29 days. Four months have 30 days: April, June, September, and November, while seven months have 31 days: January, March, May, July, August, October, and December [5].

The Ethiopian Orthodox Church holds that the world was created by God 5,500 years before Jesus was born, a period known as Amete Fida. According to Enoch 28:11, the Ethiopian Enochian calendar has 364 days. The books of Enoch reveal a 364-day calendar year that reserves the last day of the solar year [6].

The Ethiopian calendar is seven years behind the Gregorian from September 11 or 12 (during the Gregorian leap year) to December 31 in the G.C., and eight years behind from January 1 to the end of the year in the G.C., because it begins counting after Amete Fida (5500 years). An Ethiopian leap year occurs when the remainder of [5500 + a year] divided by 4 equals 3. According to the Ethiopian calendar, leap years are those that conclude in the year before a leap year in the Gregorian calendar. Pagumie will have six or seven days in the Ethiopian leap year [6].

The Ethiopian calendar is derived from the calendar of the Ethiopian Orthodox Church, which has long used and continues to use it. The church uses its calendar system to determine fasting dates and holidays. Dimetros' book, Abushaher, serves as the basis for calculating fasting dates [2].

Many countries where the majority of the population is Muslim use the Islamic (Hijrah) calendar. This calendar is entirely lunar, with 12 months of 29 or 30 days each, determined by the moon's orbit. Most Muslim nations use the calendar to determine their holidays. The dates of Muslim holidays in the Islamic calendar differ from those in the Ethiopian calendar because the latter is derived from the Ethiopian Orthodox Church calendar [6].

As mentioned earlier, Ethiopia has its own calendar, developed by the Ethiopian Orthodox Church. This calendar is one of the church's contributions to Ethiopian society. Detailed descriptions of the Ethiopian calendar are not widely known among all Ethiopians due to the lack of references, and the calendar does not provide a simple and precise method for those unfamiliar with Orthodox religion. The calendar has not established its relationship with the Gregorian calendar. Despite the good intentions behind the Ethiopian calendar's creation, it has its limitations. To address these limitations, the researcher conducted this study titled

> Mathematical Algorithms for Perpetual Ethiopian Calendar and Similar Calendars. The main objective of this study is to show techniques to develop mathematical algorithms for the perpetual Ethiopian calendar and similar calendars. This study is significant in determining the *Wengilawian* name of any year, identifying the day of September 1 (*Inkutatash*), finding the day of any given date, converting a date from E.C. to G.C., calculating the movable fasting dates of the Ethiopian Orthodox Tewahedo Church, and calculating the dates of movable holidays of the

Ethiopian Orthodox Tewahedo Church. Additionally, this research demonstrates the application of mathematics to Ethiopian indigenous knowledge.

2. Materials and method

This study used Priests' interviews and document analysis data gathering instruments to gather available data that were important to conduct the study. The priests' interviews were used to gather the spiritual preliminary concepts of this study. The following concepts were taken from different materials as the preliminary concepts of the study using document analysis.
Mathematical concepts (Arithmetic progression, Ceiling function, Congruence modulo and Floor function) and spiritual concept(*Bahre hasabe*) are basic concepts to develop Mathematical algorithms for Ethiopian calendar.

2.1. Mathematical concepts

The concept of congruence modulo was first explored by Carl Friedrich Gauss. He originated the notation $a \equiv b(mod\ n)$ in his work *Disquisitiones Arithmeticae*, published in 1801 [7].

Definition 1 (Congruence modulo)**.** : Let a and b be integers and m be a natural number. Then a is congruent to b in modulo m [denoted by $a \equiv b$ (mod m)] if m$|$(a − b) or m divides a − b [8].

Since m$|$(a − b), then a = b + km for some integer k. This shows b is a remainder when a is divided by m.

Example 1. a $40 \equiv 0(mod\ 5)$ since $5 \mid (40-0)$. b $40 \equiv -2(mod\ 3)$ since $3 \mid (40-(-2))$.

40 is not congruent to 1 in mod 8 since 8 does not divide (40 − 1).

Definition 2 (Equivalence relation)**.** : An equivalence relation R on a non - void set A is a relation that satisfies the following conditions.

a Reflexive property: For any a ∈ A, aRa.

b Symmetric property: For any a,b ∈ A, aRb $\Longrightarrow$ bRa.

c Transitive property: For any a,b,c ∈ A, if aRb and bRc, then aRc [9].

Property 3. : Congruence modulo is an equivalence relation. That means; for any a,b,c ∈ $\mathbb{Z}$ and any m ∈ $\mathbb{N}$,

a a ≡ a(mod m)

b a ≡ b(mod m) $\Longrightarrow$ b ≡ a(mod m)

c a ≡ b(mod m) and b ≡ c(mod m) $\Longrightarrow$ a ≡ c(mod m) [10].

Property 4. : Let a,b,c ∈ $\mathbb{Z}$ and m ∈ $\mathbb{N}$. If a ≡ b(mod m) and c ≡ d(mod m), then

a (a + c) ≡ (b + d)(mod m) **Addition property of congruence modulo**
b ac ≡ bd(mod m) **Multiplication property of congruence modulo**
c ka ≡ kb(mod km); ∀ k ∈ $\mathbb{N}$. **Scalar multiplication property of congruence modulo** [10].

Example 2. The remainder of 1998 ÷ 5 is $1998 \equiv x(mod\ 5)$
By property 2.1.1 (b), x ≡ 1998(mod 5) ≡ [19 ×100 + 98](mod 5)
$\Longrightarrow$ x ≡ [4 × 0 + 3](mod 5) since 19, 100 and 98 in mod 5 are equivalent to 4, 0 and 3 respectively.
$\Longrightarrow$ x ≡ 3(mod 5).
Hence, the remainder of 1998 ÷ 5 is 3.

Definition 5 (Floor function)**.** A function f is said to be the floor function if it is in the form of $f : \mathbb{R} \longrightarrow \mathbb{Z}$ such that

$$f(x) = \lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\} \text{ for any real number } x.$$

[11]

Remark 1. $\lfloor x \rfloor$ is read as the floor of x.

Example 3.

a $f(-3) = \lfloor -3 \rfloor = \max\{..., -5, -4, -3\} = -3$
b $f(3.2) = \lfloor 3.2 \rfloor = \max\{..., -2, -1, 0, 1, 2, 3\} = 3$

Property 6. If $\lfloor x \rfloor = m$, then $m \leq x < m + 1$ [12].

Example 4. If $\lfloor x \rfloor = 6$, then $6 \leq x < 7$.

Property 7. Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, then $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ [13].

Example 5. $\lfloor \frac{2003}{4} \rfloor = \lfloor 500 + 0.75 \rfloor$
Using properties 2.1.4, $\lfloor \frac{2003}{4} \rfloor = 500 + \lfloor 0.75 \rfloor = 500 + 0 = 500$

Remark 2. Let y and n be natural numbers, then

a The quotient of $y \div n$ is equal to $\lfloor \frac{y}{n} \rfloor$.
b The quotient of $-y \div n$ is equal to $\lfloor \frac{-y}{n} \rfloor + 1$.

Definition 8 (Ceiling function)**.** [11] A function f is said to be the ceiling function if it is in the form of $f : \mathbb{R} \longrightarrow \mathbb{Z}$ such that

$$f(x) = \lceil x \rceil = \min\{m \in \mathbb{Z} : m \geq x\} \text{ for any real number } x$$

Remark 3. $\lceil x \rceil$ is read as the ceiling of x.

Example 6.

a $f(-3) = \lceil -3 \rceil = \min\{-3, -2, -1, 0, 1, ...\} = -3$
b $f(3.2) = \lceil 3.2 \rceil = \min\{4, 5, 6, ...\} = 4$

Property 9. If $\lceil x \rceil = m$, then $m - 1 < x \leq m$ [11].

Example 7. If $\lceil x \rceil = 4$, then $3 < x \leq 4$.

Property 10. Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, then $\lceil x + n \rceil = \lceil x \rceil + n$ [11].

Example 8. $\lceil \frac{2003}{4} \rceil = \lceil 500 + 0.75 \rceil$
Using properties 2.1.6, $\lceil \frac{2003}{4} \rceil = 500 + \lceil 0.75 \rceil = 500 + 1 = 501$

Definition 11. A sequence $\{a_n\}$ of real numbers is a function whose domain is a set of natural numbers $(\mathbb{N})$ and whose range is a subset of the set of real numbers $(\mathbb{R})$ [14] [12].

Remark 4. The following concepts are very important for this study.

- a_n is the n^{th} term or the general term of a sequence $\{a_n\}$.
- $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$.

Definition 12. A sequence $\{a_n\}$ of real numbers is said to be Arithmetic Progression if the common difference d of any two consecutive terms of the given sequence must be equal i.e. $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots = d$ [14].

Theorem 13. *If a sequence $\{a_n\}$ of real numbers is an Arithmetic Progression, then the n^{th} term of the given sequence is calculated using*
$a_n = a_1 + (n-1)d$
where d is the common difference of any two consecutive terms of the given sequence [13].

Example 9. The sequence $1, 5, 9, 13, \ldots$ is an Arithmetic progression since $5 - 1 = 9 - 5 = 13 - 9 = \ldots = 4$ (there is a common difference of any two consecutive terms of the given sequence). The 20^{th} term of the given sequence is 77 since $a_{20} = 1 + (20-1)x4 = 1 + 19x4 = 1 + 76 = 77$.

2.2. Spiritual concepts

According to the indoctrination of Ethiopian Orthodox Tewahedo Church,

1. 4 is the length of the four year cycle in which the *Wengilawian* name of a year is repeated.
2. 7 is referred to as *Awde Ilet*. It is the length of week cycle in which a day of a week is repeated.
3. 19 is referred to as *Awde Abktie*. It is the length of a cycle in which the phase of a moon is repeated (Metonic cycle). It is also known as *Nius Kemer* in Ethiopian Orthodox Tewahedo Church.
4. 30 is referred to as *Awde Werha*. It is the length of a month cycle in which a number given to a day of a month is repeated except the 13^{th} month called *pagumie* according to the Church solar year.
5. There are 5500 years before the birth of Christ. These years are named by *Amete Feda (Amete Kunanie)*.
6. The year after the birth of Christ is named by *Amete Mihiret*.
7. The sum of *Amete Feda* and *Amete Mihiret* is named by *Amete Alem*.
8. If the remainder of [*Amete Alem* $\div$ 4] is
   a 0, then the *Wengilawian* name of a year is *Zemene Yohannes*.
   b 1, then the *Wengilawian* name of a year is *Zemene Mathewos*.
   c 2, then the *Wengilawian* name of a year is *Zemene Markos*.
   d 3, then the *Wengilawian* name of a year is *Zemene Lukas*.
9. *Metene Rabiet* is the quotient which is obtained from [*Amete Alem* $\div$ 4].
10. If the remainder of [(*Amete Alem* + *Metene Rabiet*)$\div$7] is
    a 1, then the day of *Inqutatash* will be on Tuesday.
    b 2, then the day of *Inqutatash* will be on Wednesday.
    c 3, then the day of *Inqutatash* will be on Thursday.
    d 4, then the day of *Inqutatash* will be on Friday.
    e 5, then the day of *Inqutatash* will be on Saturday.
    f 6, then the day of *Inqutatash* will be on Sunday.
    g 0, then the day of *Inqutatash* will be on Monday.

    $\Longrightarrow$ *Inqutatash* refers September 1 in E.C.
11. *Wenber* is the difference between the remainder of [*Amete Alem* $\div$19] and 1.
12. *Metik* is the remainder of [19 x *Wenber*] $\div$ 30.
    - If the *Metik* is less than 14, then the holiday of *Metik* will be in October.
    - If the *Metik* is greater than 14, then the holiday of *Metik* will be in September.
13. The daily *Tewusak* of Saturday, Sunday, Monday, Tuesday, Wednesday, Thursday and Friday are 8, 7, 6, 5, 4, 3, and 2 respectively.
14. *Mebaja Hamer* is the remainder of [(*Metik* + daily *Tewusak* of the holiday of *Metik*)$\div$30] . This number is taken for the starting date of *Nenewe,* one of fasting period for Ethiopian Orthodox Tewahedo Church .
    - If the holiday of the *Metik* falls in September, then the *Nenewe* will be in January.
    - If the holiday of the *Metik* falls in October, then the *Nenewe* will be in February.
    - If the *Metik* + daily *Tewusak* of the holiday of *Metik* is greater than 30, then the *Nenewe* will be in February.
15. Since the starting date of *Nenewe* (3 days length) and main fasting (55 days length) vary from year to year, then such types of fasting periods are named by Movable Fasting periods.

16. Since the celebration date of *Debre Zeit*, Easter, *Rikbe Kahinat*, *Irget and Bahle Amisa* vary from year to year, then such types of holidays are named by Movable Holidays.
17. *Nenewe* and main fasting are always started on Monday.
18. *Debre Zeit*, Easter and *Bahle Amisa* are always celebrated on Sunday.
19. *Irget* is always celebrated on Thursday.
20. *Rikbe Kahinat* is always on Wednesday.
21. There are

    a 14 dates after the starting date of *Nenewe* to starting date of main fasting.
    b 27 dates after the starting date of main fasting to the date of *Debre Zeit*.
    c 28 dates after the date of *Debre Zeit* to the date of Easter.
    d 24 dates after the date of Easte to the date of *Rikbe Kahinat*.
    e 39 dates after the date of Easter to the date of *Irget*.
    f 49 dates after the date of Easter to the date of *Bahle Amisa* .
22. There are 13 months in a year. The months with their numeral codes are September(1), October(2), November(3), December(4), January(5), February(6), March(7), April(8), May(9), June(10), July(11), August(12) and *Pagumie*(13).
23. 11 is known as Tinte Abektie. This number is obtained from the difference between one year of a Sun (365) and one year of a Moon (354).
24. The Abektie of a year is the remainder of (Wenber x 11) $\div$ 30. If the abektie of a year is zero, then the year is said to be a year with Albo Abektie. (From 1 to 24 are taken from [15]).

3. Results and discussion of the study

By connecting the Mathematical concepts (Arithmetic progression, Ceiling function, Congruence modulo and Floor function) to spiritual concepts (*Bahre Hasabe*), this study developed Mathematical algorithms which are applicable for

- Determining the *Wengilawian* name of any year.
- Determining the day of September 1(*Inkutatash*).
- Determining the day of any given date.
- Converting a date in E.C. to G.C.
- Calculating movable fasting dates of Ethiopian Orthodox Tewahedo Church.
- Calculating dates of movable holidays of Ethiopian Orthodox Tewahedo Church.

3.1. Determining the *Wengilawian* name of any year

Using the concept of congruence modulo and 8 in subsection 2.2, the *Wengilawian* name of any year Y is determined by using
$W \equiv [\textit{Amete Alem}](\text{mod } 4)$
$\Longrightarrow W \equiv [5500 + Y](\text{mod } 4) \equiv [0 + Y](\text{mod } 4)$ since 5500 in mod 4 is 0.
$\Longrightarrow W \equiv Y(\text{mod } 4)$
Therefore, the *Wengilawian* name of any year Y is determined by using $W \equiv Y(\text{mod } 4)$.

Remark 5. : If W is equivalent to

a 0 (mod 4), then the *Wengilawian* name of a year is *Zemene Yohannes*.
b 1 (mod 4), then the *Wengilawian* name of a year is *Zemene Mathewos*.
c 2(mod 4), then the *Wengilawian* name of a year is *Zemene Markos*.
d 3(mod 4), then the *Wengilawian* name of a year is *Zemene Lukas*.

3.2. Determining the day of September 1 (*Inkutatash*) and the day of any given date

Using the concept of congruence modulo, Floor function, and 9 & 10 in subsection 2.2, the day of September 1(*Inqutatash*) in any year Y is determined by using
$D \equiv \{\textit{Amete Alem} + \textit{Metene Rabiet}\}(\text{mod } 7)$
$\Longrightarrow D \equiv \{5500 + Y + \lfloor \frac{5500+Y}{4} \rfloor\}(\text{mod } 7)$ where D denotes numerical code for a day of *Inqutatash* and $\lfloor \frac{5500+Y}{4} \rfloor$ denotes the quotient of $(5500 + Y) \div 4$.
$\Longrightarrow D \equiv \{5500 + Y + \lfloor \frac{5500}{4} + \frac{Y}{4} \rfloor\}(\text{mod } 7)$.
$\Longrightarrow D \equiv \{5 + Y + \lfloor 1375 + \frac{Y}{4} \rfloor\}(\text{mod } 7)$ since 5500 in mod 7 is 5.
$\Longrightarrow D \equiv \{5 + Y + 1375 + \lfloor \frac{Y}{4} \rfloor\}(\text{mod } 7)$ by property 2.1.4.

$\Longrightarrow$D $\equiv$ {5 + Y + 3 + $\lfloor \frac{Y}{4} \rfloor$}(mod 7) since 1375 in mod 7 is 3.
$\Longrightarrow$D $\equiv$ {8 + Y + $\lfloor \frac{Y}{4} \rfloor$}(mod 7).
$\Longrightarrow$D $\equiv$ {1 + Y + $\lfloor \frac{Y}{4} \rfloor$}(mod 7) since 8 in mod 7 is 1.
Therefore, the day of September 1(*Inqutatash*) in any year Y is determined by using D $\equiv$ {1 + Y + $\lfloor \frac{Y}{4} \rfloor$}(mod 7).

Remark 6. : If D is equivalent to

a 1(mod 7), then the day of *Inqutatash* will be on Tuesday.
b 2(mod 7), then the day of *Inqutatash* will be on Wednesday.
c 3(mod 7), then the day of *Inqutatash* will be on Thursday.
d 4(mod 7), then the day of *Inqutatash* will be on Friday.
e 5(mod 7), then the day of *Inqutatash* will be on Saturday.
f 6(mod 7), then the day of *Inqutatash* will be on Sunday.
g 0(mod 7), then the day of *Inqutatash* will be on Monday.

Consider a certain date d in the m^{th} month of a certain year. Then the numerical code of the day of date d is

- $[D + (d-1)](mod\, 7) \equiv [D + (d-1) + 2 \times 0](mod\, 7)$ in September (1^{st} month).
- $[D + (d-1) + 2](mod\, 7) \equiv [D + (d-1) + 2 \times 1](mod\, 7)$ in October (2^{nd} month).
- $[D + (d-1) + 4](mod\, 7) \equiv [D + (d-1) + 2 \times 2](mod\, 7)$ in November (3^{rd} month).
- $[D + (d-1) + 2 \times (m-1)](mod\, 7)$ in m^{th} month where D denotes numerical code for a day of *Inqutatash* of this year.

Hence, the day of any given date d in any month m of a year Y is determined by using

$$D' \equiv [D + (d-1) + 2 \times (m-1)](mod\, 7)$$

where D denotes numerical code for a day of *Inqutatash* of a year Y.

Remark 7.

- If D' is equivalent to

  a 1(mod 7), then the day of the given date will be on Tuesday.
  b 2(mod 7), then the day of the given date will be on Wednesday.
  c 3(mod 7), then the day of the given date will be on Thursday.
  d 4(mod 7), then the day of the given date will be on Friday.
  e 5(mod 7), then the day of the given date will be on Saturday.
  f 6(mod 7), then the day of the given date will be on Sunday.
  g 0(mod 7), then the day of the given date will be on Monday.
- The value of m for September, October, November, December, January, February, March, April, May, June, July, August and *Pagumie* are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 and 13 respectively.

3.3. Calculating movable fasting dates and dates of movable holidays of Ethiopian Orthodox Tewahedo Church

Using the concept of congruence modulo and 11 in subsection 2.2, the *Wenber* of any year Y is calculated by using
$W' \equiv [Amete\ Alem](mod\, 19) - 1(mod\, 19)$ since 1 in mod 19 is 1
$\Longrightarrow W' \equiv (5500 - 1 + Y)(mod\, 19)$
$\Longrightarrow W' \equiv (9 - 1 + Y)(mod\, 19) \equiv (8 + Y)(mod\, 19)$ since 5500 in mod 19 is 9.
Hence, the *Wenber* W' of any year Y is calculated using $W' \equiv (8 + Y)(mod\, 19)$.
Using the concept of congruence modulo and 12 in subsection 2.2, the *Metik* of any year Y is calculated using
$M \equiv [19 \times Wenber](mod\, 30)$
$\Longrightarrow M \equiv [W' \times 19](mod\, 30)$
Hence, the *Metik* of any year Y is calculated using $M \equiv [W' \times 19](mod\, 30)$.

Remark 8. : If M is

a less than 14(mod 30), then the holiday of *Metik* will be in October.
b greater than 14(mod 30), then the holiday of *Metik* will be in September.

Using the concept of congruence modulo and 14 in subsection 2.2, the *Mebaja Hamer* of any year Y is calculated by using $MH \equiv [M + T](mod\ 30)$ where T denotes the daily *Tewusak* of the holiday of *Metik*.

Remark 9. :

a The starting date of Nenewe is taken from MH.
b If
- the holiday of the Metik falls in September, then the Nenewe will be in January.
- the holiday of the Metik falls in October, then the Nenewe will be in February.
- the M +T is greater than 30, then the Nenewe will be in February.

c The daily Tewusak of Saturday, Sunday, Monday, Tuesday, Wednesday, Thursday and Friday are 8, 7, 6, 5, 4, 3, and 2 respectively.

Using the concept of congruence modulo and 24 in subsection 2.2, the Abektie of any year Y is calculated using
$A \equiv [11 \times Wenber](mod 30)$
$\Rightarrow A \equiv [W' \times 11](mod 30)$
Hence, the Abektie of any year Y is calculated using
$A \equiv [W' \times 11](mod 30)$.

Remark 10. The following concepts are very important for this study

- If $W' \times 11$ is equal to 0, then $A = 0$. This year is said to be a year with Albo Abektie .
- If the year Y is a year with Albo Abektie, then
  - The next year with Albo Abektie will be $Y + 19$.
  - The preceding year with Albo Abektie will be $Y - 19$.
- $A + M = 30$.
- If the Abektie of the present year is A, then
  - the Abektie of the preceding year is $x \equiv [A - 11](mod 30)$.
  - the Abektie of the next year is $y \equiv [A + 11](mod 30)$.

Bear in mind that

- If y is equivalent to 29(mod30), then we take 0(mod30) as our wanted Abektie.
- If A is equal to 0, then we substitute 29 in A to find x.
- If [A−11] (mod30) is negative number, then we use [30 + A−11] (mod30) to find x.

For instance,

- If the Abektie of 2023 E.C is 7, then the Abektie of a year
  - 2024 E.C. is [7+11] (mod30)≡18(mod30).
  - 2025 E.C. is [18+11] (mod30)≡29(mod30)≡ 0(mod30)
  - 2026 E.C. is [0+11] (mod30) ≡ 11(mod30).
  - 2027 E.C. is [11+11] (mod30) ≡ 22(mod30).
  - 2028 E.C. is [22+11] (mod30) ≡ 33(mod30) ≡ 3(mod30).
- If the Abektie of 2008 E.C is 22, then the Abektie of a year
  - 2007 E.C. is $[22 - 11](mod 30) \equiv 11(mod 30)$.
  - 2006 E.C. is $[11 - 11](mod 30) \equiv 0(mod 30)$.
  - 2005 E.C. is $[0 - 11](mod 30) \equiv [29 - 11](mod 30) \equiv 18(mod 30)$.
  - 2004 E.C. is $[18 - 11](mod 30) \equiv 7(mod 30)$.
  - 2003 E.C. is $[7 - 11](mod 30) \equiv -4(mod 30) \equiv [30 - 4](mod 30) \equiv 26(mod 30)$.
  - 2002 E.C. is $[26 - 11](mod 30) \equiv 15(mod 30)$.
  - 2001 E.C. is $[15 - 11](mod 30) \equiv 4(mod 30)$.
  - 2000 E.C. is $[4 - 11](mod 30) \equiv -7(mod 30) \equiv 23(mod 30)$ since (30−7) in mod30.

A year 1 before the birth of Christ is a year with Albo Abektie. Then

- A year 20 before the birth of Christ is also a year with Albo Abektie. Here, we have $20 = 1 + 1 \times 19$.
- A year 39 before the birth of Christ is also a year with Albo Abektie. Here, we have $39 = 1 + 2 \times 19$.
- A year 58 before the birth of Christ is also a year with Albo Abektie. Here, we have $58 = 1 + 3 \times 19$.
- A year $1 + (n - 1) \times 19$ before or after the birth of Christ is also a year with Albo Abektie for some integer n.

So, a sequence $\{1, 20, 39, 58, \ldots, 1 + (n - 1) \times 19, \ldots\}$ forms an Arithmetic progression.
Using the concepts of Congruence modulo, Ceiling function and Arithmetic progression, we can able to develop a mathematical algorithm which is important to find the immediately preceding year with Albo Abektie of the selected year. For this, we follow the following steps.

1. Select a year Y (may be before or after the birth of Christ).
2. Find $n = \lceil \frac{Amete\ Alem\ of\ the\ selected\ year\ or\ Amete\ Fida}{19} \rceil$.
3. Find $K = [1 + (n - 1) \times 19](mod\ 5500)$.

Let us find the immediately preceding year with Albo Abektie of a year 2014 E.C.

1. $n = \lceil \frac{5500+2014}{19} \rceil = \lceil \frac{7514}{19} \rceil = \lceil 395.47 \rceil = 396$.
2. $K = [1 + (396 - 1) \times 19](mod\ 5500) = 7506(mod\ 5500) = 2006$.

Hence, 2006 is the immediately preceding year with Albo Abektie of a year 2014 E.C.
Using the concept of congruence modulo and 21 in subsection 2.2,

a The starting date of Main fasting will be taken from $[MH + 14](mod\ 30)$.
b The celebration date of *Debre Zeit* will be taken from $[MH + 14 + 27](mod\ 30) \equiv [MH + 11](mod\ 30)$ since 41 in mod 30 is 11.
c The celebration date of Easter will be taken from $[MH + 14 + 27 + 28](mod\ 30) \equiv [MH + 9](mod\ 30)$ since 69 in mod 30 is 9.
d The celebration date of *Rikbe Kahinat* will be taken from $[MH + 14 + 27 + 28 + 24](mod\ 30) \equiv [MH + 3](mod\ 30)$ since 93 in mod 30 is 3.
e The celebration date of *Irget* will be taken from $[MH + 14 + 27 + 28 + 39](mod\ 30) \equiv [MH + 18](mod\ 30)$ since 108 in mod 30 is 18.
f The celebration date of *Bahle Amisa* will be taken from $[MH + 14 + 27 + 28 + 49](mod\ 30) \equiv [MH + 28](mod\ 30)$ since 118 in mod 30 is 28.

Remark 11. : The months of the above fasting dates are depend on the month at which Nenewe falls.

Since *Nenewe* starts on Monday and the starting date of *Nenewe* is MH, then MH in mod 7 is 0. Hence, we do have the following procedures to prove 17, 18, 19 and 20 in subsection 2.2.

- Since [MH + 14](mod 7) ≡ 0(mod 7), then main fasting is always started on Monday.
- Since [MH + 14 + 27](mod 7) ≡ [MH + 41](mod 7)≡ 6(mod 7), then *Debre Zeit* is always celebrated on Sunday.
- Since [MH + 14 + 27 + 28](mod 7) ≡ [MH + 69](mod 7)≡ 6(mod 7), then Easter is always celebrated on Sunday.
- Since [MH + 14 + 27 + 28 + 24](mod 7) ≡ [MH + 93](mod 7)≡ 2(mod 7), then *Rikbe Kahinat* is always on Wednesday.
- Since [MH + 14 + 27 + 28 + 39](mod 7) ≡ [MH + 108](mod 7)≡ 3(mod 7), then *Irget* is always celebrated on Thursday.
- Since [MH + 14 + 27 + 28 + 49](mod 7) ≡ [MH + 118](mod 7)≡ 6(mod 7), then *Bahle Amisa* is always celebrated on Sunday.

Illustrative example

Consider 2012 E.C.

- The *Wengilawian* name of a year 2012 E.C. is *Zemene Yohannes*. Because W $\equiv$ 2012(mod 4)$\equiv$0(mod 4).
- A day of September 1(*Inqutatash*) in 2012 E.C. is Thursday. Because D $\equiv$ [1 + 2012 + $\lfloor\frac{2012}{4}\rfloor$](mod 7)$\equiv$[1 + 3 + 503](mod 7) $\equiv$ [4 + 6](mod 7)$\equiv$3(mod 7).
- A day of January 1 in 2012 E.C. is Friday. Because D$'$ $\equiv$ [3 + (1$-$1) + 2 $\times$(5$-$1)](mod7)$\equiv$[3 + 0 + 8](mod7)$\equiv$11(mod7)
  $\equiv$4(mod7).
- The *Wenber* of year 2012 E.C. is 6. Because W$'$$\equiv$(8 + 2012)(mod 19)$\equiv$6(mod 19).
- The *Metik* of a year 2012 E.C. is 24. Because M$\equiv$[6 $\times$ 19](mod 30)$\equiv$24(mod 30).
  Since *Metik* is greater than 14, the holiday of the *Metik* of a year 2012 E.C. will be in September 24. By the help of the formula of D$'$, September 24 will be on Saturday.
- The daily *Tewusak* of Saturday is 8.
- The *Mebaja Hamer* of 2012 E.C. is 2. Because MH $\equiv$ [24 + 8](mod 30) $\equiv$ 32(mod 30) $\equiv$ 2(mod 30).
  Since 32 is greater than 30, then *Nenewe* will be started in February 2.
- Main fasting will be started in February 16. Because [2 + 14] (mod 30)$\equiv$16(mod 30).
- *Debre Zeit* will be celebrated in March 13. Because [2 + 11] (mod 30) $\equiv$13(mod 30).
- Easter will be celebrated in April 11. Because [2 + 9] (mod 30) $\equiv$11(mod 30).
- *Rikbe Kahinat* will be in May 5. Because [2 + 3] (mod 30)$\equiv$5(mod 30).
- *Irget* will be celebrated in May 20. Because [2 + 18] (mod 30)$\equiv$20(mod 30).
- *Bahle Amisa* will be celebrated in May 30. Because [2 + 28] (mod 30)$\equiv$0(mod 30); 0 and 30 are equivalent in mod 30.

In Ethiopian Calendar, there is a fact that the day of the last date of Pagumie in any year Y is the same as the day of Christmas in the year $Y + 1$ [15].

Proof

If $n \mod 7$ is the numerical code of the day of the last date of Pagumie in any year Y, then the numerical code of the day of Inqutatash in the year $Y + 1$ will be $(n + 1) \mod 7$. The numerical code of Christmas in the year $Y + 1$ is

$$D' \equiv [n + 1 + (29 - 1) + 2 \times (4 - 1)] \mod 7 \quad \text{since } D = n + 1, d = 29, \text{ and } m = 4.$$

$$\Rightarrow D' \equiv [n + 1 + 1 - 1 + 6] \mod 7 \quad \text{since } 29 \mod 7 = 1.$$

$$\Rightarrow D' \equiv [n + 7] \mod 7$$

$$\Rightarrow D' \equiv [n + 0] \mod 7 \quad \text{since } 7 \mod 7 = 0.$$

$$\Rightarrow D' \equiv n \mod 7$$

which is also the numerical code of the day of the last date of Pagumie in any year Y. Hence, the day of the last date of Pagumie in any year Y is the same as the day of Christmas in the year $Y + 1$.

3.4. Conversion of dates in E.C. to G.C.

September is the 1^{st} month in E.C. but it is the 9^{th} month in G.C. Deep analysis of tables in the Appendix helps us to develop Mathematical algorithms which are important for converting dates in E.C. to G.C. To convert dates in E.C. to G.C., we apply the following steps.

Step 1 Determining the *Wengilawian* name of a year of a given date.
Step 2 Grouping the 13 months of E.C. into 9 parts in the following manner and determining the row and column where the month of a given date is found in Table 1
Step 3 Finding the value of date difference between E.C. and G.C in the month of a given date, denoted by B.

Table 1. Grouping months of E.C. into 9 parts

September and October	March
November and December	April and May
January	June and July
February	August
	Pagumie

Table 2. Grouping months of E.C. into 11 parts

September	
October	
November	
December	
	January
	February
	March
	April
	May
	June
	July, August, and Pagumie

Case 1 If the *Wengilawian* name of a year of a given date is not *Zemene Yohannus*, then $B = 12 - S$ where S is the sum of the row and column at which the month of a given date is found in Table 1

Case 2 If the *Wengilawian* name of a year of a given date is *Zemene Yohannus*, then B is equal to

a $12 - S + 1$ if the month of a given date is from September 01 to February 30

b $12 - S$ if the month of a given date is from March 01 to Pagumie 5 or 6

where S is the sum of the row and column at which the month of a given date is found in Table 1

Step 4 Determining the length of a month of a given date in the Gregorian calendar, denoted by N, using the following steps.

a Grouping the 13 months of E.C. into 11 parts in the following manner and determining the row and column where the month of a given date is found in Table 2

b Finding the sum of the row & column where the month of a given date is found in Table 2 denoted by M.

Then we have the following cases.

Case 1 If M is odd, then $N = 31$.

Case 2 If M is even $\setminus\{8\}$, then $N = 30$.

Case 3 If $M = 8$ and the *Wengilawian* name of a year of a given date is *Zemene Yohannus*, then $N = 29$.

Case 4 If $M = 8$ and the *Wengilawian* name of a year of a given date is not *Zemene Yohannus*, then $N = 28$.

Step 5 Converting a given date E in E.C. to G.C. using the formula

$$G = (E + B) \mod N$$

where $N = 28, 29, 30$, or 31.

Step 6 Determining the month of the converted date using the following rules.

a If $E + B \leq N$, then the month of the converted date (in G.C.) is the same as the month of a given date (in E.C.).

b If $E + B > N$, then the month of the converted date (in G.C.) is the month which comes 1^{st} after the month of a given date (in E.C.).

Step 7 Converting the year Y of a given date in E.C. to G.C. using the following facts. *If the given date is from September 01 to December 22 (21 in Zemene Yohannus) in E.C., then Y in E.C. is equal to $Y + 7$ in G.C. Otherwise, it is equal to $Y + 8$ in G.C.*

Example 10. Convert February 04/2012 E.C. to G.C.

- The *Wengilawian* name of a given year is *Zemene Yohannus* because $W \equiv 2012 \mod 4 \equiv 0 \mod 4$.
- Since the month of a given date is found in the 4^{th} row & 1^{st} column in Table 3.4.1, then $S = 4 + 1 = 5$.
- Using Case 2(a) in Step 3, $B = 12 - 5 + 1 = 8$.
- Since the month of a given date is found in the 6^{th} row & 2^{nd} column in Table 2 then $M = 6 + 2 = 8$.
- Using Case 3 in Step 4, $N = 29$.
- $G = (4 + 8) \mod 29 = 12 \mod 29$
- Using Step 6, the month of the converted date is February because $12 < 29$.
- Using Step 7, 2012 E.C. is equal to $(2012 + 8)$ G.C. i.e., 2020 G.C.

Hence, February 04/2012 E.C. is equal to February 12/2020 G.C.

4. Summary

Arithmetic progression, Ceiling function, Congruence modulo, and Floor function have applications for the perpetual Ethiopian calendar, which is developed by the Ethiopian Orthodox Tewahedo Church. They are useful to

- determine the *Wengilawian* name of any year,
- determine the day of September 1 (*Inkutatash*),
- determine the day of any given date,
- convert a date in E.C. to G.C.,
- calculate the movable fasting date of the Ethiopian Orthodox Tewahedo Church,
- calculate the date of the movable holiday of the Ethiopian Orthodox Tewahedo Church.

For these determinations and calculations, this study developed Mathematical algorithms which are stated earlier.

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References

[1] Abebe, M. (2019). *History of Ethiopian calendar*. Ethiopia: Dessie Press.

[2] Fekadu, N. (2018). *History of Ethiopian and Gregorian Calendar*. Ethiopia: Bahir Dar Press.

[3] Temesgen, K. (2020). *Comparison of Ethiopian and Gregorian Calendar*. Ethiopia: Addis Ababa Press.

[4] Chalie, A. (2017). *Contributions of Ethiopian Orthodox Church for Perpetual Ethiopian Calendar*. Ethiopia: Debre Markos Press.

[5] Manaye, B. N. (2016). *The use of Bahre Hasabe for the development of digital Ethiopian*. Ethiopia: Gondar Press.

[6] Zenebe, N. (2021). *Perpetual Ethiopian and European Calendar and Organizer System for an Android Based Smart Phones*. Ethiopia: AAU – ETD.
[7] Proofwiki. (2018). *Definition: Congruence (number theory)*. Retrieved on November 11, 2018, from https://proofwiki.org/wiki/.
[8] Cleveland, T. (2018). *Number Theory. Scientific e-Resources*. Retrieved from http://books.google.ie/books?id=o-PEDwAAQBAJ$&$printsec=frontcover$&$dq=number+theory$&$hl=$&$cd=5$&$source=gbs$_$api
[9] Nagell, T. (2021). *Introduction to Number Theory. American Mathematical Society*. Retrieved from http://books.google.ie/books?id=Znc5EAAAQBAJ$&$printsec=frontcover$&$dq=number+theory$&$hl=$&$cd=3$&$source=gbs$_$api
[10] Fine, B., Gaglione, A., Moldenhauer, A., Rosenberger, G., & Spellman, D. (2017). *Algebra and Number Theory. Walter de Gruyter GmbH & Co KG*. Retrieved from http://books.google.ie/books?id=7-Y3DwAAQBAJ$&$printsec=frontcover$&$dq=number+theory$&$hl=$&$cd=1$&$source=gbs$_$api
[11] Jongsma, C. (2020). *Introduction to Discrete Mathematics via Logic and Proof*. Springer. Retrieved from http://books.google.ie/books?id=1NUKzgEACAAJ$&$dq=discrete+mathematics$&$hl
[12] Levin, O. (2018). *Discrete Mathematics*. Createspace Independent Publishing Platform. Retrieved from http://books.google.ie/books?id=BWhCugEACAAJ$&$dq=discrete+mathematics$&$hl=$&$cd=1$&$source=gbs$_$api
[13] Shah, D., Sahni, M., Sahni, R., León-Castro, E., & Olazabal-Lugo, M. (2022). *Series of Floor and Ceiling Function—Part I: Partial Summations*. Mathematics, 10(7), 1178. https://doi.org/10.3390/math10071178.
[14] Yadav, D. (2020). *Multi-Dimensional Arithmetic Progression*. Retrieved from http://books.google.ie/books?id=ImVuzQEACAAJ$&$dq=arithematic+progression$&$hl=$&$cd=1$&$source=gbs$_$api
[15] Tesifa, G. (2020). *Abushaher (yekene mekuteria)*. Ethiopia: Aksum Publishing House.
[16] Cai, L. (2022). *Arithmetic Progression_Geometric Progression (A′level H2 Math)*. AcesMath! Retrieved from http://books.google.ie/books?id=HX9sEAAAQBAJ$&$printsec=frontcover$&$dq=Arithmetic+progression$&$hl=$&$cd=4$&$source=gbs$_$api