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A blended numerical procedure for quadratic riccati differential equations utilizing ramadan group transform and variations of adomian decomposition

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Abstract: To solve the approximate analytic solutions of the quadratic Riccati differential equations, this study introduces a hybrid method that combines an accelerated variant of the Adomian decomposition method (AADM) proposed by I. El-Kalla with the Ramadan Group transform (RGT). This hybrid technique produces accurate and dependable results, outperforming the regular Adomian decomposition method (RADM) and the Newton- Raphson version of Adomian polynomials in terms of accuracy. Three examples are provided here to demonstrate good accuracy and fast convergence when compared to the exact solution and other recent analytical methods using Shifted Chebyshev polynomials, Variation of Parameters Method (VPM), Bezier polynomials, homotopy analysis method (HAM), and Newton - Raphson based modified Laplace Adomian decomposition method.

Keywords: Ramdan group transform, adomian polynomials, Newton- Raphson, accelerated adomian, accuracy.

1. Introduction

One of the most significant types of nonlinear differential equations is the *Quadratic Riccati Differential Equation* (QRDE), which is given by

$$\frac{dy}{dt} = q(t)\,y + r(t)\,y^2 + p(t), \quad y(0) = a, \tag{1}$$

where $q(t)$, $r(t)$, and $p(t)$ are known scalar functions, and a is an arbitrary constant. This equation is named after the Italian aristocrat, Count Jacopo Francesco Riccati (1676–1754) [1]. The Riccati equation has a wide range of applications in engineering and scientific research, including robust stabilization, stochastic implementation theory, network synthesis, optimal control, and financial mathematics [2]. Consequently, it has received extensive attention and been studied by numerous authors.

Recently, many methods have been proposed for solving QRDEs. For instance, the Bézier curves method constructs a Bézier polynomial of degree n [3], whereas the multistage variational iteration method has been shown to be an efficient technique for solving QRDEs [4]. In [5], the desired approximate solution is expanded in terms of Legendre scaling functions and an integral operational matrix, reducing the problem to a system of algebraic equations. In addition, iterative decomposition methods are employed to approximate the solutions of generalized Riccati differential equations [6].

Various iterative approaches are available for solving such equations both numerically and analytically, including the Adomian Decomposition Method (ADM) [7,8], Homotopy Perturbation Method (HPM) [9], Variational Iteration Method (VIM) [10], and Differential Transform Method (DTM) [11]. Liao [12] demonstrated that the HPM and Homotopy Analysis Method (HAM) equations are essentially equivalent. In [13], two numerical approaches based on cubic B-spline scaling functions and Chebyshev cardinal functions are presented for the Riccati equation, converting the original problem into a system of algebraic equations through an operational matrix of derivatives. Moreover, the standard RK4 scheme for the nonlinear Riccati differential equation is discussed in [14], along with an examination of its stability. Numerical comparisons therein illustrate the behavior of the approximate versus the exact solutions for different mesh sizes at chosen nodal points.

Zheng et al. [15] introduced a technique using the Bernstein-Bézier control points to solve differential equations numerically. Evrenosoglu and Somali [16] applied this approach to singularly perturbed two-point boundary value problems. In [17], the QRDE is treated by the cubic B-spline approach, and error estimates are provided. Their findings show that the method offers reliable solutions and compares favorably with other recent techniques.

2. Mathematical Preliminaries and Notions

In this section, we present the fundamental definitions and theorems that will enable readers to understand the Riccati-Galerkin technique (RGT) and its core concepts.

2.1. Adomian polynomials [18]

The Adomian Decomposition Method (ADM) has proven effective for providing analytic approximate solutions to a wide range of linear and nonlinear functional equations. Consider the general nonlinear operator equation:

$$P(y) + R(y) + Q(y) = g, \tag{2}$$

where P is the highest-order derivative operator (assumed to be invertible), R is a linear operator of lower order than P, Q is a nonlinear operator, and g is the source term. The ADM expresses the solution y as an infinite series:

$$y = \sum_{k=0}^{\infty} y_k, \tag{3}$$

and expands the nonlinear term $Q(y)$ as

$$Q(y) = \sum_{n=0}^{\infty} A_n, \tag{4}$$

where $\{A_n\}$ are the *Adomian polynomials*, used to approximate the nonlinear terms in ordinary or partial differential equations.

We focus on three main versions of Adomian polynomials.

2.1.1. Regular adomian method (RAP)

These polynomials are determined formally by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(N\left(\sum_{i=0}^{\infty} \lambda^i y_i\right)\right)\right]_{\lambda=0}, \tag{5}$$

where $N(y_i)$ denotes the nonlinear function in the problem (differential or integral equation). For $N(y) = y^2$, the first few Adomian polynomials are

$$A_0 = y_0^2, \quad A_1 = 2\,y_0\,y_1, \quad A_2 = y_1^2 + 2\,y_0\,y_2,$$

$$A_3 = 2\,y_1\,y_2 + 2\,y_0\,y_3, \quad A_4 = y_2^2 + 2\,y_1\,y_3 + 2\,y_0\,y_4.$$

2.1.2. Newton–Raphson adomian polynomials (N-R adomian version)

A second variant employs the Newton–Raphson technique to refine the Adomian polynomials. In this version, the unknown function $N(y)$ is replaced by its Newton–Raphson form, thereby enhancing the polynomials [19]:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda_n^n}\left(f\sum_{i=0}^{n} \lambda^i\left(u_i - \frac{f(y_i)}{f'(y_i)}\right)\right)\right]_{\lambda=0}. \tag{6}$$

For instance, if $N(y) = y^2$, then the first few Newton–Raphson Adomian polynomials can be expressed as:

$$A_0 = \left(\tfrac{1}{2}\right)^2 (y_0^2), \quad A_1 = 2\left(\tfrac{1}{2}\right)^2 (y_0\, y_1),$$

$$A_2 = \left(\tfrac{1}{2}\right)^2 (2\, y_0\, y_2 + y_1^2).$$

2.1.3. El-Kalla adomian polynomials (accelerated adomian)

The third variant, introduced by El-Kalla [20], is known as the *Accelerated Adomian* method. El-Kalla demonstrated that this new formula achieves faster convergence. The El-Kalla polynomials are defined by

$$\overline{A}_n = N(s_n) - \sum_{i=0}^{n-1} \overline{A}_i, \tag{7}$$

where $N(\cdot)$ is the same nonlinear operator. For $f(y) = y^2$, some of the first terms of $\{\overline{A}_n\}$ are

$$\overline{A}_0 = N(s_0) = y_0^2,$$

$$\overline{A}_1 = N(s_1) - \overline{A}_0 = (y_0 + y_1)^2 - y_0^2 = 2\, y_0\, y_1 + y_1^2,$$

$$\overline{A}_2 = N(s_2) - \overline{A}_0 - \overline{A}_1 = (y_0 + y_1 + y_2)^2 - y_0^2 - 2y_0y_1 - y_1^2 = 2\, y_0\, y_2 + 2\, y_1\, y_2 + y_2^2.$$

2.2. Ramadan group integral transform (RGT) [21]

A new integral transform, called the *Ramadan Group transform (RGT)*, was introduced in 2016 by M. A. Ramadan et al. [21]. One of the essential features of this transform is that it generalizes well-known transforms such as the Laplace and Sumudu transforms.

We consider functions in the set A, defined by

$$A = \left\{ f(t) : \exists\, M > 0,\ t_1 > 0,\ t_2 > 0 \text{ such that } \left|f(t)\right| < M \exp\!\left(\tfrac{|t|}{t_n}\right),\ \text{if } t \in (-1)^n \times [0, \infty) \right\}.$$

The Ramadan Group (RG) transform of a function $f(t)$ is given by:

$$K(s,u) \;=\; RG\big[f(t)\,;\,(s,u)\big] \;=\; \begin{cases} \displaystyle\int_0^\infty e^{-st} f(ut)\, dt, & \text{if } -t_1 < u \le 0, \\ \displaystyle\int_0^\infty e^{-st} f(ut)\, dt, & \text{if } 0 \le u < t_2. \end{cases}$$

Remark 1.

1. **Special Case (Laplace Transform).** Setting $u = 1$ reduces the RG transform to the Laplace transform:
$$F(s) \;=\; L\big[f(t); s\big] \;=\; \int_0^\infty e^{-st} f(t)\, dt.$$
2. **Special Case (Sumudu Transform).** Setting $s = 1$ reduces the RG transform to the Sumudu transform:
$$G(u) \;=\; S\big[f(t); u\big] \;=\; \int_0^\infty e^{-t} f(ut)\, dt, \quad u \in (-\tau_1, \tau_2).$$

By comparing these definitions with the RG transform, one obtains the following relationships:

$$K(s,1) \;=\; F(s), \quad K(1,u) \;=\; G(u), \quad K(s,u) \;=\; \frac{1}{u} F(\tfrac{s}{u}).$$

Theorem 1. [21] *The RG transform satisfies:*

$$K(s,1) = F(s), \quad K(1,u) = G(u), \quad K(s,u) = \frac{1}{u} F\!\left(\frac{s}{u}\right).$$

Proof. The first two portions follow directly from the definitions of the Laplace and Sumudu transforms. The third part,

$$K(s,u) = \int_0^\infty e^{-st} f(ut)\,dt = \frac{1}{u} F\Big(\frac{s}{u}\Big),$$

can be shown by the change of variable $w = ut$, yielding

$$\int_0^\infty e^{-\frac{s}{u}w} f(w)\,\frac{dw}{u} = \frac{1}{u} F\Big(\frac{s}{u}\Big).$$

□

Theorem 2. [21] *Let $k(s,u)$ be the RG transform of $f(t)$. Then:*

$$RG\Big(\frac{d}{dt} f(t)\Big) = \frac{s\,RG\big(f(t)\big) - s\,f(0)}{u},$$

$$RG\Big(\frac{d^2}{dt^2} f(t)\Big) = \frac{s^2\,RG\big(f(t)\big) - s\,f(0) - u\,\acute{f}(t)}{u^2},$$

and in general,

$$RG\Big(\frac{d^n f(t)}{dt^n}\Big) = \frac{s^n\,RG\big(f(t)\big)}{u^n} - \sum_{k=0}^{n-1} \frac{s^{n-k-1} f^{(k)}(0)}{u^{n-k}}.$$

Table 1. Ramadan group transform of common functions

Function $f(t)$	$RG[f(t)] = K(s,u)$
1	$\frac{1}{s}$
t	$\frac{u}{s^2}$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{u^{n-1}}{s^n}$
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{su}}$
e^{at}	$\frac{1}{s-au}$
$t\,e^{at}$	$\frac{u}{(s-au)^2}$
$\frac{\sin(\omega t)}{\omega}$	$\frac{u}{s^2+u^2\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+u^2\omega^2}$
$\frac{\sin(at)}{a}$	$\frac{u}{s^2+u^2a^2}$

3. Analysis of the Hybrid Ramadan Group Accelerated Adomian Method

Consider the Riccati differential equation (RDE) with initial conditions described in (1). To solve this RDE, we apply the Ramadan Group transform to both sides of (1):

$$RG\Big[\frac{dy}{dt}\Big] = RG\Big[q(t)\,y + r(t)\,y^2 + p(t)\Big]. \tag{8}$$

Using the linearity and differential properties of the RG transform, we obtain:

$$RG[y] = \frac{a}{s} + \frac{u}{s}\,RG\big[q(t)\,y\big] + \frac{u}{s}\,RG\big[r(t)\,y^2\big] + \frac{u}{s}\,RG\big[p(t)\big]. \tag{9}$$

The Adomian method expresses the solution as an infinite series

$$y = \sum_{n=0}^{\infty} y_n, \tag{10}$$

with the nonlinear term $f(y) = y^2$ decomposed as

$$f(y) = y^2 = \sum_{n=0}^{\infty} A_n. \tag{11}$$

3.1. Regular adomian polynomials

They are computed by

$$A_n = \frac{1}{n!}\,\frac{d^n}{d\lambda^n}\Big[f\big(\textstyle\sum_{i=0}^{\infty}\lambda^i y_i\big)\Big]\Big|_{\lambda=0}. \tag{12}$$

A few terms for $f(y) = y^2$ are:

$$A_0 = y_0^2, \quad A_1 = 2\,y_0\,y_1, \quad A_2 = y_1^2 + 2\,y_0\,y_2.$$

3.2. Newton–Raphson adomian polynomials

They are given by

$$A_n = \frac{1}{n!}\,\frac{d^n}{d\lambda^n}\Big[f\big(\textstyle\sum_{i=0}^{n}\lambda^i\big[y_i - \frac{f(y_i)}{f'(y_i)}\big]\big)\Big]\Big|_{\lambda=0}. \tag{13}$$

The first few terms are:

$$A_0 = \left(\tfrac{1}{2}\right)^2 y_0^2, \quad A_1 = 2\left(\tfrac{1}{2}\right)^2 (y_0\,y_1), \quad A_2 = \left(\tfrac{1}{2}\right)^2 \left(2\,y_0\,y_2 + y_1^2\right).$$

3.3. El-Kalla adomian polynomials

They are given by

$$\overline{A}_n = N(s_n) - \sum_{i=0}^{n-1} \overline{A}_i, \tag{14}$$

where a few terms are:

$$\overline{A}_0 = y_0^2, \quad \overline{A}_1 = 2\,y_0\,y_1 + y_1^2, \quad \overline{A}_2 = 2\,y_0\,y_2 + 2\,y_1\,y_2 + y_2^2.$$

Substituting the series expansions (10) and (11) into (9), and applying linearity, yields:

$$\sum_{n=0}^{\infty} RG[y] = \frac{a}{s} + \frac{u}{s}\sum_{n=0}^{\infty} RG\big[q(t)\,y_n\big] + \frac{u}{s}\sum_{n=0}^{\infty} RG\big[r(t)\,A_n\big] + \frac{u}{s}\,RG\big[p(t)\big]. \tag{15}$$

By matching terms, the recursive relationships become:

$$RG[y_0] = \frac{a}{s} + \frac{u}{s}\,RG\big[p(t)\big], \tag{16}$$

$$RG[y_1] = \frac{a}{s} + \frac{u}{s}\,RG\big[q(t)\,y_0\big] + \frac{u}{s}\,RG\big[r(t)\,A_0\big], \tag{17}$$

$$RG[y_2] = \frac{a}{s} + \frac{u}{s}\,RG\big[q(t)\,y_1\big] + \frac{u}{s}\,RG\big[r(t)\,A_1\big], \tag{18}$$

and in general:

$$RG[y_{n+1}] = \frac{a}{s} + \frac{u}{s}\,RG\big[q(t)\,y_n\big] + \frac{u}{s}\,RG\big[r(t)\,A_n\big]. \tag{19}$$

Each y_n can be recovered by applying the *inverse* Ramadan Group transform to the right-hand side. Substituting back into (10) provides the solution to the quadratic RDE.

4. Examples

Example 1. Consider the following nonlinear quadratic Riccati equation [22,23]:

$$\begin{cases} y'(t) = 1 + 2y(t) - y^2(t), \\ y(0) = 0. \end{cases} \tag{20}$$

The exact solution to (20) is

$$y(t) = 1 + \sqrt{2}\tanh\Big[\sqrt{2}t \;+\; \tfrac{1}{2}\ln\Big(\frac{\sqrt{2}-1}{\sqrt{2}+1}\Big)\Big]. \tag{21}$$

This problem has been extensively studied by various authors. For instance, Ghomanjani and Khorram [22] applied a Bézier polynomial of degree $n = 7$. In that study, the approximate solution was given by

$$\begin{aligned} y(t) &= 0.8795413035\,t^7 - 3.611374466\,t^6 + 0.4836486195 + 6.005471116\,t^5 \\ &\quad - 4.458236528\,t^4 + 0.2583047925\,t^3 + 1.733381250\,t + 0.8045502275\,t^2. \end{aligned} \tag{22}$$

They reported the absolute errors (see Table 2) to be less than 10^{-4}.

Table 2. Absolute error for $y(t)$ in Example 4.1 using a degree-7 Bézier polynomial [22]

t	Error of method in [22]
0.0	9.6780×10^{-11}
0.1	0.000248944564121
0.3	0.0004481946615024
0.5	2.89441×10^{-10}
0.7	0.000374115023553
0.9	0.00017849475908
1.0	3.2516×10^{-10}

4.1. Fractional-order extension

The fractional-order version of this problem,

$$D_t^\alpha y(t) \;-\; 2\,y(t) \;+\; y^2(t) \;=\; 1, \quad 0 < \alpha \leq 1, \quad t \geq 0, \tag{23}$$

has been investigated by Ul Haq et al. [23], who used the variation of parameters method (VPM). They employed three terms to obtain analytical solutions, comparing their results with the homotopy analysis method (HAM) proposed by Abbasbandy [9]. Table 3 shows the comparison for the case $\alpha = 1$.

Table 3. Comparison of numerical results for problem (23) at $\alpha = 1$. VPM results are from [23], HPM results are from [9]

t	Exact	VPM [23]	HPM [9]	Error of VPM
0.0	0.0000000	0.0000000	0.0000000	0.0000000
0.2	0.2419744004	0.2419499764	0.2419648204	2.4424×10^{-5}
0.4	0.5678068604	0.5673979034	0.5681149562	4.089×10^{-4}
0.6	0.9535582813	0.9525886597	0.9582588343	9.696×10^{-4}
0.8	1.3463542580	1.3457899840	1.3652395490	5.64×10^{-4}
1.0	1.6894889740	1.6886513080	1.7238095240	8.37×10^{-4}

The same problem was also solved using shifted Chebyshev polynomials by Ezz-Eldien et al. [24], who presented numerical results for $m = 4$ and $m = 8$ basis functions. Table 4 shows the absolute errors at selected points.

Table 4. Absolute errors using shifted Chebyshev polynomials [24] for $\alpha = 1$ with $m = 4$ and $m = 8$

t	$m = 4$	$m = 8$
0.1	7.2143×10^{-5}	2.0536×10^{-6}
0.2	2.1479×10^{-3}	1.3148×10^{-6}
0.3	3.8702×10^{-3}	9.5441×10^{-7}
0.4	3.9104×10^{-3}	3.8234×10^{-6}
0.5	2.2581×10^{-3}	1.9730×10^{-6}

4.2. Proposed approach: three variants of the Ramadan group method

Next, we apply the proposed method (Ramadan Group transform combined with three variants of the Adomian Decomposition Method) to the quadratic Riccati differential equation:

$$\begin{cases} y'(t) = 1 + 2\,y(t) - y^2(t), \\ y(0) = 0. \end{cases} \tag{24}$$

Its exact solution is

$$y(t) = 1 + \sqrt{2}\,\tanh\Big[\sqrt{2}t + \tfrac{1}{2}\,\ln\Big(\tfrac{\sqrt{2}-1}{\sqrt{2}+1}\Big)\Big].$$

Taking the Ramadan Group (RG) transform of (24) yields

$$\mathrm{RG}[y'(t)] = \mathrm{RG}[1] + \mathrm{RG}[2y(t)] - \mathrm{RG}[y^2(t)].$$

$$\frac{s}{u}Y(s,u) - \frac{1}{u}y(0) = \frac{1}{s} + 2\,Y(s,u) - \mathrm{RG}[\,y^2(t)\,],$$

$$\frac{s}{u}Y(s,u) - 2\,Y(s,u) = \frac{1}{s} - \mathrm{RG}[\,y^2(t)\,],$$

$$\frac{s-2u}{u}\,Y(s,u) = \frac{1}{s} - \mathrm{RG}[\,y^2(t)\,],$$

$$Y(s,u) = \frac{u}{s(s-2u)} - \frac{u}{s-2u}\,\mathrm{RG}[\,y^2(t)\,].$$

Taking the inverse Ramadan Group transform provides the solution in an infinite series form:

$$y(t) = \sum_{n=0}^{\infty} y_n(t),$$

where the nonlinear term $y^2(t)$ is decomposed by the Adomian polynomials.

4.3. Adomian decomposition variants for solving a Riccati-type equation

This document presents three variants of the "Combined Ramadan Group" approach to the Adomian decomposition method for solving a Riccati-type differential equation. Each variant provides a sequence of partial solutions $y_0, y_1, y_2, \ldots$, along with the corresponding Adomian polynomials A_n. The approximate solution for each method is given by summing the first few terms,

$$y(t) \approx y_0 + y_1 + y_2 + y_3 + y_4.$$

4.3.1. First variant: regular adomian decomposition method

Below are the initial terms and Adomian polynomials for the standard Adomian method.

$$A_0 = (y_0\,y_0) = \frac{1}{4}\big(-1 + e^{2t}\big)^2,$$

$$y_1 = \frac{1}{8}\big(-1 + e^{4t} - 4\,e^{2t}\,t\big),$$

$$A_1 = 2\,(y_0\,y_1) = \frac{1}{8}\left(-1+e^{2t}\right)\left(-1+e^{4t}-4\,e^{2t}\,t\right),$$

$$y_2 = \frac{1}{32}\Big(-2-e^{2t}+2\,e^{4t}+e^{6t}-4\,e^{2t}\,t-8\,e^{4t}\,t+8\,e^{2t}\,t^2\Big),$$

$$A_2 = 2\,(y_0\,y_2)+(y_1\,y_1),$$

$$A_2 = \frac{1}{64}\left(1-e^{4t}+4\,e^{2t}\,t\right)^2+\frac{1}{32}\left(-1+e^{2t}\right)\Big(-2-e^{2t}+2\,e^{4t}+e^{6t} -4\,e^{2t}\,t-8\,e^{4t}\,t+8\,e^{2t}\,t^2\Big),$$

$$y_3 = \frac{1}{384}\Big(15+12\,e^{2t}-12\,e^{4t}-12\,e^{6t}-3\,e^{8t}+12\,e^{2t}\,t+72\,e^{4t}\,t+36\,e^{6t}\,t -48\,e^{2t}\,t^2-96\,e^{4t}\,t^2+32\,e^{2t}\,t^3\Big),$$

$$A_3 = 2\,(y_1\,y_2)+2\,(y_0\,y_3),$$

$$A_3 = \frac{1}{128}\left(1-e^{4t}+4\,e^{2t}\,t\right)\Big(-2-e^{2t}+2\,e^{4t}+e^{6t}-4\,e^{2t}\,t-8\,e^{4t}\,t+8\,e^{2t}\,t^2\Big) +\frac{1}{384}\left(-1+e^{2t}\right)\Big(15+12\,e^{2t}-12\,e^{4t}-12\,e^{6t}-3\,e^{8t}+12\,e^{2t}\,t +72\,e^{4t}\,t+36\,e^{6t}\,t-48\,e^{2t}\,t^2-96\,e^{4t}\,t^2+32\,e^{2t}\,t^3\Big),$$

$$y_4 = \frac{1}{1536}\Big(-42-42\,e^{2t}+24\,e^{4t}+39\,e^{6t}+18\,e^{8t}+3\,e^{10t}-192\,e^{4t}\,t-180\,e^{6t}\,t -48\,e^{8t}\,t+96\,e^{2t}\,t^2+384\,e^{4t}\,t^2+216\,e^{6t}\,t^2-96\,e^{2t}\,t^3-256\,e^{4t}\,t^3+32\,e^{2t}\,t^4\Big).$$

Hence, the approximate solution using this first variant is:

$$y(t)\,(\text{approx.}) = y_0+y_1+y_2+y_3+y_4. \tag{25}$$

4.3.2. Second variant: accelerated adomian (El-Kalla-Based) method

In this approach, an accelerated Adomian method (based on El-Kalla's formulation) is applied:

$$A_0 = (y_0\,y_0) = \frac{1}{4}\left(-1+e^{2t}\right)^2,$$

$$y_1 = \frac{1}{8}\left(-1+e^{4t}-4\,e^{2t}\,t\right),$$

$$A_1 = (2\,y_0\,y_1)+(y_1\,y_1),$$

$$y_2 = \frac{1}{384}\Big(-21+7\,e^{2t}+6\,e^{4t}+9\,e^{6t}-e^{8t}-48\,e^{2t}\,t-48\,e^{4t}\,t+12\,e^{6t}\,t +72\,e^{2t}\,t^2-48\,e^{4t}\,t^2\Big),$$

$$A_2 = (2\,y_0\,y_2)+(2\,y_1\,y_2)+(y_2\,y_2),$$

$$y_3 = \frac{1}{928972800}\Big(20440350+9759493\,e^{2t}-29827350\,e^{4t}+1020600\,e^{6t}-2052400\,e^{8t}+738675\,e^{10t} -87318\,e^{12t}+8400\,e^{14t}-450\,e^{16t}+65356200\,e^{2t}\,t+42184800\,e^{4t}\,t+3817800\,e^{6t}\,t -8929200\,e^{8t}\,t+1360800\,e^{10t}\,t-166320\,e^{12t}\,t+12600\,e^{14t}\,t -24494400\,e^{2t}\,t^2-40370400\,e^{4t}\,t^2+45057600\,e^{6t}\,t^2-9122400\,e^{8t}\,t^2 +1360800\,e^{10t}\,t^2-151200\,e^{12t}\,t^2+49896000\,e^{2t}\,t^3-32659200\,e^{4t}\,t^3 +21772800\,e^{6t}\,t^3-5040000\,e^{8t}\,t^3+907200\,e^{10t}\,t^3 -16329600\,e^{4t}\,t^4+10886400\,e^{6t}\,t^4-2419200\,e^{8t}\,t^4\Big),$$

$$A_3 = (2\,y_0\,y_3) + (2\,y_1\,y_3) + (2\,y_2\,y_3) + (y_3\,y_3),$$
$$y_4 \quad \text{(similarly derived)},$$
$$y(t)\,(\text{approx.}) = y_0 + y_1 + y_2 + y_3 + y_4.$$

4.3.3. Third variant: Newton–Raphson-based adomian decomposition

Finally, this variant incorporates a Newton–Raphson step to improve convergence:

$$\begin{aligned}
A_0 &= \left(\tfrac{1}{2}\right)^2 (y_0\,y_0) \\
&= \frac{1}{16}\left(-1 + e^{2t}\right)^2, \\
y_1 &= \frac{1}{32}\left(1 - e^{4t} + 4\,e^{2t}\,t\right), \\
A_1 &= 2\left(\tfrac{1}{2}\right)^2 (y_0\,y_1) \\
&= \frac{1}{128}\left(-1 + e^{2t}\right)\left(1 - e^{4t} + 4\,e^{2t}\,t\right), \\
y_2 &= \frac{1}{512}\left(-2 - e^{2t} + 2\,e^{4t} + e^{6t} - 4\,e^{2t}\,t - 8\,e^{4t}\,t + 8\,e^{2t}\,t^2\right), \\
A_2 &= \left(\tfrac{1}{2}\right)^2 \left[2\,(y_0\,y_2) + (y_1\,y_1)\right], \\
y_3 &= \frac{15 + 12\,e^{2t} - 12\,e^{4t} - 12\,e^{6t} - 3\,e^{8t} + 12\,e^{2t}t + 72\,e^{4t}t + \dots}{24576},
\end{aligned}$$

$$\begin{aligned}
A_3 &= \left(\tfrac{1}{2}\right)\left[(y_0\,y_3) + (y_1\,y_2)\right] \\
&= \frac{1}{2}\Bigg[\frac{\left(1 - e^{4t} + 4\,e^{2t}\,t\right)\left(-2 - e^{2t} + 2\,e^{4t} + e^{6t} - 4\,e^{2t}\,t - 8\,e^{4t}\,t + 8\,e^{2t}\,t^2\right)}{16384} \\
&\quad + \frac{1}{49152}\left(-1 + e^{2t}\right)\left(15 + 12\,e^{2t} - 12\,e^{4t} - 12\,e^{6t} - 3\,e^{8t} + 12\,e^{2t}\,t + 72\,e^{4t}\,t + 36\,e^{6t}\,t\right. \\
&\quad \left. - 48\,e^{2t}\,t^2 - 96\,e^{4t}\,t^2 + 32\,e^{2t}\,t^3\right)\Bigg], \\
y_4 &= \frac{1}{393216}\left(-42 - 42\,e^{2t} + 24\,e^{4t} + \dots\right).
\end{aligned}$$

Finally, the approximate solution is:

$$y(t)\,(\text{approx.}) = y_0 + y_1 + y_2 + y_3 + y_4.$$

Table 5. Approximate solutions using RG + regular Adomian (RG+RA), RG+Newton–Raphson (RG+N-R), and RG+El-Kalla for four iterations, compared with the exact solution

t	Exact	RG+(N-R)	RG+RA	RG+(El-Kalla)
0.1	0.110295	0.110600	0.110295	0.110295
0.2	0.241977	0.244915	0.241977	0.241977
0.3	0.395105	0.406933	0.395105	0.395105
0.4	0.567812	0.600786	0.567812	0.567813
0.5	0.756014	0.830483	0.756014	0.756022

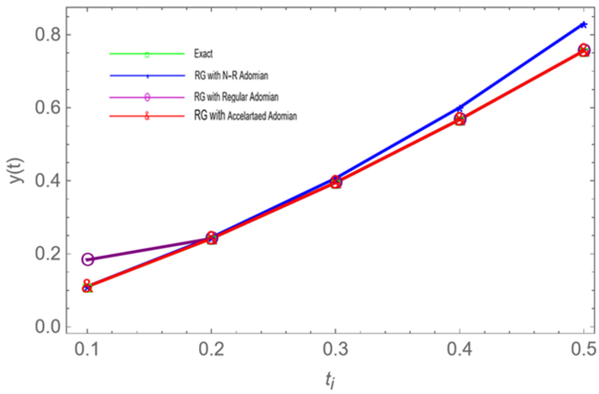

Figure 1. Comparison of approximate solutions using the three proposed methods (RG+RA, RG+N-R, RG+El-Kalla) from Table 5

Table 6. Absolute error using RG + regular Adomian, RG + Newton–Raphson Adomian, and RG + El-Kalla for four iterations

t	RG+(N-R)	RG+RA	RG+(El-Kalla)
0.1	0.000304306	1.50116×10^{-13}	2.52992×10^{-14}
0.2	0.00293774	5.21961×10^{-10}	8.50531×10^{-11}
0.3	0.0118284	7.64785×10^{-8}	1.18058×10^{-8}
0.4	0.0329739	3.05733×10^{-6}	4.38331×10^{-7}
0.5	0.0744688	5.98×10^{-5}	7.77579×10^{-6}

Table 7. Comparison of the absolute error between our method (RG+El-Kalla with four iterations) and the shifted Chebyshev polynomials approach ($m = 4, 8$) from [24]

t	RG+(El-Kalla)	Shifted Chebyshev [24]	
		$m = 4$	$m = 8$
0.1	2.52992×10^{-14}	7.2143×10^{-5}	2.0536×10^{-6}
0.2	8.50531×10^{-11}	2.1479×10^{-3}	1.3148×10^{-6}
0.3	1.18058×10^{-8}	3.8702×10^{-3}	9.5441×10^{-7}
0.4	4.38331×10^{-7}	3.9104×10^{-3}	3.8234×10^{-6}
0.5	7.77579×10^{-6}	2.2581×10^{-3}	1.9730×10^{-6}

Example 2. Consider the following first-order nonlinear differential equation:

$$y'(t) = 1 + y^2(t), \quad y(0) = 0,$$

which has the exact solution

$$y(t) = \tan(t).$$

This problem was solved by Vind Mishra and Dimple Rani [25] using a Newton–Raphson-based modified Laplace Adomian decomposition method for quadratic Riccati differential equations. Their numerical results, obtained after three iterations, are presented in Table 8.
Next, we applied three variants of the Adomian decomposition method combined with the Ramadan Group (RG) transform to solve the same problem. The resulting numerical solutions are shown in Tables 9–10.

Based on the numerical results presented in Tables 8–12 for Example 2, we conclude that the Ramadan Group (RG) transform combined with the accelerated Adomian (El-Kalla) version demonstrates higher accuracy compared to RG with the regular Adomian approach, RG with the Newton–Raphson variant, and the method in [25] (i.e., the Newton–Raphson-based modified Laplace Adomian decomposition method).

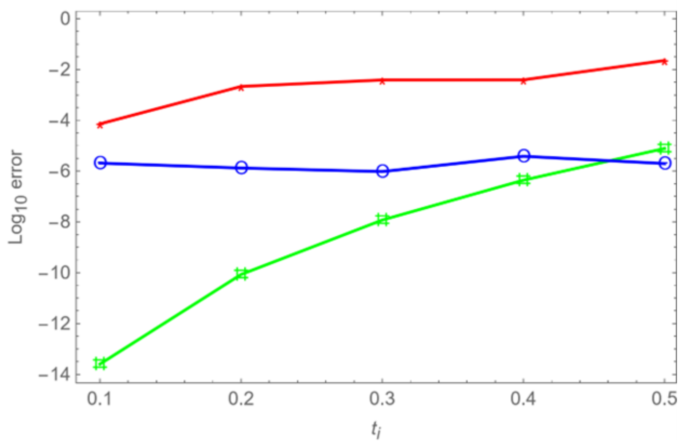

Figure 2. Comparison of absolute errors between the proposed RG+El-Kalla method and shifted Chebyshev ($m = 4, 8$) from [24]

Table 8. Exact and approximate solutions for three iterations using the method in [25]

t	Exact	Approx. Solution	Absolute Error
0	0	0	0
0.01	0.010000333	0.010000083	2.5×10^{-7}
0.02	0.020002667	0.020000667	2.0×10^{-6}
0.03	0.030009003	0.030002250	6.753×10^{-6}
0.04	0.040021347	0.040005334	1.601×10^{-5}
0.05	0.050041708	0.050010419	3.129×10^{-5}
0.06	0.060072104	0.060018006	5.410×10^{-5}
0.07	0.070114558	0.070028597	8.596×10^{-5}
0.08	0.080171105	0.080042694	1.284×10^{-4}
0.09	0.090243790	0.090060799	1.830×10^{-4}
0.10	0.100334672	0.100083417	2.513×10^{-4}

Table 9. Numerical results for the problem using RG with Adomian (accelerated El-Kalla version) with three iterations

t	Exact	Approx. (El-Kalla)	Absolute Error
0.00	0.000000000000000	0.000000000000000	0.000000000000000
0.01	0.010000333346667207	0.010000333346667205	1.73×10^{-18}
0.02	0.020002667093402423	0.020002667093402423	0
0.03	0.030009003241180714	0.030009003241180547	1.67×10^{-16}
0.04	0.040021346995514566	0.040021346995512340	2.23×10^{-15}
0.05	0.050041708375538790	0.050041708375522230	1.66×10^{-14}
0.06	0.060072103831297280	0.060072103831211750	8.55×10^{-14}
0.07	0.070114557872002710	0.070114557871659850	3.43×10^{-13}
0.08	0.080171104708072550	0.080171104706930910	1.14×10^{-12}
0.09	0.090243789909785450	0.090243789906485880	3.30×10^{-12}
0.10	0.100334672085450550	0.100334672076921770	8.53×10^{-12}

Example 3. Consider the following quadratic Riccati differential equation:

$$\dot{y}(t) = e^t - e^{3t} + e^{2t}y(t) - e^t\, y^2(t), \quad 0 \leq t \leq 1, \tag{26}$$

where the exact solution is $y(t) = e^t$.

Table 10. Numerical results for the problem using RG with Adomian (regular version) for three iterations

t	Exact	Approx. (Regular)	Absolute Error
0.00	0.000000000000000	0.000000000000000	0.000000000000000
0.01	0.010000333346667207	0.010000333346667207	0.000000000000000
0.02	0.020002667093402423	0.020002667093402413	1.04×10^{-17}
0.03	0.030009003241180714	0.030009003241180283	4.30×10^{-16}
0.04	0.040021346995514566	0.040021346995508820	5.75×10^{-15}
0.05	0.050041708375538790	0.050041708375496034	4.28×10^{-14}
0.06	0.060072103831297280	0.060072103831076570	2.21×10^{-13}
0.07	0.070114557872002710	0.070114557871118450	8.84×10^{-13}
0.08	0.080171104708072550	0.080171104705129660	2.94×10^{-12}
0.09	0.090243789909785450	0.090243789901284840	8.50×10^{-12}
0.10	0.100334672085450550	0.100334672063492070	2.20×10^{-11}

Table 11. Numerical results for the problem using RG with Adomian (Newton–Raphson version) for three iterations

t	Exact	Approx. Solution	Absolute Error
0	0	0	0.0000
0.01	0.01	0.01	2.5×10^{-7}
0.02	0.020003	0.020001	2.0×10^{-6}
0.03	0.030009	0.030002	6.753×10^{-6}
0.04	0.040021	0.040005	1.601×10^{-5}
0.05	0.050042	0.05001	3.129×10^{-5}
0.06	0.060072	0.060018	5.410×10^{-5}
0.07	0.070115	0.070029	8.596×10^{-5}
0.08	0.080171	0.080043	1.284×10^{-4}
0.09	0.090244	0.090061	1.830×10^{-4}
0.10	0.100335	0.100083	2.513×10^{-4}

Table 12. Comparison of absolute errors for the three proposed variants (RA, N-R, El-Kalla) versus the method in [25] using three iterations

t	Newton–Raphson (3 iterations)	Regular Adomian (3 iterations)	Accelerated El-Kalla (3 iterations)	Method in [25] (3 iterations)
0.00	0	0	0	0
0.01	2.5×10^{-7}	0	1.73×10^{-18}	2.50×10^{-7}
0.02	2.0×10^{-6}	1.04×10^{-17}	0	2.00×10^{-6}
0.03	6.75×10^{-6}	4.30×10^{-16}	1.67×10^{-16}	6.75×10^{-6}
0.04	1.60×10^{-5}	5.75×10^{-15}	2.23×10^{-15}	1.60×10^{-5}
0.05	3.13×10^{-5}	4.28×10^{-14}	1.66×10^{-14}	3.13×10^{-5}
0.06	5.41×10^{-5}	2.21×10^{-13}	8.55×10^{-14}	5.41×10^{-5}
0.07	8.60×10^{-5}	8.84×10^{-13}	3.43×10^{-13}	8.60×10^{-5}
0.08	1.28×10^{-4}	2.94×10^{-12}	1.14×10^{-12}	1.28×10^{-4}
0.09	1.83×10^{-4}	8.50×10^{-12}	3.30×10^{-12}	1.83×10^{-4}
0.10	2.51×10^{-4}	2.20×10^{-11}	8.53×10^{-12}	2.51×10^{-4}

By applying the Ramadan group (RG) transform to (26), we obtain:

$$\mathrm{RG}[\dot{y}(t)] = \mathrm{RG}[e^t] - \mathrm{RG}[e^{3t}] + \mathrm{RG}[e^{2t}y(t)] - \mathrm{RG}[e^t y^2(t)].$$

That is,

$$\frac{s}{u}Y(s,u) - \frac{1}{u}y(0) = \frac{1}{s-u} - \frac{1}{s-3u} + \mathrm{RG}[e^{2t}y(t)] - \mathrm{RG}[e^t y^2(t)],$$

$$\frac{s}{u}Y(s,u) = \frac{1}{u} + \frac{1}{s-u} - \frac{1}{s-3u} + \mathrm{RG}[e^{2t}y(t)] - \mathrm{RG}[e^t y^2(t)],$$

$$Y(s,u) = \frac{1}{s} + \frac{u}{s^2-us} - \frac{u}{s^2-3us} + \frac{u}{s}\,\mathrm{RG}[e^{2t}y(t)] - \frac{u}{s}\,\mathrm{RG}[e^t y^2(t)].$$

Taking the inverse RG transform, we obtain:

$$\mathrm{RG}^{-1}[Y(s,u)] = \mathrm{RG}^{-1}\left[\frac{1}{s}\right] + \mathrm{RG}^{-1}\left[\frac{u}{s^2-us}\right] - \mathrm{RG}^{-1}\left[\frac{u}{s^2-3us}\right] + \mathrm{RG}^{-1}\left[\frac{u}{s}\mathrm{RG}[e^{2t}y(t)]\right] - \mathrm{RG}^{-1}\left[\frac{u}{s}\mathrm{RG}[e^{t}y^2(t)]\right],$$

$$y(t) = \mathrm{RG}^{-1}\left[\frac{1}{s}\right] + \mathrm{RG}^{-1}\left[\frac{u}{s^2-us}\right] - \mathrm{RG}^{-1}\left[\frac{u}{s^2-3us}\right] + \mathrm{RG}^{-1}\left[\frac{u}{s}\mathrm{RG}[e^{2t}y(t)]\right] - \mathrm{RG}^{-1}\left[\frac{u}{s}\mathrm{RG}[e^{t}y^2(t)]\right].$$

In this approach, the solution is represented as an infinite series $y(t) = \sum_{n=0}^{\infty} y_n$, where the terms y_n are computed recursively. The nonlinear term $y^2(t)$ is decomposed as

$$y^2(t) = \sum_{n=0}^{\infty} A_n.$$

Hence,

$$\sum_{n=0}^{\infty} y_n = \mathrm{RG}^{-1}\left[\frac{1}{s}\right] + \mathrm{RG}^{-1}\left[\frac{u}{s^2-us}\right] - \mathrm{RG}^{-1}\left[\frac{u}{s^2-3us}\right] + \mathrm{RG}^{-1}\left[\frac{u}{s}\mathrm{RG}[e^{2t}\sum_{n=0}^{\infty} y_n]\right] - \mathrm{RG}^{-1}\left[\frac{u}{s}\mathrm{RG}[e^{t}\sum_{n=0}^{\infty} A_n]\right].$$

The zeroth-order term is given by

$$y_0 = \mathrm{RG}^{-1}\left[\frac{1}{s}\right] + \mathrm{RG}^{-1}\left[\frac{u}{s^2-us}\right] - \mathrm{RG}^{-1}\left[\frac{u}{s^2-3us}\right],$$

and the subsequent terms satisfy

$$y_n = \mathrm{RG}^{-1}\left[\frac{u}{s}\mathrm{RG}[e^{2t}\sum_{n=0}^{\infty} y_n]\right] - \mathrm{RG}^{-1}\left[\frac{u}{s}\mathrm{RG}[e^{t}\sum_{n=0}^{\infty} A_n]\right].$$

Table 13. Approximate solutions for the proposed method (RG with El-Kalla) after seven iterations

Absolute Error	Approximate Solution	Exact	t
0.0	1.105170918075648	1.105170918075648	0.1
2.2204×10^{-16}	1.22140275816017	1.22140275816017	0.2
2.2204×10^{-16}	1.349858807576003	1.349858807576003	0.3
4.4409×10^{-16}	1.491824697641271	1.49182469764127	0.4
4.4409×10^{-16}	1.648721270700128	1.648721270700128	0.5
2.2204×10^{-16}	1.822118800390509	1.822118800390509	0.6
0.0	2.013752707470477	2.013752707470477	0.7
2.2204×10^{-15}	2.22554092849247	2.225540928492468	0.8
2.5668×10^{-13}	2.459603111156693	2.45960311115695	0.9
4.8046×10^{7}	$-4.804574717620919 \times 10^{7}$	2.718281828459045	1.0

Based on the numerical results shown in Table 13 (and additional data in Tables 4.12 and 4.13), we conclude that our proposed method outperforms the techniques presented in [14,17,22], which employ the conventional RK4 method, the Bézier curves method, and the cubic B-spline approach, respectively.

5. Conclusions

In this study, the Ramadan group transform is paired with several forms of Adomian decomposition. The three Adomian polynomial variations approximate the unknown quadratic function. Regular Adomian, Newton-Raphson formula, and accelerated El-Kalla are three examples. A numerical approach for solving Quadratic RDEs is developed. Tables of numerical results reveal that the novel technique is effective and simple in identifying the approximate solution of Q RDEs. In terms of accuracy, using a hybrid Ramadan group with the El-Kalla version of the Adomian polynomials exceeds all other techniques, as seen in the tables. For all calculations, MATHEMAICA12 is utilized.

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