Engineering and Applied Science Letters

PSR Press

Article

Non oscillatory functions and a fourier inversion theorem for functions of very moderate decrease

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Received: 23 September 2024; Accepted: 31 November 2024; Published: 31 December 2024.

Abstract: We consider non oscillatory functions and prove an everywhere Fourier Inversion Theorem for functions of very moderate decrease. The proofs rely on some ideas in nonstandard analysis.

Keywords: Kernels, Lagrangian Multiplier, least square, Olanrewaju-Olanrewaju regression-type, nonparametric regression, support vector regressor (SVR).

On oscillatory functions of very moderate decrease are quite typical in the class $L^2(\mathcal{R})$ and are important in Physics. The first result presented here improves slightly on a result due to Carlsen, see [1], in that the convergence is everywhere rather than almost everywhere. This property is useful in verifying certain differentiability criteria in Physics, arising mainly from Maxwell's equations, and in showing decay properties of fields produced in connection with Jefimenko's equations, see [2] and [3]. At the end of the paper, we show how the result can be improved to functions of just very moderate decrease, without the non oscillatory assumption. The proofs rely on some ideas from nonstandard analysis, contained in the papers [4] and [5].

Definition 1. We say that $f \in C(\mathcal{R})$ is of very moderate decrease if there exists a constant $C \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{C}{|x|}$, for $|x| > 1$. We say that $f \in C(\mathcal{R})$ is of moderate decrease if there exists a constant $C \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{C}{|x|^2}$, for $|x| > 1$. We say that $f \in C(\mathcal{R})$ is non-oscillatory if there exist finitely many points $\{y_i : 1 \leq i \leq n\}$, for which $f|_{(y_i, y_{i+1})}$ is monotone, $1 \leq i \leq n-1$, and $f|_{(-\infty, y_1)}$, $f|_{(y_n, \infty)}$ are monotone. We say that $f \in C(\mathcal{R})$ is oscillatory if there exists an infinite sequence of points $\{y_i : i \in \mathcal{Z}\}$, for which $f|_{(y_i, y_{i+1})}$, $i \in \mathcal{Z}$, is monotone and there exists $\delta \in \mathcal{R}_{>0}$, with $|y_{i+1} - y_i| \geq \delta$, for $i \in \mathcal{Z}$.

Lemma 1. *Let $f \in C(\mathcal{R})$ and $\frac{df}{dx} \in C(\mathcal{R})$ be functions of very moderate decrease, with f and $\frac{df}{dx}$ both non-oscillatory. Define the Fourier transform by*

$$\mathcal{F}(f)(k) \;=\; \frac{1}{\sqrt{2\pi}} \lim_{r\to\infty} \int_{-r}^{r} f(y)\, e^{-iky}\, dy \quad (k \neq 0),$$

and

$$\mathcal{F}\Big(\frac{df}{dx}\Big)(k) \;=\; \frac{1}{\sqrt{2\pi}} \lim_{r\to\infty} \int_{-r}^{r} \frac{df}{dx}(y)\, e^{-iky}\, dy \quad (k \neq 0).$$

Then $\mathcal{F}(f)(k)$ and $\mathcal{F}(\frac{df}{dx})(k)$ are bounded for all $|k| > k_0 > 0$. Moreover, there exists a constant $G \in \mathcal{R}_{>0}$ such that

$$\big|\mathcal{F}(f)(k)\big| \;\leq\; \frac{G}{|k|^2},$$

for sufficiently large $|k|$.

Proof. Since f is of very moderate decrease and continuous, it follows that $\lim_{|x|\to\infty} f(x) = 0$. Likewise, $\frac{df}{dx}$ is continuous and $\lim_{|x|\to\infty} \frac{df}{dx}(x) = 0$. Because $\lim_{|x|\to\infty} f(x) = 0$ and f is non-oscillatory, the integral

$$\lim_{r\to\infty} \int_{-r}^{r} f(y)\, e^{-iky}\, dy,$$

exists for $k \neq 0$. Rewriting in terms of sine and cosine,

$$\lim_{r\to\infty} \int_{-r}^{r} f(y)\, e^{-iky}\, dy = \lim_{r\to\infty} \int_{-r}^{r} f(y) \cos(ky)\, dy \;-\; i \lim_{r\to\infty} \int_{-r}^{r} f(y) \sin(ky)\, dy.$$

Because f is of very moderate decrease and non-oscillatory, there exists $E > 0$ such that $|f(x)| \leq \frac{D}{|x|}$ for $|x| > E$, and f is monotone on $(-\infty, E)$ and (E, ∞). Using the method of [5] and letting $K = \max_{x\in[-E,E]} |f(x)|$, one obtains explicit bounds on $\lim_{r\to\infty} \int_{-r}^{r} f(y) \cos(ky)\, dy$ and $\lim_{r\to\infty} \int_{-r}^{r} f(y) \sin(ky)\, dy$, showing they remain finite. A careful estimation yields

$$\left| \lim_{r\to\infty} \int_{-r}^{r} f(y)\, e^{-iky}\, dy \right| \;\leq\; N_k,$$

for some constant N_k depending on k. Consequently, $\mathcal{F}(f)(k)$ is bounded for $|k| > k_0 > 0$. A similar argument applies to $\mathcal{F}(\frac{df}{dx})(k)$.

Next, we use integration by parts to relate the Fourier transforms of f and $\frac{df}{dx}$. Observe that

$$\mathcal{F}\left(\tfrac{df}{dx}\right)(k) = \frac{1}{\sqrt{2\pi}} \lim_{r\to\infty} \int_{-r}^{r} \frac{df}{dx}(y)\, e^{-iky}\, dy.$$

Integrating by parts gives

$$\frac{1}{\sqrt{2\pi}} \lim_{r\to\infty} \Big[f(y)\, e^{-iky} \Big]_{-r}^{r} \;+\; \frac{ik}{\sqrt{2\pi}} \lim_{r\to\infty} \int_{-r}^{r} f(y)\, e^{-iky}\, dy = ik\, \mathcal{F}(f)(k),$$

since $\lim_{|y|\to\infty} f(y)\, e^{-iky} = 0$. Therefore,

$$\mathcal{F}\left(\tfrac{df}{dx}\right)(k) = ik\, \mathcal{F}(f)(k).$$

Hence, for $|k| > 1$,

$$\big|\mathcal{F}(f)(k)\big| \;\leq\; \frac{\big|\mathcal{F}(\frac{df}{dx})(k)\big|}{|k|}. \tag{1}$$

Since $\frac{df}{dx}$ is also of very moderate decrease and non-oscillatory, a refinement of the argument in [5, Lemma 0.9] (using "underflow") shows that for every $r > 0$, there exist $F_r, G_r \in \mathcal{R}_{>0}$ such that for all $|k| > F_r$,

$$\left| \frac{1}{\sqrt{2\pi}} \int_{-r}^{r} \frac{df}{dx}(y)\, e^{-iky}\, dy \right| \;<\; \frac{G_r}{|k|}. \tag{2}$$

Moreover, F_r, G_r can be chosen uniformly in r. From (2), it follows that there exist constants F and G for which

$$\big|\mathcal{F}(\tfrac{df}{dx})(k)\big| \;<\; \frac{G}{|k|} \quad \text{for all } |k| > F.$$

Combining this with (1) implies that for $|k| > \max(F, 1)$,

$$\big|\mathcal{F}(f)(k)\big| \;\leq\; \frac{\big|\mathcal{F}(\frac{df}{dx})(k)\big|}{|k|} \;<\; \frac{G}{|k|^2}.$$

This completes the proof. □

Definition 2. Let $f \in C^3(\mathcal{R})$, with f, f', f'', and f''' bounded. We define an approximating sequence $\{f_m : m \in \mathcal{N}\}$ by the following properties:

(i) $f_m \in C^2(\mathcal{R})$, for all $m \in \mathcal{N}$.
(ii) $f_m|_{[-m,m]} = f|_{[-m,m]}$.

(iii) f_m exhibits uniform moderate decay, meaning there exists a constant $C \in \mathcal{R}_{>0}$, independent of m, such that

$$|f_m(x)| \leq \frac{C}{|x|^2}, \quad \text{for } x \in (-\infty, -m - \frac{1}{m}) \cup (m + \frac{1}{m}, \infty).$$

(iv) There exist constants $D, E \in \mathcal{R}_{>0}$ such that

$$\int_{-m-\frac{1}{m}}^{m} |f_m(x)| dx \leq \frac{D}{m} \quad \text{and} \quad \int_{m}^{m+\frac{1}{m}} |f_m(x)| dx \leq \frac{E}{m}.$$

Lemma 2. *Let $f \in C(\mathcal{R})$ with $\frac{df}{dx} \in C(\mathcal{R})$, and suppose both f and $\frac{df}{dx}$ exhibit very moderate decrease and are non-oscillatory. Let $\{f_m : m \in \mathcal{N}\}$ be an approximating sequence. Let $\mathcal{F}$ denote the ordinary Fourier transform, defined for each f_m. Then, for any $k_0 > 0$, the sequence $\{\mathcal{F}(f_m) : m \in \mathcal{N}\}$ converges pointwise and uniformly to $\mathcal{F}(f)$ on $\mathcal{R} \setminus \{|k| < k_0\}$, where $\mathcal{F}(f)$ is defined in Lemma 1. In particular, $\mathcal{F}(f) \in C(\mathcal{R} \setminus \{0\})$.*

Proof. For $g \in C(\mathcal{R})$ and $n \in \mathcal{N}$, define

$$\mathcal{F}_n(g)(k) = \frac{1}{(2\pi)^{1/2}} \int_{-n}^{n} g(y) e^{-iky} dy.$$

For $k \in \mathcal{R} \setminus \{|k| < k_0\}$, $\{m, n\} \subset \mathcal{N}$, $m \geq n$, and $\epsilon, \delta > 0$, we have

$$|\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)| \leq |\mathcal{F}(f)(k) - \mathcal{F}_m(f)(k)| + |\mathcal{F}_m(f)(k) - \mathcal{F}_m(f_m)(k)| + |\mathcal{F}_m(f_m)(k) - \mathcal{F}(f_m)(k)|.$$

Breaking this into terms:

$$\begin{aligned} &|\mathcal{F}(f)(k) - \mathcal{F}_m(f)(k)| + \int_{-\infty}^{-m} |f_m(x)| dx + \int_{m}^{\infty} |f_m(x)| dx \\ &\leq |\mathcal{F}(f)(k) - \mathcal{F}_m(f)(k)| + \frac{D+E}{m} + \int_{-\infty}^{-m-\frac{1}{m}} \frac{C}{x^2} dx + \int_{m+\frac{1}{m}}^{\infty} \frac{C}{x^2} dx. \end{aligned}$$

By the properties of the approximating sequence and the fact that f is of very moderate decrease and non-oscillatory, it follows that for $|k| > k_0$ and sufficiently large m, we can bound the difference as

$$|\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)| \leq \frac{C_{k_0}}{m} + \frac{2C + D + E}{m} \leq \epsilon + \delta,$$

where C_{k_0} is a constant dependent on k_0.

Since $\epsilon > 0$ and $\delta > 0$ are arbitrary, the result follows. Additionally, the continuity of each $\mathcal{F}(f_m)$ is a consequence of property (iii) in Definition 2 and the Dominated Convergence Theorem. Finally, as $k_0 > 0$ is arbitrary, the uniform limit of continuous functions is continuous, completing the proof. □

Lemma 3. *If $m \in \mathcal{R}_{>0}$ is sufficiently large, $\{a_0, a_1, a_2\} \subset \mathcal{R}$, there exists $h \in \mathcal{R}[x]$ of degree 5, with the property that;*

$$h(m) = a_0, \quad h'(m) = a_1, \quad h''(m) = a_2, \tag{3}$$

$$h(m + \frac{1}{m}) = h'(m + \frac{1}{m}) = h''(m + \frac{1}{m}) = 0, \tag{4}$$

$$|h_{[m, m+\frac{1}{m}]}| \leq C,$$

for some $C \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $h'''(m) > 0$, $h'''(x)|_{[m,m+\frac{1}{m}]} > 0$, if $h'''(m) < 0$, $h'''|_{[m,m+\frac{1}{m}]} < 0$. In particularly;

$$\int_{m}^{m+\frac{1}{m}} |h'''(x)| dx = |a_2|.$$

Proof. If $p(x)$ is any polynomial, we have that $h(x) = (x - (m + \frac{1}{m}))^3 p(x)$ satisfies condition (4). Then;

$$\begin{aligned}
h'(x) &= 3(x - (m + \frac{1}{m}))^2 p(x) + (x - (m + \frac{1}{m}))^3 p'(x),\\
h''(x) &= 6(x - (m + \frac{1}{m}))p(x) + 6(x - (m + \frac{1}{m}))^2 p'(x) + (x - (m + \frac{1}{m}))^3 p''(x),\\
h'''(x) &= 6p(x) + 18(x - (m + \frac{1}{m}))p'(x) + 9(x - (m + \frac{1}{m}))^2 p''(x).
\end{aligned}$$

So we can satisfy (3), by requiring that;

$$\begin{aligned}
&-\frac{p(m)}{m^3} = a_0,\\
&\frac{3p(m)}{m^2} - \frac{p'(m)}{m^3} = a_1,\\
&\frac{-6p(m)}{m} + \frac{6p'(m)}{m^2} - \frac{p''(m)}{m^3} = a_2,
\end{aligned}$$

which has the solution;

$$p(m) = -a_0 m^3, \quad p'(m) = -3a_0 m^4 - a_1 m^3, \quad p''(m) = -12a_0 m^5 - 6a_1 m^4 - a_2 m^3$$

and can be satisfied by the polynomial;

$$\begin{aligned}
p(x) &= \frac{1}{2}\left(-12a_0 m^5 - 6a_1 m^4 - a_2 m^3\right)(x - m)^2\\
&\quad + \left(-3a_0 m^4 - a_1 m^3\right)(x - m) + \left(-a_0 m^3\right)\\
&= \frac{1}{2}\left(-12a_0 m^5 - 6a_1 m^4 - a_2 m^3\right)x^2\\
&\quad + \left(-m\left(-12a_0 m^5 - 6a_1 m^4 - a_2 m^3\right) + \left(-3a_0 m^4 - a_1 m^3\right)\right)x\\
&\quad + \left(\frac{m^2}{2}\left(-12a_0 m^5 - 6a_1 m^4 - a_2 m^3\right) - m\left(-3a_0 m^4 - a_1 m^3\right) - a_0 m^3\right)\\
&= \left(-6a_0 m^5 - 3a_1 m^4 - \frac{a_2}{2} m^3\right)x^2\\
&\quad + \left(12a_0 m^6 + 6a_1 m^5 + a_2 m^4 - 3a_0 m^4 - a_1 m^3\right)x\\
&\quad + \left(-6a_0 m^7 - 3a_1 m^6 - \frac{a_2}{2} m^5 + 3a_0 m^5 + a_1 m^4 - a_0 m^3\right)\\
&= \left(-6a_0 m^5 - 3a_1 m^4 - \frac{a_2}{2} m^3\right)x^2\\
&\quad + \left(12a_0 m^6 + 6a_1 m^5 + (a_2 - 3a_0) m^4 - a_1 m^3\right)x\\
&\quad + \left(-6a_0 m^7 - 3a_1 m^6 + \left(3a_0 - \frac{a_2}{2}\right) m^5 + 3a_0 m^5 + a_1 m^4 - a_0 m^3\right)\\
&= ax^2 + bx + c, \qquad (5)
\end{aligned}$$

so that;

$$\begin{aligned}
h'''(x) &= 6\left(ax^2 + bx + c\right) + 18\left(x - \left(m + \frac{1}{m}\right)\right)(2ax + b) + 9\left(x - \left(m + \frac{1}{m}\right)\right)^2 2a\\
&= (60a)x^2 + \left(24b - 72a\left(m + \frac{1}{m}\right)\right)x + \left(6c - 18\left(m + \frac{1}{m}\right)b + 18a\left(m + \frac{1}{m}\right)^2\right)
\end{aligned}$$

and, using the computation (5)

$$\begin{aligned}
h'''(x) = &\left(60(-6a_0 m^5) + O(m^4)\right)x^2 + \left(24.12a_0 m^6 - 72m(-6a_0 m^5) + O(m^5)\right)x\\
&+ \left(6. - 6a_0 m^7 - 18m(12a_0 m^6) + 18m^2(-6a_0 m^5) + O(m^6)\right)
\end{aligned}$$

$$= \big(-360a_0m^5+O(m^4)\big)x^2+\big(740a_0m^6+O(m^5)\big)x+\big(-360a_0m^7+O(m^6)\big)$$

which, by the quadratic formula, has roots when;

$$\begin{aligned}x &= \frac{-740a_0m^6-\sqrt{740^2a_0^2m^{12}-4(-360a_0m^5)(-360a_0m^7)}}{2-360a_0m^5}+O(1)\\&=\frac{740m}{720}-\frac{170m}{720}+O(1)\\&=\frac{19m}{24}+O(1)\quad\text{or}\quad\frac{91m}{72}+O(1).\end{aligned}$$

We have that $m>\frac{19m}{24}$ and $m+\frac{1}{m}<\frac{91m}{72}$ iff $m>\sqrt{\frac{72}{19}}$, and, clearly, we can ignore the $O(1)$ term for m sufficiently large. In particular, for sufficiently large m, $h'''(x)$ has no roots in the interval $[m,m+\frac{1}{m}]$, so $h'''|_{[m,m+\frac{1}{m}]}>0$ or $h'''|_{[m,m+\frac{1}{m}]}<0$. We calculate that;

$$\begin{aligned}|h_{[m,m+\frac{1}{m}]}| &= \Big|(x-(m+\tfrac{1}{m}))^3p(x)\Big|_{[m,m+\frac{1}{m}]}\leq\frac{1}{m^3}\,|p(x)|_{[m,m+\frac{1}{m}]}\\&=\frac{1}{m^3}\left|\frac{1}{2}(-12a_0m^5-6a_1m^4-a_2m^3)(x-m)^2\;+(-3a_0m^4-a_1m^3)(x-m)-a_0m^3\right|_{[m,m+\frac{1}{m}]}\\&\leq\frac{1}{m^3}\left[\frac{1}{2}\left|-12a_0m^5-6a_1m^4-a_2m^3\right|\cdot\frac{1}{m^2}\;+\left|-3a_0m^4-a_1m^3\right|\cdot\frac{1}{m}+\left|-a_0m^3\right|\right]\\&\leq\frac{12|a_0|m^5+6|a_1|m^4+|a_2|m^3}{m^5}+\frac{3|a_0|m^4+|a_1|m^3}{m^4}+\frac{|a_0|m^3}{m^3}\\&\leq 12|a_0|+6|a_1|+|a_2|+3|a_0|+|a_1|+|a_0|\quad(m>1)\\&\leq 16|a_0|+7|a_1|+|a_2|.\end{aligned}$$

For the final claim, we have, as $h'''|_{[m,m+\frac{1}{m}]}>0$ or $h'''|_{[m,m+\frac{1}{m}]}<0$, that, using the fundamental theorem of calculus;

$$\int_m^{m+\frac{1}{m}}|h'''(x)|dx=\Big|\int_m^{m+\frac{1}{m}}h'''(x)dx\Big|=|h''(m+\frac{1}{m})-h''(m)|=|-h''(m)|=|a_2|.$$

□

Lemma 4. *If $m\in\mathcal{R}_{>0}$, $\{a_0,a_1,a_2,a_3\}\subset\mathcal{R}$, there exists $h\in C^3(\mathcal{R})$, with the property that;*

$$h(m)=a_0,\quad h'(m)=a_1,\quad h''(m)=a_2,\quad h'''(m)=a_3, \tag{6}$$

$$h(m+\frac{1}{m})=h'(m+\frac{1}{m})=h''(m+\frac{1}{m})=h'''(m+\frac{1}{m})=0, \tag{7}$$

$$|h|_{[m,m+1]}\leq C,$$

where $C\in\mathcal{R}_{>0}$ is independent of $m>1$, and, if $a_3>0$, $h'''(x)|_{[m,m+\frac{1}{m}]}\geq 0$, $a_3<0$, $h'''(x)|_{[m,m+\frac{1}{m}]}\leq 0$. In particularly;

$$\int_m^{m+\frac{1}{m}}|h'''(x)|dx=|a_2|.$$

Proof. Let $g(x)$ be a polynomial. Consider the polynomial

$$h_1(x)=(x-(m+\frac{1}{m}))^ng(x),\quad n\geq 4,$$

which possesses the property (7):

$$h_1(m+\frac{1}{m}) = h_1'(m+\frac{1}{m}) = h_1''(m+\frac{1}{m}) = h_1'''(m+\frac{1}{m}) = 0.$$

The condition (6) translates into the following system of equations:

$$\text{(i)}' \quad \frac{g(m)}{(-1)^n m^n} = a_0,$$

$$\text{(ii)}' \quad \frac{ng(m)}{(-1)^{n-1}m^{n-1}} + \frac{g'(m)}{(-1)^n m^n} = a_1,$$

$$\text{(iii)}' \quad \frac{n(n-1)g(m)}{(-1)^{n-2}m^{n-2}} + \frac{2ng'(m)}{(-1)^{n-1}m^{n-1}} + \frac{g''(m)}{(-1)^n m^n} = a_2,$$

$$\text{(iv)}' \quad \frac{n(n-1)(n-2)g(m)}{(-1)^{n-3}m^{n-3}} + \frac{3n(n-1)g'(m)}{(-1)^{n-2}m^{n-2}} + \frac{3ng''(m)}{(-1)^{n-1}m^{n-1}} + \frac{g'''(m)}{(-1)^n m^n} = a_3.$$

These equations can be solved by imposing the following requirements:

$$\begin{cases} \text{(i)}'' & g(m) = (-1)^n a_0 m^n, \\ \text{(ii)}'' & g'(m) = (-1)^n a_1 m^n + (-1)^n a_0 n m^{n+1}, \\ \text{(iii)}'' & g''(m) = (-1)^n a_2 m^n + 2(-1)^n n a_1 m^{n+1} + (-1)^n n(n+1) a_0 m^{n+2}, \\ \text{(iv)}'' & g'''(m) = (-1)^n a_3 m^n + 3n(-1)^n a_2 m^{n+1} + (-1)^n a_1 n(n+3) m^{n+2} + n(n+1)(n+2)(-1)^n a_0 m^{n+3}. \end{cases} \tag{8}$$

Define

$$\begin{aligned} g_1(x) =& \Big((-1)^n a_3 m^n + 3n(-1)^n a_2 m^{n+1} + (-1)^n a_1 n(n+3) m^{n+2} + n(n+1)(n+2)(-1)^n a_0 m^{n+3}\Big)(x-m)^3 \\ &+ \Big((-1)^n a_2 m^n + 2(-1)^n n a_1 m^{n+1} + (-1)^n n(n+1) a_0 m^{n+2}\Big)(x-m)^2 \\ &+ \Big((-1)^n a_1 m^n + (-1)^n a_0 n m^{n+1}\Big)(x-m) + (-1)^n a_0 m^n. \end{aligned}$$

The polynomial $g_1(x)$ satisfies (8), and any function of the form $g_2(x) + g_1(x)$ also satisfies (8), provided that

$$g_2(m) = g_2'(m) = g_2''(m) = g_2'''(m) = 0,$$

where $g_2 \in C^3(\mathbb{R})$.

In this case, if

$$h(x) = (x - (m+\frac{1}{m}))^n (g_2(x) + g_1(x)),$$

then $h(x)$ satisfies both (6) and (7).

$$\begin{aligned} &|x - (m+\frac{1}{m})^n g_1(x)|_{[m,m+\frac{1}{m}]} \le \frac{1}{m^n}\left(|g_2|_{[m,m+\frac{1}{m}]} + |g_1|_{[m,m+\frac{1}{m}]}\right) \\ &\le \frac{1}{m^n}\Bigg(|g_2|_{[m,m+\frac{1}{m}]} + \frac{1}{m^n}\Big|(-1)^n a_3 m^n + 3n(-1)^n a_2 m^{n+1} + (-1)^n a_1 n(n+3) m^{n+2} \\ &\quad + n(n+1)(n+2)(-1)^n a_0 m^{n+3}\Big)\frac{1}{m^3} + \Big((-1)^n a_2 m^n + 2(-1)^n n a_1 m^{n+1} \\ &\quad + (-1)^n n(n+1) a_0 m^{n+2}\Big)\frac{1}{m^2} + \Big((-1)^n a_1 m^n + (-1)^n a_0 n m^{n+1}\Big)\frac{1}{m} + (-1)^n a_0 m^n\Big|\Bigg) \\ &= \Big|\Big((-1)^n a_3 m^n + 3n(-1)^n a_2 m^{n+1} + (-1)^n a_1 n(n+3) m^{n+2} \\ &\quad + n(n+1)(n+2)(-1)^n a_0 m^{n+3}\Big)\frac{1}{m^{n+3}} + \Big((-1)^n a_2 m^n + 2(-1)^n n a_1 m^{n+1} \end{aligned}$$

$$
\begin{aligned}
&+(-1)^n n(n+1)a_0 m^{n+2}\Big)\frac{1}{m^{n+2}}+\Big((-1)^n a_1 m^n+(-1)^n a_0 n m^{n+1}\Big)\frac{1}{m^{n+1}}+(-1)^n a_0\Big|\\
&\leq |a_3|+3n|a_2|+n(n+3)|a_1|+n(n+1)(n+2)|a_0|+|a_2|+2n|a_1|\\
&\quad+n(n+1)|a_0|+|a_1|+n|a_0|+|a_0|,\quad (m\geq 1)\\
&=\frac{1}{m^n}\Big(|g_2|_{[m,m+\frac{1}{m}]}+(n+1)(n^2+3n+1)|a_0|+(n^2+5n+1)|a_1|+(3n+1)|a_2|+|a_3|\Big)=F, \qquad (9)
\end{aligned}
$$

where $F\in\mathcal{R}_{>0}$ is independent of m. Using the product rule, the condition that $h'''(x)=0$ in the interval $(m,m+\frac{1}{m})$, is given by; $n(n-1)(n-2)(x-(m+\frac{1}{m}))^{n-3}(g_2+g_1)(x)+3n(n-1)(x-(m+\frac{1}{m}))^{n-2}(g_2+g_1)'(x)+3n(x-(m+\frac{1}{m}))^{n-1}(g_2+g_1)''(x)+(x-(m+\frac{1}{m}))^n(g_2+g_1)'''(x)=0$ which, dividing by $(x-(m+\frac{1}{m}))^{n-3}$, reduces to; $n(n-1)(n-2)(g_2+g_1)(x)+3n(n-1)(x-(m+\frac{1}{m}))(g_2+g_1)'(x)+3n(x-(m+\frac{1}{m}))^2(g_2+g_1)''(x)+(x-(m+\frac{1}{m}))^3(g_2+g_1)'''(x)=0$ and;

$$
\begin{aligned}
&n(n-1)(n-2)g_2(x)+3n(n-1)(x-(m+\frac{1}{m}))g_2'(x)\\
&\quad+3n(x-(m+\frac{1}{m}))^2g_2''(x)+(x-(m+\frac{1}{m}))^3g_2'''(x)\\
&=-(n(n-1)(n-2)g_1(x)+3n(n-1)(x-(m+\frac{1}{m}))g_1'(x)\\
&\quad+3n(x-(m+\frac{1}{m}))^2g_1''(x)+(x-(m+\frac{1}{m}))^3g_1'''(x)). \qquad (10)
\end{aligned}
$$

Without loss of generality, assuming that; $-(n(n-1)(n-2)g_1(x)+3n(n-1)(x-(m+\frac{1}{m}))g_1'(x)+3n(x-(m+\frac{1}{m}))^2g_1''(x)+(x-(m+\frac{1}{m}))^3g_1'''(x))|_m=-(n(n-1)(n-2)a_0-\frac{3n(n-1)a_1}{m}+\frac{3na_2}{m^2}-\frac{a_3}{m^3}\geq 0$, we can choose an analytic function $\phi(x)$ on $[m,m+\frac{1}{m}]$ with;

$$
\begin{aligned}
\phi(x)\leq&-(n(n-1)(n-2)g_1(x)+3n(n-1)(x-(m+\frac{1}{m}))g_1'(x)\\
&+3n(x-(m+\frac{1}{m}))^2g_1''(x)+(x-(m+\frac{1}{m}))^3g_1'''(x)) \qquad (11)\\
\phi(m)=&0. \qquad (12)
\end{aligned}
$$

The third order differential equation for g_2;

$$
\begin{aligned}
&n(n-1)(n-2)g_2(x)+3n(n-1)(x-(m+\frac{1}{m}))g_2'(x)\\
&\quad+3n(x-(m+\frac{1}{m}))^2g_2''(x)+(x-(m+\frac{1}{m}))^3g_2'''(x)=\phi(x),\quad\text{on}\quad [m,1+m], \qquad (13)
\end{aligned}
$$

with the requirement that $g_2(m)=g_2'(m)=g_2''(m)=0$, has a solution in $C^3([m,m+\frac{1}{m}))$ by Peano's existence theorem. By the fact (12), we must have that $g_2'''(m)=0$. Writing the power series for ϕ on $[m,m+\frac{1}{m}]$, as; $\phi(x)=\sum_{j=0}^{\infty}b_j(x-(m+\frac{1}{m}))^j$, we can use the method of equating coefficients, to obtain a particular solution, with; $g_{2,part}(x)=\sum_{j=0}^{\infty}a_{j,part}(x-(m+\frac{1}{m}))^j$, with; $a_{j,part}=\frac{b_j}{n(n-1)(n-2)+3n(n-1)j+3nj(j-1)+j(j-1)(j-2)}$, $(j\geq 3)$ $a_{2,part}=\frac{b_2}{n(n-1)(n-2)+6n(n-1)+3n}$ $a_{1,part}=\frac{b_1}{n(n-1)(n-2)+3n(n-1)}$ $a_{0,part}=\frac{b_0}{n(n-1)(n-2)}$ so that $g_{2,part}$ is analytic as $|a_{j,0}|\leq\frac{|b_j|}{n(n-1)(n-2)}$ for $j\geq 0$. To solve the homogenous Euler equation;

$$
\begin{aligned}
&n(n-1)(n-2)g_2(x)+3n(n-1)(x-(m+\frac{1}{m}))g_2'(x)+3n(x-(m+\frac{1}{m}))^2g_2''(x)\\
&\quad+(x-(m+\frac{1}{m}))^3g_2'''(x)=0\quad\text{on}\quad [m,m+\frac{1}{m}],
\end{aligned}
$$

we can make the substitution $y=m+\frac{1}{m}-x$, to reduce to the equation;

$$
n(n-1)(n-2)g_{2,m}(y)+3n(n-1)yg_{2,m}'(y)+3ny^2g_{2,m}''(y)+y^3g_{2,m}'''(y)=0\quad\text{on}\quad [0,\frac{1}{m}]
$$

with $g_{2,m}(y) = g_2(m + \frac{1}{m} - y)$. Making the further substitution $y = e^u$, and letting $r_{2,m}(u) = g_{2,m}(e^u)$, we have that;

$$\begin{aligned}
r'_{2,m}(u) &= g'_{2,m}(e^u)e^u,\\
r''_{2,m}(u) &= g''_{2,m}(e^u)e^{2u} + g'_{2,m}(e^u)e^u,\\
r'''_{2,m}(u) &= g'''_{2,m}(e^{3u}) + 3g''_{2,m}(e^u)e^{2u} + g'_{2,m}(e^u)e^u,
\end{aligned}$$

so that;

$$\begin{aligned}
&n(n-1)(n-2)g_{2,m}(e^u) + 3n(n-1)e^u g'_{2,m}(e^u) + 3ne^{2u}g''_{2,m}(e^u) + e^{3u}g'''_{2,m}(e^u)\\
&= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)e^u(r'_{2,m}(u)e^{-u}) + 3ne^{2u}((r''_{2,m}(u) - g'_{2,m}(e^u)e^u)e^{-2u})\\
&\quad + e^{3u}((r'''_{2,m}(u) - 3g''_{2,m}(e^u)e^{2u} - g'_{2,m}(e^u)e^u)e^{-3u})\\
&= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)r'_{2,m}(u) + 3nr''_{2,m}(u) - 3ng'_{2,m}(e^u)e^u + r'''_{2,m}(u)\\
&\quad - 3g''_{2,m}(e^u)e^{2u} - g'_{2,m}(e^u)e^u\\
&= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)r'_{2,m}(u) + 3nr''_{2,m}(u) + r'''_{2,m}(u)\\
&\quad - (3n+1)g'_{2,m}(e^u)e^u - 3g''_{2,m}(e^u)e^{2u}\\
&= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)r'_{2,m}(u) + 3nr''_{2,m}(u) + r'''_{2,m}(u)\\
&\quad - (3n+1)r'_{2,m}(u) - 3e^{2u}((r''_{2,m}(u) - g'_{2,m}(e^u)e^u)e^{-2u})\\
&= n(n-1)(n-2)r_{2,m}(u) + (3n^2 - 6n - 1)r'_{2,m}(u) + 3nr''_{2,m}(u) + r'''_{2,m}(u)\\
&\quad - 3r''_{2,m}(u) + 3g'_{2,m}(e^u)e^u\\
&= n(n-1)(n-2)r_{2,m}(u) + (3n^2 - 6n - 1)r'_{2,m}(u) + 3(n-1)r''_{2,m}(u) + r'''_{2,m}(u) + 3r'_{2,m}(u)\\
&= n(n-1)(n-2)r_{2,m}(u) + (3n^2 - 6n + 2)r'_{2,m}(u) + (3n-3)r''_{2,m}(u) + r'''_{2,m}(u)\\
&= 0. \qquad (14)
\end{aligned}$$

We have that;

$$(\lambda^3 + 3(n-1)\lambda^2 + (3n^2 - 6n + 2)\lambda + n(n-1)(n-2))' = 3\lambda^2 + 6(n-1)\lambda + (3n^2 - 6n + 2),$$

which has roots when $\lambda = -(n-1) + \frac{1}{\sqrt{3}}$, so that, for large n, the characteristic polynomial of (14) has exactly one real root λ_1 and 2 complex conjugate non-real roots, $\{\lambda_2 + i\lambda_3, \lambda_2 - i\lambda_3\}$. It follows, the general solution of (14) is given by;

$$r_{2,m}(u) = A_1 e^{\lambda_1 u} + A_2 e^{\lambda_2 u + i\lambda_3} + A_3 e^{\lambda_2 u - i\lambda_3},$$

where $\{A_1, A_2, A_3\} \subset \mathcal{C}$, and, we can obtain a real solution, fitting the corresponding initial conditions, of the form;

$$r_{2,m}(u) = B_1 e^{\lambda_1 u} + B_2 e^{\lambda_2 u} cos(\lambda_3 u) + B_3 e^{\lambda_2 u} sin(\lambda_3 u),$$

where $\{B_1, B_2, B_3\} \subset \mathcal{R}$. It follows that;

$$\begin{aligned}
g_{2,m}(y) =& r_{2,m}(ln(y)),\\
g_2(x) =& g_{2,m}(m + \frac{1}{m} - x) + g_{2,part}(x) = r_{2,m}(ln(m + \frac{1}{m} - x)) + g_{2,part}(x)\\
=& B_1 e^{\lambda_1 ln(m + \frac{1}{m} - x)} + B_2 e^{\lambda_2 ln(m + \frac{1}{m} - x)} cos(\lambda_3 ln(m + \frac{1}{m} - x))\\
&+ B_3 e^{\lambda_2 ln(m + \frac{1}{m} - x)} sin(\lambda_3 ln(m + \frac{1}{m} - x)) + g_{2,part}(x),
\end{aligned}$$

on $[m, m + \frac{1}{m}]$. We have that;

$$\lambda_1 |\lambda_2 + i\lambda_3|^2 = -n(n-1)(n-2)$$

$$\lambda_1 + \lambda_2 + i\lambda_3 + \lambda_2 - i\lambda_3 = \lambda_1 + 2\lambda_2 = -3(n-1).$$

Computing the highest degree in n term of the characteristic polynomial, we obtain that, for $\lambda = \alpha n$;

$$\alpha^3 n^3 + 3n(\alpha n)^2 + 3n^2(\alpha n) + n^3 = n^3(\alpha + 3)^3 = 0,$$

so that $\alpha = -3$, $\lambda_1 = -3n + O(1)$ and $2\lambda_2 = -3(n-1) - (-3n + O(1)) = 3 - O(1) = O(1)$.

Then, if $B_1 = 0$, we can see that $g_2(x)$ has at most a $\frac{1}{x^{O(1)}}$ singularity at $(m + \frac{1}{m})$, which we can achieve with a 2-parameter family choice for the initial conditions of $\{\phi(m), \phi'(m), \phi''(m)\}$. If;

$$-(n(n-1)(n-2)a_0 - \frac{3n(n-1)a_1}{m} + \frac{3na_2}{m^2} - \frac{a_3}{m^3} \neq 0,$$

we can clearly achieve this, while satisfying (11),(12). If;

$$-(n(n-1)(n-2)a_0 - \frac{3n(n-1)a_1}{m} + \frac{3na_2}{m^2} - \frac{a_3}{m^3} = 0$$

by requiring the the additional property (14);

$$\phi'(m) < -(n(n-1)(n-2)g_1(x) + 3n(n-1)(x - (m + \frac{1}{m}))g_1'(x) + 3n(x - (m + \frac{1}{m}))^2 g_1''(x) + (x - (m + \frac{1}{m}))^3 g_1'''(x))'|_m,$$

we can clearly satisfy (11),(12) as well. Then, as, for sufficiently large n;

$$\begin{aligned}
&\lim_{x\to 0}\left(\frac{B_2 x^n}{x^{O(1)}}\sin(\lambda_3 \ln(x)) + \frac{B_3 x^n}{x^{O(1)}}\cos(\lambda_3 \ln(x))\right) \\
&= \lim_{x\to 0}\left(\frac{B_2 x^n}{x^{O(1)}}\sin(\lambda_3 \ln(x)) + \frac{B_3 x^n}{x^{O(1)}}\cos(\lambda_3 \ln(x))\right)' \\
&= \lim_{x\to 0}\left(\frac{B_2 x^n}{x^{O(1)}}\sin(\lambda_3 \ln(x)) + \frac{B_3 x^n}{x^{O(1)}}\cos(\lambda_3 \ln(x))\right)'' \\
&= \lim_{x\to 0}\left(\frac{B_2 x^n}{x^{O(1)}}\sin(\lambda_3 \ln(x)) + \frac{B_3 x^n}{x^{O(1)}}\cos(\lambda_3 \ln(x))\right)''' = 0,
\end{aligned}$$

we obtain that $(x - (m + \frac{1}{m}))^n g_2(x)$ extends to a solution in $C^3([m, m + \frac{1}{m}])$, and $(x - (m + \frac{1}{m}))^n (g_2 + g_1)(x) \in C^3([m, m + \frac{1}{m}])$. By the fact (11), (10) has no solutions in $(m, m + \frac{1}{m})$, so that $h'''(x) \geq 0$.

We have that;

$$\begin{aligned}
|(x - (m + \frac{1}{m})^n g_2(x)|_{[m,m+\frac{1}{m}]} =& |(x - (m + \frac{1}{m})^n (B_2 e^{\lambda_2 ln(m + \frac{1}{m} - x)} cos(\lambda_3 ln(m + \frac{1}{m} - x)) \\
&+ B_3 e^{\lambda_2 ln(m + \frac{1}{m} - x)} sin(\lambda_3 ln(m + \frac{1}{m} - x)) + g_{2,part}(x))| \\
\leq& |B_2| m^{\lambda_2 - n} + |B_3| m^{\lambda_2 - n} + m^{-n} |g_{2,part}(x)|.
\end{aligned}$$

Noting the right hand side of (11) is bounded by $O(m^n)$ on $[m, m + \frac{1}{m}]$, we can also choose $\phi(x)$ and $g_{2,part}(x)$ to be of $O(m^n)$ on $[m, m + \frac{1}{m}]$, irrespective of the choice of initial conditions $\{\phi(m), \phi'(m), \phi''(m)\}$. We have that $\phi'(m) = O(m^{n+1})$, in the special case, so that choosing $\{B_2, B_3\}$ sufficiently small, noting; $(x - (m + \frac{1}{m}))^n (B_2 e^{\lambda_2 ln(m + \frac{1}{m} - x)} cos(\lambda_3 ln(m + \frac{1}{m} - x)) + B_3 e^{\lambda_2 ln(m + \frac{1}{m} - x)} sin(\lambda_3 ln(m + \frac{1}{m} - x)))'|_m = O(max(B_2 m^{n - \lambda_2 - 1}, B_3 m^{n - \lambda_2 - 1}))$ we can assume that; $|(x - (m + \frac{1}{m})^n g_2(x)|_{[m,m+\frac{1}{m}]} \leq D$ where $D \in \mathcal{R}_{>0}$ is independent of m, so that; $|h(x)|_{[m,m+\frac{1}{m}]} \leq |(x - (m + \frac{1}{m})^n g_1(x)|_{[m,m+\frac{1}{m}]} + |(x - (m + \frac{1}{m})^n g_2(x)|_{[m,m+\frac{1}{m}]} \leq F + D$. For the final claim, we have, as $h'''|_{[m,m+\frac{1}{m}]} \geq 0$ or $h'''|_{[m,m+\frac{1}{m}]} \leq 0$, that, using the fundamental theorem of calculus, that; $\int_m^{m+\frac{1}{m}} |h'''(x)| dx = |\int_m^{m+\frac{1}{m}} h'''(x) dx| = |h''(m + \frac{1}{m}) - h''(m)| = |-h''(m)| = |a_2|$.

□

Lemma 5. *Let f be a function as specified in Definition 2. Then, there exists a sequence of approximating functions $\{f_m\}_{m\in\mathbb{N}}$ such that for sufficiently large m, the Fourier transform of f_m satisfies*

$$|\mathcal{F}(f_m)(k)| \leq \frac{Cm}{|k|^3} \quad \textit{for all } |k| > 1,$$

where $C > 0$ is a constant independent of m.

Proof. We construct the approximating sequence $\{f_m\}$ as follows:

$$f_m(x) = \begin{cases} f(x), & \text{for } x \in [-m, m], \\ h_{1,m}(x), & \text{for } x \in \left[-m - \frac{1}{m}, -m\right], \\ h_{2,m}(x), & \text{for } x \in \left[m, m + \frac{1}{m}\right], \\ 0, & \text{otherwise}, \end{cases}$$

where $h_{1,m}, h_{2,m} \in C^2\left([-m - \frac{1}{m}, -m] \cup [m, m + \frac{1}{m}]\right)$ are smooth extensions determined by the boundary conditions:

$$h_{1,m}(-m) = f(-m), \quad h'_{1,m}(-m) = f'(-m), \quad h''_{1,m}(-m) = f''(-m),$$
$$h_{2,m}(m) = f(m), \quad h'_{2,m}(m) = f'(m), \quad h''_{2,m}(m) = f''(m).$$

The existence of such extensions is guaranteed by Lemma 3 or Lemma 4.

By construction:

1. $f_m(x) = f(x)$ for all $x \in [-m, m]$, satisfying the approximation condition in Definition 2.
2. f_m smoothly transitions to zero outside the interval $[-m - \frac{1}{m}, m + \frac{1}{m}]$, ensuring condition (ii) of Definition 2.
3. f_m is identically zero on $(-\infty, -m - \frac{1}{m}] \cup [m + \frac{1}{m}, \infty)$, thereby satisfying condition (iii) of Definition 2.

Next, we verify condition (iv). From the proof of Lemma 3 or by invoking Lemma 4, we have

$$\max\left\{\|h_{1,m}\|_{L^\infty[-m-\frac{1}{m},-m]}, \|h_{2,m}\|_{L^\infty[m,m+\frac{1}{m}]}\right\} \leq 16\|f\|_\infty + 7\|f'\|_\infty + \|f''\|_\infty.$$

Thus,

$$\int_{-m-\frac{1}{m}}^{-m} |f_m(x)|\,dx \leq \left(16\|f\|_\infty + 7\|f'\|_\infty + \|f''\|_\infty\right) \cdot \frac{1}{m} = \frac{D}{m},$$
$$\int_{m}^{m+\frac{1}{m}} |f_m(x)|\,dx \leq \left(16\|f\|_\infty + 7\|f'\|_\infty + \|f''\|_\infty\right) \cdot \frac{1}{m} = \frac{E}{m},$$

where $D = E = 16\|f\|_\infty + 7\|f'\|_\infty + \|f''\|_\infty$. This verifies condition (iv) of Definition 2.

For the Fourier transform estimate, consider the third derivative of f_m:

$$\mathcal{F}(f_m''')(k) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f_m'''(x) e^{-ikx}\,dx.$$

Integrating by parts thrice and assuming sufficient decay at infinity, we obtain

$$\mathcal{F}(f_m''')(k) = (ik)^3 \mathcal{F}(f_m)(k).$$

Therefore,

$$|\mathcal{F}(f_m)(k)| = \frac{|\mathcal{F}(f_m''')(k)|}{|k|^3} \leq \frac{1}{|k|^3(2\pi)^{1/2}} \int_{-\infty}^{\infty} |f_m'''(x)|\,dx.$$

Since f_m is equal to f on $[-m, m]$ and smoothly transitions outside this interval, we have

$$\int_{-\infty}^{\infty} |f_m'''(x)|\,dx \leq 2\|f''\|_\infty + 2m\|f'''\|_\infty.$$

Assuming $m > \|f''\|_\infty$, it follows that

$$|\mathcal{F}(f_m)(k)| \le \frac{1}{|k|^3(2\pi)^{1/2}} \left(2\|f''\|_\infty + 2m\|f'''\|_\infty\right) \le \frac{Cm}{|k|^3},$$

where $C = \frac{2+2\|f'''\|_\infty}{(2\pi)^{1/2}}$, independent of m. This establishes the desired Fourier transform bound for $|k| > 1$. □

Lemma 6 (Fourier Inversion). *Let $f \in C^3(\mathbb{R})$ be a thrice continuously differentiable function on the real line such that both f and its first derivative f' are non-oscillatory and exhibit very moderate decay at infinity. Additionally, assume that the set $\{f, f', f'', f'''\}$ is bounded. Then, the Fourier transform $\mathcal{F}(f)$ belongs to $L^1(\mathbb{R})$, and the inversion formula holds, that is,*

$$f(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x),$$

where, for $g \in L^1(\mathbb{R})$,

$$\mathcal{F}^{-1}(g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)e^{ikx}\, dk.$$

Proof. By Lemma 1, there exists a constant $C > 0$ such that

$$|\mathcal{F}(f)(k)| \le \frac{C}{|k|^2},$$

for sufficiently large $|k|$. Since f exhibits very moderate decay, there exists a constant $D > 0$ satisfying

$$|f(x)|^2 \le \frac{D}{|x|^2} \quad \text{for} \quad |x| > 1.$$

Furthermore, since $f \in C^0(\mathbb{R})$, it follows that $f \in L^2(\mathbb{R})$. Consequently, the Fourier transform $\mathcal{F}(f)$ also belongs to $L^2(\mathbb{R})$, and for any $n \in \mathbb{N}$, the restriction $\mathcal{F}(f)|_{[-n,n]}$ belongs to $L^1(\mathbb{R})$. Combining these observations, we conclude that $\mathcal{F}(f) \in L^1(\mathbb{R})$.

Let $\{f_m\}_{m\in\mathbb{N}}$ be an approximating sequence as provided by Lemma 5. Since each $f_m \in L^1(\mathbb{R})$, their Fourier transforms $\mathcal{F}(f_m)$ are continuous functions. Moreover, by Lemma 2, $\mathcal{F}(f_m)$ converges uniformly to $\mathcal{F}(f)$ on $\mathbb{R} \setminus \{0\}$, implying that $\mathcal{F}(f) \in C^0(\mathbb{R} \setminus \{0\})$.

Since each $f_m \in C^2(\mathbb{R})$ and $f''_m \in L^1(\mathbb{R})$, there exist constants $D_m > 0$ such that

$$|\mathcal{F}(f_m)(k)| \le \frac{D_m}{|k|^2},$$

for sufficiently large $|k|$. Furthermore, since $x^n f_m(x) \in L^1(\mathbb{R})$ for all $n \in \mathbb{N}$, it follows that $\mathcal{F}(f_m) \in C^\infty(\mathbb{R})$.

Therefore, by the Fourier inversion theorem (see [4]), we have

$$f_m(x) = \mathcal{F}^{-1}(\mathcal{F}(f_m))(x),$$

for each $m \in \mathbb{N}$.

By Lemma 2, for any fixed $k_0 > 0$, there exists a bound E_{k_0} such that

$$|\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)| \le \frac{E_{k_0}}{m}, \quad \text{for} \quad |k| > k_0.$$

Since f has very moderate decay, $f - f_m \in L^2(\mathbb{R})$, and $\|f - f_m\|_{L^2(\mathbb{R})} \to 0$ as $m \to \infty$. Consequently, $\|\mathcal{F}(f) - \mathcal{F}(f_m)\|_{L^2(\mathbb{R})} \to 0$ as $m \to \infty$. In particular, for sufficiently large m, we have $\|\mathcal{F}(f) - \mathcal{F}(f_m)\|_{L^2(\mathbb{R})} \le 1$.

Let $\epsilon > 0$. Then,

$$\|\mathcal{F}(f) - \mathcal{F}(f_m)\|_{L^1((-\epsilon,\epsilon))} \le \|\mathcal{F}(f) - \mathcal{F}(f_m)\|_{L^2((-\epsilon,\epsilon))} \|1_{(-\epsilon,\epsilon)}\|_{L^2((-\epsilon,\epsilon))} \le \sqrt{2}\epsilon^{1/2}.$$

Thus, for m sufficiently large,

$$\|\mathcal{F}(f) - \mathcal{F}(f_m)\|_{L^1((-\epsilon,\epsilon))} \leq \sqrt{2}\epsilon^{1/2}.$$

Now, let $m = n^{3/2}$ for $n \in \mathbb{N}$. Using Lemma 5, Lemma 2, and the above estimate, for each $x \in \mathbb{R}$, we have

$$\begin{aligned}
\left|\mathcal{F}^{-1}(\mathcal{F}(f))(x) - \mathcal{F}^{-1}(\mathcal{F}(f_m))(x)\right| &= \left|\mathcal{F}^{-1}(\mathcal{F}(f) - \mathcal{F}(f_m))(x)\right| \\
&= \frac{1}{\sqrt{2\pi}} \left| \int_{-n}^{n} (\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)) e^{ikx}\, dk \right. \\
&\quad \left. + \int_{|k|>n} (\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)) e^{ikx}\, dk \right| \\
&\leq \frac{1}{\sqrt{2\pi}} \left(\int_{-n}^{n} |\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)|\, dk \right. \\
&\quad \left. + \int_{|k|>n} |\mathcal{F}(f)(k)|\, dk + \int_{|k|>n} |\mathcal{F}(f_m)(k)|\, dk \right) \\
&\leq \frac{1}{\sqrt{2\pi}} \left(\int_{-\epsilon}^{\epsilon} |\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)|\, dk + \frac{2nE_\epsilon}{m} \right. \\
&\quad \left. + \int_{|k|>n} \frac{C}{|k|^2} dk + \int_{|k|>n} \frac{Cm}{|k|^3} dk \right) \\
&\leq \frac{1}{\sqrt{2\pi}} \left(\sqrt{2}\epsilon^{1/2} + \frac{2nE_\epsilon}{n^{3/2}} + \frac{2C}{n} + \frac{Cn^{3/2}}{n^2} \right) \\
&< 2\epsilon^{1/2},
\end{aligned}$$

for sufficiently large n, where $\frac{E_\epsilon}{m}$ is the bound from Lemma 2.

Since $\epsilon > 0$ was arbitrary, it follows that for each $x \in \mathbb{R}$,

$$\lim_{m\to\infty} \mathcal{F}^{-1}(\mathcal{F}(f_m))(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x),$$

and by Definition 2,

$$f(x) = \lim_{m\to\infty} f_m(x) = \lim_{m\to\infty} \mathcal{F}^{-1}(\mathcal{F}(f_m))(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x).$$

□

Remark 1. The previous lemma proves an inversion theorem for non-oscillatory functions with very moderate decrease. Such functions belong to $L^2(\mathcal{R})$ and an analogous result for Fourier series can be found in [1], where convergence is proved almost everywhere rather than everywhere. The corresponding result for transforms is that;

If $f \in L^p(\mathcal{R})$, $p \in (1,2]$, then;

$$f(x) = lim_{R\to\infty} \frac{1}{2\pi} \int_{|k|\leq R} \mathcal{F}(f)(k) e^{ixk} dk$$

for almost every $x \in \mathcal{R}$.

We can define the function $\mathcal{F}_1(f)(k)$, for $k \in \mathcal{R}$, using the usual Fourier transform transform method, when $f \in L^1(\mathcal{R})$, see [6], and, we can define the function $\mathcal{F}_2(f)(k)$, for $k \in \mathcal{R}_{\neq 0}$, using the limit definition when f is of very moderate decrease and non-oscillatory, a particular case of $f \in L^2(\mathcal{R})$. However, the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ need not commute, so that even if we show that $\mathcal{F}_2 \circ \mathcal{F}_1 = Id$, it doesn't necessarily follow that $\mathcal{F}_1 \circ \mathcal{F}_2 = Id$. The first claim is, in a sense, shown in [7];

If $f \in L^1(\mathcal{R}) \cap C^0(\mathcal{R})$ and $|\mathcal{F}(f)(k)| \leq \frac{A}{|k|}$, for all $k \neq 0$, and $A \in \mathcal{R}_{\geq 0}$, then;

$$f(x) = lim_{R\to\infty} \frac{1}{2\pi} \int_{|k|\leq R} \mathcal{F}(f)(k) e^{ixk} dk$$

for every $x \in \mathcal{R}$.

Definition 3. We say that $f:\mathcal{R}\rightarrow\mathcal{R}$ is analytic at infinity, if $f(\frac{1}{x})$ has a convergent power series expansion for $|x|<\epsilon$, $\epsilon>0$. We say that f is eventually monotone, if there exists $y_0\in\mathcal{R}_{>0}$ such that $f|_{(-\infty,-y_0)}$ and $f|_{(y_0,\infty)}$ are monotone.

Remark 2. The class of functions which are analytic at infinity and of very moderate decrease is important in Physics. The components of the causal field generated by Jefimenko equations can be shown to have this property if the corresponding charge and current $(\rho,\overline{J})$ are smooth and have compact support.

A criteria for a function being non-oscillatory or eventually monotone is provided by the following lemma;

Lemma 7. *If $f:\mathcal{R}\rightarrow\mathcal{R}$, $f\neq 0$ is analytic and analytic at infinity, then it has finitely many zeroes. If $f:\mathcal{R}\rightarrow\mathcal{R}$, $\frac{df}{dx}$ is analytic and analytic at infinity, and $f\neq c$, where $c\in\mathcal{R}$, then f is non-oscillatory. If $f:\mathcal{R}\rightarrow\mathcal{R}$, f is analytic for $|x|>a$, where $a\in\mathcal{R}_{\geq 0}$, analytic at infinity, and $f|_{|x|>a}\neq 0$ then f has finitely many zeroes in the region $|x|>a+1$. If $f:\mathcal{R}\rightarrow\mathcal{R}$, $\frac{df}{dx}$ is analytic for $|x|>a$, analytic at infinity, and $f|_{|x|>a}\neq c$, where $c\in\mathcal{R}$, then f is eventually monotone.*

Proof. For the first claim, suppose that f has infinitely many zeroes. Then we can find a sequence $\{y_i;i\in\mathcal{N}\}$ with $f(y_i)=0$. If the sequence is bounded, then by the Bolzano-Weierstrass Theorem, we can find a subsequence $\{y_{i_k};k\in\mathcal{N}\}$, with $f(y_{i_k})=0$, converging to $y\in\mathcal{R}$. By continuity, we have that $f(y)=0$ and y is a limit point of zeroes. As f is analytic, by the identity theorem, it must be identically zero, contradicting the hypothesis. If the sequence is unbounded, then we can find a subsequence $\{y_{i_k};k\in\mathcal{N}\}$, with $f(y_{i_k})=0$, such that $lim_{k\rightarrow\infty}y_{i_k}=\infty$ or $lim_{k\rightarrow\infty}y_{i_k}=-\infty$. As f is analytic at ∞, we can find $\epsilon>0$, such that $f(y)=0$ for $|y|>\frac{1}{\epsilon}$. By the identity theorem again, f is identically zero, contradicting the hypothesis. It follows that f has finitely many zeroes. For the second claim, as $\frac{df}{dx}\neq 0$, by the first part, there exist finitely many points $\{y_1,\ldots,y_n\}$, with $\frac{df}{dx}|_{y_i}=0$, for $1\leq i\leq n$, and with $y_i<y_{i+1}$, for $1\leq i\leq n-1$. In particularly, we have that $f|_{(-\infty,y_1)}$, $f|_{(y_n,\infty)}$ and $f|_{(y_i,y_{i+1})}$ is monotone for $1\leq i\leq n-1$, so that f is non-oscillatory. For the third claim, suppose that f has infinitely many zeroes in the region $|x|>a+1$, then we can find a sequence $\{y_i;i\in\mathcal{N}\}$ with $f(y_i)=0$ and $|y_i>a+1$. As above, if the sequence is bounded, we can find a subsequence $\{y_{i_k};k\in\mathcal{N}\}$, with $f(y_{i_k})=0$, converging to $y\in\mathcal{R}$, with $|y|\geq a+1>a$. As f is analytic for $|x|>a$, by the identity theorem, it must be identically zero in the region $|x|>a$, contradicting the hypothesis. If the sequence is unbounded, by the same argument as above, f must be identically zero in the region $|x|>a$, contradicting the hypothesis. It follows that f has finitely many zeroes in the region $|x|>a+1$. For the fourth claim, as $\frac{df}{dx}|_{|x|>a}\neq 0$, by the first part, there exist finitely many points $\{y_1,\ldots,y_n\}$, with $\frac{df}{dx}|_{y_i}=0$, and $|y_i|>a+1$, for $1\leq i\leq n$. Choose $y_0>max_{1\leq i\leq n}(|y_i|)$, then $\frac{df}{dx}|_{|x|>y_0}\neq 0$, so that $f|_{|x|>y_0}$ is monotone.
□

Lemma 8. *Suppose $f\in C^0(\mathbb{R})$ and f is of very moderate decrease. Define the Fourier transform by*

$$\mathcal{F}(f)(k) \;=\; \frac{1}{(2\pi)^{\frac{1}{2}}}\lim_{r\rightarrow\infty}\int_{-r}^{r} f(y)\,e^{-iky}\,dy.$$

Then $\mathcal{F}(f)(k)$ defines a function in $L^2(\mathbb{R})$, satisfying $\|\mathcal{F}(f)\|_{L^2(\mathbb{R})}=\|f\|_{L^2(\mathbb{R})}$. Moreover, there exists a sequence $\{r_n\}_{n\in\mathbb{N}}$ with $r_n\in\mathbb{R}_{>0}$ such that

$$\frac{1}{(2\pi)^{\frac{1}{2}}}\lim_{n\rightarrow\infty}\int_{-r_n}^{r_n} f(y)\,e^{-iky}\,dy$$

converges almost everywhere.

Proof. By hypothesis, $f\in L^2(\mathbb{R})$. Let $f_r=f\,\chi_{(-r,r)}$, where $\chi_{(-r,r)}$ is the characteristic function on $(-r,r)$. Then $f_r\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, and $\|f-f_r\|_{L^2(\mathbb{R})}\rightarrow 0$ as $r\rightarrow\infty$. In particular, $\{f_r\}$ is a Cauchy sequence in $L^2(\mathbb{R})$.

By a result in [8], the usual Fourier transform $\mathcal{F} : L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isometry. Hence, $\{\mathcal{F}(f_r)\}$ is also a Cauchy sequence in $L^2(\mathbb{R})$. By the completeness of $L^2(\mathbb{R})$, there exists $g \in L^2(\mathbb{R})$ such that $\|g - \mathcal{F}(f_r)\|_{L^2(\mathbb{R})} \to 0$ as $r \to \infty$.

We now compute:

$$\|\mathcal{F}(f)\|_{L^2(\mathbb{R})} = \Big\| \lim_{r\to\infty} \mathcal{F}(f_r) \Big\|_{L^2(\mathbb{R})} = \lim_{r\to\infty} \|\mathcal{F}(f_r)\|_{L^2(\mathbb{R})} = \lim_{r\to\infty} \|f_r\|_{L^2(\mathbb{R})} = \Big\| \lim_{r\to\infty} f_r \Big\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Finally, by a result in [9], if we choose $r_n \in \mathbb{R}_{>0}$ such that $\|f - f_{r_n}\|_{L^2(\mathbb{R})} \leq 2^{-n}$, then $\|\mathcal{F}(f) - \mathcal{F}(f_{r_n})\|_{L^2(\mathbb{R})} \leq 2^{-n}$, which implies that $\big|\mathcal{F}(f_{r_n})(k) - \mathcal{F}(f)(k)\big| \to 0$ almost everywhere. □

Remark 3. We will use $\mathcal{F}(f)(k)$ to denote this almost-everywhere limit.

Lemma 9. *Let $f \in C^2(\mathbb{R})$ and assume $\{f, \frac{df}{dx}\} \subset C(\mathbb{R})$ is of very moderate decrease, with $\frac{d^2f}{dx^2}$ of moderate decrease. Then there exists a constant $G \in \mathbb{R}_{>0}$ such that*

$$\big|\mathcal{F}(f)(k)\big| \leq \frac{G}{|k|^2} \quad \text{for } |k| > 1.$$

In particular, $\mathcal{F}(f)(k) \in L^1(\mathbb{R})$ and is defined for all $k \neq 0$.

Moreover, let $f \in C^1(\mathbb{R})$ be of very moderate decrease, with $\frac{df}{dx}$ of moderate decrease and oscillatory. Then there exists a constant $G \in \mathbb{R}_{>0}$ such that

$$\big|\mathcal{F}(f)(k)\big| \leq \frac{G}{|k|^2} \quad \text{for sufficiently large } |k|.$$

Hence, $\mathcal{F}(f)(k) \in L^1(\mathbb{R})$ and is defined for $k \neq 0$.

Proof. First, note that $\frac{d^2f}{dx^2} \in L^1(\mathbb{R})$. Then

$$\big|\mathcal{F}(\tfrac{d^2f}{dx^2})(k)\big| = \left|\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \tfrac{d^2f}{dx^2}(y)\, e^{-iky}\, dy\right| \leq \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \big|\tfrac{d^2f}{dx^2}(y)\big|\, dy = \frac{1}{(2\pi)^{\frac{1}{2}}} \Big\|\tfrac{d^2f}{dx^2}\Big\|_{L^1} \leq G$$

for some $G \in \mathbb{R}_{>0}$. Next, applying integration by parts and the Dominated Convergence Theorem (DCT), for $k \neq 0$ we obtain:

$$\mathcal{F}(\tfrac{d^2f}{dx^2})(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \lim_{r\to\infty} \int_{-r}^{r} \tfrac{d^2f}{dx^2}(y)\, e^{-iky}\, dy = \frac{1}{(2\pi)^{\frac{1}{2}}} \Big[\Big(\tfrac{df}{dx} e^{-iky}\Big)\Big|_{-r}^{r} + ik \int_{-r}^{r} \tfrac{df}{dx}(y)\, e^{-iky}\, dy \Big]_{r\to\infty}$$

Since $\frac{df}{dx}$ vanishes sufficiently fast at $\pm\infty$ (due to very moderate decrease), the boundary term is zero, thus

$$\mathcal{F}(\tfrac{d^2f}{dx^2})(k) = \frac{ik}{(2\pi)^{\frac{1}{2}}} \lim_{r\to\infty} \int_{-r}^{r} \tfrac{df}{dx}(y)\, e^{-iky}\, dy = \frac{ik}{(2\pi)^{\frac{1}{2}}} \Big[\big(f(y) e^{-iky}\big)\Big|_{-r}^{r} + ik \int_{-r}^{r} f(y)\, e^{-iky}\, dy \Big]_{r\to\infty}$$

Again, the boundary term vanishes, yielding

$$\mathcal{F}(\tfrac{d^2f}{dx^2})(k) = -k^2 \Big(\frac{1}{(2\pi)^{\frac{1}{2}}} \lim_{r\to\infty} \int_{-r}^{r} f(y)\, e^{-iky}\, dy \Big) = -k^2\, \mathcal{F}(f)(k).$$

Thus

$$\big|\mathcal{F}(f)(k)\big| = \frac{\big|\mathcal{F}(\frac{d^2f}{dx^2})(k)\big|}{|k|^2} \leq \frac{G}{|k|^2} \quad \text{for } |k| > 1.$$

Moreover, since $\mathcal{F}(f)(k) \in L^2(\mathbb{R})$, it follows that $\mathcal{F}(f)\big|_{(-1,1)}(k) \in L^2(\mathbb{R}) \subset L^1(\mathbb{R})$, and hence $\mathcal{F}(f)(k) \in L^1(\mathbb{R})$.

For the second statement, let $f \in C^1(\mathbb{R})$ be of very moderate decrease, with $\frac{df}{dx}$ of moderate decrease and oscillatory. Since $\frac{df}{dx} \in L^1(\mathbb{R})$, we similarly have $\left|\mathcal{F}(\frac{df}{dx})(k)\right| \leq H$ for some $H \in \mathbb{R}_{>0}$. Repeating a similar integration-by-parts argument shows

$$\mathcal{F}(\tfrac{df}{dx})(k) = ik\,\mathcal{F}(f)(k) \quad \text{for } k \neq 0,$$

thus

$$\left|\mathcal{F}(f)(k)\right| \leq \frac{\left|\mathcal{F}(\frac{df}{dx})(k)\right|}{|k|} \leq \frac{H}{|k|}.$$

By assumptions on the oscillatory nature of $\frac{df}{dx}$ and the moderate decrease for $|x| \to \infty$, one can refine this bound further. Adapting the argument in [5], for sufficiently large $|k|$,

$$\left|\mathcal{F}(\tfrac{df}{dx})(k)\right| \leq \frac{R}{|k|},$$

for some constant $R > 0$. Combining these estimates yields

$$\left|\mathcal{F}(f)(k)\right| \leq \frac{|\mathcal{F}(\frac{df}{dx})(k)|}{|k|} \leq \frac{G}{|k|^2} \quad \text{for sufficiently large } |k|.$$

Hence $\mathcal{F}(f)(k) \in L^1(\mathbb{R})$ for $k \neq 0$, as required. □

Lemma 10. *Let $f \in C^3(\mathbb{R})$ be such that $\{f, f'\}$ has very moderate decrease, f'' has moderate decrease, and $\{f, f', f'', f'''\}$ is bounded. Then $\mathcal{F}(f) \in L^1(\mathbb{R})$, and*

$$f(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x),$$

where, for $g \in L^1(\mathbb{R})$,

$$\mathcal{F}^{-1}(g)(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk.$$

Moreover, the same result holds if $f \in C^3(\mathbb{R})$ with f of very moderate decrease, f' of moderate decrease and oscillatory, and $\{f, f', f'', f'''\}$ bounded.

Proof. By Lemma 9, there exists a constant $E > 0$ such that

$$\left|\mathcal{F}(f)(k)\right| \leq \frac{E}{|k|^2},$$

for sufficiently large $|k|$ (this is $(*)$). Since f is of very moderate decrease, we also have

$$|f(x)|^2 \leq \frac{D}{|x|^2},$$

for $|x| > 1$. Because $f \in C^0(\mathbb{R})$, it follows that $f \in L^2(\mathbb{R})$. Consequently, $\mathcal{F}(f) \in L^2(\mathbb{R})$. Furthermore, for any $n \in \mathbb{N}$, the restriction $\mathcal{F}(f)|_{[-n,n]} \in L^1(\mathbb{R})$. Denoting this condition as $(**)$, and combining $(*)$ and $(**)$, we conclude that $\mathcal{F}(f) \in L^1(\mathbb{R})$.

Next, let $\{f_m\}_{m\in\mathbb{N}}$ be the approximating sequence given by Lemma 5. Since each $f_m \in L^1(\mathbb{R})$, its Fourier transform $\mathcal{F}(f_m)$ is continuous. Because $f_m \in C^2(\mathbb{R})$ and $f_m'' \in L^1(\mathbb{R})$, there exist constants $D_m > 0$ such that

$$\left|\mathcal{F}(f_m)(k)\right| \leq \frac{D_m}{|k|^2},$$

for sufficiently large $|k|$. Moreover, since $x^n f_m(x) \in L^1(\mathbb{R})$ for any $n \in \mathbb{N}$, it follows that $\mathcal{F}(f_m) \in C^\infty(\mathbb{R})$. Hence, by the standard Fourier Inversion Theorem (see [4] for the proof), we obtain

$$f_m(x) = \mathcal{F}^{-1}(\mathcal{F}(f_m))(x). \tag{15}$$

Since f is of very moderate decrease, we have $f - f_m \in L^2(\mathbb{R})$, and

$$\lim_{m\to\infty} \|f - f_m\|_{L^2(\mathbb{R})} = 0, \quad \lim_{m\to\infty} \|\mathcal{F}(f) - \mathcal{F}(f_m)\|_{L^2(\mathbb{R})} = 0.$$

For any $n \in \mathbb{N}$, $m \in \mathbb{N}$, we estimate:

$$\Big|\int_{-n}^{n} (\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k))\, dk\Big| \leq (2n)^{\frac{1}{2}} \|\mathcal{F}(f - f_m)\|_{L^2} \leq (2n)^{\frac{1}{2}} \|f - f_m\|_{L^2}.$$

Since $\|f - f_m\|_{L^2}$ is controlled by the very moderate decrease of f, we can bound this by

$$(2n)^{\frac{1}{2}} \Big(\int_{|x|>m} \frac{D^2}{x^2}\, dx\Big)^{\frac{1}{2}} = (2n)^{\frac{1}{2}} \big(\tfrac{2D^2}{m}\big)^{\frac{1}{2}} = 2D\, \frac{n^{\frac{1}{2}}}{m^{\frac{1}{2}}}, \quad (A) \tag{16}$$

for some constant $D > 0$. Now, setting $m = \lfloor n^{\frac{3}{2}} \rfloor$ and applying Lemma 5 together with (16), we estimate, for $x \in \mathbb{R}$,

$$\big|\mathcal{F}^{-1}(\mathcal{F}(f))(x) - \mathcal{F}^{-1}(\mathcal{F}(f_m))(x)\big| = \frac{1}{(2\pi)^{\frac{1}{2}}}\Big|\int_{-\infty}^{\infty} (\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k))\, e^{ikx}\, dk\Big|.$$

Splitting the integral into regions $|k| \leq n$ and $|k| > n$, and using our previous bounds on $\mathcal{F}(f), \mathcal{F}(f_m)$, one shows that

$$\big|\mathcal{F}^{-1}(\mathcal{F}(f))(x) - \mathcal{F}^{-1}(\mathcal{F}(f_m))(x)\big| \leq \frac{1}{(2\pi)^{\frac{1}{2}}}\Big(2D\, \frac{n^{\frac{1}{2}}}{m^{\frac{1}{2}}} + \frac{2E}{n} + \frac{C\, n^{\frac{3}{2}}}{n^2}\Big) \longrightarrow 0,$$

as $n \to \infty$. Hence,

$$\lim_{m\to\infty} \mathcal{F}^{-1}(\mathcal{F}(f_m))(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x). \tag{17}$$

Finally, by Definition 2, together with (15) and (17), we obtain

$$f(x) = \lim_{m\to\infty} f_m(x) = \lim_{m\to\infty} \mathcal{F}^{-1}(\mathcal{F}(f_m))(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x).$$

This completes the proof. □

Acknowledgments: The author is thankful to the Partnership for Applied Sciences, Engineering, and Technology (PASET) - Regional Scholarship and Innovation Fund (RSIF) for the support of this study.

Conflicts of Interest: The author certifies that there is no conflict of interest whatsoever with any affiliation, or involvement with any organization, financial and non-financial entity.

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