Engineering and Applied Science Letters

PSR Press

Article

Trapezoidal type inequalities via (s,m)-preinvex functions for Caputo fractional derivatives

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Received: 25 September 2021; Accepted: 03 January 2023; Published: 21 March 2025.

Abstract: In this paper, by using a new identity we establish some trapezoidal type inequalities for functions whose modulus of the first derivatives are (s,m)-preinvex via Caputo fractional derivatives.

Keywords: Hermite-Hadamard inequality, Hölder inequality, Caputo fractional derivatives, (s,m)-preinvex functions

1. Introduction

One of the famous inequalities for the class of convex functions is the so-called Hermite-Hadamard inequality (see [1,2]), which can be stated as follows: For every convex function f on the interval $[a,b]$ with $a<b$, we have

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{1}$$

Several generalizations and variants of inequality (1), have been obtained see [3–14] and references therein.

In [15], Dragomir and Agarwal gave some results which are in relationship with inequality (1) as follows:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{b-a}{8}\left(|f'(a)|+|f'(b)|\right), \tag{2}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{3}$$

Pearce and Pečarić [16], gave a variant of the result given in the inequality (3) as follows:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{b-a}{4}\left(|f'(a)|^q+|f'(b)|^q\right). \tag{4}$$

Noor [17], established the anologue preinvex of inequality (1) as follows:

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\leq\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\,du\leq\frac{f(a)+f(b)}{2}. \tag{5}$$

In [18], Barani et al., investigated some trapezium type inequalities for differentiable preinvex functions

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right|\leq\frac{|\eta(b,a)|}{8}\left(|f'(a)|+|f'(b)|\right), \tag{6}$$

and
$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{|\eta(b,a)|}{2(p+1)^{\frac{1}{p}}}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}. \tag{7}$$

Recently, Farid et al. [19], gave the followig Hermite-Hadamard inequality for Caputo fractional derivatives
$$f^{(n)}\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left(\left({}^cD_{a+}^{\alpha}f^{(n)}\right)(b)+(-1)^n\left({}^cD_{b-}^{\alpha}f^{(n)}\right)(a)\right) \leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}. \tag{8}$$

Also, in the same paper, they established the following result
$$\begin{aligned}&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left(\left({}^cD_{a+}^{\alpha}f^{(n)}\right)(b)+(-1)^n\left({}^cD_{b-}^{\alpha}f^{(n)}\right)(a)\right)\right|\\ &\leq \frac{b-a}{2(n-\alpha+1)}\left(1-\frac{1}{2^{n-\alpha}}\right)\left(\left|f^{(n+1)}(a)\right|+\left|f^{(n+1)}(b)\right|\right).\end{aligned} \tag{9}$$

Motivated by the results of the paper [19], in this paper, we use a new identity in order to establish some trapezoidal type inequalities for functions whose modulus of the first derivatives are (s,m)-preinvex in the second sens via fractional derivatives at Caputo meaning.

2. Preliminaries

This section recalls some known definitions

Definition 1. [20] A function $f: I \to \mathbb{R}$ is said to be convex, if
$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y), \tag{10}$$
holds for all $x,y \in I$ and all $t \in [0,1]$.

Definition 2. [21] A function $f: I=[a,b] \subset [0,+\infty) \to \mathbb{R}$ is called s-convex, if
$$f(tx+(1-t)y) \leq t^s f(x)+(1-t)^s f(y), \tag{11}$$
holds for all $x,y \in I$ and all $t \in [0,1]$.

Definition 3. [22] A function $f: [0,b] \to \mathbb{R}$ is said to be m-convex, where $m \in (0,1]$,if
$$f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y), \tag{12}$$
holds for all $x,y \in I$, and $t \in [0,1]$.

Definition 4. [23] A function $f: [0,b] \to \mathbb{R}$ is said to be (s,m)-convex, where $s,m \in (0,1]$,if
$$f(tx+m(1-t)y) \leq t^s f(x)+m(1-t)^s f(y), \tag{13}$$
holds for all $x,y \in I$, and $t \in [0,1]$.

Definition 5. [24] A set $K \subseteq \mathbb{R}^n$ is said an invex with respect to the bi-function $\eta: K \times K \to \mathbb{R}^n$, if for all $x,y \in K$, we have
$$x+t\eta(y,x) \in K. \tag{14}$$

Definition 6. [24] A function $f: K \to \mathbb{R}$ is said to be preinvex, if
$$f(x+t\eta(y,x)) \leq (1-t)f(x)+tf(y), \tag{15}$$

holds for all $x, y \in K$ and all $t \in [0,1]$.

Definition 7. [25] A function $f : K = [a, a + \eta(b,a)] \subset [0, +\infty) \to \mathbb{R}$ is called s-preinvex, if

$$f(x + t\eta(y,x)) \leq (1-t)^s f(x) + t^s f(y), \tag{16}$$

holds for all $x, y \in K$ and all $t \in [0,1]$.

Definition 8. [26] A function $f : K \subset [0, b^*] \to \mathbb{R}$ is said to be m-preinvex with respect to η, where $b^* > 0$ and $m \in (0,1]$, if

$$f(x + t\eta(y,x)) \leq (1-t) f(x) + mtf\left(\frac{y}{m}\right), \tag{17}$$

holds for all $x, y \in K$, and $t \in [0,1]$.

Definition 9. [27] A function $f : K \subset [0, b^*] \to \mathbb{R}$ is said to be (s,m)-preinvex with respect to η for some fixed $s, m \in (0,1]$, where $b^* > 0$, if

$$f(x + t\eta(y,x)) \leq (1-t)^s f(x) + mt^s f\left(\frac{y}{m}\right), \tag{18}$$

holds for all $x, y \in K$, and $t \in [0,1]$.

Condition C. ([28]) Let K be an invex set with respect to the bi-function $\eta(.,.)$ then for any $a, b \in K$ and $t \in [0.1]$, we have

$$\eta(a, a + t\eta(b,a)) = -t\eta(b,a) \text{ and } \eta(b, a + t\eta(b,a)) = (1-t)\eta(b,a). \tag{19}$$

From Condition C, it follows that

$$\eta(a + t_2\eta(b,a), a + t_1\eta(b,a)) = (t_2 - t_1)\eta(b,a), \tag{20}$$

holds for any $a, b \in K$ and $t_1, t_2 \in [0.1]$.

Definition 10. [29] Let $\alpha > 0$ and $\alpha \notin \{1,2,3,...\}, n = [\alpha] + 1, f \in AC^n[a,b]$ which is the space of functions having n^{th} derivatives absolutely continuous. The right-side and left-side Caputo fractional derivatives of order α are defined as follows:

$$\left({}^cD^\alpha_{a^+} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt, x > a, \tag{21}$$

and

$$\left({}^cD^\alpha_{b^-} f\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)\, dt, b > x. \tag{22}$$

If $\alpha = n \in \{1,2,3,...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $\left({}^cD^\alpha_{a^+} f\right)(x)$ coincides with $f^{(n)}(x)$ whereas $\left({}^cD^\alpha_{b^-} f\right)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$.

In particular we have for $n = 1$ and $\alpha = 0$

$$\left({}^cD^0_{a^+} f\right)(x) = \left({}^cD^0_{b^-} f\right)(x) = f(x). \tag{23}$$

Definition 11. [30] The incomplete beta function is defined for all complex numbers x and y with $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$ and $0 \leq a < 1$ as follows

$$B_a(x,y) = \int_0^a t^{x-1}(1-t)^{y-1}\,dt. \tag{24}$$

Lemma 1. [31] *For any $0 \leq a < b$ in $\mathbb{R}$ and $0 < \alpha \leq 1$, we have*

$$b^\alpha - a^\alpha \leq (b-a)^\alpha. \tag{25}$$

3. Main results

Lemma 2. *Let $f : [a, a+\eta(b,a)] \subset [0, b^*] \to \mathbb{R}$ be the function such that $f \in C^{n+1}([a, a+\eta(b,a)])$ with $\eta(b,a) > 0$ and $b^* > 0$. Then the following equality for Caputo fractional derivatives holds*

$$\begin{aligned}&\frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)\left[\left({}^cD^\alpha_{a+}f\right)(a+\eta(b,a)) + (-1)^n\left({}^cD^\alpha_{(a+\eta(b,a))^-}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\\ &= \frac{\eta(b,a)}{2}\int_0^1\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)f^{(n+1)}(a+t\eta(b,a))\,dt.\end{aligned} \tag{26}$$

Proof. Let

$$I_1 = \int_0^1 t^{n-\alpha} f^{(n+1)}(a+t\eta(b,a))\,dt, \tag{27}$$

and

$$I_2 = \int_0^1 (1-t)^{n-\alpha} f^{(n+1)}(a+t\eta(b,a))\,dt. \tag{28}$$

Integrating by parts I_1 we get

$$\begin{aligned} I_1 &= \frac{1}{\eta(b,a)}f^{(n)}(a+\eta(b,a)) - \frac{n-\alpha}{\eta(b,a)}\int_0^1 t^{n-\alpha-1}f^{(n)}(a+t\eta(b,a))\,dt,\\ &= \frac{1}{\eta(b,a)}f^{(n)}(a+\eta(b,a)) - \frac{n-\alpha}{(\eta(b,a))^2}\int_a^{a+\eta(b,a)}\left(\frac{u-a}{\eta(b,a)}\right)^{n-\alpha-1}f^{(n)}(u)\,du,\\ &= \frac{1}{\eta(b,a)}f^{(n)}(a+\eta(b,a)) - \frac{(n-\alpha)\Gamma(n-\alpha)}{(\eta(b,a))^{n-\alpha+1}}(-1)^n\left({}^cD^\alpha_{(a+\eta(b,a))^-}f\right)(a). \end{aligned} \tag{29}$$

Similarly, we have

$$\begin{aligned} I_2 &= -\frac{1}{\eta(b,a)}f^{(n)}(a) + \frac{n-\alpha}{\eta(b,a)}\int_0^1 (1-t)^{n-\alpha-1}f^{(n)}(a+t\eta(b,a))\,dt,\\ &= -\frac{1}{\eta(b,a)}f^{(n)}(a) + \frac{n-\alpha}{(\eta(b,a))^{n-\alpha+1}}\int_a^{a+\eta(b,a)}(a+\eta(b,a)-u)^{n-\alpha-1}f^{(n)}(u)\,du,\\ &= -\frac{1}{\eta(b,a)}f^{(n)}(a) + \frac{(n-\alpha)\Gamma(n-\alpha)}{(\eta(b,a))^{n-\alpha+1}}\left({}^cD^\alpha_{a+}f\right)(a+\eta(b,a)). \end{aligned} \tag{30}$$

Subtracting (30) from (29), and then multiplying the resulting equality by $\frac{\eta(b,a)}{2}$ we get the desired result. □

Theorem 1. *Let $f : [a, a+\eta(b,a)] \subset [0, b^*] \to \mathbb{R}$ be the function such that $f \in C^{n+1}([a, a+\eta(b,a)])$ with $b^* > 0$ and $\eta(b,a) > 0$. If $\left|f^{(n+1)}\right|$ is (s,m)-preinvex for some fixed $s, m \in (0,1]$, then the following inequality for Caputo fractional derivatives holds*

$$\left|\frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)\left[\left({}^{c}D^{\alpha}_{a^+}f\right)(a+\eta(b,a)) + (-1)^n\left({}^{c}D^{\alpha}_{(a+\eta(b,a))^-}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\right|$$
$$\le \frac{\eta(b,a)}{2}\left(\left|f^{(n+1)}(a)\right| + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|\right)\left(\frac{1}{n-\alpha+s+1}\left(1 - \frac{1}{2^{n-\alpha+s}}\right)\right.$$
$$\left. + B_{\frac{1}{2}}(s+1, n-\alpha+1) - B_{\frac{1}{2}}(n-\alpha+1, s+1)\right), \tag{31}$$

where $B_{\frac{1}{2}}(.,.)$ is the incomplete beta function.

Proof. From Lemma 2, properties of modulus, and (s,m)-preinvexity of $\left|f^{(n+1)}\right|$ we have

$$\left|\frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)\left[\left({}^{c}D^{\alpha}_{a^+}f\right)(a+\eta(b,a)) + (-1)^n\left({}^{c}D^{\alpha}_{(a+\eta(b,a))^-}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\right|$$
$$\le \frac{\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha} - t^{n-\alpha}\right)\left|f^{(n+1)}(a+t\eta(b,a))\right| dt\right.$$
$$\left. + \int_{\frac{1}{2}}^{1}\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)\left|f^{(n+1)}(a+t\eta(b,a))\right| dt\right)$$
$$\le \frac{\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha} - t^{n-\alpha}\right)\left((1-t)^s\left|f^{(n+1)}(a)\right| + mt^s\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|\right) dt\right.$$
$$\left. + \int_{\frac{1}{2}}^{1}\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)\left((1-t)^s\left|f^{(n+1)}(a)\right| + mt^s\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|\right) dt\right)$$
$$= \frac{\eta(b,a)}{2}\left(\left|f^{(n+1)}(a)\right|\int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha} - t^{n-\alpha}\right)(1-t)^s dt + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|\int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha} - t^{n-\alpha}\right)t^s dt\right.$$
$$\left. + \left|f^{(n+1)}(a)\right|\int_{\frac{1}{2}}^{1}\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)(1-t)^s dt + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|\int_{\frac{1}{2}}^{1}\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)t^s dt\right),$$
$$= \frac{\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha} - t^{n-\alpha}\right)(1-t)^s dt + \int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha} - t^{n-\alpha}\right)t^s dt\right)$$
$$\times\left(\left|f^{(n+1)}(a)\right| + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|\right)$$
$$= \frac{\eta(b,a)}{2}\left(\left|f^{(n+1)}(a)\right| + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|\right)\left(\frac{1}{n-\alpha+s+1}\left(1 - \frac{1}{2^{n-\alpha+s}}\right)\right.$$
$$\left. + B_{\frac{1}{2}}(s+1, n-\alpha+1) - B_{\frac{1}{2}}(n-\alpha+1, s+1)\right). \tag{32}$$

The proof is completed. □

Corollary 1. *In Theorem 1, if we choose $\eta(b,a) = b - a$, we obtain*

$$\left| \frac{f^{(n)}(a) + f^{(n)}(b)}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)}{2(b-a)^{n-\alpha}} \left[\left({}^{c}D^{\alpha}_{a+} f\right)(b) + (-1)^{n} \left({}^{c}D^{\alpha}_{b-} f\right)(a) \right] \right|$$
$$\leq \frac{b-a}{2} \left(\left| f^{(n+1)}(a) \right| + m \left| f^{(n+1)}\left(\frac{b}{m}\right) \right| \right) \left(\frac{1}{n-\alpha+s+1} \left(1 - \frac{1}{2^{n-\alpha+s}}\right) \right.$$
$$\left. + B_{\frac{1}{2}}(s+1, n-\alpha+1) - B_{\frac{1}{2}}(n-\alpha+1, s+1) \right). \tag{33}$$

Corollary 2. *In Theorem 1, if we take $m = 1$, we obtain*

$$\left| \frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha) \left[\left({}^{c}D^{\alpha}_{a+} f\right)(a+\eta(b,a)) + (-1)^{n} \left({}^{c}D^{\alpha}_{(a+\eta(b,a))^{-}} f\right)(a) \right]}{2(\eta(b,a))^{n-\alpha}} \right|$$
$$\leq \frac{\eta(b,a)}{2} \left(\left| f^{(n+1)}(a) \right| + \left| f^{(n+1)}(b) \right| \right) \left(\frac{1}{n-\alpha+s+1} \left(1 - \frac{1}{2^{n-\alpha+s}}\right) \right.$$
$$\left. + B_{\frac{1}{2}}(s+1, n-\alpha+1) - B_{\frac{1}{2}}(n-\alpha+1, s+1) \right). \tag{34}$$

Moreover, if we take $\eta(b,a) = b - a$, we get

$$\left| \frac{f^{(n)}(a) + f^{(n)}(b)}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)}{2(b-a)^{n-\alpha}} \left[\left({}^{c}D^{\alpha}_{a+} f\right)(b) + (-1)^{n} \left({}^{c}D^{\alpha}_{b-} f\right)(a) \right] \right|$$
$$\leq \frac{b-a}{2} \left(\left| f^{(n+1)}(a) \right| + \left| f^{(n+1)}(b) \right| \right) \left(\frac{1}{n-\alpha+s+1} \left(1 - \frac{1}{2^{n-\alpha+s}}\right) \right.$$
$$\left. + B_{\frac{1}{2}}(s+1, n-\alpha+1) - B_{\frac{1}{2}}(n-\alpha+1, s+1) \right). \tag{35}$$

Corollary 3. *In Theorem 1, if we take $s = 1$, we obtain*

$$\left| \frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha) \left[\left({}^{c}D^{\alpha}_{a+} f\right)(a+\eta(b,a)) + (-1)^{n} \left({}^{c}D^{\alpha}_{(a+\eta(b,a))^{-}} f\right)(a) \right]}{2(\eta(b,a))^{n-\alpha}} \right|$$
$$\leq \frac{\eta(b,a)}{2^{n-\alpha+1}} \left(\frac{2^{n-\alpha}-1}{n-\alpha+1} \right) \left(\left| f^{(n+1)}(a) \right| + m \left| f^{(n+1)}\left(\frac{b}{m}\right) \right| \right). \tag{36}$$

Moreover, if we take $\eta(b,a) = b - a$, we get

$$\left| \frac{f^{(n)}(a) + f^{(n)}(b)}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)}{2(b-a)^{n-\alpha}} \left[\left({}^{c}D^{\alpha}_{a+} f\right)(b) + (-1)^{n} \left({}^{c}D^{\alpha}_{b-} f\right)(a) \right] \right|$$
$$\leq \frac{b-a}{2^{n-\alpha+1}} \left(\frac{2^{n-\alpha}-1}{n-\alpha+1} \right) \left(\left| f^{(n+1)}(a) \right| + m \left| f^{(n+1)}\left(\frac{b}{m}\right) \right| \right). \tag{37}$$

Corollary 4. *In Theorem 1, if we take $s = m = 1$, we obtain*

$$\left| \frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha) \left[\left({}^{c}D^{\alpha}_{a+} f\right)(a+\eta(b,a)) + (-1)^{n} \left({}^{c}D^{\alpha}_{(a+\eta(b,a))^{-}} f\right)(a) \right]}{2(\eta(b,a))^{n-\alpha}} \right|$$
$$\leq \frac{\eta(b,a)}{2^{n-\alpha+1}} \left(\frac{2^{n-\alpha}-1}{n-\alpha+1} \right) \left(\left| f^{(n+1)}(a) \right| + \left| f^{(n+1)}(b) \right| \right). \tag{38}$$

Moreover, if we take $\eta(b,a) = b - a$, we get Theorem 2.4 from [19].

Theorem 2. *Let $f : [a, a+\eta(b,a)] \subset [0, b^*] \to \mathbb{R}$ be the function such that $f \in C^{n+1}([a, a+\eta(b,a)])$ with $b^* > 0$ and $\eta(b,a) > 0$. If $\left|f^{(n+1)}\right|^q$ is (s,m)-preinvex for some fixed $s, m \in (0,1]$ where $q > 1$ with $p^{-1} + q^{-1} = 1$, then the following inequality for Caputo fractional derivatives holds*

$$\left|\frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)\left[\left({}^cD^\alpha_{a^+}f\right)(a+\eta(b,a)) + (-1)^n\left({}^cD^\alpha_{(a+\eta(b,a))^-}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\frac{1}{pn-p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n+1)}(a)\right|^q + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}. \tag{39}$$

Proof. From Lemma 2, properties of modulus, Hölder's inequality, Lemma 1, and (s,m)-preinvexity of $\left|f^{(n+1)}\right|^q$, we have

$$\left|\frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)\left[\left({}^cD^\alpha_{a^+}f\right)(a+\eta(b,a)) + (-1)^n\left({}^cD^\alpha_{(a+\eta(b,a))^-}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1\left|t^{n-\alpha} - (1-t)^{n-\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n+1)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha} - t^{n-\alpha}\right)^p dt + \int_{\frac{1}{2}}^1\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)^p dt\right)^{\frac{1}{p}} \times \left(\int_0^1\left|f^{(n+1)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}}(1-2t)^{pn-p\alpha}dt + \int_{\frac{1}{2}}^1(2t-1)^{pn-p\alpha}dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left((1-t)^s\left|f^{(n+1)}(a)\right|^q + mt^s\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta(b,a)}{2}\left(\frac{1}{pn-p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n+1)}(a)\right|^q + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}. \tag{40}$$

The proof is completed. □

Corollary 5. *In Theorem 2, if we choose $\eta(b,a) = b-a$, we obtain*

$$\left|\frac{f^{(n)}(a) + f^{(n)}(b)}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)}{2(b-a)^{n-\alpha}}\left[\left({}^cD^\alpha_{a^+}f\right)(b) + (-1)^n\left({}^cD^\alpha_{b^-}f\right)(a)\right]\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{pn-p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n+1)}(a)\right|^q + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}. \tag{41}$$

Corollary 6. *In Theorem 2, if we take $m = 1$, we obtain*

$$\left|\frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)\left[\left({}^cD^\alpha_{a^+}f\right)(a+\eta(b,a)) + (-1)^n\left({}^cD^\alpha_{(a+\eta(b,a))^-}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\right|$$

$$\leq \frac{\eta(b,a)}{2}\left(\frac{1}{pn-p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n+1)}(a)\right|^{q}+\left|f^{(n+1)}(b)\right|^{q}}{s+1}\right)^{\frac{1}{q}}. \tag{42}$$

Moreover, if we take $\eta(b,a)=b-a$, we get

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{(n-\alpha)\Gamma(n-\alpha)}{2(b-a)^{n-\alpha}}\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(b)+(-1)^{n}\left({}^{c}D^{\alpha}_{b^{-}}f\right)(a)\right]\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{pn-p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n+1)}(a)\right|^{q}+m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}. \tag{43}$$

Corollary 7. *In Theorem 2, if we take $s=1$, we obtain*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(a+\eta(b,a))}{2}-\frac{(n-\alpha)\Gamma(n-\alpha)\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(a+\eta(b,a))+(-1)^{n}\left({}^{c}D^{\alpha}_{(a+\eta(b,a))^{-}}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\frac{1}{pn-p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n+1)}(a)\right|^{q}+m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{44}$$

Moreover, if we take $\eta(b,a)=b-a$, we get

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{(n-\alpha)\Gamma(n-\alpha)}{2(b-a)^{n-\alpha}}\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(b)+(-1)^{n}\left({}^{c}D^{\alpha}_{b^{-}}f\right)(a)\right]\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{pn-p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n+1)}(a)\right|^{q}+m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{45}$$

Corollary 8. *In Theorem 2, if we take $s=m=1$, we obtain*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(a+\eta(b,a))}{2}-\frac{(n-\alpha)\Gamma(n-\alpha)\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(a+\eta(b,a))+(-1)^{n}\left({}^{c}D^{\alpha}_{(a+\eta(b,a))^{-}}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\frac{1}{pn-p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n+1)}(a)\right|^{q}+\left|f^{(n+1)}(b)\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{46}$$

Moreover, if we take $\eta(b,a)=b-a$, we get

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{(n-\alpha)\Gamma(n-\alpha)}{2(b-a)^{n-\alpha}}\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(b)+(-1)^{n}\left({}^{c}D^{\alpha}_{b^{-}}f\right)(a)\right]\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{pn-p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n+1)}(a)\right|^{q}+\left|f^{(n+1)}(b)\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{47}$$

Theorem 3. *Let $f : [a, a+\eta(b,a)] \subset [0, b^*] \to \mathbb{R}$ be the function such that $f \in C^{n+1}([a, a+\eta(b,a)])$ with $b^* > 0$ and $\eta(b,a) > 0$. If $\left|f^{(n+1)}\right|^q$ is (s,m)-preinvex for some fixed $s, m \in (0,1]$, where $q \geq 1$, then the following inequality for Caputo fractional derivatives holds*

$$\left|\frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)\left[\left({}^cD^{\alpha}_{a^+}f\right)(a+\eta(b,a)) + (-1)^n\left({}^cD^{\alpha}_{(a+\eta(b,a))^-}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\frac{2^{n-\alpha}-1}{(n-\alpha+1)2^{n-\alpha-1}}\right)^{1-\frac{1}{q}}\left(\left|f^{(n+1)}(a)\right|^q + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}} \tag{48}$$
$$\times\left(\frac{2^{n-\alpha+s}-1}{(n-\alpha+s+1)2^{n-\alpha+s}} + B_{\frac{1}{2}}(s+1, n-\alpha+1) - B_{\frac{1}{2}}(n-\alpha+1, s+1)\right)^{\frac{1}{q}},$$

where $B_{\frac{1}{2}}(.,.)$ is the incomplete beta function.

Proof. From Lemma 2, properties of modulus, power mean inequality, and (s,m)-preinvexity of $\left|f^{(n+1)}\right|^q$, we have

$$\left|\frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)\left[\left({}^cD^{\alpha}_{a^+}f\right)(a+\eta(b,a)) + (-1)^n\left({}^cD^{\alpha}_{(a+\eta(b,a))^-}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1\left|t^{n-\alpha} - (1-t)^{n-\alpha}\right|dt\right)^{1-\frac{1}{q}} \times \left(\int_0^1\left|t^{n-\alpha} - (1-t)^{n-\alpha}\right|\left|f^{(n+1)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1\left|t^{n-\alpha} - (1-t)^{n-\alpha}\right|dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|t^{n-\alpha} - (1-t)^{n-\alpha}\right|\left((1-t)^s\left|f^{(n+1)}(a)\right|^q + mt^s\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha} - t^{n-\alpha}\right)dt + \int_{\frac{1}{2}}^1\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha} - t^{n-\alpha}\right)\left((1-t)^s\left|f^{(n+1)}(a)\right|^q + mt^s\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q\right)dt\right.$$
$$\left. + \int_{\frac{1}{2}}^1\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)\left((1-t)^s\left|f^{(n+1)}(a)\right|^q + mt^s\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta(b,a)}{2}\left(\frac{2^{n-\alpha}-1}{(n-\alpha+1)2^{n-\alpha-1}}\right)^{1-\frac{1}{q}}\left(\left|f^{(n+1)}(a)\right|^q + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}$$
$$\times\left(\int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha} - t^{n-\alpha}\right)(1-t)^s dt + \int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha} - t^{n-\alpha}\right)t^s dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta(b,a)}{2}\left(\frac{2^{n-\alpha}-1}{(n-\alpha+1)2^{n-\alpha-1}}\right)^{1-\frac{1}{q}}\left(\left|f^{(n+1)}(a)\right|^q + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}$$

$$\times\left(\frac{2^{n-\alpha+s}-1}{(n-\alpha+s+1)\,2^{n-\alpha+s}}+B_{\frac{1}{2}}\left(s+1,n-\alpha+1\right)-B_{\frac{1}{2}}\left(n-\alpha+1,s+1\right)\right)^{\frac{1}{q}}. \tag{49}$$

The proof is completed. □

Corollary 9. *In Theorem 3, if we choose $\eta(b,a)=b-a$, we obtain*

$$\begin{aligned}&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{(n-\alpha)\,\Gamma(n-\alpha)}{2(b-a)^{n-\alpha}}\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(b)+(-1)^{n}\left({}^{c}D^{\alpha}_{b^{-}}f\right)(a)\right]\right|\\&\leq\frac{b-a}{2}\left(\frac{2^{n-\alpha}-1}{(n-\alpha+1)\,2^{n-\alpha-1}}\right)^{1-\frac{1}{q}}\left(\left|f^{(n+1)}(a)\right|^{q}+m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^{q}\right)^{\frac{1}{q}}\\&\quad\times\left(\frac{2^{n-\alpha+s}-1}{(n-\alpha+s+1)\,2^{n-\alpha+s}}+B_{\frac{1}{2}}\left(s+1,n-\alpha+1\right)-B_{\frac{1}{2}}\left(n-\alpha+1,s+1\right)\right)^{\frac{1}{q}}.\end{aligned} \tag{50}$$

Corollary 10. *In Theorem 3, if we take $m=1$, we obtain*

$$\begin{aligned}&\left|\frac{f^{(n)}(a)+f^{(n)}(a+\eta(b,a))}{2}-\frac{(n-\alpha)\,\Gamma(n-\alpha)\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(a+\eta(b,a))+(-1)^{n}\left({}^{c}D^{\alpha}_{(a+\eta(b,a))^{-}}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\right|\\&\leq\frac{\eta(b,a)}{2}\left(\frac{2^{n-\alpha}-1}{(n-\alpha+1)\,2^{n-\alpha-1}}\right)^{1-\frac{1}{q}}\left(\left|f^{(n+1)}(a)\right|^{q}+\left|f^{(n+1)}(b)\right|^{q}\right)^{\frac{1}{q}}\\&\quad\times\left(\frac{2^{n-\alpha+s}-1}{(n-\alpha+s+1)\,2^{n-\alpha+s}}+B_{\frac{1}{2}}\left(s+1,n-\alpha+1\right)-B_{\frac{1}{2}}\left(n-\alpha+1,s+1\right)\right)^{\frac{1}{q}}.\end{aligned} \tag{51}$$

Moreover, if we take $\eta(b,a)=b-a$, we get

$$\begin{aligned}&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{(n-\alpha)\,\Gamma(n-\alpha)}{2(b-a)^{n-\alpha}}\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(b)+(-1)^{n}\left({}^{c}D^{\alpha}_{b^{-}}f\right)(a)\right]\right|\\&\leq\frac{b-a}{2}\left(\frac{2^{n-\alpha}-1}{(n-\alpha+1)\,2^{n-\alpha-1}}\right)^{1-\frac{1}{q}}\left(\left|f^{(n+1)}(a)\right|^{q}+\left|f^{(n+1)}(b)\right|^{q}\right)^{\frac{1}{q}}\\&\quad\times\left(\frac{2^{n-\alpha+s}-1}{(n-\alpha+s+1)\,2^{n-\alpha+s}}+B_{\frac{1}{2}}\left(s+1,n-\alpha+1\right)-B_{\frac{1}{2}}\left(n-\alpha+1,s+1\right)\right)^{\frac{1}{q}}.\end{aligned} \tag{52}$$

Corollary 11. *In Theorem 3, if we take $s=1$, we obtain*

$$\begin{aligned}&\left|\frac{f^{(n)}(a)+f^{(n)}(a+\eta(b,a))}{2}-\frac{(n-\alpha)\,\Gamma(n-\alpha)\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(a+\eta(b,a))+(-1)^{n}\left({}^{c}D^{\alpha}_{(a+\eta(b,a))^{-}}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}}\right|\\&\leq\frac{\eta(b,a)}{2}\left(\frac{2^{n-\alpha}-1}{(n-\alpha+1)\,2^{n-\alpha-1}}\right)\left(\frac{\left|f^{(n+1)}(a)\right|^{q}+m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}.\end{aligned} \tag{53}$$

Moreover, if we take $\eta(b,a)=b-a$, we get

$$\begin{aligned}&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{(n-\alpha)\,\Gamma(n-\alpha)}{2(b-a)^{n-\alpha}}\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(b)+(-1)^{n}\left({}^{c}D^{\alpha}_{b^{-}}f\right)(a)\right]\right|\\&\leq\frac{b-a}{2}\left(\frac{2^{n-\alpha}-1}{(n-\alpha+1)\,2^{n-\alpha-1}}\right)\left(\frac{\left|f^{(n+1)}(a)\right|^{q}+m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}.\end{aligned} \tag{54}$$

Corollary 12. *In Theorem 3, if we take $s = m = 1$, we obtain*

$$\left| \frac{f^{(n)}(a) + f^{(n)}(a+\eta(b,a))}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(a+\eta(b,a)) + (-1)^{n}\left({}^{c}D^{\alpha}_{(a+\eta(b,a))^{-}}f\right)(a)\right]}{2(\eta(b,a))^{n-\alpha}} \right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\frac{2^{n-\alpha}-1}{(n-\alpha+1)2^{n-\alpha-1}}\right)\left(\frac{\left|f^{(n+1)}(a)\right|^{q} + \left|f^{(n+1)}(b)\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{55}$$

Moreover, if we take $\eta(b,a) = b - a$, we get

$$\left| \frac{f^{(n)}(a) + f^{(n)}(b)}{2} - \frac{(n-\alpha)\Gamma(n-\alpha)}{2(b-a)^{n-\alpha}}\left[\left({}^{c}D^{\alpha}_{a^{+}}f\right)(b) + (-1)^{n}\left({}^{c}D^{\alpha}_{b^{-}}f\right)(a)\right] \right|$$
$$\leq \frac{b-a}{2}\left(\frac{2^{n-\alpha}-1}{(n-\alpha+1)2^{n-\alpha-1}}\right)\left(\frac{\left|f^{(n+1)}(a)\right|^{q} + \left|f^{(n+1)}(b)\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{56}$$

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