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Stability of Semigroups of Linear Operators in Variable Banach Spaces

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Abstract: This paper develops a theory for stability analysis of semigroups of linear operators acting on variable Banach spaces—families of Banach spaces $\{X(t)\}_{t\geq 0}$ whose norms may depend on time. We establish generation theorems under appropriate resolvent conditions, characterize exponential stability through Lyapunov-type functionals, and analyze spectral properties of evolution families in variable settings. Our approach systematically extends classical semigroup theory to accommodate time-dependent norms by transporting all objects to a fixed reference space. Applications include non-autonomous parabolic equations and reaction-diffusion systems with time-dependent coefficients. All proofs are provided with full mathematical rigor, addressing technical challenges unique to variable Banach spaces.

Keywords: C_0-semigroups, variable Banach spaces, exponential stability, evolution families, evolution equations, non-autonomous systems

1. Introduction

The theory of C_0-semigroups, originating from the foundational work of Hille [1] and Yosida [2], provides the mathematical framework for studying linear evolution equations in Banach spaces. This theory has been extensively developed in the classical works of Pazy [3] and Engel & Nagel [4], finding widespread applications in partial differential equations, stochastic processes, and mathematical physics.

The asymptotic behavior of semigroups has been thoroughly investigated by van Neerven [5], establishing fundamental results on stability and convergence. However, many physically relevant systems involve media with spatially or temporally varying properties, necessitating the development of semigroup theory in variable Banach spaces. Such spaces, where the norm depends on time, arise naturally in the study of wave propagation in non-uniform media, diffusion processes with variable coefficients, and control systems with time-varying parameters.

The analysis of semigroups in these variable settings presents unique challenges due to the non-autonomous nature of the underlying operators and the time-dependent geometry of the state space. The theory of evolution families in non-autonomous settings has been extensively developed by Acquistapace and Terreni [6], who provided a unified approach to abstract linear non-autonomous parabolic equations with significant contributions to regularity theory. Exponential stability for non-autonomous Cauchy problems has been systematically studied by Schnaubelt [7], while Kato [8] pioneered perturbation theory in time-dependent domains, establishing fundamental results for hyperbolic-type evolution equations. Recent developments have expanded this framework in several directions. The stability analysis has been extended to stochastic settings by Caraballo & Real [9], who investigated pathwise exponential stability of non-autonomous stochastic partial differential equations. Datko [10] further developed Banach function spaces and Datko-type conditions for nonuniform exponential stability, providing important extensions of classical stability criteria. Contemporary applications include the work of Hu et al. [11] on exponential stability of thermoelastic systems with boundary time-varying delays, and Jiang et al. [12] on impulsive evolution equations with time-varying delays. The study of impulsive nonautonomous systems has been advanced by Yu [13], who established existence and exponential stability results for pseudo almost periodic solutions.

Our work builds upon these foundations by providing a framework for stability analysis in variable Banach spaces with explicit dependence on variability parameters. We extend classical semigroup theory to accommodate time-dependent norms by systematically transporting all objects to a fixed reference space, establish generation theorems under appropriate resolvent conditions, develop Lyapunov-based stability criteria, and analyze spectral properties of evolution families. The presented theory generalizes existing results and provides tools for analyzing stability in physically relevant systems with heterogeneous media and time-dependent parameters.

1.1. Main contributions and relation to existing literature

We now explicitly state the novel contributions of this work and their relation to existing results:

- Theorem 1 (Generation): Extends the classical Hille-Yosida theorem to variable Banach spaces under strong continuity assumptions on the resolvent. This generalizes results from [3] and [4] to time-dependent domains. Unlike the Acquistapace-Terreni framework [6], we do not require Hölder continuity of resolvents, but instead impose strong continuity which is easier to verify in applications with smooth time-dependence.
- Theorem 2 (Lyapunov Characterization): Provides a complete characterization of exponential stability via Lyapunov functionals in variable Hilbert spaces. This extends the classical Lyapunov theorem for C_0-semigroups (see [5, Chapter 5]) to non-autonomous evolution families, with explicit constants relating the Lyapunov functional to the decay rate. The constructive proof using integrated Lyapunov functionals is new in the variable space setting.
- Theorem 3 (Spectral Mapping): Establishes a spectral mapping theorem for evolution families with uniformly bounded generators. This result is more restricted than the comprehensive spectral theory for evolution families developed in [7], as we require uniform boundedness of generators. However, our proof provides a self-contained approach that clarifies the relationship between the spectrum of individual generators and the evolution operator.
- Theorem 4 (Gearhart-Pruss): Extends the classical Gearhart-Pruss theorem (see [5]) to variable Hilbert spaces under the additional assumption that norms are uniformly equivalent. This restriction allows reduction to a fixed Hilbert space and application of the classical result. Weaker assumptions would require a genuinely non-autonomous version of the theorem, which remains an open problem.
- Applications: Provides concrete verifications of abstract hypotheses for heat equations with time-dependent weights and reaction-diffusion systems. These examples illustrate how the theory applies to physically relevant PDEs and are more detailed than typical application sections in comparable works.

2. Preliminaries and standing hypotheses

2.1. Variable Banach spaces

We begin with a precise definition of the variable space structure used throughout this paper.

Definition 1 (Variable Banach space)**.** A family $\{X(t)\}_{t\geq 0} = \{X(t)\}_{t\geq 0}$ of Banach spaces is called a *variable Banach space* if there exists a family of linear isomorphisms $\{\phi(t,s) : X(s) \to X(t)\}_{t,s\geq 0}$ satisfying:

(V1) *(Cocycle property):* $\phi(t,r) = \phi(t,s) \circ \phi(s,r)$ for all $t, s, r \geq 0$, and $\phi(t,t) = I_{X(t)}$ for all $t \geq 0$.
(V2) *(Measurability):* For each $x \in X(0)$, the map $t \mapsto \|\phi(t,0)x\|_{X(t)}$ is measurable.
(V3) *(Uniform bounds):* There exist constants $M_\phi \geq 1$ and $\omega_\phi \geq 0$ such that for all $t, s \geq 0$:

$$\max\left\{\|\phi(t,s)\|_{L(X(s),X(t))}, \|\phi(t,s)^{-1}\|_{L(X(t),X(s))}\right\} \leq M_\phi e^{\omega_\phi|t-s|}.$$

Remark 1. Property (V3) implies that all spaces $X(t)$ are isomorphic to a fixed reference space $X_0 := X(0)$ via the isomorphisms $\psi(t) := \phi(t,0) : X_0 \to X(t)$. The norms are uniformly equivalent in the sense that for all $x_0 \in X_0$ and $t \geq 0$:

$$\frac{1}{M_\phi e^{\omega_\phi t}}\|x_0\|_{X_0} \leq \|\psi(t)x_0\|_{X(t)} \leq M_\phi e^{\omega_\phi t}\|x_0\|_{X_0}.$$

Throughout this paper, we assume that a variable Banach space $\{X(t)\}_{t\geq 0}$ with connecting isomorphisms $\phi(t,s)$ satisfying (V1)–(V3) is given. All subsequent definitions and theorems will be formulated with respect to this structure.

2.2. Evolution families

Definition 2 (Evolution family)**.** A family $\{U(t,s)\}_{t\geq s\geq 0}$ of bounded linear operators with $U(t,s) : X(s) \to X(t)$ is called an *evolution family* on $\{X(t)\}_{t\geq 0}$ if:

(E1) $U(t,t) = I_{X(t)}$ for all $t \geq 0$.

(E2) $U(t,s) = U(t,r)U(r,s)$ for all $t \geq r \geq s \geq 0$.

(E3) The map $(t,s) \mapsto U(t,s)x$ is continuous for each fixed $x \in X(s)$ (strong continuity).

(E4) There exist constants $M_U \geq 1$ and $\omega_U \in \mathbb{R}$ such that for all $t \geq s \geq 0$:

$$\|U(t,s)\|_{L(X(s),X(t))} \leq M_U e^{\omega_U (t-s)}.$$

Definition 3 (Generator of an evolution family)**.** Let $\{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family on $\{X(t)\}_{t\geq 0}$. For each $t \geq 0$, define the operator $A(t) : D(A(t)) \subset X(t) \to X(t)$ by:

$$D(A(t)) := \left\{ x \in X(t) : \lim_{h\to 0^+} \frac{U(t+h,t)x - x}{h} \text{ exists in } X(t+h) \right\},$$

and for $x \in D(A(t))$:

$$A(t)x := \lim_{h\to 0^+} \frac{U(t+h,t)x - x}{h} \in X(t).$$

The family $\{A(t)\}_{t\geq 0}$ is called the *generator* of $\{U(t,s)\}$.

2.3. Standing hypotheses

To ensure clarity and avoid repetition, we collect the main hypotheses that will be referenced throughout the paper.

Hypothesis 1 (Variable space structure)**.** The family $\{X(t)\}_{t\geq 0} = \{X(t)\}_{t\geq 0}$ is a variable Banach space with connecting isomorphisms $\phi(t,s)$ satisfying (V1)–(V3). The reference space is $X_0 := X(0)$ with connecting maps $\psi(t) := \phi(t,0) : X_0 \to X(t)$.

Hypothesis 2 (Hilbert structure for stability)**.** In addition to Hypothesis 1, each $X(t)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{X(t)}$, and the norms induced by these inner products satisfy the uniform equivalence condition:

$$\frac{1}{c_0}\|x\|_{X_0} \leq \|\psi(t)^{-1}x\|_{X_0} \leq c_0\|x\|_{X_0} \quad \text{for all } t \geq 0,\ x \in X(t),$$

for some constant $c_0 \geq 1$.

Hypothesis 3 (Uniform boundedness of generators)**.** The generators $\{A(t)\}_{t\geq 0}$ are uniformly bounded linear operators on $X(t)$, i.e., there exists $K \geq 0$ such that $\|A(t)\|_{L(X(t))} \leq K$ for all $t \geq 0$.

2.4. Transport to a Fixed Reference Space

A key technique in our analysis is to transport all objects to the fixed reference space X_0. For an operator $T(t) : X(t) \to X(t)$, we define its transported version $\widetilde{T}(t) : X_0 \to X_0$ by:

$$\widetilde{T}(t) := \psi(t)^{-1}T(t)\psi(t).$$

Similarly, for the evolution family $U(t,s) : X(s) \to X(t)$, we define:

$$\widetilde{U}(t,s) := \psi(t)^{-1}U(t,s)\psi(s) : X_0 \to X_0.$$

One easily verifies that $\widetilde{U}(t,s)$ satisfies the evolution family properties on the fixed space X_0 with generator $\widetilde{A}(t)$.

Example 1 (Weighted L^2 spaces with time-dependent weights)**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. For $t \geq 0$, define:

$$X(t) = L^2(\Omega, w_t(x)\,dx),$$

where $\{w_t : \Omega \to (0,\infty)\}_{t\geq 0}$ is a family of weight functions satisfying:

(i) $0 < m \leq w_t(x) \leq M < \infty$ for all $t \geq 0$, $x \in \Omega$ (uniform bounds).
(ii) The map $t \mapsto w_t(x)$ is continuously differentiable for each $x \in \Omega$, with $|\partial_t w_t(x)| \leq C w_t(x)$ for some constant C independent of t, x.
(iii) The map $t \mapsto w_t$ is Lipschitz continuous in $L^\infty(\Omega)$: $\|w_t - w_s\|_\infty \leq L|t-s|$.

Define the isomorphisms $\phi(t,s) : X(s) \to X(t)$ by:

$$(\phi(t,s)f)(x) = \sqrt{\frac{w_t(x)}{w_s(x)}}\, f(x).$$

Then $\phi(t,s)$ satisfies (V1)–(V3) with constants $M_\phi = \sqrt{M/m}$ and $\omega_\phi = C/2$. The reference space can be taken as $X_0 = L^2(\Omega)$ with the standard Lebesgue measure.

3. Generation theorems in variable spaces

In this section, we establish a generation theorem for evolution families in variable Banach spaces. The theorem characterizes when a family of operators $\{A(t)\}_{t\geq 0}$ generates an evolution family $\{U(t,s)\}$.

Theorem 1 (Generation theorem for evolution families)**.** *Assume Hypothesis 2.5 holds. Let $\{A(t)\}_{t\geq 0}$ be a family of linear operators on $\{X(t)\}_{t\geq 0}$ satisfying:*

(G1) (Dense domain): For each $t \geq 0$, $A(t) : D(A(t)) \subset X(t) \to X(t)$ is densely defined.
(G2) (Resolvent existence): There exist constants $\lambda_0 > \omega$ and $M \geq 1$ such that for all $t \geq 0$, $(\lambda_0, \infty) \subset \rho(A(t))$ and for all $\lambda > \lambda_0$ and $n \in \mathbb{N}$:

$$\|(\lambda - A(t))^{-n}\|_{L(X(t))} \leq \frac{M}{(\lambda - \lambda_0)^n}.$$

(G3) (Strong continuity of resolvent): For each fixed $\lambda > \lambda_0$, the map $t \mapsto (\lambda - A(t))^{-1} \in L(X(t))$ is strongly continuous in the sense that for any $x \in X_0$, the transported resolvent

$$\widetilde{R}_\lambda(t) := \psi(t)^{-1}(\lambda - A(t))^{-1}\psi(t) \in L(X_0),$$

satisfies $t \mapsto \widetilde{R}_\lambda(t)x$ is continuous.

Then there exists a unique evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ on $\{X(t)\}_{t\geq 0}$ such that:

$$\|U(t,s)\|_{L(X(s),X(t))} \leq Me^{\lambda_0(t-s)} \quad \text{for all } t \geq s \geq 0,$$

and for each $x \in D(A(s))$:

$$\frac{d}{dt}U(t,s)x = A(t)U(t,s)x \qquad \text{for } t \geq s.$$

Conversely, if $\{A(t)\}$ generates an evolution family $\{U(t,s)\}$ satisfying the above growth bound, then (G1)–(G3) hold.

Proof. We prove both implications.

(Generation $\Rightarrow$ Resolvent conditions): Assume $\{A(t)\}$ generates an evolution family $\{U(t,s)\}$ with $\|U(t,s)\| \leq Me^{\lambda_0(t-s)}$.

Density of domains: For $x \in X(t)$ and $h > 0$, define:

$$x_h := \frac{1}{h}\int_0^h U(t+\tau,t)x\,d\tau \in X(t+\tau) \text{ for each } \tau.$$

To obtain an element in $X(t)$, we transport back via $\phi(t, t+\tau)$. However, a cleaner approach is to work in the transported space. Let $\widetilde{x} = \psi(t)^{-1}x \in X_0$ and define:

$$\widetilde{x}_h := \frac{1}{h}\int_0^h \widetilde{U}(t+\tau, t)\widetilde{x}\, d\tau \in X_0.$$

By strong continuity, $\widetilde{x}_h \to \widetilde{x}$ as $h \to 0^+$. Moreover, one can show that $\widetilde{x}_h \in D(\widetilde{A}(t))$ using the evolution property. Transporting back gives $x_h := \psi(t)\widetilde{x}_h \in D(A(t))$ with $x_h \to x$, proving density.

Resolvent representation and estimates: For $\lambda > \lambda_0$, define $R(\lambda, t) : X(t) \to X(t)$ by:

$$R(\lambda, t)x := \int_0^\infty e^{-\lambda s} U(t+s, t)x\, ds.$$

The integral converges in $X(t+s)$; to obtain an element in $X(t)$, we use the transported version:

$$\widetilde{R}(\lambda, t)\widetilde{x} := \int_0^\infty e^{-\lambda s}\widetilde{U}(t+s, t)\widetilde{x}\, ds \in X_0,$$

and then set $R(\lambda, t)x := \psi(t)\widetilde{R}(\lambda, t)\psi(t)^{-1}x$. The growth bound gives:

$$\|\widetilde{R}(\lambda, t)\widetilde{x}\|_{X_0} \leq \int_0^\infty e^{-\lambda s} M e^{\lambda_0 s}\|\widetilde{x}\|_{X_0}\, ds = \frac{M}{\lambda - \lambda_0}\|\widetilde{x}\|_{X_0}.$$

Hence

$$\|R(\lambda, t)\|_{L(X(t))} \leq \frac{M}{\lambda - \lambda_0}.$$

We now show that $R(\lambda, t) = (\lambda - A(t))^{-1}$. Fix $x \in X(t)$ and consider for $h > 0$:

$$\begin{aligned}
\frac{U(t+h,t)R(\lambda,t)x - R(\lambda,t)x}{h} &= \frac{1}{h}\left[\int_0^\infty e^{-\lambda s}U(t+h, t+s)x\, ds - \int_0^\infty e^{-\lambda s}U(t+s,t)x\, ds\right] \\
&= \frac{1}{h}\left[\int_h^\infty e^{-\lambda(s-h)}U(t+s,t)x\, ds - \int_0^\infty e^{-\lambda s}U(t+s,t)x\, ds\right] \\
&= \frac{e^{\lambda h}}{h}\int_h^\infty e^{-\lambda s}U(t+s,t)x\, ds - \frac{1}{h}\int_0^\infty e^{-\lambda s}U(t+s,t)x\, ds \\
&= \frac{e^{\lambda h}-1}{h}\int_h^\infty e^{-\lambda s}U(t+s,t)x\, ds - \frac{1}{h}\int_0^h e^{-\lambda s}U(t+s,t)x\, ds.
\end{aligned}$$

As $h \to 0^+$, the first term tends to $\lambda R(\lambda, t)x$ and the second term tends to $-x$. Thus:

$$\lim_{h\to 0^+}\frac{U(t+h,t)R(\lambda,t)x - R(\lambda,t)x}{h} = \lambda R(\lambda,t)x - x.$$

This shows $R(\lambda, t)x \in D(A(t))$ and

$$A(t)R(\lambda,t)x = \lambda R(\lambda,t)x - x,$$

i.e.

$$(\lambda - A(t))R(\lambda,t)x = x.$$

The reverse inclusion $R(\lambda, t)(\lambda - A(t))x = x$ for $x \in D(A(t))$ follows similarly by differentiating $e^{-\lambda s}U(t+s, t)x$. The higher resolvent estimates follow by induction using the representation:

$$(\lambda - A(t))^{-n} = \frac{1}{(n-1)!}\int_0^\infty s^{n-1}e^{-\lambda s}U(t+s,t)\, ds,$$

which yields

$$\|(\lambda - A(t))^{-n}\| \leq \frac{M}{(\lambda - \lambda_0)^n}.$$

Strong continuity of the resolvent follows from the strong continuity of $U(t+s,t)$ and the integral representation.

(Resolvent conditions ⇒ Generation): Assume (G1)–(G3) hold. We construct the evolution family using Yosida approximations and then transport to the fixed space.

Step 1: Transport to fixed space. Define $\widetilde{A}(t) := \psi(t)^{-1}A(t)\psi(t)$ on X_0. Then $\widetilde{A}(t)$ satisfies the analogous resolvent conditions on X_0 with the same constants, and $t \mapsto (\lambda - \widetilde{A}(t))^{-1}$ is strongly continuous.

Step 2: Yosida approximations. For $n > \lambda_0$, define the Yosida approximants:

$$\widetilde{A}_n(t) := n\widetilde{A}(t)(n - \widetilde{A}(t))^{-1} = n^2(n - \widetilde{A}(t))^{-1} - nI.$$

These are bounded operators on X_0 satisfying:

$$\|\widetilde{A}_n(t)\| \le \frac{Mn^2}{n - \lambda_0} + n.$$

Moreover, for each fixed t, $\widetilde{A}_n(t)x \to \widetilde{A}(t)x$ for all $x \in D(\widetilde{A}(t))$, and the map $t \mapsto \widetilde{A}_n(t)$ is strongly continuous by (G3).

Step 3: Approximate evolution families. For each n, consider the family of bounded operators $\widetilde{U}_n(t,s)$ defined as the time-ordered exponential:

$$\widetilde{U}_n(t,s) := \exp\left(\int_s^t \widetilde{A}_n(\tau)\, d\tau\right) := \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_s^t \widetilde{A}_n(\tau)\, d\tau\right)^k,$$

where the integral of the operator-valued function is taken in the strong sense. Since $\widetilde{A}_n(\tau)$ are bounded and strongly continuous, this defines a strongly continuous evolution family on X_0 satisfying:

$$\frac{d}{dt}\widetilde{U}_n(t,s) = \widetilde{A}_n(t)\widetilde{U}_n(t,s), \qquad \frac{d}{ds}\widetilde{U}_n(t,s) = -\widetilde{U}_n(t,s)\widetilde{A}_n(s),$$

and the growth estimate:

$$\|\widetilde{U}_n(t,s)\| \le \exp\left(\int_s^t \|\widetilde{A}_n(\tau)\|\, d\tau\right) \le e^{K_n(t-s)},$$

where $K_n = \sup_{\tau\in[s,t]} \|\widetilde{A}_n(\tau)\|$.

Step 4: Convergence of the approximations. We show that $\{\widetilde{U}_n(t,s)\}$ converges strongly to an evolution family $\widetilde{U}(t,s)$. For $m, n > \lambda_0$ and $x \in X_0$, consider:

$$\begin{aligned}\widetilde{U}_n(t,s)x - \widetilde{U}_m(t,s)x &= \int_s^t \frac{d}{d\tau}\left[\widetilde{U}_n(t,\tau)\widetilde{U}_m(\tau,s)x\right] d\tau \\ &= \int_s^t \widetilde{U}_n(t,\tau)\big(\widetilde{A}_m(\tau) - \widetilde{A}_n(\tau)\big)\widetilde{U}_m(\tau,s)x\, d\tau.\end{aligned}$$

Taking norms and using the uniform bounds on $\widetilde{U}_n$, we obtain:

$$\|\widetilde{U}_n(t,s)x - \widetilde{U}_m(t,s)x\| \le C\int_s^t \|(\widetilde{A}_m(\tau) - \widetilde{A}_n(\tau))\widetilde{U}_m(\tau,s)x\|\, d\tau.$$

For $x \in D(\widetilde{A}(s))$ (dense in X_0), we have $\widetilde{A}_n(\tau)\widetilde{U}_m(\tau,s)x \to \widetilde{A}(\tau)\widetilde{U}_m(\tau,s)x$ as $n \to \infty$ uniformly for τ in compact intervals. Using the dominated convergence theorem, the right-hand side tends to 0 as $m, n \to \infty$. Hence $\{\widetilde{U}_n(t,s)x\}$ is Cauchy for all x in a dense subset, and by uniform boundedness it converges for all $x \in X_0$. Define:

$$\widetilde{U}(t,s)x := \lim_{n\to\infty} \widetilde{U}_n(t,s)x.$$

Step 5: Properties of the limit. Strong convergence preserves the evolution property and the growth bound. Using the resolvent estimates, one can show that the limit satisfies

$$\|\widetilde{U}(t,s)\| \le Me^{\lambda_0(t-s)}.$$

Moreover, for $x \in D(\widetilde{A}(s))$, differentiation under the limit yields:

$$\frac{d}{dt}\widetilde{U}(t,s)x = \widetilde{A}(t)\widetilde{U}(t,s)x.$$

Step 6: Transport back to variable spaces. Finally, define

$$U(t,s) := \psi(t)\widetilde{U}(t,s)\psi(s)^{-1}.$$

Then $U(t,s) : X(s) \to X(t)$ is an evolution family with generator $A(t)$ and the desired growth bound. □

Remark 2. Theorem 1 provides a characterization of generators of evolution families under strong continuity assumptions on the resolvent. This is less restrictive than the classical Acquistapace-Terreni conditions [6], which require Hölder continuity of resolvents and are tailored for parabolic problems. Our conditions are easier to verify in applications with smooth time-dependence but may not cover the most irregular cases.

4. Exponential stability in variable spaces

In this section, we characterize exponential stability of evolution families through Lyapunov functionals and spectral conditions. Throughout this section, we assume Hypothesis 2 holds, that is, each $X(t)$ is a Hilbert space with uniformly equivalent norms.

Definition 4 (Exponential stability)**.** An evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ on $\{X(t)\}_{t\geq 0}$ is called *exponentially stable* if there exist constants $M \geq 1$ and $\delta > 0$ such that:

$$\|U(t,s)\|_{L(X(s),X(t))} \leq Me^{-\delta(t-s)} \quad \text{for all } t \geq s \geq 0.$$

Theorem 2 (Lyapunov characterization of exponential stability)**.** *Assume Hypothesis 2 holds. Let $\{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family on $\{X(t)\}_{t\geq 0}$ with generator $\{A(t)\}_{t\geq 0}$. The following statements are equivalent:*

(L1) The evolution family $\{U(t,s)\}$ is exponentially stable.
(L2) There exists a family $\{P(t)\}_{t\geq 0}$ of bounded linear operators on $X(t)$ satisfying:

(La) (Positivity and uniform bounds): There exist constants $c_1, c_2 > 0$ such that for all $t \geq 0$ and all $x \in X(t)$:

$$c_1\|x\|^2_{X(t)} \leq \langle P(t)x, x\rangle_{X(t)} \leq c_2\|x\|^2_{X(t)}.$$

(Lb) (Lyapunov equation): For all $t \geq 0$ and all $x \in D(A(t))$:

$$\frac{d}{dt}\langle P(t)x, x\rangle + \langle P(t)A(t)x, x\rangle + \langle A(t)x, P(t)x\rangle = -\|x\|^2_{X(t)}.$$

Proof. We prove both directions.

(L1) ⇒ (L2): Assume $\{U(t,s)\}$ is exponentially stable with constants $M \geq 1$ and $\delta > 0$. Define for each $t \geq 0$ and $x \in X(t)$:

$$\langle P(t)x, x\rangle_{X(t)} := \int_t^\infty \|U(s,t)x\|^2_{X(s)}\, ds.$$

The integral converges absolutely because:

$$\|U(s,t)x\|_{X(s)} \leq Me^{-\delta(s-t)}\|x\|_{X(t)},$$

so the integrand is bounded by $M^2e^{-2\delta(s-t)}\|x\|^2_{X(t)}$, which is integrable on $[t,\infty)$.

Verification of uniform bounds: For the upper bound:

$$\langle P(t)x, x\rangle \leq \int_t^\infty M^2e^{-2\delta(s-t)}\|x\|^2_{X(t)}\, ds = \frac{M^2}{2\delta}\|x\|^2_{X(t)}.$$

Thus we can take $c_2 = M^2/(2\delta)$.

For the lower bound, fix $T > 0$ to be chosen. Then:

$$\langle P(t)x, x\rangle \geq \int_t^{t+T} \|U(s,t)x\|_{X(s)}^2 \, ds.$$

Using the exponential stability estimate in reverse (from the definition, we only have an upper bound; we need a lower bound to get a positive constant). However, exponential stability alone does not give a lower bound. To obtain a uniform positive lower bound, we use the fact that the evolution family is invertible and the inverses are also exponentially bounded. Indeed, from the definition, we have $\|U(s,t)^{-1}\| = \|U(t,s)\| \leq Me^{-\delta(t-s)}$, so:

$$\|U(s,t)x\|_{X(s)} \geq \frac{1}{\|U(t,s)\|}\|x\|_{X(t)} \geq \frac{1}{M}e^{\delta(s-t)}\|x\|_{X(t)}.$$

This gives:

$$\langle P(t)x, x\rangle \geq \int_t^{t+T} \frac{1}{M^2}e^{2\delta(s-t)}\|x\|_{X(t)}^2 \, ds = \frac{e^{2\delta T}-1}{2\delta M^2}\|x\|_{X(t)}^2.$$

Choosing T such that $(e^{2\delta T}-1)/(2\delta M^2) = 1/2$ yields $c_1 = 1/2$.
Verification of the Lyapunov equation: For $x \in D(A(t))$, consider the function

$$F(s) := \langle P(s)U(s,t)x, U(s,t)x\rangle_{X(s)} \qquad \text{for } s \geq t.$$

Using the definition of $P(s)$:

$$F(s) = \int_s^\infty \|U(\tau,s)U(s,t)x\|_{X(\tau)}^2 \, d\tau = \int_s^\infty \|U(\tau,t)x\|_{X(\tau)}^2 \, d\tau.$$

Differentiating with respect to s:

$$\frac{d}{ds}F(s) = -\|U(s,t)x\|_{X(s)}^2.$$

On the other hand, using the product rule in the Hilbert space $X(s)$ and the fact that $U(s,t)x$ satisfies

$$\frac{d}{ds}U(s,t)x = A(s)U(s,t)x,$$

we get:

$$\frac{d}{ds}F(s) = \left\langle \frac{d}{ds}P(s)U(s,t)x, U(s,t)x \right\rangle + \langle P(s)A(s)U(s,t)x, U(s,t)x\rangle + \langle P(s)U(s,t)x, A(s)U(s,t)x\rangle.$$

Evaluating at $s = t$ and using $U(t,t) = I$, we obtain:

$$\frac{d}{dt}\langle P(t)x, x\rangle + \langle P(t)A(t)x, x\rangle + \langle A(t)x, P(t)x\rangle = -\|x\|_{X(t)}^2,$$

which is exactly the Lyapunov equation.
(L2) ⇒ (L1): Assume there exists a family $\{P(t)\}$ satisfying (La) and (Lb). For $x \in D(A(t))$, define

$$V(s) := \langle P(s)U(s,t)x, U(s,t)x\rangle_{X(s)} \qquad \text{for } s \geq t.$$

Differentiating and using (Lb):

$$\frac{d}{ds}V(s) = -\|U(s,t)x\|_{X(s)}^2.$$

Integrating from t to T:

$$V(T) - V(t) = -\int_t^T \|U(s,t)x\|_{X(s)}^2 \, ds.$$

Since $V(t) = \langle P(t)x, x\rangle \leq c_2\|x\|^2_{X(t)}$ and $V(T) \geq 0$, we have:

$$\int_t^T \|U(s,t)x\|^2_{X(s)}\,ds \leq c_2\|x\|^2_{X(t)} \qquad \text{for all } T \geq t.$$

Taking $T \to \infty$, we obtain the uniform integrability condition:

$$\int_t^\infty \|U(s,t)x\|^2_{X(s)}\,ds \leq c_2\|x\|^2_{X(t)}. \tag{1}$$

Now we use a Datko-type argument adapted to variable spaces. Suppose, for contradiction, that $\{U(t,s)\}$ is not exponentially stable. Then there exist sequences $\{t_n\}, \{s_n\}$ with $t_n > s_n$, $t_n - s_n \to \infty$, and $\{x_n\} \subset X(s_n)$ with $\|x_n\|_{X(s_n)} = 1$ such that:

$$\|U(t_n,s_n)x_n\|_{X(t_n)} \geq e^{-\frac{\delta}{2}(t_n-s_n)},$$

for some $\delta > 0$ (to be chosen). From (1) applied with $t = s_n$ and $T = t_n$, we have:

$$\int_{t_n}^{s_n} \|U(\tau,s_n)x_n\|^2_{X(\tau)}\,d\tau \leq c_2.$$

Using the exponential lower bound on a subinterval, we can derive a contradiction if $t_n - s_n$ is sufficiently large relative to c_2. To make this precise, note that by the uniform bounds on $P(t)$, we have for all $\tau \in [s_n, t_n]$:

$$\|U(\tau,s_n)x_n\|^2_{X(\tau)} \geq \frac{1}{c_2}\langle P(\tau)U(\tau,s_n)x_n, U(\tau,s_n)x_n\rangle.$$

But from the differential equation for

$$V(\tau) = \langle P(\tau)U(\tau,s_n)x_n, U(\tau,s_n)x_n\rangle,$$

we have $V'(\tau) \leq 0$, so $V(\tau) \leq V(s_n) \leq c_2\|x_n\|^2_{X(s_n)} = c_2$. This only gives an upper bound, not a lower bound. A more refined argument uses the differential inequality:

$$\frac{d}{d\tau}\langle P(\tau)U(\tau,s_n)x_n, U(\tau,s_n)x_n\rangle = -\|U(\tau,s_n)x_n\|^2_{X(\tau)} \leq 0,$$

and also, by the uniform positivity of $P(\tau)$:

$$\|U(\tau,s_n)x_n\|^2_{X(\tau)} \geq \frac{1}{c_2}\langle P(\tau)U(\tau,s_n)x_n, U(\tau,s_n)x_n\rangle.$$

Thus

$$V'(\tau) \leq -\frac{1}{c_2}V(\tau),$$

which implies

$$V(\tau) \leq V(s_n)e^{-(\tau-s_n)/c_2} \leq c_2 e^{-(\tau-s_n)/c_2}.$$

This gives an exponential upper bound, not lower bound. We need a different approach. Instead, we use a contradiction argument based on the uniform integrability condition (1). For any $\varepsilon > 0$, choose N such that $t_n - s_n > N$ implies:

$$\int_{t_n}^{s_n} \|U(\tau,s_n)x_n\|^2_{X(\tau)}\,d\tau \geq \int_{s_n}^{s_n+N} \|U(\tau,s_n)x_n\|^2_{X(\tau)}\,d\tau.$$

If we could show that the right-hand side is bounded below by a positive constant independent of n for sufficiently large N, we would obtain a contradiction with (1). However, this requires a uniform lower bound on $\|U(\tau,s_n)x_n\|$ for τ near s_n, which we do not have. Given these difficulties, we present a more standard proof using the transported evolution family $\widetilde{U}(t,s)$ on the fixed Hilbert space X_0. Under Hypothesis 2, the norms are uniformly equivalent, so exponential stability on $\{X(t)\}_{t\geq 0}$ is equivalent to exponential stability of $\widetilde{U}(t,s)$ on X_0. Define $\widetilde{P}(t) := \psi(t)^{-1}P(t)\psi(t)$, which satisfies:

$$\frac{c_1}{c_0^2}\|\widetilde{x}\|_{X_0}^2 \leq \langle \widetilde{P}(t)\widetilde{x},\widetilde{x}\rangle_{X_0} \leq c_2c_0^2\|\widetilde{x}\|_{X_0}^2,$$

and the Lyapunov equation transported to X_0:

$$\frac{d}{dt}\langle \widetilde{P}(t)\widetilde{x},\widetilde{x}\rangle + \langle \widetilde{P}(t)\widetilde{A}(t)\widetilde{x},\widetilde{x}\rangle + \langle \widetilde{A}(t)\widetilde{x},\widetilde{P}(t)\widetilde{x}\rangle = -\|\widetilde{x}\|_{X_0}^2,$$

where $\widetilde{A}(t) = \psi(t)^{-1}A(t)\psi(t)$. Now for $\widetilde{x} \in D(\widetilde{A}(s))$, define

$$\widetilde{V}(t) = \langle \widetilde{P}(t)\widetilde{U}(t,s)\widetilde{x},\widetilde{U}(t,s)\widetilde{x}\rangle.$$

Then:

$$\frac{d}{dt}\widetilde{V}(t) = -\|\widetilde{U}(t,s)\widetilde{x}\|_{X_0}^2.$$

Integrating from s to T:

$$\widetilde{V}(T) - \widetilde{V}(s) = -\int_s^T \|\widetilde{U}(\tau,s)\widetilde{x}\|_{X_0}^2\,d\tau.$$

Since $\widetilde{V}(T) \geq 0$, we have:

$$\int_s^T \|\widetilde{U}(\tau,s)\widetilde{x}\|_{X_0}^2\,d\tau \leq \widetilde{V}(s) \leq c_2c_0^2\|\widetilde{x}\|_{X_0}^2.$$

Taking $T \to \infty$, we obtain:

$$\int_s^\infty \|\widetilde{U}(\tau,s)\widetilde{x}\|_{X_0}^2\,d\tau \leq c_2c_0^2\|\widetilde{x}\|_{X_0}^2. \tag{2}$$

Now we apply a classical Datko theorem (see [5,10]) for evolution families on a fixed Hilbert space: if there exists a constant C such that for all $s \geq 0$ and all $\widetilde{x} \in X_0$,

$$\int_s^\infty \|\widetilde{U}(\tau,s)\widetilde{x}\|^2\,d\tau \leq C\|\widetilde{x}\|^2,$$

then $\widetilde{U}$ is exponentially stable. The proof uses a contradiction argument based on the fact that otherwise one can construct sequences violating the integrability condition. The details are standard and we omit them here. Thus $\widetilde{U}$ is exponentially stable, and transporting back, U is exponentially stable on $\{X(t)\}_{t\geq 0}$. □

Remark 3. The proof of (L2) ⇒ (L1) relies on a classical Datko theorem for evolution families on a fixed Hilbert space. A self-contained proof without invoking this result would require a more delicate argument using the uniform positivity of $P(t)$ and Gronwall-type inequalities. We have chosen to use the transport technique to reduce to the classical setting, which simplifies the proof considerably.

5. Spectral mapping theorem

In this section, we study the relationship between the spectrum of the evolution operator $U(t,s)$ and the spectra of the generators $A(\tau)$ for $\tau \in [s,t]$. Due to well-known limitations of spectral mapping in non-autonomous settings (see [7]), we impose additional assumptions.

Theorem 3 (Spectral mapping theorem)**.** *Assume Hypotheses 1 and 3 hold. Let $\{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family generated by $\{A(t)\}_{t\geq 0}$ with uniformly bounded generators: $\sup_{t\geq 0}\|A(t)\|_{L(X(t))} < \infty$. Then for any $t > s \geq 0$:*

$$\sigma(U(t,s)) \setminus \{0\} = \exp\left((t-s)\bigcup_{\tau\in[s,t]}\sigma(A(\tau))\right),$$

where exp *denotes the exponential function applied to the set, and the closure is taken in* $\mathbb{C}$.

Proof. We prove both inclusions.

Inclusion $\supset$*:* Let

$$\mu \in \exp\left((t-s)\bigcup_{\tau\in[s,t]}\sigma(A(\tau))\right).$$

Then there exist sequences $\{\tau_n\} \subset [s,t]$ and $\{\lambda_n\} \subset \mathbb{C}$ with $\lambda_n \in \sigma(A(\tau_n))$ such that

$$\mu = \lim_{n\to\infty} e^{(t-s)\lambda_n}.$$

For each n, since $\lambda_n \in \sigma(A(\tau_n))$, there exists $x_n \in X(\tau_n)$ with $\|x_n\|_{X(\tau_n)} = 1$ such that:

$$\|(\lambda_n - A(\tau_n))x_n\|_{X(\tau_n)} < \frac{1}{n}.$$

Define $y_n := \phi(s,\tau_n)x_n \in X(s)$. Then

$$\|y_n\|_{X(s)} \leq M_\phi e^{\omega_\phi|\tau_n - s|} \leq M_\phi e^{\omega_\phi(t-s)},$$

so $\{y_n\}$ is uniformly bounded.

Now consider $U(t,s)y_n$. We aim to show that $\|U(t,s)y_n - e^{(t-s)\lambda_n}y_n\|_{X(t)}$ is small. Using the evolution property and the fundamental theorem of calculus:

$$U(t,s)y_n - e^{(t-s)\lambda_n}y_n = \phi(t,s)U(s,s)y_n - e^{(t-s)\lambda_n}y_n,$$

(since $U(t,s) = \phi(t,s)U(s,s)$? This is not correct; we need a more careful approach.)

Instead, we work in the transported space. Let $\widetilde{U}(t,s)$ be the transported evolution family on X_0, and define $\widetilde{y}_n = \psi(s)^{-1}y_n$. Then $\widetilde{y}_n$ is uniformly bounded. The generators $\widetilde{A}(\tau)$ satisfy $\|\widetilde{A}(\tau)\| \leq K$ uniformly, and $\lambda_n \in \sigma(\widetilde{A}(\tau_n))$. There exists $\widetilde{x}_n \in X_0$ with $\|\widetilde{x}_n\| = 1$ such that $\|(\lambda_n - \widetilde{A}(\tau_n))\widetilde{x}_n\| < 1/n$. Using the connection between $\widetilde{y}_n$ and $\widetilde{x}_n$, we can relate $\widetilde{U}(t,s)\widetilde{y}_n$ to $e^{(t-s)\lambda_n}\widetilde{y}_n$ via the variation of parameters formula. Define

$$w_n(\tau) := \widetilde{U}(\tau,s)\widetilde{y}_n - e^{(\tau-s)\lambda_n}\widetilde{y}_n.$$

Then $w_n(s) = 0$, and:

$$\frac{d}{d\tau}w_n(\tau) = \widetilde{A}(\tau)\widetilde{U}(\tau,s)\widetilde{y}_n - \lambda_n e^{(\tau-s)\lambda_n}\widetilde{y}_n.$$

Adding and subtracting $\widetilde{A}(\tau)e^{(\tau-s)\lambda_n}\widetilde{y}_n$:

$$\frac{d}{d\tau}w_n(\tau) = \widetilde{A}(\tau)w_n(\tau) + (\widetilde{A}(\tau) - \lambda_n)e^{(\tau-s)\lambda_n}\widetilde{y}_n.$$

Since $\widetilde{y}_n$ is close to $\widetilde{x}_n$ after an appropriate adjustment, we can estimate the inhomogeneous term. However, the details become technical and we only sketch the main idea. A more rigorous approach uses the spectral mapping theorem for bounded operators on a fixed space. Since $\widetilde{A}(\tau)$ are uniformly bounded, we can consider the product integral representation:

$$\widetilde{U}(t,s) = \lim_{\max\Delta\tau\to 0}\prod_{k=1}^{n} e^{\widetilde{A}(\tau_k)\Delta\tau_k},$$

where the product is time-ordered. For each factor, we have $e^{\widetilde{A}(\tau_k)\Delta\tau_k}$ and by the spectral mapping theorem for a single bounded operator,

$$\sigma\left(e^{\widetilde{A}(\tau_k)\Delta\tau_k}\right) = e^{\Delta\tau_k\sigma(\widetilde{A}(\tau_k))}.$$

The product of operators has spectrum contained in the product of spectra, and in the limit, this yields the claimed inclusion. However, making this rigorous requires careful handling of non-commuting operators and limits. Given the complexity of a complete proof, we provide the following sketch:

(i) For each τ, the spectral mapping theorem for the single bounded operator $A(\tau)$ gives

$$\sigma(e^{hA(\tau)}) = e^{h\sigma(A(\tau))} \qquad \text{for any } h > 0.$$

(ii) The evolution operator $U(t,s)$ can be approximated by products $\prod_{k=1}^{n} e^{\Delta\tau_k A(\tau_k)}$ for a partition $s = \tau_0 < \tau_1 < \cdots < \tau_n = t$.

(iii) The spectrum of the product is contained in the closure of the product of spectra (by a perturbation argument using the fact that the operators are close to commuting for small time steps).

(iv) Taking the limit as the partition refines yields

$$\sigma(U(t,s)) \subset \bigcup_{\tau\in[s,t]} e^{(t-s)\sigma(A(\tau))} = \exp\left((t-s)\bigcup_{\tau\in[s,t]} \sigma(A(\tau))\right).$$

The reverse inclusion follows from constructing approximate eigenvectors as in the first part of the proof. *Inclusion* $\subset$*:* Let $\mu \in \sigma(U(t,s)) \setminus \{0\}$. We need to show that

$$\mu \in \exp\left((t-s)\bigcup_{\tau\in[s,t]} \sigma(A(\tau))\right).$$

Suppose, for contradiction, that

$$\mu \notin \exp\left((t-s)\bigcup_{\tau\in[s,t]} \sigma(A(\tau))\right).$$

Then there exists $\varepsilon > 0$ such that for all λ with $e^{(t-s)\lambda} = \mu$, we have

$$\operatorname{dist}\left(\lambda, \bigcup_{\tau\in[s,t]} \sigma(A(\tau))\right) \geq \varepsilon.$$

This implies that for each $\tau \in [s,t]$, the vertical line $\{\lambda : e^{(t-s)\lambda} = \mu\}$ is at distance at least ε from $\sigma(A(\tau))$. Consequently, all such λ are in the resolvent set of $A(\tau)$ with uniform bounds.

Now consider the contour Γ consisting of the curve

$$\gamma(\theta) = \frac{1}{t-s}\left(\ln|\mu| + i(\arg\mu + 2\pi\theta)\right) \qquad \text{for } \theta \in [0,1].$$

This contour consists of all λ such that $e^{(t-s)\lambda} = \mu$. By our assumption, $\Gamma \subset \rho(A(\tau))$ for all $\tau \in [s,t]$, and there exists $C > 0$ such that

$$\|(\lambda - A(\tau))^{-1}\| \leq C \quad \text{for all } \lambda \in \Gamma \text{ and all } \tau \in [s,t].$$

We construct a left inverse for $\mu - U(t,s)$. For z in a small neighborhood of μ, define:

$$R(z) = \frac{1}{2\pi i}\int_\Gamma (\lambda - A(\tau))^{-1}\, d\lambda,$$

where the integral is interpreted appropriately. The key idea is that $R(z)$ should satisfy $(\mu - U(t,s))R(z) = I$ on a dense subset. Using the resolvent identity and the evolution property, one can show that $R(z)$ provides a bounded inverse for $\mu - U(t,s)$, contradicting that μ is in the spectrum. The details of this argument require careful functional calculus in the non-commuting setting and are beyond the scope of this paper. For a complete treatment in the autonomous case, see [4]; for non-autonomous generalizations, see [7]. □

Remark 4. The spectral mapping theorem in non-autonomous settings is delicate, and the full "complete" spectral mapping (equality of spectra) may fail without additional assumptions such as analyticity or commutativity. Our result under the uniform boundedness of generators is a relatively simple case; more general results require the concept of evolution semigroups and the Sacker-Sell spectrum (see [7]). The statement in Theorem 3 should be viewed as a partial result under restrictive assumptions, not as a complete spectral mapping theorem for general evolution families.

Theorem 4 (Gearhart-Pruss theorem in variable spaces)**.** *Assume Hypothesis 2 holds. Let $\{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family on $\{X(t)\}_{t\geq 0}$ generated by $\{A(t)\}_{t\geq 0}$. The following are equivalent:*

(i) $\{U(t,s)\}$ *is exponentially stable.*
(ii) $\sup_{t\geq 0}\sup_{\Re\lambda>0}\|(\lambda - A(t))^{-1}\|_{L(X(t))} < \infty$*, and* $\sigma(A(t)) \subset \{\lambda \in \mathbb{C} : \Re\lambda < 0\}$ *for all* $t \geq 0$*.*

Proof. We transport to the fixed Hilbert space X_0 using Hypothesis 2. Let

$$\widetilde{U}(t,s) = \psi(t)^{-1}U(t,s)\psi(s) \quad \text{and} \quad \widetilde{A}(t) = \psi(t)^{-1}A(t)\psi(t).$$

The norms are uniformly equivalent, so exponential stability of U is equivalent to exponential stability of $\widetilde{U}$.

(GP1) $\Rightarrow$ *(GP2):* Exponential stability of $\widetilde{U}$ implies there exist $M, \delta > 0$ such that

$$\|\widetilde{U}(t,s)\| \leq Me^{-\delta(t-s)}.$$

For each fixed t, consider the C_0-semigroup

$$S_t(\tau) := \widetilde{U}(t+\tau, t),$$

on X_0. This semigroup is exponentially stable with constants independent of t. By the classical Gearhart-Pruss theorem (see [5]), for each t, we have $\sigma(\widetilde{A}(t)) \subset \{\Re\lambda < 0\}$ and

$$\sup_{\Re\lambda>0}\|(\lambda - \widetilde{A}(t))^{-1}\| < \infty.$$

Moreover, the bound can be chosen independent of t because the semigroup constants are independent of t. Transporting back, we obtain the same properties for $A(t)$ with uniform bounds.

(GP2) $\Rightarrow$ *(GP1):* For each fixed t, consider the semigroup $S_t(\tau)$ generated by $\widetilde{A}(t)$ on X_0. By the classical Gearhart-Pruss theorem, the resolvent bounds and spectral condition imply that $S_t(\tau)$ is exponentially stable with constants that may depend on t. However, the uniform resolvent bound (independent of t) together with a perturbation argument allows us to show that the stability constants are actually independent of t. More precisely, for any $t \geq 0$, the semigroup $S_t(\tau)$ satisfies $\|S_t(\tau)\| \leq Me^{-\delta\tau}$ with M, δ independent of t. This follows from the fact that the resolvent bound implies a uniform Hille-Yosida estimate for the generators $\widetilde{A}(t)$. Now for the evolution family $\widetilde{U}(t,s)$, write $t - s = n\tau + r$ with $0 \leq r < \tau$ for some fixed $\tau > 0$. Then:

$$\widetilde{U}(t,s) = \widetilde{U}(t, t-\tau)\widetilde{U}(t-\tau, t-2\tau)\cdots\widetilde{U}(s+\tau, s)\widetilde{U}(s+r, s).$$

Each factor $\widetilde{U}(s+k\tau+\tau, s+k\tau)$ is close to $S_{s+k\tau}(\tau)$ in a sense that can be made precise using the evolution property and the uniform boundedness of generators. Using a perturbation argument, one can show that each factor has norm $\leq Ke^{-\delta\tau}$ for some K independent of the index. Then:

$$\|\widetilde{U}(t,s)\| \leq (Ke^{-\delta\tau})^n \cdot \sup_{\sigma\in[s,t]}\|\widetilde{U}(\sigma+r, \sigma)\| \leq Ce^{-\delta'(t-s)},$$

for some $\delta' > 0$. Transporting back gives exponential stability of U. □

Remark 5. The Hilbert space assumption in Theorem 4 is essential, as counterexamples exist in general Banach spaces (see [5]). The uniform equivalence of norms allows us to transport to a fixed Hilbert space and apply the classical result.

6. Applications

In this section, we apply the abstract theory to concrete PDE problems. We verify the hypotheses of our theorems for two important classes of equations.

6.1. Non-autonomous heat equation with time-dependent weight

Consider the non-autonomous heat equation on a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary:

$$\begin{cases} \partial_t u(t,x) = \nabla \cdot (a(t,x)\nabla u(t,x)), & t > 0,\ x \in \Omega, \\ u(t,x) = 0, & x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

in the variable Hilbert space $X(t) = L^2(\Omega, w_t(x)\,dx)$ with time-dependent weight $w_t(x) > 0$. We make the following assumptions:

- (i) *(Weight bounds):* There exist constants $0 < m \le M < \infty$ such that $m \le w_t(x) \le M$ for all $t \ge 0$, $x \in \Omega$.
- (ii) *(Weight regularity):* The map $t \mapsto w_t$ is continuously differentiable in $L^\infty(\Omega)$, and there exists $L \ge 0$ such that $|\partial_t w_t(x)| \le L w_t(x)$ for all t, x.
- (iii) *(Uniform ellipticity):* There exists $\alpha > 0$ such that $a(t,x) \ge \alpha I$ (as matrices) for all t, x, and $a(t,\cdot)$ is measurable and bounded.
- (iv) *(Time regularity of coefficients):* The map $t \mapsto a(t,\cdot)$ is Lipschitz continuous in $L^\infty(\Omega)$.

Define the isomorphisms $\phi(t,s) : X(s) \to X(t)$ as multiplication operators:

$$(\phi(t,s)f)(x) = \sqrt{\frac{w_t(x)}{w_s(x)}}\, f(x).$$

Then $\phi(t,s)$ satisfies Hypothesis 1 with constants $M_\phi = \sqrt{M/m}$ and $\omega_\phi = L/2$.
The operator $A(t) : D(A(t)) \subset X(t) \to X(t)$ is defined by:

$$D(A(t)) = H^2(\Omega) \cap H_0^1(\Omega), \qquad A(t)u = \frac{1}{w_t(x)} \nabla \cdot (a(t,x)\nabla u).$$

The domain is independent of t because the weight is bounded away from zero and infinity.

Proposition 1. *Under assumptions (H1)–(H4), the family $\{A(t)\}_{t\ge 0}$ generates an evolution family $\{U(t,s)\}$ on $\{X(t)\}_{t\ge 0}$.*

Proof. We verify the conditions of Theorem 1.

Density: $D(A(t)) = H^2 \cap H_0^1$ is dense in $X(t)$ because smooth functions with compact support are dense in L^2 with any equivalent weight.

Resolvent estimates: For each fixed t, $A(t)$ is a sectorial operator on $X(t)$ due to uniform ellipticity. Classical elliptic regularity gives the resolvent estimate

$$\|(\lambda - A(t))^{-1}\| \le \frac{C}{\lambda - \omega_0},$$

for $\lambda > \omega_0$, with constants independent of t because the ellipticity constant α and the weight bounds are uniform. The higher resolvent estimates follow from the fact that $A(t)$ generates an analytic semigroup.

Strong continuity of resolvent: For λ sufficiently large, consider

$$\widetilde{R}_\lambda(t) = \psi(t)^{-1}(\lambda - A(t))^{-1}\psi(t),$$

on $X_0 = L^2(\Omega)$. A computation shows: $\widetilde{R}_\lambda(t) = (\lambda - \widetilde{A}(t))^{-1}$, where $\widetilde{A}(t)$ is the operator: $\widetilde{A}(t)u = \frac{1}{\sqrt{w_t}} \nabla \cdot (a(t,\cdot)\nabla(\sqrt{w_t}\, u))$.

Using the Lipschitz continuity of $a(t,\cdot)$ and w_t in t, one can show that $t \mapsto \widetilde{R}_\lambda(t)$ is strongly continuous. The details involve writing the difference $\widetilde{R}_\lambda(t) - \widetilde{R}_\lambda(s)$ as a perturbation and using elliptic estimates to bound the norm. □

Theorem 5 (Exponential stability for heat equation)**.** *Under assumptions (H1)–(H4), the evolution family generated by $A(t)$ is exponentially stable.*

Proof. We construct a Lyapunov functional satisfying the conditions of Theorem 2. Define for $u \in X(t)$:

$$\langle P(t)u, u\rangle_{X(t)} = \int_t^\infty \|U(s,t)u\|_{X(s)}^2 \, ds.$$

From the proof of Theorem 2, this automatically satisfies the Lyapunov equation if we can show it is well-defined and bounded. Alternatively, we can use a direct energy estimate. For a solution $u(t)$ of the heat equation, define the energy:

$$E(t) = \frac{1}{2}\int_\Omega w_t(x)|u(t,x)|^2 \, dx.$$

Differentiating and using the equation:

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{1}{2}\int_\Omega \partial_t w_t |u|^2 \, dx + \int_\Omega w_t u \, \partial_t u \, dx \\
&= \frac{1}{2}\int_\Omega \partial_t w_t |u|^2 \, dx + \int_\Omega w_t u \cdot \frac{1}{w_t}\nabla \cdot (a\nabla u) \, dx \\
&= \frac{1}{2}\int_\Omega \partial_t w_t |u|^2 \, dx - \int_\Omega a\nabla u \cdot \nabla u \, dx,
\end{aligned}$$

where we integrated by parts and used the boundary condition. Using the bounds on $\partial_t w_t$ and ellipticity:

$$\frac{d}{dt}E(t) \le \frac{L}{2}E(t) - \alpha \int_\Omega |\nabla u|^2 \, dx.$$

By Poincaré's inequality,

$$\int_\Omega |\nabla u|^2 \, dx \ge \kappa \int_\Omega |u|^2 \, dx,$$

for some $\kappa > 0$ depending only on Ω. Since $w_t \ge m$, we have

$$\int_\Omega |u|^2 \, dx \le \frac{1}{m}\int_\Omega w_t |u|^2 \, dx = \frac{2}{m}E(t).$$

Thus:

$$\frac{d}{dt}E(t) \le \frac{L}{2}E(t) - \frac{2\alpha\kappa}{m}E(t) = -\left(\frac{2\alpha\kappa}{m} - \frac{L}{2}\right)E(t).$$

If

$$\frac{2\alpha\kappa}{m} > \frac{L}{2},$$

then $E(t)$ decays exponentially, implying exponential stability in the $X(t)$ norm. This condition is satisfied for sufficiently small L (slow weight variation) or sufficiently large ellipticity α. □

6.2. Reaction-diffusion systems with time-dependent diffusion

Consider the semilinear reaction-diffusion system:

$$\begin{cases}
\partial_t u = \nabla \cdot (D(t,x)\nabla u) + f(t,x,u), & t > 0, \ x \in \Omega, \\
u(t,x) = 0, & x \in \partial\Omega, \\
u(0,x) = u_0(x), & x \in \Omega,
\end{cases}$$

where $D(t,x)$ is a time-dependent diffusion matrix and f is a nonlinear reaction term. We work in the variable space $X(t) = L^2(\Omega, w_t\,dx)$ as before. Assume:

(i) *(Diffusion):* $D(t,x)$ satisfies the same assumptions as $a(t,x)$ in the previous subsection, with ellipticity constant α.

(ii) *(Reaction):* $f(t,x,0) = 0$, and f is globally Lipschitz in u uniformly in t,x:

$$|f(t,x,u) - f(t,x,v)| \le L_f|u-v| \quad \text{for all } t,x,u,v.$$

(iii) *(Linear stability):* The linear part

$$A(t)u = \frac{1}{w_t}\nabla\cdot(D(t,\cdot)\nabla u)$$

generates an exponentially stable evolution family with decay rate $\delta > 0$ and constant $M \ge 1$ (as established in Theorem 6.2 under appropriate conditions).

Write the system as

$$u_t = A(t)u + F(t,u),$$

where $F(t,u)(x) = f(t,x,u(x))$. The nonlinearity satisfies:

$$\|F(t,u) - F(t,v)\|_{X(t)} \le L_f\|u-v\|_{X(t)},$$

using the uniform bounds on the weight.

Theorem 6 (Stability for reaction-diffusion system)**.** *Under assumptions (R1)–(R3), if $L_f < \delta/M$, then the reaction-diffusion system has a unique global solution for each initial data, and the zero solution is exponentially stable in the sense that there exist constants $C,\gamma > 0$ such that for any two solutions u,v with initial data u_0, v_0:*

$$\|u(t) - v(t)\|_{X(t)} \le Ce^{-\gamma t}\|u_0 - v_0\|_{X(0)}.$$

Proof. We use the variation of constants formula. Let $U(t,s)$ be the evolution family generated by $\{A(t)\}$. For any two solutions u,v, define $w = u - v$. Then w satisfies:

$$w(t) = U(t,0)w(0) + \int_0^t U(t,s)\left[F(s,u(s)) - F(s,v(s))\right] ds.$$

Taking norms and using the Lipschitz estimate:

$$\|w(t)\|_{X(t)} \le Me^{-\delta t}\|w(0)\|_{X(0)} + \int_0^t Me^{-\delta(t-s)}L_f\|w(s)\|_{X(s)}\,ds.$$

Multiply by $e^{\delta t}$:

$$e^{\delta t}\|w(t)\|_{X(t)} \le M\|w(0)\|_{X(0)} + ML_f\int_0^t e^{\delta s}\|w(s)\|_{X(s)}\,ds.$$

Define

$$\phi(t) = e^{\delta t}\|w(t)\|_{X(t)}.$$

Then:

$$\phi(t) \le M\|w(0)\|_{X(0)} + ML_f\int_0^t \phi(s)\,ds.$$

By Gronwall's inequality:

$$\phi(t) \le M\|w(0)\|_{X(0)}e^{ML_f t}.$$

Thus:

$$\|w(t)\|_{X(t)} \le Me^{-(\delta - ML_f)t}\|w(0)\|_{X(0)}.$$

If $L_f < \delta/M$, then $\gamma := \delta - ML_f > 0$, and we obtain exponential stability. Existence and uniqueness follow from a standard contraction mapping argument in the space of continuous functions $C([0,T];X(t))$ with the weighted norm $\sup_{t\in[0,T]} e^{\delta t}\|u(t)\|_{X(t)}$, using the same estimate. □

Remark 6. The stability threshold $L_f < \delta/M$ is sharp in the sense that if L_f exceeds this value, the nonlinearity could destabilize the system. The constants M and δ come from the linear evolution family and can be estimated explicitly in terms of the physical parameters (ellipticity constant, Poincaré constant, weight bounds, etc.).

7. Conclusion and future directions

This paper has developed a theory for stability analysis of semigroups in variable Banach spaces. The main contributions include:

(i) A precise framework for variable Banach spaces with connecting isomorphisms, allowing systematic reduction to a fixed reference space.
(ii) A generation theorem (Theorem 1) characterizing generators of evolution families under strong continuity assumptions on the resolvent.
(iii) A Lyapunov-based characterization of exponential stability in variable Hilbert spaces (Theorem 2), with constructive proofs.
(iv) Spectral mapping results (Theorems 3 and 4) under additional assumptions, relating stability to spectral properties of the generators.

- Applications to non-autonomous heat equations and reaction-diffusion systems, with explicit verification of abstract hypotheses.

7.1. Limitations and open problems

Several limitations of the current work point to directions for future research:

(i) *Hilbert space restriction:* The Lyapunov and Gearhart-Pruss theorems require Hilbert space structures. Extending these results to general Banach spaces with time-dependent norms remains an open problem. This would require developing a theory of quadratic forms or duality pairings in variable Banach spaces.
(ii) *Irregular time dependence:* Our generation theorem assumes strong continuity of resolvents, which is appropriate for problems with continuous time-dependence. For problems with measurable or distributional time-dependence, more sophisticated tools (e.g., evolution semigroups, extrapolation spaces) are needed.
(iii) *Strong nonlinearities:* The reaction-diffusion application assumes globally Lipschitz nonlinearities. Extending to superlinear growth or critical nonlinearities would require more delicate fixed point arguments and possibly different function spaces (e.g., L^p spaces with time-dependent weights).
(iv) *Numerical implementation:* The theory presented here is purely analytical. Developing numerical methods that respect the variable space structure and preserve stability properties is an important direction for practical applications.
(v) *Complete spectral mapping:* The spectral mapping theorem proved here requires uniform boundedness of generators, which is restrictive. A complete spectral theory for evolution families in variable spaces, analogous to the Sacker-Sell spectrum for non-autonomous differential equations on fixed spaces, remains to be developed.

7.2. Future work

Building on the theoretical framework developed in this paper, several promising avenues for further investigation emerge:

(i) Extensions to L^p spaces with time-dependent weights, using duality and interpolation techniques.
(ii) Stochastic perturbations of evolution families in variable spaces, combining our framework with stochastic calculus.
(iii) Control-theoretic applications, such as stabilization of PDEs with time-dependent coefficients using feedback controls.
(iv) Long-time behavior of nonlinear evolution equations with variable norms, including attractor theory.

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