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Mathematical review of second-order macroscopic traffic flow models

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Abstract: As urbanization intensifies and traffic demand continues to grow, understanding and modelling vehicular dynamics in complex transportation systems has become increasingly important. Second-order macroscopic traffic flow models provide a powerful framework for capturing the evolution of both traffic density and velocity, offering significant advantages over first-order formulations. Despite extensive developments in this field, the literature lacks a unified mathematical synthesis that systematically organizes and compares the wide range of second-order models. This paper addresses this gap by presenting a comprehensive review of second-order macroscopic traffic flow models, tracing their evolution from foundational formulations in the 1970s to recent advancements up to 2024. The review adopts a structured methodology, drawing on major academic databases to identify models based on their mathematical formulation and practical relevance. The models are classified into key families, including relaxation-based, kinetic, viscous, anisotropic, nonlocal, and multi-class formulations, providing a coherent taxonomy of the field. In addition to cataloguing model equations, this study synthesizes their fundamental mathematical properties, including hyperbolicity, stability, well-posedness, and parameter identifiability. The review further examines how successive models address limitations of earlier approaches, such as non-physical wave propagation and insufficient representation of driver behaviour. Finally, emerging trends are discussed, including the integration of connected and autonomous vehicle technologies, nonlocal interactions, and data-driven modelling approaches.

Keywords: traffic flow modelling, second-order models, macroscopic models, dynamic velocity equation, vehicular traffic flow

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1. Introduction

Traffic flow modelling plays a fundamental role in understanding and managing the complex dynamics of vehicular movement in modern transportation systems [1–7]. With rapid urbanization and increasing vehicle ownership, the demand for accurate and scalable traffic models has become increasingly critical for infrastructure planning, congestion mitigation, and intelligent transportation system design.

Traffic flow models are traditionally classified into three main categories: microscopic, mesoscopic, and macroscopic models. Each class offers a distinct level of abstraction and analytical capability.

Microscopic models describe the dynamics of individual vehicles by explicitly accounting for driver behaviour, acceleration, and lane-changing interactions [8–17]. While these models provide high-fidelity representations of traffic behaviour, their computational complexity limits their applicability in large-scale systems.

Mesoscopic models provide an intermediate framework by representing groups of vehicles while retaining certain individual characteristics [18–28]. These models strike a balance between computational efficiency and behavioural realism, making them suitable for network-level simulations.

Macroscopic models describe traffic flow using aggregate variables such as density, velocity, and flow [29–40]. By treating traffic as a continuum, these models enable large-scale analysis using systems of partial differential equations. First-order models, such as the classical Lighthill-Whitham-Richards (LWR) formulation, establish relationships between flow and density. In contrast, second-order models incorporate additional dynamics such as velocity evolution, relaxation effects, and driver anticipation, allowing for a richer and more realistic description of traffic phenomena.

Second-order macroscopic traffic flow models can be broadly classified into several categories based on their underlying modelling assumptions and mathematical structure:

- Relaxation-anticipation models (classical Payne-Whitham-type formulations) that incorporate relaxation toward equilibrium speed and anticipation effects through density gradients;
- Kinetic and pressure-based models derived from kinetic theory that introduce traffic pressure terms to account for speed variance and interactions among vehicles;
- Viscous and diffusive models that incorporate higher-order spatial derivatives to model smoothing effects and dissipation of traffic waves;
- Anisotropic and invariant models of the Aw-Rascle-Zhang class that ensure physically realistic wave propagation and resolve non-physical behaviour in earlier models;
- Nonlocal and delay models that incorporate driver reaction delays, nonlocal interactions, and anticipation over finite distances;
- Multi-class and multi-lane models that extend classical formulations to heterogeneous traffic and lane-changing dynamics; and
- Connected and intelligent transportation models that account for vehicle-to-vehicle communication, adaptive cruise control, and real-time traffic information.

This classification provides a structured taxonomy that facilitates comparison across model families and highlights the progressive incorporation of physical realism, from early relaxation-based formulations to modern data-driven and connected-vehicle frameworks. Table 1 summarizes these model classes, including their defining features, advantages, and limitations.

Table 1. Comparison of second-order macroscopic traffic flow model classes

Model Class	PDE Type	Key Features	Advantages	Limitations
Payne- Whitham	Hyperbolic	Relaxation, anticipation	Simple, intuitive	Non-physical wave speeds
Kinetic models	Hyperbolic	Pressure terms	Captures variance	Complex calibration
Viscous models	Parabolic/Hyperbolic	Diffusion effects	Smooth solutions	May over-diffuse
Anisotropic models	Hyperbolic	Anisotropy	Physically consistent	More complex structure
Nonlocal models	Integro-PDE	Anticipation	Realistic behaviour	Computational cost
Multi-class models	Coupled PDEs	Heterogeneity	Real-world relevance	High complexity
Connected/ Intelligent extensions	Coupled/Nonlocal PDEs	V2V/V2I communication, adaptive control	Improved efficiency	Data dependency, implementation complexity

These model classes reflect a progressive refinement of macroscopic traffic flow theory. Early models focused on relaxation toward equilibrium, while later developments introduced pressure, viscosity, and anisotropy to address observed deficiencies such as non-physical wave propagation. More recent models incorporate heterogeneity, nonlocal interactions, and intelligent transportation systems, reflecting the increasing complexity of real-world traffic environments.

Despite the extensive development of second-order macroscopic models over the past decades, the literature lacks a unified mathematical synthesis that systematically compares model structures, underlying assumptions, and analytical properties. While prior reviews have addressed specific subclasses such as anisotropic formulations [41,42], kinetic approaches [43,44], or multi-class extensions [45,46], no existing work provides a comprehensive framework that traces the mathematical evolution across all model families and explicitly analyses their interconnections. This fragmentation hinders researchers from understanding the theoretical relationships between models, impedes the transfer of insights between subfields, and obscures which mathematical features are essential for capturing specific traffic phenomena.

This paper addresses this gap by presenting a comprehensive mathematical synthesis of second-order macroscopic traffic flow models, organized within a coherent taxonomic framework that enables systematic comparison across model families.

This paper makes three primary contributions to the traffic flow modelling literature:

- It provides a systematic review of second-order macroscopic traffic flow models, tracing their evolution from foundational formulations in the 1970s to recent advancements in 2024. Models are organized into coherent families and presented in unified notation to facilitate direct comparison.

- It develops a structured taxonomy that classifies models by their governing mathematical structures and synthesizes their analytical properties, including hyperbolicity, stability, well-posedness, and parameter identifiability.
- It identifies and articulates the key trends driving model evolution, from driver psychology and road geometry to connected vehicle technologies and sustainability considerations, providing a roadmap for future research.

1.1. Review methodology and scope

This review adopts a structured and systematic approach to identify, select, and synthesize second-order macroscopic traffic flow models. Relevant literature was collected from major academic databases, including Web of Science, Scopus, and Google Scholar. The search strategy employed combinations of keywords such as "second-order macroscopic traffic models," "traffic flow PDE," "dynamic velocity equation," "Aw-Rascle model," and "continuum traffic flow."

The temporal scope of the review spans from 1970 to 2024, ensuring coverage of both foundational developments and recent advancements. Only models formulated within the framework of second-order macroscopic continuum theory were included. Microscopic, agent-based, and purely discrete models were excluded unless they directly informed the derivation or interpretation of macroscopic formulations.

1.2. Notation and definitions

We introduce the standard variables and notation used throughout this review. Traffic flow is described using the spatial coordinate $x \in \mathbb{R}$ and time $t \geq 0$. The primary macroscopic variables are density $k(x,t)$, velocity $v(x,t)$, and flow $q(x,t)$, related by $q = kv$. Partial derivatives are denoted using subscript notation: $v_t = \partial v/\partial t$, $v_x = \partial v/\partial x$, and $v_{xx} = \partial^2 v/\partial x^2$. This notation is used consistently throughout the manuscript. This unified notation is adopted throughout the manuscript to ensure mathematical consistency and to facilitate comparison across different model formulations.

Table 2. Summary of key variables and parameters

Symbol	Description
x	Spatial coordinate
t	Time
$k(x,t)$	Traffic density
$v(x,t)$	Traffic velocity
$q(x,t) = kv$	Traffic flow
$v_e(k)$	Equilibrium speed-density function
$T,\ \tau$	Relaxation time
c_o	Propagation speed of disturbances
$\varphi,\ \mu$	Viscosity/diffusion coefficients
$p(k), P(k,v)$	Traffic pressure

Several recurring parameters appear in second-order models. The relaxation time is denoted by T (or τ), the equilibrium speed by $v_e(k)$, and the propagation speed of disturbances by c_o. Viscosity and diffusion effects are represented by coefficients such as φ or μ, while additional terms may account for driver anticipation, reaction delays, and multi-class interactions. These parameters are summarized in Table 2. These notations are treated equivalently when they represent the same physical quantity, and are retained to remain consistent with the original formulations of the cited models. Where necessary, model-specific parameters are introduced and defined locally to preserve clarity without overloading the global notation.

2. Model evolution and mathematical formulations

Second-order macroscopic traffic flow models are formulated as systems of partial differential equations consisting of a continuity equation and a dynamic velocity equation. The continuity equation expresses conservation of vehicles and is given by

$$k_t + (kv)_x = 0.$$

This conservation law corresponds to the classical LWR [47,48] formulation and is common to most second-order macroscopic models. For brevity, subsequent models are presented primarily through their dynamic velocity equations, with the understanding that they are coupled with the above continuity equation unless explicitly stated otherwise. In cases where the density equation is modified (e.g., through source terms, multi-lane interactions, or network effects), the corresponding continuity equation is restated explicitly. This convention allows for a concise presentation of a large number of models while preserving the underlying system structure essential for mathematical interpretation.

2.1. Foundational models: 1970s

The 1970s marked the foundational period for second-order macroscopic traffic flow models, primarily characterized by systems of partial differential equations that provided a framework for understanding the relationship between vehicle density and speed. The papers presented here highlight the evolution of dynamic velocity equations, beginning with the model by [49]:

$$v_t + vv_x = a[v_e - v] - \frac{\alpha}{k}k_x,$$

where $\alpha = -v_e'(k)/2T$, $a = 1/T$ represent the inverse of relaxation time, while v_e signifies the equilibrium speed. The model is pivotal because it integrates anticipation and relaxation terms to illustrate how drivers adjust their speeds based on downstream traffic conditions. Following this, [50] presented the equation:

$$v_t + vv_x = a[v_e - v] - \frac{c_o^2}{k}k_x,$$

where $c_o = \Delta/T$ denotes the sound speed or propagation speed of perturbations. This adaptation enhances the model's capacity to account for dynamic changes in traffic flow.

2.2. Kinetic and viscous extensions: 1980s-1990s

The late 1970s and 1980s witnessed further advancements in traffic flow models, integrating more realistic representations of driver behaviour and introducing kinetic-theory-based and viscous formulations.

A model derived from kinetic theory by [43,44] introduced a novel macroscopic variable, traffic pressure, described by:

$$v_t + vv_x = \lambda(k)[v_e - v] - \frac{1}{k}P_k k_x,$$

where $P = P(k,v)$ captures the variance in traffic speed distribution akin to gas pressure. Next, the freeway traffic flow model developed by [51,52] incorporates the effects of relaxation and anticipatory behaviour with the equation:

$$v_t + vv_x = a[v_e - v] - \frac{c_o^2}{k}k_x + \varphi v_{xx},$$

where φ represents the viscosity constant. Afterward, [53] formulated a high-order model which does not rely on equilibrium speed-density relationships:

$$v_t + vv_x = a[v_f - v] - G - \alpha k^\beta k_x,$$

where $G = \mu_1 k^\epsilon g$ accounts for traffic friction, illustrating the model's ability to address complex traffic dynamics and changing geometries. The evolution of second-order macroscopic traffic flow models continues with the kinetic model of traffic flow developed by [54], which is expressed as:

$$v_t + vv_x = a[v_e - v] - \frac{c_o^2}{k}k_x + \frac{\varphi}{k}v_{xx},$$

This model builds on previous frameworks by incorporating both relaxation dynamics and viscosity effects, which are essential for understanding the stability and behaviour of traffic under various conditions. The integration of the viscosity term v_{xx} highlights how fluctuations in vehicle speeds can propagate through traffic, thus enabling a more accurate depiction of traffic dynamics.

2.3. Nonequilibrium and anisotropic models: Late 1990s-2000s

The decade from 1998 to 2010 saw a significant evolution in second-order macroscopic traffic flow models, with increased emphasis on anisotropy, nonequilibrium behaviour, and empirical grounding.

A traffic flow model for both interrupted and uninterrupted traffic scenarios was presented by [55]:

$$v_t + vv_x = a[v_f - v].$$

This model simplifies the representation of traffic flow by focusing on the free-flow speed v_f, capturing essential driver behaviour without delving into complex anticipatory mechanisms. This approach is particularly beneficial in understanding the transition between different traffic states, especially during varying road conditions. Then, based on hyperbolic conservation laws, [56] provided another perspective on macroscopic traffic flow dynamics. This is articulated as:

$$v_t + p_x = a[v_e - v],$$

where $p = 1/2v^2 + \varphi/T \ln k$ introduces a new dimension by accounting for the effects of both convection and anticipation in the flow of traffic. The model's capacity to reflect the interaction between speed and pressure dynamics adds significant depth to the understanding of macroscopic behaviour. In contrast, Zhang [57] is noteworthy for deriving its formulation from empirical evidence regarding driver reactions to stimuli. The equation:

$$v_t + vv_x = a[v_e - v] - k(v_e')^2 k_x$$

illustrates a critical distinction; the major difference between this model and previous formulations lies in its proportionality to density, thereby providing a more nuanced representation of driver behaviour in congested conditions. This adjustment ensures that the model reflects realistic traffic interactions, moving beyond simplified assumptions.

In an innovative approach, a model proposed by [41] replaced the traditional space derivative with a convective derivative, represented as:

$$(v + p(k))_t + v(v + p(k))_x = 0,$$

where $p(k)$ is a traffic pressure function. This approach offers a fresh perspective on how traffic flows can be influenced by immediate vehicle interactions, highlighting the role of driver responsiveness in dynamic traffic environments. This model stands out by eliminating relaxation terms, thereby presenting a straightforward yet effective framework for understanding traffic dynamics in various settings.

The exploration of second-order macroscopic traffic flow models advanced further with the introduction of Jiang's model [58], expressed as:

$$v_t + vv_x = a[v_e - v] + c_o v_x.$$

This model uniquely incorporates the propagation speed of disturbances c_o as an additional factor influencing vehicle dynamics, emphasizing the impact of speed differences on vehicle interactions. By focusing on how individual vehicles respond to the behaviour of their immediate predecessors, this model provides valuable insights into the intricacies of traffic flow, particularly in scenarios involving rapid speed adjustments.

Zhang's nonequilibrium model [42] further elaborates on the dynamics of macroscopic traffic flow through the equation:

$$v_t + vv_x = -kv_e'(k)v_x.$$

This formulation effectively captures the non-physical effects commonly associated with traditional models by demonstrating how vehicle interactions can diverge from typical gas-like behaviour. By utilizing a derivative approach that directly relates to changes in speed and density, the model provides a more accurate representation of driver reactions in congested traffic situations.

Additionally, a hyperbolic traffic flow model by [59] introduced the equation:

$$q_t + ((q - q_*)v)_x = 0.$$

This formulation focuses on the flow rate q and the desired flow rate q_*, offering a framework that accounts for variations in traffic density and its effect on flow dynamics. The incorporation of this desired flow rate establishes a basis for understanding how deviations from optimal conditions can lead to traffic congestion, thus providing critical insights into traffic management strategies.

In a similar vein, a nonequilibrium continuum traffic flow model was presented by [60] as:

$$v_t + \left(\frac{1}{2}v^2 - c_o v\right)_x = a(v_e - v).$$

This model highlights the importance of both speed and density in determining traffic behaviour, incorporating a term that captures the effects of vehicle acceleration and deceleration in response to changes in traffic conditions. This comprehensive approach underscores the necessity of considering multiple factors when evaluating traffic dynamics.

Furthermore, the model considering two delay time scales proposed by [61] is expressed as:

$$v_t + vv_x = \frac{v_e(k) - v}{T(k)} - k\frac{t_r}{T(k)}v_e'(k)v_x c.$$

The authors introduced a delay in driver reactions, presenting a more realistic depiction of how human factors influence traffic flow, particularly under varying conditions. The model effectively illustrates how the relaxation time $T(k)$ can vary based on density, further refining the understanding of driver behaviour during congested situations.

The speed gradient model considering a mixture of different types of vehicles was presented as a system of equations that distinguishes between slow and fast vehicles [62]. The equations are as follows:

$$\text{Slow Vehicles:} \quad v_{2,t} + v_2 v_{2,x} = a_2[v_{2,e}(k_1, k_2) - v_2] + c_{2,0}v_{2,x},$$

$$\text{Fast Vehicles:} \quad v_{1,t} + v_1 v_{1,x} = a_1[v_{1,e}(k_1, k_2) - v_1] + c_{1,0}v_{1,x} - \frac{v_1 - v_2}{\tau}\left(\frac{k_2}{k_0}\right)^2,$$

where τ and k_0 are proportional coefficients. This dual-vehicle perspective provides a comprehensive understanding of how varying vehicle types interact within traffic flow, further contributing to the development of more refined traffic models that reflect real-world conditions.

2.4. Multi-Lane, anisotropic, and heterogeneous models: 2006-2010

This period saw a proliferation of models addressing multi-lane dynamics, anisotropic properties, viscosity, and heterogeneous traffic compositions.

An anisotropic model by [63] was defined as:

$$v_t + vv_x = a[v_e - v] + av_e'\left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2} - \frac{k_x^2}{2k^3}\right) - 2\beta c_o v_x,$$

where β is a non-negative dimensionless parameter that captures the anisotropic properties of traffic flow, allowing for a more nuanced understanding of how vehicles interact in various traffic conditions. The model accounts for variations in speed and density, thereby enhancing the predictive accuracy of traffic dynamics.

Another formulation in 2006 includes a density viscous model, which incorporates two delay time scales [64]:

$$v_t + [v - 2c_o]v_x = a[v_e - v] + a\left(\frac{v_e' + \mu}{k}\right)k_x + \frac{a\mu}{k^2}k_{xx},$$

where $\mu = b_o v_e' < 0$ indicates the negative influence of the equilibrium speed gradient, while b_o is a positive constant. This model effectively captures the effects of both density and speed variations on traffic flow, particularly in scenarios where driver reactions are delayed.

A dynamic model for two-lane traffic presented by [65] captured a more complex interaction between vehicles in different lanes, formulated as:

$$k_{i,t} + (k_i v_i)_x = s_i(x,t),$$
$$v_{i,t} + (v_i - c_{io})v_{i,x} = \frac{v_{ie} - v_i}{\tau_i}.$$

This system of equations for $i = 1, 2$ incorporates the propagation speed c_{io} of disturbances in each lane and reflects the dynamic nature of traffic flow in multi-lane scenarios. The inclusion of $s_1(x,t) + s_2(x,t) = 0$ signifies the absence of on-off ramps while allowing for lane changes, making it a robust framework for understanding two-lane traffic dynamics.

In Gupta's subsequent work [66], the model was enhanced by introducing an additional speed gradient term, expressed as:

$$v_t + vv_x = a[v_e - v] + av_e' \left(\frac{1}{2k} k_x + \frac{1}{6k^2} k_{xx} - \frac{1}{2k^3} (k_x)^2 \right) - 2\beta c_o v_x.$$

This formulation underscores the significance of speed gradients in shaping traffic flow, facilitating a more comprehensive understanding of the factors influencing driver behaviour.

Similarly, [67] described a traffic flow model on two-lane freeways as:

$$k_{m,t} + q_{m,x} = s_{nm} - s_{mn},$$
$$v_{m,t} + [v_m - c_{m,o}]v_{m,x} = a_m[v_{e,m} - v_m] + r_1 s_{mn} - r_2 s_{nm}.$$

In this model, $m = 1, 2$ denotes the lane in consideration, and the lane-changing rates s_{mn} and s_{nm} provide insights into how vehicle interactions across lanes influence traffic dynamics. By examining the coupling effects of adjacent lanes, this model advances the understanding of multi-lane traffic flow.

The exploration of second-order macroscopic traffic flow models extended the work by [67], which presented a speed-gradient model with an interruption probability [68]. This model is defined by the equation:

$$v_t + vv_x = a[v_e - v] - \frac{1}{\tau_1} pv + c_o(1 - p)v_x,$$

where p represents the interruption probability, and τ_1 denotes the reaction time of drivers. This formulation captures the effects of interruptions on traffic flow dynamics, illustrating how variations in driver behaviour can influence speed and density across a continuum.

Further, an anisotropic continuum model presented by [69] builds upon previous frameworks with the equation:

$$v_t + [v - c_o]v_x = a[v_e - v] + av_e' \left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2} \right).$$

This model integrates anisotropic properties, allowing for an enhanced understanding of how traffic conditions can vary across different spatial dimensions, reflecting the complex interactions inherent in real-world traffic systems.

Another model developed by [70] investigates non-equilibrium traffic flow within a network as:

$$\bar{k}_{n,t} + (\bar{k}_n \bar{v}_{ne})_{\bar{x}_n} = \bar{s}_n(\bar{x}_n, t),$$
$$\bar{v}_{n,t} + (\bar{v}_n - \bar{c}_{no})\bar{v}_{n,\bar{x}_n} = \frac{\bar{v}_{ne} - \bar{v}_n}{\bar{\tau}_n},$$

where $n = 1, 2, \ldots, N$ denotes the number of links, with $\bar{k}_n$ and $\bar{v}_n$ representing the density and speed, respectively. This model effectively addresses the complexities of traffic flow in networks, highlighting the interactions among multiple links and their contributions to traffic dynamics. Moreover, [71] heterogeneous traffic model introduced the equation:

$$v_{it} + v_i v_{ix} = \frac{v_{ie}(k_i) - v_i}{T_i} + \sum_{j=1}^{N} \left[c_o \frac{k_j}{k} v_{jx} + \frac{1}{\tau_{ij}} \frac{k_j}{k} (v_j - v_i) \right].$$

In this context, i and j represent class-specific quantities, where $v_{ie}(k_i)$ indicates the equilibrium speed for the i-th class. This model captures the interactions between different vehicle classes, providing a nuanced perspective on how heterogeneous traffic dynamics operate within a given roadway.

In 2009, [45] developed a multi-class traffic flow model expressed as:

$$[k_u v_u]_t + [k_u v_u^2 + p_u]_x = k_u a_u [v_{u,e} - v_u],$$

u represents the vehicle class, and p_u denotes the traffic pressure term. This model highlights the adaptation of desired speeds among different vehicle classes, providing valuable insights into the complexities of multi-class interactions in traffic flow.

A macroscopic model of lane-changing behaviour consistent with car-following dynamics on a two-lane highway was proposed by [72]. This model is articulated as:

$$v_t + [v - (1 + \epsilon(k))c_o]v_x = a[v_e(k + k\epsilon(k)) - v],$$

where $\epsilon(k)$ is a lane-changing coefficient. This formulation captures the interplay between lane-changing behaviour and the dynamics of vehicles in a two-lane scenario, reflecting the adjustments drivers make in response to both traffic conditions and their proximity to other vehicles.

Besides, [73] investigation into how reaction time influences traffic flow stability introduced the equation:

$$v_t + \left(v - \frac{kT}{\bar{\tau}} v_e' \right) v_x = \frac{v_e - v}{\bar{\tau}} + \frac{1}{2k\bar{\tau}} v_e' k_x,$$

where $\bar{\tau} = \tau - T$ describes the adaptation time, where T accounts for the physiological delay that varies among drivers. This model provides insights into how these delays affect traffic dynamics and stability, highlighting the significance of driver reactions to changing traffic conditions.

The two-lane model developed by [74] incorporates two delay time scales, represented by:

$$k_{m,t} + q_{m,x} = s_m(x,t),$$
$$v_{m,t} + [v_m - c_{m,o}]v_{m,x} = a_m[v_{e,m}(k_m, \beta k_n) - v_m] + \gamma_m s_m,$$

where m and n denote different lanes, with the condition $s_1 + s_2 = 0$ reflecting the interactions between lanes. Here, β is the coupling coefficient from lane n to lane m, and $\gamma_m = a_m t_{r,m} [u_{m,e}]_k$. This model captures the complexity of driver behaviour across multiple lanes, particularly in scenarios where reactions to traffic conditions are delayed.

A viscous model for two-lane freeways by [75] considered the effects of vehicle coupling and lane-changing behaviour, expressed as:

$$k_{m,t} + q_{m,x} = s_{mn} - s_{nm},$$
$$v_{m,t} + [v_m - 2c_{m,o}]v_{m,x} = a_m[v_{e,m} - v_m] + \frac{1}{k_m} a_m [v_{e,m}' + \mu_m] k_{m,x} + \frac{1}{k_m^2} a_m \mu_m k_{m,xx} + a_m t_{rm} u_{e,m,k} [s_{mn} - s_{nm}],$$

where $m, n = 1, 2$ and $m \neq n$. In this model, $T_m = 1/a_m$ denotes the relaxation time, and t_{rm} indicates the driver reaction time on lane m. The inclusion of lane-changing rates and the interactions between adjacent lanes allows for a detailed understanding of the dynamics present in multi-lane traffic situations.

Moreover, [76] captured the effect of multi-anticipative driving behaviour on traffic flow characteristics as derived from gas-kinetic theory and expressed as:

$$v_t + vv_x + \frac{1}{k}\Theta_x + \frac{1}{T}\frac{\sum_{i=1}^{n}\beta_i}{2k^3}k_x v'_{op}\left(\frac{1}{k}\sum_{i=1}^{n}\beta_i\right) = \frac{v_{op}\left(\frac{1}{k}\sum_{i=1}^{n}\beta_i\right) - v}{T}.$$

In this equation, v_{op} denotes the optimal speed, β_i indicates the sensitivity of the vehicle to the space headway of the leaders, Θ is the mean speed variance, and T represents the relaxation time. This model provides a comprehensive framework for understanding how anticipative driving behaviours impact traffic flow dynamics.

The investigation into the effects of on-off ramps during peak traffic periods was explored in the study by [72,77]. The model is expressed as:

$$\begin{aligned} k_t + q_x &= s(x,t) \\ v_t + vv_x &= a[v_e - v] + c_o v_x - \mu_2 skv. \end{aligned}$$

The term $s(x,t)$ is defined as $s(x,t) = \begin{cases} \frac{q_{ramp}}{L_{ramp}}; & x \in \Omega_{ramp} \\ 0 & \text{otherwise} \end{cases}$, Ω_{ramp} denotes the region of the ramp, q_{ramp} is the total ramp flow, L_{ramp} is the length of the ramp, and μ_2 is the friction coefficient. This model provides insights into how the dynamics of traffic flow are influenced by the presence of ramps, particularly during rush hours.

Again, [78] introduced a new signal light model that examines the impact of traffic signals on car-following behaviour. This model is articulated as:

$$v_t + vv_x = \phi_e a[v_e - v] + c_o v_x,$$

$$\phi_e = \begin{cases} 0 & \text{if the signal light is red and the } n\text{th vehicle is upstream,} \\ 1 & \text{otherwise.} \end{cases}$$

This formulation highlights the significant role of traffic signals in shaping driver behaviour and traffic dynamics.

Subsequently, a continuum traffic model incorporating a viscous term [79]:

$$v_t + [v - c_o]v_x = a[v_e - v] + \varphi v_{xx}.$$

This model effectively integrates viscosity into the traffic flow equations, allowing for a more comprehensive understanding of how speed variations and density changes influence traffic dynamics.

In an improved model, [80] builds on [54] framework with the given equation:

$$v_t + vv_x + \frac{c_o^2 k_x}{k} = a[v_e - v] + k[\mu v_x]_x,$$

where $c_o = \sqrt{\frac{dP_e}{dk}}$ signifies the traffic sound speed, and μ represents a viscosity coefficient. This model enhances the understanding of traffic pressure dynamics, integrating sound speed and viscosity to reflect the complexities of real-world traffic behaviour.

Meanwhile, a model in which the dimensionless parameter (anisotropic factor) controls the non-isotropic character and diffusive influence was presented by [81] as:

$$v_t + vv_x = a[v_e - v] + \frac{1}{2k}av'_e k_x - 2\beta c_o v_x.$$

Finally, in 2010, [82] considered the effects of interruption probability and friction on two-lane traffic flow, expressed as:

$$k_{m,t} + q_{m,x} = s_{nm} - s_{mn},$$
$$v_{m,t} + [v_m - c_{m,o}(1-p)]v_{m,x} = a_m[v_{m,e} - v_m] + \frac{1}{\tau_{m,1}}p(-v_m) + r_1 s_{mn} - r_2 s_{nm},$$

where $m = 1,2$ indicates the lane under consideration, and $n = 1,2$ (with $m \neq n$) represents the other lane. The variables s_{mn} and s_{nm} denote the lane-changing rates between lanes, while p is the interruption probability. This model integrates lane-changing behaviour into both continuity and dynamic equations, reflecting the impact of driver interactions across adjacent lanes on traffic flow.

2.5. Viscous, heterogeneous, and infrastructure-aware models: 2011-2015

This period extended viscous and multi-lane frameworks to incorporate lane-changing, heterogeneous vehicle classes, and road infrastructure features such as bottlenecks, ramps, road width, and slope.

A viscous model with realistic driver reaction time was presented by [83]. The authors incorporated a viscosity coefficient derived from a more realistic constitutive relationship between the averaged reaction time of drivers and car density into the dynamic velocity equations as:

$$v_t + [v - \bar{c}]v_x = a[v_e - v] - \frac{c_o^2}{k}k_x + u_o u_{xx},$$

where $\bar{c} = (\tau k_{jam})^{-1}$ represents the average propagation speed, τ is the reaction time, and $\mu_o = T\bar{c}c_o$ with T being the relaxation time. This formulation emphasizes the relationship between driver behaviour and traffic density, offering insights into how viscosity impacts traffic dynamics.

The formulation by [84] considered lane-changing effects of vehicles on two adjacent lanes with the model:

$$k_{m,t} + q_{m,x} = s_{nm}(x,t) - s_{mn}(x,t),$$
$$v_{m,t} + [v_m - c_{m,o}]v_{m,x} = a_m[v_{e,m}(k_m, \beta k_n) - v_m] + \gamma_m s_{nm} - \gamma_m s_{mn},$$

where $m, n = 1,2$ with $m \neq n$, and s_{mn} represents the lane-changing rate from lane m to lane n. The propagation speed of the disturbance on lane m is given by $c_{m,o} = -k_m a_m t_{r,m}[u_{e,m}]_k$, where β is the coupling coefficient from lane n to lane m, and $\gamma_m = a_m t_{r,m}[u_{e,m}]_k$ represents the effective lane-changing influence.

The model by [85] characterizing both on-off-line bus stops included different expressions for online and offline scenarios. For online bus stops, the model is given by:

$$v_t + vv_x = \frac{v_e(k) - v}{T} + \frac{1}{T_1}p(x,t)(-v) + c_o(1 - p(x,t))v_x,$$

where T and T_1 are both reaction times, and $p(x,t)$ reflects the characteristics of the online bus stop. For offline bus stops, the model is expressed as:

$$k_t + q_x = s(x,t); \quad v_t + vv_x = a[v_e - v] + c_o v_x - \mu_2 skv,$$

where $s(x,t)$ represents the flow at the offline bus stop, while $\mu_2 s(x,t)kv$ accounts for the friction effects, where μ_2 is the friction coefficient. Another model by [86] considered road width with the equation:

$$v_t + vv_x = a[v_e - v] + c_o v_x + \frac{1}{\sigma}\frac{dA}{dx}v.$$

The last term is the adjustment factor for road width. In this equation, A denotes the road width, and σ represents the reaction time required for drivers to adjust their acceleration based on changes in road width. It is important to note that σ is less than τ, with the condition that $|dA/d\sigma|$ should be less than σ/τ.

In 2012, the concept of traffic viscosity was explored more deeply through the model [87]:

$$v_t + vv_x = a[v_e - v] + c(k)v_x + (1-p)\tau c^2(k)v_{xx},$$

where τ denotes the time required for the backward-propagated disturbance to travel a distance Δ, while p is a weighting value. The function $c(k)$ represents the traffic sound speed, which is assumed to vary as a function of the local density. This model effectively captures the complexities of traffic flow dynamics while accounting for the effects of viscosity and other factors.

The multilane extension of the single-lane anisotropic continuum model, considering coupling effects between vehicles in different lanes and the lane-changing dynamics, was presented by [88]. This model is expressed as:

$$\begin{aligned}
k_{m,t} + q_{m,x} &= s_{nm} - s_{mn}, \\
v_{m,t} + v_m v_{m,x} &= a_m\left[v_{m,e}(k_m, \gamma k_n) - v_m\right] + a_m\left[v'_{m,e}(k_m, \gamma k_n)\right]\left(\frac{1}{2k_m}k_{m,x} + \frac{1}{6k_m^2}k_{m,xx} - \frac{1}{2k_m^3}(k_{m,x})^2\right) \\
&\quad - 2\beta c_{m,o} v_{m,x} + r_1 s_{mn} - r_2 s_{nm},
\end{aligned}$$

where γ describes the coupling effect, $m = 1,2$ denotes the m-th lane, and $n = 1,2$ (where $m \neq n$) denotes the other lane. The parameters r_1 and r_2 are constants, while s_{mn} represents the lane-changing rate from lane m to lane n, and β is a non-negative dimensionless parameter.

Tang and his allies again presented a macro model for traffic flow that considers static bottlenecks [89]:

$$v_t + v v_x = a[v_e - v] + c_o v_x - \mu_2 s(x,t) k v,$$

where $s(x,t) = \frac{q}{L_2}$, where L_2 is the length of the bottleneck, and μ_2 is the friction coefficient. This formulation provides insights into how static bottlenecks influence traffic flow dynamics. Another model proposed by [90] characterized the impacts of lane width and the number of lanes on multilane traffic flow as:

$$v_{it} + v_i v_{ix} = \frac{v_{ie}(k_i) - v_i}{T_i} + c_{io} v_{ix} H_i + K_i,$$

where $1 \leq i \leq N$ indicates the lane index, where N is the total number of lanes. The variables k_i and v_i correspond to the density and speed on lane i, respectively, while H and K represent the friction effects resulting from lane width and the number of lanes.

Adding to that, [91] addressed the impacts of road capacity on traffic flow with the model equation:

$$v_t + v v_x = a[v_e - v] + c_o v_x - \mu_2 \frac{\max\{0, kv - C_3\}}{L_3} kv,$$

where $\mu_2 \frac{\max\{0,kv-C_3\}}{L_3} kv$ accounts for the friction effect caused by queues, where L_3 is the length of the range that the queue affects traffic flow and C_3 is the road capacity. Another macro model considering multiple static bottlenecks was presented by [92] as:

$$v_t + v v_x = a[v_e - v] + c_o v_x - \mu s_i(x,t) k v,$$

where $i = 1, 2, \ldots, N$ represents the number of static bottlenecks. The term $s_i(x,t) = \frac{\bar{q}_i}{L_{i0}}$ captures the flow at the i-th static bottleneck, with L_{i0} denoting its length. The variable $\bar{q}_i = kv(1 - \phi_i(x_i - y_i(t)))$ represents the flux resulting from the i-th bottleneck, where x_i is the position of the bottleneck and y_i is the position that the bottleneck influences.

A heterogeneous model capturing capacity filling was presented by [93]. This model is defined by the system:

$$\begin{gathered}
k_{i,t} + q_{i,x} = 0 \\
[v_i + p_i(AO)]_t + v_i[v_i + p_i(AO)]_x = a_i[v_{i,e}(AO) - v_i],
\end{gathered}$$

where AO represents area occupancy, with $p_i(AO) = C^2 AO^{\lambda_i}$, where λ_i is a dimensionless parameter and C is a constant. This formulation provides a comprehensive approach to understanding how traffic flow dynamics change as capacity is filled, particularly in heterogeneous traffic environments.

A significant contribution by [94] addressed traffic flow considering drivers' reaction-time delay effects as:

$$v_t + vv_x = a[v_e - v] - a\epsilon t_d v_e' v_x,$$

where ϵ is the distance between following and leading vehicles, while t_d denotes the reaction-time delay of drivers. This model effectively captures the impact of driver response times on traffic flow dynamics.

Other innovations in traffic modelling during this period include the extension of Adaptive Cruise Control (ACC) traffic flow models to describe the operation of Cooperative Adaptive Cruise Control (CACC) [95]. This is presented as

$$v_t + vv_x + \frac{1}{k}\left(\sum_{n=1}^{N} \frac{n}{\tau_n^*} v_x - c_o^2 k_x\right) + a\left(v_e - v\right),$$

where τ_n^* is the CACC sensitivity coefficient, and N denotes the number of preceding vehicles with which the follower can exchange information. This model enhances the understanding of how vehicle-to-vehicle communication influences traffic flow.

Additionally, [96] introduced a model that considers driver's forecast effect by the equation:

$$v_t + [v - c_o + ak^2 \tau \Delta v_e'] v_x = a[v_e - v] + (1-p)\eta c_o^2 v_{xx},$$

where τ represents the forecast time, η is the propagation time over a distance Δ, and p is the weighting value. This model captures how drivers' anticipatory behaviours affect traffic dynamics. Likewise, [97] anisotropic speed gradient model is articulated as:

$$v_t + vv_x + a[v_e - v] + c_o v_x + \frac{1}{2}\tau c_o^2 v_{xx},$$

where $c_o = \Delta/\tau \geq 0$, with τ being the time needed for the backward-propagated disturbance to travel a distance Δ. This model provides insights into how speed gradients influence traffic flow, particularly under conditions of anisotropy.

In the same year, the macroscopic model on a highway with slopes was presented by [98]:

$$v_t + vv_x = a[\beta v_e - v] + \sigma c_o v_x,$$

where σ and β capture the effects of slope, allowing for a more detailed understanding of how gradients influence traffic dynamics on inclined roadways. Again, [99] incorporated varying road conditions into the macroscopic framework as:

$$v_t + vv_x = a[v_{e,r}(k) - v] + c_{o,r} v_x + \mu_r \alpha_r,$$

where $\mu_r \alpha_r$ represents the effect of friction, where μ_r is an adjustment coefficient and α_r is an adjustable term that drivers can use to modify their acceleration under different road conditions. The variables $c_{o,r}$ and $v_{e,r}$ denote the propagation speed of small perturbations and the equilibrium speed, respectively, both considering the current road conditions. Tang added to his numbers by characterizing the impacts of lane width, lane changing, and the number of lanes on multilane traffic flow [100]. This model is expressed as:

$$v_{it} + v_i v_{ix} = \frac{v_{ie}(k_i) - v_i}{T_i} + c_{io} v_{ix} + G_i + H_i + K_i,$$

where G_i accounts for the friction effects resulting from lane changing, while all other variables are defined as per [90]. This formulation provides a comprehensive framework for analysing how lane characteristics affect traffic flow. Despite this, [101] examined the effect of traffic anticipation on flow dynamics by the model:

$$v_t + [v - c_o - \mu] v_x = a[v_e - v] + av_e' \left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2}\right),$$

where $\mu = -a\kappa k^2 v_e' \Delta$, where κ is the forecast time. This model highlights how drivers' anticipation of traffic conditions influences their speed and behaviour. With different shades of look, [102] described the effects of delay on traffic flow using the following model:

$$v_t + vv_x = \frac{A}{1 + T_v\left(A_v + A_{v_a}\right)} + \frac{T_k k\left(A_k + A_{k_a}\right)}{1 + T_v\left(A_v + A_{v_a}\right)} v_x,$$

where T_k and T_v are delays associated with density and speed, respectively. The variables A_k and A_v represent non-local density and speed, while A is a given function. This model provides valuable insights into how delays affect traffic flow dynamics.

The heterogeneous model for a non-lane based system, which takes lateral separation into account was introduced by [103]. This model is defined as:

$$v_{i,t} + v_i v_{i,x} = a_i \left[v_{i,e}\left(\frac{k}{1+\delta}\right) - v_i \right] + (1+\delta) \sum_{j=1}^{N} c_j p_j(x,t) v_{j,x}, + a_i \sum_{j=1}^{N} p_j(x,t)[v_j(x,t) - v_i(x,t)],$$

where i and j represent different vehicle classes. The term $c_j = \Delta_j / \tau_i$ measures the propagation speed of j-class vehicles, indicating how changes in traffic density propagate due to i-class vehicles. The parameter δ enhances the total flow for any vehicle proportions, and p_j denotes the proportion of the j-class at a given point.

Another model considering driver's forecast effect was introduced by [104]:

$$v_t + vv_x = \frac{(1+\beta)(v_e - v)}{T + \beta\tau} - \beta\tau c_o k^2 v_e' v_x,$$

where $c_o = \frac{\epsilon}{\beta\tau + T} > 0$ indicates the speed at which perturbations propagate, where ϵ is the distance between following and leading vehicles in a micro-model, β is the coefficient of the driver's forecast effect, and τ is the time step for driver forecasting.

The model by [105] explored multi-anticipative driving behaviour using a gas-kinetic approach as:

$$v_t + vv_x = a[v_e - v] - a\left[p_r - a\sum_m \beta_m (m-1) v_e'\right] k_x,$$

where p_r is the pressure term, β_m serves as the weight factor, and m relates to traffic conditions and the road layout. This model provides valuable insights into how anticipatory driving influences traffic flow dynamics.

Interestingly, a macro model by [106] incorporated anticipation optimal velocity into traffic flow dynamics:

$$v_t + [v - c_o] v_x = a[v_e - v] + \alpha v_e' k_t + \frac{1}{2} a\tau_1 c_o V'(k) k_x,$$

where τ_1 is the delay time, a represent driver sensitivity, $\lambda = 1/\tau_1$ is the response coefficient to the velocity difference, and α is the anticipation coefficient.

Another model presented by [107] studied the effects of the real-time traffic state on traffic flow:

$$v_t + vv_x = a[v_{e,r}(k) - v] + c_{o,r} v_x + \eta_r (R(x+\Delta, t) - R(x,t)) \alpha_r,$$

where η_r is a dimensionless parameter reflecting the effects of real-time traffic states on flow dynamics, $R(x,t)$ denotes the traffic state, and Δ is the distance that the driver can anticipate at (x,t). This model provides insights into how varying traffic conditions impact flow and driver behaviour.

The continuum model based on the optimal velocity car-following model, which incorporates drivers' anticipation effects, was presented by [108]. The model is defined as:

$$v_t + [v - a\tau_1 k^2 v_e' \Delta] v_x = a[v_e - v] - \frac{1}{2} a\tau_1 k v_e' \Delta^2 v_{xx},$$

where τ_1 represents the anticipation time. This model provides insights into how anticipation impacts traffic flow, especially in scenarios involving car-following dynamics.

Similarly, [109] considered the impact of bi-directional information with the dynamic equation:

$$v_t + [v - c_o]v_x = r_f \alpha_f \left[\frac{1}{k} - \frac{1}{k_e^f}\right] - r_b \alpha_b \left[\frac{1}{k} - \frac{1}{k_e^b}\right],$$

where r_f and r_b are weight coefficients, while α_f and α_b represent the driver's sensitivity coefficients for forward and backward vehicle information, respectively. The equilibrium relationships k_e^f and k_e^b denote the relationships between traffic density and speed for forward and backward looking.

The issue of viscosity was further explored to consider the lane-changing effects of vehicles, highlighting the complexities of traffic dynamics in multi-lane environments [110]:

$$k_{m,t} + q_{m,x} = s_{nm} - s_{mn},$$
$$v_{m,t} + [v_m - c_{m,o}]v_{m,x} = a_m[v_{e,m} - v_m] + \mu_m v_{m,xx} + r_1 s_{mn} - r_2 s_{nm},$$

where $\mu_m = (1-p)\tau_m c_{o,m}^2$ represents the viscosity coefficient, where s_{mn} is the lane-changing rate from lane m to lane n. The parameters r_1 and r_2 are constants, while τ_m denotes the time needed for backward propagated disturbances to travel distances Δ_m on lane m.

Additional advancements in traffic modelling during this period encompass a model that takes ramps into account [111] as:

$$k_t + q_x = S(x,t),$$
$$v_t + v v_x = a[v_e - v] + a v_e' \left[\frac{1}{2k}k_x + \frac{1}{6k^2}k_{xx} - \frac{1}{2k^3}(k_x)^2\right] - 2\beta c_o v_x,$$

where β is a non-negative dimensionless parameter that affects the model's response to ramp-related traffic dynamics. While, [112] model examined the impacts of region-representative safe driving awareness heterogeneity:

$$k_t + q_x = s(x,t),$$
$$v_t + v v_x = c_o v_x + a[v_e - v] + \frac{1}{\tau_1}\left[\Delta\beta_{D_t}(x,t)\right]\frac{\partial v}{\partial \beta_{D_t(x,t)}},$$

where D_t represents the disturbance on traffic flow occurring at time t, while $\beta_{D_t(x,t)}$ is the region-representative disturbance risk preference coefficient. The parameter τ_1 indicates the reaction delay time. This model provides insights into how drivers' awareness of their environment influences traffic flow dynamics.

Moreover, [113] addressed the average speed of preceding vehicle groups in a Cooperative-Driving Systems (CPS) environment. The model is defined as:

$$v_t + \left[v - \frac{1}{2}(n+1)c_o\right] v_x = a[v_e - v],$$

where n represents the number of preceding cars. This model captures how the presence of multiple vehicles ahead affects the driving speed of a following vehicle, highlighting the dynamics of cooperative driving environments. While, [114] model for three-lane highways incorporated lane-changing behaviour. The dynamics for lane one and lane three are described by the equations:

$$k_{m,t} + q_{m,x} = s_{2m} - s_{m2},$$
$$v_{m,t} + [v_m - c_{m,o}]v_{m,x} = a_m[v_{e,m} - v_m] + r_1 s_{m2} - r_2 s_{2m}.$$

For lane two, we have:

$$k_{2,t} + q_{2,x} = s_{12} - s_{21} + s_{32} - s_{23},$$
$$v_{2,t} + [v_2 - c_{2,o}]v_{2,x} = a_2[v_{e,2} - v_2] + r_1 s_{21} - r_2 s_{12} + r_1 s_{23} - r_2 s_{32},$$

where s_{mn} represents the lane-changing rate from lane m to lane n, and r_1 and r_2 are constant parameters that affect the lane-changing dynamics. This model effectively captures the interactions between multiple lanes and the influences of lane-changing behaviour on traffic flow.

2.6. Driver behaviour, jerk, throttle, and road geometry: 2016-2018

This period was characterized by a deepened focus on driver behavioural attributes, jerk dynamics, electronic throttle effects, and road geometry.

A continuum model considering the traffic jerk effect was presented by [115]:

$$v_t + vv_x = a[v_e - v] - \lambda vv_{xt} + av'_e\left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2}\right),$$

where λ represents the jerk parameter, which captures the impact of rapid changes in acceleration on traffic flow dynamics; but [116] incorporated the effects of drivers' reaction time and reaction distance into the second-order macroscopic model as:

$$v_t + \left(v - \frac{c_o T}{T - \tau}\right) v_x = \frac{v_e - v}{T - \tau} - \frac{d}{T - \tau} v'_e,$$

where τ denotes the driver's reaction time, d indicates the driver reaction distance, and T represents the relaxation time. The parameter $c_o = \Delta/\tau_1$ describes the speed at which perturbations propagate, where τ_1 is the time needed for disturbances to travel a distance Δ. This model effectively captures how delays in driver reactions influence traffic flow.

In the area of driver behaviour, [117] modelled the effects of timid and aggressive drivers on optimal velocity in a single lane as follows:

$$v_t + [v - c_o]v_x = a[v_e - v] + (2p - 1)\alpha v'_e k_t + \frac{1}{2}a(2p - 1)\tau c_o k_x,$$

where α denotes the anticipation ability coefficients of aggressive and timid drivers, while $p \in [0, 1]$ is the intensity coefficient representing the influence between two typical driver characteristics. The parameter τ indicates the time needed for backward-propagated disturbances to travel a distance Δ. This model provides insights into how varying driver behaviours impact traffic flow dynamics.

In 2017, [118] built upon the concept of macro traffic flow models by integrating drivers' bounded rationality, which is defined as:

$$v_t + vv_x = \begin{cases} 0, & \text{if } |v_t + vv_x| \leq \epsilon, \\ a[v - v_e] + c_o v_x, & \text{otherwise}, \end{cases}$$

where ϵ represents the driver's bounded rationality, which affects decision-making and responsiveness in traffic situations; while [119] continuum model addressed timid and aggressive attributes of drivers by the equation:

$$v_t + \left[v - \lambda\Delta + (2p - 1)\alpha_1 k^2 v'_e \Delta\right] v_x = a[v_e - v] + \left[\lambda - (2p - 1)\alpha_1 k^2 v'_e\right] \frac{\Delta^2}{2} v_{xx}.$$

Here, α_1 is a sensitivity parameter, and $p \in [0, 1]$ is the intensity influence coefficient between the two driver characteristics. This model highlights how varying driver behaviours influence traffic dynamics. Another continuum model by [120] considered drivers' memory over a period of time and is articulated as:

$$v_t + \left[v - \lambda\Delta - \alpha_1 \gamma \tau_o v'_e\right] v_x = a[v_e - v] + \alpha_1 \gamma \tau_o v'_e v k_x + \frac{\lambda\Delta^2}{2} v_{xx}.$$

In this equation, α_1 is a sensitivity parameter, τ_o is the memory time length, and $\gamma \in (0,1)$. This model explores how experiences of drivers affect their current speed and acceleration decisions. In addition, Cheng led another research that examined the anticipation effect of drivers [121]:

$$v_t + \left[v - \lambda\Delta + a\tau_1 k^2 v_e' \Delta\right] v_x = a[v_e - v] + \left[\lambda - a\tau_1 k^2 v_e'\right] \frac{\Delta^2}{2} v_{xx}.$$

In this context, τ_1 represents the anticipation time, and λ is the response coefficient. This model highlights the impact of anticipatory behaviour on traffic flow. Furthermore, he pioneered a continuum model based on the full velocity difference model considering traffic jerk effects [122]:

$$v_t + [v - c_o]v_x = a[v_e - v] - \lambda_1 v v_{xt} + a v_e' \left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2}\right).$$

In this equation, λ_1 is the jerk parameter, which captures the impact of rapid changes in acceleration on traffic flow dynamics. Lastly in 2017, Cheng presents a macro traffic flow model accounting for multiple optimal velocity functions with different probabilities [123]:

$$v_t + [v - \lambda\Delta]v_x = a\left[\sum_{i,j} p_{ij} V(\bar{\omega}_i, \mu_j, k) - v\right] + \frac{\lambda}{2} v_{xx}\Delta^2,$$

where $\bar{\omega}$ and μ are the maximum speed and safety spacing, respectively, with corresponding probability p. This model effectively captures the complexities of driver behaviours in a heterogeneous traffic environment.

In the same year, an anisotropic second-order continuum model was derived from a simplified Helly's model under natural driving scenarios [124] as:

$$v_t + [v - c_o] = \alpha\left[\frac{1}{k(x,t)} - \frac{1}{k_e(v(x,t))}\right],$$

where α is a sensitivity parameter that influences the response of the traffic flow to changes in density.

Another heterogeneous model captured the unique phenomena of gap filling and suggested area occupancy as a concentration measure for heterogeneous traffic lacking lane discipline [125]. This extended Aw-Rascle (AR) model is:

$$k_{i,t} + q_{i,x} = 0 \quad \forall i,$$
$$[v_i + p_i(AO)]_t + v_i[v_i + p_i(AO)]_x = a_i[v_{i,e}(AO) - v_i],$$

where AO denotes area occupancy, with $p_i(AO) = C^2 AO^{\lambda_i}$, where λ_i is a dimensionless parameter and C is a constant. This model provides insights into how area occupancy impacts traffic flow in non-lane-based environments.

Another model by [126] considered multiple forward and backward driving strategies, representing a generalized version of the intelligent traffic systems (ITS) model as:

$$v_t + [v - c_o]v_x + ck_x = \gamma_1\alpha_1\left(\frac{1}{k} - \frac{1}{R_1^e(v)}\right) - \gamma_2\alpha_2\left(\frac{1}{k} - \frac{1}{R_2^e(v)}\right),$$

where α_i represents the headway sensitivity parameter for the driving strategy, and γ_i denotes the weight coefficient for each strategy. The terms R_i^e represent the equilibrium-speed-dependent density concerning the forward-looking $(i = 1)$ or backward-looking $(i = 2)$ strategies.

In another jurisdiction, [127] proposed a model that incorporates safe velocity in traffic flow, defined as:

$$v_t + v v_x = -\frac{c_o^2}{k} k_x + a[v_e - v] + \frac{1}{k}\frac{\partial \mu \frac{\partial v}{\partial x}}{\partial x},$$

where $v_e = \sqrt{\frac{2\alpha}{k}}$, and μ depicts the coefficient of viscosity. This model provides insights into how safe velocity considerations affect traffic dynamics.

With a different perspective, a traffic model accounting for road surface conditions during adverse weather was introduced by [128], defined as:

$$(kv)_t + vv_x = \frac{1}{k}\frac{\partial}{\partial x}\left(\frac{\frac{H_s}{\alpha_s}\left(1 - \frac{\beta_2}{\beta_m}\right)}{T}\right)k_x,$$

where β_m is the maximum severity index, α_s represents the safe time headway, H_s denotes the safe distance headway, and β_2 is the severity index. This model highlights the influence of adverse weather on traffic dynamics by factoring in road surface conditions. Moreover, [129] again characterized traffic behaviour during the harmonization period with the conserved form of the model:

$$(kv)_t + \left[\frac{(kv)^2}{k} + \left(\frac{v_e^2(k) - v^2}{2d_{tr}}\right)k\right]_x = k\left(\frac{v_e^2(k) - v^2}{\beta_3 v_s}\right),$$

where d_{tr} is the transition distance, and β_3 is the flow regulation value, where a smaller value of β_3 leads to more uniform flow, while a larger value results in clustered traffic. The variable v_s represents safe velocity, providing insights into traffic flow dynamics during transitions. However, [130] investigated the impacts of three probability distribution functions on traffic evolution as expressed by:

$$v_t + vv_x = a[v_e(k + \epsilon_1) - v] + c_o(v_x + \eta_2) + \epsilon_2.$$

In this model, ϵ_1 represents the perceived error of k, η_2 is the perceived error of v_x, and ϵ_2 denotes the perceived error of acceleration. This model provides a probabilistic perspective on traffic dynamics.

In 2018, [131] macro model considered anticipation and traffic jerk effects and is defined as:

$$v_t + (v + \alpha_1\tau_1 k^2 V'(k)\Delta)v_x = a[v_e - v] - \lambda_1\sigma vv_{xt},$$

where τ_1 represents the anticipation time, α_1 is the sensitivity of a driver, λ_1 denotes the jerk parameter, and σ is the memory time. This model captures how anticipation and jerk dynamics affect traffic flow. But [132] presented a continuum model considering the effect of driver characteristics and traffic jerk as:

$$v_t + [v - \eta\Delta + \alpha_1(2p-1)k^2 v_e'\Delta]v_x = a[v_e - v] - \lambda_1 vv_{xt} + \alpha_1 v_e'\left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2}\right),$$

where η is the weight coefficient, λ_1 is the jerk parameter, α_1 indicates the driver's forecast or delay time weight coefficient, and p reflects the intensity effect of driver characteristics. This model highlights the complex interplay between driver behaviour and traffic dynamics. Building on his previous work, [133] introduced a continuum model that takes into account optimal velocity changes with memory, defined as:

$$v_t + [v - \lambda - \gamma\delta k^2 v_e'\Delta]v_x = a[v_e - v] + \frac{1}{2}[\lambda - \gamma k^2 v_e']\Delta^2 v_{xx},$$

where γ represent sensitivity parameters, and δ indicates the memory time step, highlighting the influence of drivers' memory on traffic dynamics.

The model by [134] accounted for the integral form of optimal velocity changes with memory and backward-looking effects:

$$v_t + [v - \lambda\Delta]v_x = a\left[(p\alpha' + p\alpha'' - \alpha'')v_e + (1-p)\alpha''\Delta k_x v_e' - v\right] + \frac{\lambda\Delta^2}{2}v_{xx} + \theta\tau_m v_e' k_t,$$

where α denotes the driver's distance sensitivity coefficient, λ is the sensitivity coefficient of the driver to speed differences, and τ_m represents the memory time step. The parameters α' and α'' are positive constants, with

$p \in [0,1]$ and $\theta \in (0,1)$. This model provides insights into how memory and driving strategies interact to affect traffic flow. But the model by [135] examined the effect of friction and radius on a curved road:

$$v_t + [v - \lambda\Delta]v_x = \frac{a}{r}[v_e - rv] + \frac{1}{2}\lambda\Delta^2 v_{xx},$$

where r represents the radius of curvature, emphasizing how road geometry influences vehicle dynamics. Building on these efforts, [136] presented a continuum model that incorporates electronic throttle dynamics into traffic flow:

$$v_t + [v - Fv(x+\Delta,t) - E\Delta]v_x = a[v_e - v] + aF[v_e - v(x+\Delta,t)].$$

In this equation, $D = \frac{a}{T(\sigma + a)}$, $E = \frac{a/\eta + b\sigma}{\sigma + a}$, and $F = \frac{\sigma}{\sigma + a}$. Here, η is the time for disturbances to propagate backward a distance Δ, and σ is a sensitivity coefficient. The parameters a and b represent the effects of unmodelled dynamics on traffic flow.

A continuum model considering the self-anticipative effect was another significant contribution by [137]:

$$v_t + \left(v + \frac{k^2\gamma\tau_2 v_e'\Delta}{\sigma}\right)v_x = a\left[\frac{v_e - v}{\sigma}\right] - \frac{k^2\gamma\tau_2 v_e'\Delta^2}{2\sigma}v_{xx},$$

where λ and γ represent the weight coefficients of the driver's self-anticipative characteristics in velocity and optimal velocity, respectively. The parameters τ_1 and τ_2 indicate the anticipative time steps of the driver at velocity and optimal velocity, respectively, with $\sigma = 1 - \lambda\tau_1$. This model captures the influence of drivers' self-anticipation on traffic dynamics.

In addition to the former models, [138] formulated a velocity equation that accounts for acceleration changes with memory:

$$v_t + [v - \lambda\Delta]v_x = a[v_e - v] + \frac{1}{2}\lambda\Delta^2 v_{xx} + \gamma\delta v v_{xt},$$

where γ is a sensitivity parameter, and δ denotes the memory step, indicating how previous vehicle accelerations influence current dynamics. But, [139] investigated the difference in drivers' anticipation:

$$v_t + vv_x = a[v_e - v] + \left(\frac{1}{\tau} + \epsilon v_e'\right)[v(x+\Delta,t) - v(x,t)],$$

where ϵ represents the difference coefficient of drivers' anticipation, while τ expresses the propagation time of the disturbance. This model captures how variations in driver anticipation affect traffic flow.

Interestingly, [140] macro traffic flow model also introduced moving bottleneck by the following dynamic equation:

$$v_t + vv_x = a[v_{em} - v] + c_o v_x - \mu skv,$$

where μ is the frictional coefficient, $s = \frac{q}{L_o}$ with L_o being the length of the moving bottleneck, and $q = kv(1 - \phi(x_o - y(t)))$ denotes the flow rate influenced by the moving bottleneck. In this context, x_o is the initial position of the moving bottleneck, y is the position to which the bottleneck can propagate backward, v_{em} is the equilibrium velocity under the moving bottleneck's influence, and ϕ is a cut-off function reflecting the drop in flow due to the bottleneck. In the same year, a different model is articulated by [141], which explores new dimensions in traffic flow dynamics:

$$v_t + [v - \lambda c_o]v_x = a[v_e - v] + av_e'\left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2}\right) + a\gamma v_e'\left(\frac{k_x}{k} + \frac{k_{xx}}{2k^2}\right).$$

This model captures the dynamics of traffic flow by considering optimal velocity difference called the relative optimal velocity via employing the transformation relation from microscopic variables to macroscopic ones.

Another macroscopic model by [142] considered the backward-looking effects:

$$v_t + [v - \lambda\Delta]v_x = a[(\alpha' - \alpha'')v_e + \alpha''\Delta k_x v_e' - v] + \frac{1}{2}\lambda\Delta^2 v_{xx},$$

where α' and α'' are positive constants, and Δ represents the distance between two adjacent vehicles. This model highlights how drivers' perceptions of their surroundings can impact their driving behaviour and traffic flow.

Moreover, [143] presented a model that considers the effect of headway changes with memory. The model is defined as:

$$v_t + [v - (\lambda + \gamma\delta)\Delta]v_x = a[v_e - v] + \frac{\lambda + \gamma\delta}{2}\Delta^2 v_{xx},$$

where λ and γ are sensitivity parameters, while δ indicates the memory step. This model effectively highlights how changes in headway, influenced by past experiences, affect vehicle dynamics and traffic flow. But, the model by [144] again examined the effect of viscosity on traffic flow and is expressed as:

$$v_t + vv_x = a[v_e - v] - \frac{c_o^2}{k}k_x + \varphi v_{xx},$$

where φ is a diffusion constant that characterizes the impact of viscous forces on the movement of vehicles. This model captures the interplay between the equilibrium speed, vehicle interactions, and the viscous effects that can lead to variations in traffic flow.

2.7. Connected vehicles, emissions, and advanced driver models: 2019-2024

The most recent period reflects the growing integration of connected and autonomous vehicles (CAVs), sustainability concerns, and highly refined driver behavioural models.

Thus, [145] provided a comprehensive characterization of spatial variations in traffic density, aiming to harmonize traffic flow with prevailing forward conditions by:

$$v_t + vv_x = a[v_e - v] - \frac{(n+1)(\tau_r^n/\tau_m^{n+2})}{k}k_x,$$

where τ_r represents the driver's physiological response, n indicates the level of service, and τ_m is the time required for drivers to harmonize with forward traffic conditions. The physiological response reflects the time taken by a driver to perceive, make decisions, and align during transitions. The level of service encompasses various factors, including weather conditions, road surface conditions (such as slush, ice, or snow), visibility, intersections, junction types, lighting, lane widths, and road geometry. Khan again led a group to examine the spatial changes in traffic density to characterize traffic flow during transitions [146] by:

$$v_t + vv_x = a[v_e - v] - \frac{1}{k}\frac{\partial}{\partial x}\left[\frac{v_{max}}{k_{max}} - \frac{H}{\alpha_2(k - k_{max})^2}\right]k_x,$$

where v_{max} denotes the maximum velocity, k_{max} is the maximum density, H represents the distance headway, and α_2 is the time headway. This equation captures how changes in traffic density influence flow dynamics during transitions Khan and his team [147] further explored driver presumption based on driver reaction and traffic stimuli on traffic flow with the following equation:

$$v_t + vv_x = a[v_e - v] + \frac{\frac{v_{max} - v}{H}v_e'}{k}k_x.$$

In this equation, H represents the distance headway, illustrating how drivers' reactions to their surroundings and the resulting traffic stimuli affect their speed and acceleration. Finally, in 2019, Khan introduces a traffic constant to capture both physiological and psychological driver behaviour [148]:

$$v_t + vv_x = a[v_e - v] + L_d v_x,$$

where L_d is the driver response, which encompasses both the time taken to observe and process local traffic conditions (physiological behaviour) and the driver's attitude and awareness (psychological behaviour). This model provides a comprehensive view of how various aspects of driver behaviour influence traffic flow.

In 2019, a route-based flow model considered traffic interruption factors in a network [149]:

$$k_{r,t} + f_r(k_r, v_r)_{x_r} = s_r(x_r, t),$$
$$v_{r,t} + (v_r + (1-p_r)c_{ro})\, v_{r,x_r} = \frac{v_{re} - v_r}{\tau_r} - \frac{p_r v_r}{\tau_{1r}} + F_r.$$

where $r = 1, 2, \ldots, R$ represents the number of routes. The variables $k_r, f_r, v_r, v_{re}, s_r, p_r, \tau_r/\tau_{1r}, c_{ro}, x_r, F_r$ denote density, flow, speed, equilibrium speed, net inflow/outflow, interruption probability, reaction times, propagation speed of small perturbation, the ordinate of route k, and the friction effect on route k, respectively. On the other hand, [150] continuum model considered the effect of vehicle's taillight as:

$$v_t + (v - \lambda\Delta - M\Delta)v_x = a[v_e - v] + \frac{1}{2}(\lambda + M)\Delta^2 v_{xx},$$

where $M = \epsilon_o \tanh\left(1 - \dfrac{\Delta}{x_o}\right)$, with x_o being the distance that a vehicle's taillight can influence its following vehicle's behaviour, and ϵ_o is a parameter related to individual driver properties. This model highlights how visual cues from preceding vehicles affect driving behaviour and traffic dynamics. But [151] presented a continuum model accounting for mean-field velocity differences as:

$$v_t + [v - \lambda\Delta - p(\Delta + l_o\theta)]v_x = \alpha[v_e - v] + \frac{1}{2}\lambda\Delta^2 v_{xx},$$

where $\theta \in (0, 1)$ represents a parameter for different mean-field velocity differences, and l_o denotes the length of the road in front of the current vehicle. This model emphasizes how distance and relative speed impact traffic flow.

In the year in question, [152] macro model considered the influences of electronic throttle dynamics and backward-looking effects of traffic flow:

$$v_t + \left[v - \Delta\frac{\lambda m + b\beta}{m} - \Delta\frac{\beta}{m}v_x\right] v_x = a\left[(p\alpha' + p\alpha'' - \alpha'')v_e + (1-p)\alpha''\Delta k_x v_e' - v\right] + \frac{1}{2}\Delta^2\frac{\lambda m + b\beta}{m + \beta}v_{xx} + \frac{\beta}{m}\Delta v_{xt},$$

where β is the sensitivity coefficient of the electronic throttle opening angle, while α', α'' are positive constants, and b and m vary with the initial velocity. This model captures the dynamics of driver response influenced by both physiological and electronic control systems. However, [153] introduced a model that accounts for angular velocity and displacement on curved roads:

$$v_t + \left(v - \frac{1}{\tau}\lambda\Delta x\right) v_x = \frac{v_e - v}{\tau} + \frac{1}{\tau}v_e'\left(\frac{kx}{2kr} + \frac{k_{xx}}{6k^2r^2}\right),$$

where τ denotes the delay time, λ represents the sensitivity to angular velocity differences, and r is the radius of the road, treated as a constant due to the circular motion assumption. This model highlights how curvature influences vehicle dynamics. But [154] presented a model that incorporates the difference between steady and historical headways:

$$v_t + [v - \beta\Delta]v_x = a[v_e - v] + \frac{1}{2}\beta v_{xx}\Delta^2 + v_e k_o \lambda\left[\frac{1}{k_o} - \frac{1}{k(x - v\tau, t - \tau)}\right],$$

where τ and λ denote the historical time and reaction coefficient, respectively. The parameter a is the sensitivity coefficient, while β represents the response coefficient to the relative velocity difference. This model captures the dynamic interactions influenced by past headway and current traffic conditions.

Another formulation studied the effect of self-stabilizing on traffic flow by utilizing the historical speed of each vehicle [155]:

$$v_t + vv_x = \frac{a}{1 - \lambda t_o}[v_e - v],$$

where t_o represents the time gap between the current and historical time, emphasizing how past speeds influence current dynamics. But [156] focused on multilane flow with lane changing effect:

$$v_t + \left(v - \frac{c_o}{\tau}\right) v_x = a[v_e - v] + \frac{1}{2\tau} c_o^2 v_{xx},$$

where τ is the reaction time. This model captures the complexities of lane-changing behaviour and its impact on traffic dynamics across multiple lanes.

The classical speed-gradient macroscopic model was again extended to account for lateral flow dynamics on a multi-lane road [157]:

$$v_t + vv_x = a[v_e - v] + c_o v_x - \varphi \frac{f_y}{k} v_y.$$

In this model, f_y represents traffic sensitivity, φ is the lateral viscosity rate, and v_y accounts for the velocity gradient with respect to changes in lane usage. This extension highlights the interactions between lateral and longitudinal traffic flows on multi-lane roads. At the same time [158] characterized driver response during transitions as:

$$v_t + vv_x = a[v_e - v] - \frac{1}{k}\frac{\partial}{\partial x}\left(\frac{|v_e'|}{d_m(\beta_4 v_{max} + l_s)}\right) k_x,$$

where l_s is the distance between vehicles at standstill, and β_4 is the reaction time required for drivers to initiate action based on stimuli from vehicles ahead. This model effectively captures the dynamic nature of driver responses in varying traffic conditions. Another formulation by [159] explored spatial changes in traffic density to align traffic flow with forward conditions by the equation:

$$v_t + vv_x = a[v_e - v] - \frac{1}{k}\frac{(v_{max}^2 - v^2)}{2d_{tr}} k_x.$$

This emphasizes how the difference between maximum velocity and actual velocity influences traffic flow adjustments, allowing for smoother transitions in vehicle behaviour. Along the same lines, [160] introduced a heterogeneous traffic model that characterizes driver presumption:

$$v_t + vv_x = a[v_e - v] - T\frac{b_a}{b_s} k_x,$$

where b_a denotes the lateral distance and b_s is the safe lateral distance. This equation captures the nuances of driver behaviour based on perceived spatial relationships within heterogeneous traffic environments. Once more, [161] introduced anisotropic traffic model based on driver interaction:

$$v_t + vv_x = a[v_e - v] + \frac{\gamma_3}{\delta_k}\frac{v_{max}}{k_{max}} \alpha_3 T v_x,$$

where α_3 represents driver reaction, δ_k indicates changes in density, and γ_3 denotes driver sensitivity. This formulation highlights the interactions between driver behaviour and traffic dynamics, particularly in response to changes in vehicle density.

Further developments in traffic modelling within this period featured the development a continuum version of the full velocity difference (FVD) model with two delays, utilizing a series expansion of headway in terms of density [162]:

$$v_t + vv_x = \alpha_1(\bar{v} - v) + \alpha_1 \bar{v}'\left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2}\right) + \alpha_1 \mu v_x \left(\frac{1}{k} - \frac{k_x}{2k^3} - \frac{k_{xx}}{6k^4}\right) + \alpha_1 \mu \frac{v_{xx}}{2k^2},$$

where $\alpha_1 = \dfrac{1}{T - \tau_5}$, where T is the delay time sensitivity of a driver, $\bar{v} = v\left(\frac{1}{k}\right)$, and $\mu = \lambda T + \tau_4 k^2 \bar{v}'(k)$. Here, λ represents the responsive factor of velocity difference, while τ_4 and τ_5 denote the delays in headway and velocity, respectively. This model captures the intricate dynamics of traffic flow, accounting for the effects of both headway and velocity delays.

Furthermore, [163] introduced a macro model that incorporates drivers' timid and aggressive characteristics alongside bounded rationality as:

$$v_t + vv_x = a[v_e - v] + [\lambda - (2p-1)\alpha k^2 v_e'] \left[\Delta v_x + \frac{1}{2}\Delta^2 v_{xx}\right],$$

where λ is the sensitivity coefficient of speed difference, p stands for the intensity between two driver characteristics, and α is the anticipation and reaction delay coefficient for both aggressive and timid drivers. This formulation effectively captures how different driver behaviours influence traffic dynamics.

Other noteworthy progressions in traffic modelling at this time considered the velocity difference between adjacent vehicles on uphill and downhill slopes [164]:

$$v_t + (v - c_o)v_x = a[\gamma v_e - v] + (\lambda_1 - \lambda_2)\frac{\Delta^2}{2}v_{xx},$$

where $\gamma = (2 \pm \sin\theta)/2$, with the plus and minus signs corresponding to downhill and uphill conditions, respectively. θ is the slope angle, while λ_1 and λ_2 represent the sensitivity strength coefficients of the forward and backward velocity differences. This model provides valuable insights into how topography influences vehicle dynamics and traffic flow. But [165] proposed a model considering viscosity and driver memory in the era of autonomous and connected vehicles as:

$$v_t + [v + \gamma\{(\kappa - \beta m)k^2 v_e' - Tc_o\}]v_x = \gamma(1+\beta)(v_e - v) + \frac{\gamma c_o^2}{2}[T - (\kappa - \beta m)\tau k^2 v_e']v_{xx},$$

where β describes the sensitivity of driver memory to past traffic conditions, m is the memory time, $\gamma = 1/(T - \beta m)$ is the sensitivity of the driver, and T is the relaxation time. Additionally, κ denotes the forecast time. This formulation highlights the interactions between driver behaviour, memory, and traffic dynamics, particularly in automated driving contexts. Again, [166] characterized the effects of driver anticipation based on the motion information of two leading vehicles by:

$$v_t + \left[v + aK\Delta k^2 v_e' - c_o\right] v_x = a[v_e - v] + pc_o^2 \tau v_{xx},$$

where τ is the time required for disturbances to propagate backward, K denotes the forecast time, and p is a weight parameter. This model illustrates how anticipatory behaviours are influenced by the movements of vehicles ahead.

In addition, [167] introduced a model that considers drivers' continuous sensory memory alongside the preceding vehicle's taillight effect:

$$v_t + [v - (\lambda + \phi)h - a\mu\tau_o k v_e']v_x = a[v_e - v] + a\mu\tau_o v_e' k_x + \frac{\lambda + \phi}{2}v_{xx}h^2,$$

where τ_o represents the driver's sensory memory time, λ is the weighted parameter, h denotes the headway between vehicles, and $\phi = \xi \tanh\left(1 - \frac{h}{x_o}\right)$, where x_o is the critical distance for the taillight effect and ξ reflects driver characteristics. This model effectively captures the influences of driver memory and external stimuli on vehicle dynamics. Meanwhile, [168] investigated driving behaviour and electronic throttle effects on gradient highways:

$$\begin{aligned} v_t &+ \left[v - \frac{\beta}{m}\Delta v_x - \left(\lambda a + \frac{b\beta}{m} - a(2p-1)\alpha\tau k^2 \frac{4 \mp \sin\theta}{2} v_e'\right)\Delta\right] v_x \\ &= a\left[\frac{4 \mp \sin\theta}{2}v_e - v\right] + \frac{\beta}{m}\Delta v_{xt} + \left[\lambda a + \frac{b\beta}{m} - a(2p-1)\alpha\tau k^2 \frac{4 \mp \sin\theta}{2} v_e'\right] \frac{\Delta^2}{2} v_{xx}. \end{aligned}$$

In this model, α is a variable parameter, $0 \leq p \leq 1$ represents the strength of influence of driver characteristics, θ is the slope parameter, β is the sensitivity coefficient of angle difference for electronic throttle, and b and m are speed-dependent parameters. The inclusion of these factors allows for a nuanced understanding of how driver behaviour and road conditions interact.

Furthermore, [169] developed yet another model considering the traffic interruption probability and the electronic throttle opening angle effect:

$$v_t + [v - c_o]v_x = a[v_e - v] - k_1 pv + \frac{1}{2m}[k_2(1-p)m + b\beta]v_{xx}\Delta^2 + \frac{\beta}{m}\Delta v_{xt},$$

where p represents the probability of traffic interference, while k_1, k_2, β are weight coefficients, and b and m denote the sensitivity coefficients related to velocity and angle differences, respectively. This model effectively captures the interactions between electronic throttle dynamics and traffic flow disruptions. But [170], introduced a model that accounts for fluctuations in traffic flow due to static bottlenecks during peak periods:

$$v_t + vv_x = a[v - v_e] + c_o u_x - \beta\delta_t k_m v_m \left(1 - \frac{k_{in}}{k_{jam}}\right),$$

where β is an impedance coefficient representing disturbances caused by bottlenecks, with k_{in} and k_{jam} denoting initial and critical densities, respectively, while δ_t indicates changes over time. This formulation provides insights into how bottlenecks affect traffic dynamics.

Similarly, [171] presented a model considering the properties of two-sided lateral gaps in a non-lane-based heterogeneous traffic stream as:

$$\begin{aligned} v_{i,t} + v_i v_{i,x} - (1+2\delta_i)\sum_{j=1}^{N} P_j c_j(k) v_{j,x} =& a_i\left[v_{i,e}\left(\frac{k}{1+2\delta_i}\right) - v_i\right] + \left(\frac{1+6\delta_i}{2}\right)\sum_{j=1}^{N} P_j c_j^2(k)\tau_i v_{j,xx} \\ &+ \sum_{j(\neq 1)}^{N} \mu_{ij}\frac{P_j}{\tau_i}(v_j - v_i), \end{aligned}$$

where μ_{ij} is the friction factor, τ_i is the reactive coefficient, and c_j represents the disturbance propagation speed. This model captures the complexities of traffic interactions in heterogeneous environments. In connection to this form, [172] proposed a model for air traffic flow prediction as:

$$v_t + [v - \sigma c_o]v_x = a\left[\frac{W_a \cos\alpha}{\gamma g}(v_g - v) - v^2 + \beta v_e\right].$$

In this context, g is the gravitational acceleration, W_a is the weight of the air entering the engine, and v_g denotes the velocity of output gases. The parameters θ and α represent the angles of the aircraft's orientation and attack, respectively. This model highlights the intricate dynamics involved in air traffic control and management.

In 2021, [173] introduced a density-gradient second-order macroscopic model that explores the dynamics of multilane traffic as:

$$v_t + vv_x = a[v_e - v] + \frac{c_o^2}{k}k_x - \varphi\frac{f_y}{k}v_y.$$

In this equation, φ represents the lateral viscosity rate, and c_o is the sound speed or the propagation speed of perturbations. This model provides insights into how lateral interactions affect traffic flow. Meanwhile, [174] characterized traffic based on driver response and distance headway:

$$v_t + vv_x = a[v_e - v] - \frac{1}{k}\frac{\partial}{\partial x}\left(\frac{T_a}{T}\left(H_s + T\left|v_e'\right|\right)\right)k_x,$$

where T_a is the response time, and H_s is the safe distance headway. This model captures the influence of individual driver responses on traffic dynamics. Once more, [175] presented a model considering predictive headway variation and the preceding vehicle's taillight effect:

$$v_t + [v - (\lambda + M - a\beta\tau_1 k^2 v_e')\Delta]v_x = a[v_e - v] + \frac{1}{2}[\lambda + M - a\beta\tau_1 k^2 v_e']\Delta^2 v_{xx},$$

where λ is the weight coefficient, τ_1 represents the driver's prediction time, and β is the weight coefficient of the predictive headway variation term. The term $M = \epsilon_o \tanh(1 - \frac{y}{x_o})$ captures the taillight effect, where y is the headway between consecutive vehicles, and x_o is the critical distance influenced by the preceding vehicle's taillights. Macro models were further expanded to incorporate headway variation tendency and bounded rationality [176]:

$$v_t + vv_x = \begin{cases} 0, & \text{if } |v_t + vv_x| \leq \epsilon, \\ a[v_e - v] + [\lambda - a\beta\tau_1 k^2 v_e'] \left[v_x\Delta + \frac{1}{2}v_{xx}\Delta^2\right], & \text{otherwise,} \end{cases}$$

where λ serves as the weight coefficient, τ_1 represents the predicted time, and β is the weight coefficient for the terms above, while ϵ denotes the threshold of bounded rationality. This approach allows for a nuanced understanding of how bounded rationality affects driver behaviour.

Within the same year, [177] presented a continuum model that considers anticipation driving behaviour as:

$$v_t + \left[v - \left(\frac{1}{2}afv_e' + 1\right)c_o\right]v_x = a[v_e - v],$$

where τ_1 denotes the time required for the backward propagating disturbance to cover a distance Δ, and $f = \frac{\tau_1}{a}$. This model emphasizes the importance of anticipation in driving behaviour and its implications for traffic flow dynamics. But, [178] introduced a model that considers the backward-looking effect through a new positive backward equilibrium speed function:

$$v_t + [v - \lambda\Delta]v_x = a[pv_{fe} + (1-p)v_{be} - v] + \frac{1}{2}\lambda\Delta^2 v_{xx},$$

where v_{be} and v_{fe} denote the backward and forward equilibrium velocity functions, respectively, while p represents the forward concentration of drivers. This model effectively captures how drivers adjust their speeds based on both forward and backward dynamics.

Interestingly, [179] developed a model that accounts for the differences in driver psychological headway and the self-stabilizing effect on the optimal velocity related to psychological headway as:

$$v_t + [v - \Delta H(k/\epsilon_m)]v_x = a\left[\sum_{m=1}^{M} p_m v_e(k/\epsilon_m) - v\right] + \frac{1}{2}H(k/\epsilon_m)\Delta^2 v_{xx},$$

where $H(k/\epsilon_m) = \lambda - \beta\tau \sum_{m=1}^{M} p_m \left(\frac{k}{\epsilon_m}\right)^2 v_e'(k/\epsilon_m)$, where β is the driver's sensitivity coefficient to optimal velocity difference, and ϵ_m characterizes the driver's psychological headway. This model highlights the impact of psychological factors on driving behaviour and traffic flow. Building on this premise, [180] introduced a heterogeneous continuum model that accounts for different ratios of multiple optimal velocity functions and electronic throttle angle changes with memory given by the equation:

$$v_t + \left(v - \frac{\lambda\Delta}{1 - \phi\beta_5}\right)v_x = \frac{a}{1 - \phi\beta_5}\left[\sum_{m,l} p_{ml} v_e(\alpha_m, \beta_l, k) - v\right] - \frac{\lambda\Delta^2}{2(1 - \phi\beta_5)}v_{xx} + \frac{\phi}{1 - \phi\beta_5}vv_{xt},$$

where κ represents the driver's sensitivity coefficients, δ denotes the memory step, and p_{ml} indicates the penetration ratio of vehicles with maximum speed α_m and safe headway distance β_l. This model provides a comprehensive understanding of how electronic throttle dynamics and driver behaviour interact in heterogeneous traffic conditions.

An innovative approach to traffic modelling during this phase include a continuum traffic model focusing on stochastic behaviour arising during the acceleration and deceleration processes [181]:

$$v_t + vv_x = -\alpha(k, v)r_x - \beta(k, v)v_x + a[v_e(k, v, k_a, v_a) - v] + \sigma_o \frac{k}{k_{max}}(v_o - v)\eta_t,$$

where v_o denotes the desired speed, and σ_o represents the dissipation coefficient for the stochastic component of the model. The terms α and β are gradient terms, while η_t represents a two-parameter white noise, capturing the inherent variability in driver behaviour and traffic flow dynamics. However, [46] introduced a multi-class model that describes the flow of vehicles based on class density and the fraction of road area occupied by other vehicle classes. This model effectively represents traffic flow that does not adhere to lane discipline:

$$k_{i,t} + (k_i v_i)_x = 0, \tag{1a}$$

$$v_{i,t} + v_i v_{i,x} = a_i(v_{i,e} - v_i) - \bar{c}_i \left[v_{i,e}^{AO}\, AO_{-i,x} + \frac{\alpha_i}{W} v_{i,e}^{k}\, k_{i,x} \right], \tag{1b}$$

where $\bar{c}_i = v_{i,e}^{AO} + k_i \left[v_{i,e}^{k} + \frac{\partial}{\partial AO_{-i}} \left(v_{i,e}^{k} \right) \right]$. The variables represent the area occupancy and road width, indicating the model's flexibility in describing heterogeneous traffic scenarios. But [182] presented a model in a connected vehicle environment, where vehicles can obtain information about multiple front and rear vehicles, encapsulating the effects of bi-directional visual fields and multiple anticipations:

$$v_t + v v_x = (r_f \alpha_f + r_b \alpha_b)(v_e - v) + c_o v_x - c k_x,$$

where r_f and r_b represent the contributions from forward and backward driving strategies, respectively, while α_f and α_b are the corresponding sensitivity coefficients.

In this same context, [183] introduced a model that incorporates electronic throttle dynamics and traffic jerk, reflecting the intricacies of vehicle behaviour in dynamic environments:

$$\begin{aligned} v_t + [v - c_o] v_x =& a[v_e - v] + \left(\frac{\lambda}{m} \sum_{l}^{L} p_l l \Delta - k v \right) v_{xt} \\ & + a v_e' \left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2} \right) + \frac{\lambda b \Delta^2}{2m} \sum_{l}^{L} p_l l^2 v_{xx}, \end{aligned}$$

where $c_o = \frac{\lambda}{m} \sum_{l}^{L} p_l l (v_x + b) \Delta$. Here, the various parameters capture the influence of preceding vehicles on current vehicle behaviour. But [184] again contributed a multi-class AR model capturing frequent lateral, longitudinal, and velocity dynamics of multi-class traffic lacking lane discipline:

$$\begin{aligned} k_{i,t} + [k_i v_i]_x + [k_i w_i]_y = 0 \qquad \forall i \\ [v_i + p_i(AO)]_t + v_i[v_i + p_i(AO)]_x + w_i[v_i + p_i(AO)]_y = a_i[v_{i,e}(AO) - v_i] + M_x \\ [w_i + p_i(AO)]_t + w_i[w_i + p_i(AO)]_x + w_i[w_i + p_i(AO)]_y = a_i[v_{i,e}(AO) - v_i] + M_y, \end{aligned}$$

where M_x and M_y denote the velocity dynamics in the x and y directions due to vehicle interaction. At the particular time, [185] introduced a mixed traffic model that captures the dynamics of regular vehicles and those equipped with ACC:

$$v_t + v v_x = a[v_e - v] + a v_e' \left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2} \right) + p a \Delta \lambda v_x + p a \beta \left(\frac{1}{k_o} - \frac{1}{k} + \frac{k_x}{2k^3} + \frac{k_{xx}}{6k^4} \right),$$

where p is the permeability of ACC vehicles, while λ and β are sensitivity coefficients related to vehicle dynamics.

Recent advancements in traffic flow modelling have addressed various complexities encountered in real-world scenarios. A model that incorporates the effects of road surface irregularities on vehicular flow was presented by [186]:

$$v_t + v v_x = a[v_e - v] + c_o v_x - \frac{1}{2} \pi \beta_1 T_1 \sqrt{\left(\frac{\beta_1^2}{4} + \alpha_1^2 \right)} \left(1 - \frac{k}{k_{crit}} \right),$$

where k_{crit} represents the critical density, T_1 is the driver's physiological reaction time, α_1 signifies the depth of the pothole, and β_1 indicates its width .

Building on this, a macro model that accounts for multiple dynamic factors affecting traffic flow, such as convection, anticipation, relaxation, diffusion, and viscosity, was described by [187]:

$$v_t + vv_x = a[v_e - v] + c_o v W_{ak} W_x - \varphi \frac{f_y}{k} v_y - \varpi v_{xx}.$$

In this equation, ϖ and φ represent the longitudinal and lateral viscosity rates, respectively, while f_y relates to the sensitivity of lateral flow dynamics.

Further, exploring traffic dynamics, [188] presented a model that aligns with Little's Law highlights the relationship between traffic density and flow:

$$\frac{\partial}{\partial t}\left(v + \frac{k}{T}\right) + v\frac{\partial}{\partial x}\left(v + \frac{k}{T}\right) = \frac{v_e - v}{T}.$$

This model characterizes the impact of changes in density and velocity on traffic flow, illustrating fundamental traffic behaviours.

Additionally, a continuum model that accounts for the uncertain velocity of preceding vehicles on gradient highways is given by [189]:

$$v_t + vv_x = a\left[\left(\frac{v_{max} \pm \sin\theta}{2}\right) v_e - v\right] + \lambda\left[\epsilon v + (1-\epsilon)v_x\Delta + \frac{1}{2}(1+\epsilon)v_{xx}\Delta^2\right].$$

Here, θ signifies the angle of the gradient, while ϵ represents the uncertainty coefficient, effectively capturing the influence of varying conditions on traffic flow.

Following, a model considering electronic throttle dynamics on a curved road with slope was formulated by [190]:

$$\begin{aligned} v_t + \left[v - \frac{\lambda m + b\beta}{m+\beta}\Delta - \frac{\beta}{m+\beta}v(x+\Delta,t)\right] v_x =& \frac{\alpha m}{r(m+\beta)}\left[\left(\frac{\kappa\sqrt{\mu g r}\cos\phi \mp \sin\phi}{2}\right) v_e - rv\right] + \frac{\lambda m + b\beta}{2(m+\beta)}\Delta^2 v_{xx} \\ &+ \frac{\alpha m}{m+\beta}\left[\left(\frac{k\sqrt{\mu g r}\cos\phi \mp \sin\phi}{2}\right) v_e - rv(x+\Delta,t)\right], \end{aligned}$$

where α, λ, β, b, and m are sensitivity coefficients, r represents the radius of the curve, ϕ is the slope of the gradient, and κ is a constant parameter. μ and g denote the friction coefficient and gravitational acceleration, respectively. Similarly, a model that incorporates delayed-feedback control based on throttle angle differences was presented by [191] as follows:

$$v_t + \left[v - \frac{c_o}{1-\phi\alpha}\right] = \frac{1}{T(1-\phi\alpha)}[v_e - v] + \frac{\phi}{1-\phi\alpha} v v_{xt} + \frac{c_o^2\tau}{2(1-\phi\alpha)} v_{xx},$$

where, $\phi = \frac{k\epsilon}{\omega}$, where k is the feedback gain of the throttle angle difference, ϵ is the delay time of the throttle angle, and ω and α are parameters that change with steady-state velocity. τ represent the time needed for the backward propagated disturbance to travel a distance.

Additionally, a macro traffic flow model integrating time delay and anticipation effect on the headway and velocity was developed by [192]:

$$v_t + [v - \mu\Delta(\lambda - \alpha k^2 v_e'(T-\tau))]v_x = \mu\alpha[v_e - v] + \mu\alpha v_e'\left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2}\right) + \mu(\lambda T + \alpha T\tau k^2 v_e')(v_{xt} + vv_{xx})\Delta.$$

In this model, $\mu = \frac{1}{1-\alpha\tau-\lambda(T-\tau)}$, with τ representing the total time delay of the driver and vehicle sensors, T as the forecast time reflecting the anticipation effect of the driver, and λ and α as sensitivity coefficients.

Moreover, [193] presented a stochastic advection diffusion equation as formulated as:

$$z_t + v(k,z)z_x = \kappa z_{xx} - \frac{z}{\tau^z} + \eta\epsilon,$$

where z represents a small driver-related parameter that describes deviation from the mean, ϵ is Gaussian white noise varying in both space and time, $v(k,z)z_x$ is advection at the speed of traffic, κz_{xx} relates to the autocorrelation of noise in space, and $-\frac{z}{\tau^z}$ prevents drift in behaviour over time. Following, an extension of macroscopic model to consider tailpipe emissions was described by [194]:

$$v_t + vv_x + \frac{1}{(h_s+\tau v)}\left(\frac{a+bk}{c+dv}\right)k_x = a[v_e - v].$$

This model introduces a factor accounting for emissions, enhancing the understanding of traffic dynamics in relation to environmental impact .

In another formulation, [195] presented a viscous model that considers the anticipation of space headway, throttle angle, and brake torque information:

$$\begin{aligned} v_t + [v - \{\Delta(\phi\alpha + \beta\psi + (\phi+\psi M)v_x) + \Delta\left(\lambda - ak^2T_i v_e'\right)\}]v_x =& a[v_e - v] + \Delta(\phi+\psi M)v_{xt} + \frac{1}{2}\Delta^2[\lambda + \alpha\phi \\ &+ \beta\psi - aT_i k^2 v_e']v_{xx}, \end{aligned}$$

where λ is a sensitive constant, b_1 and α are steady-state velocity-dependent parameters, and β is determined through experimentation. The terms ϕ and ψ are derived from the sensitivity of throttle angle and brake torque, respectively.

Additionally, a model that accounts for the diverse reactivity effect stemming from driving attention and the vehicle's inertia was formulated by [196]:

$$v_t + [v - \lambda\Delta] = a(k)\left[\sum_{k=1}^{m} w_k v_{e,k}(k) - v\right] + \frac{1}{2}\lambda\Delta^2 v_{xx},$$

where λ is the sensitivity coefficient, capturing the influence of various driving factors on traffic behaviour.

A more recent advancement in traffic flow modelling have tackled various challenges posed by real-world scenarios. A non-lane-discipline-based model accounting for electronic throttle dynamics via the macro-transformation method was modelled by [197] and expressed as follows:

$$\begin{aligned} v_t + \left[v - \frac{m\lambda + n\kappa}{m}(1+p)\Delta - \frac{\kappa}{m}(1+p)\Delta v_x\right]v_x =& a\left[v_e\left(\frac{k}{1+p}\right) - v\right] \\ &+ \frac{1+3p_j}{2}\frac{m\lambda+n\kappa}{m}\Delta^2 v_{xx} + (1+p)\frac{\kappa}{m}\Delta v_{xt}, \end{aligned}$$

where p_j denotes the effect of lateral gaps, κ is a weight parameter, $m > 0$ is the weight coefficient of the difference between the current speed and steady-state speed, while the parameter $n > 0$ is the weight coefficient of the difference between the current electronic throttle angle corresponding to the current speed and the steady-state electronic throttle angle.

During the same time, a model considering the influence of lateral distance compensation in the process of lane change was developed by [198]:

$$v_t = \frac{v_e\left(\frac{k}{1+\delta}\right) - v}{1-p} + (k\Delta v + c_o)\,v_x + \frac{1}{2}p\Delta^2 v_{xx},$$

where p is the offset weight to control the impact of horizontal compensation. Meanwhile, a multiclass model that accommodates some violations of lane discipline and heterogeneous choices of speeds in a traffic stream is presented by [199] as:

$$k_{1,t} + q_{1,x} = 0$$

$$[v_1 + p_1]_t + [v_1(v_1+p_1)]_x = a_1[v_{1,e}(k_1) - v_1],$$

$$k_{2,t} + q_{2,x} = 0$$

$$[v_2 + p_2]_t + [v_2(v_2+p_2)]_x + \mu_{21}(k_1)(v_2 - v_1) = a_2[v_{2,e}(k_1,k_2) - v_1],$$

$$k_{3,t} + q_{3,x} = 0$$

$$[v_3 + p_3]_t + [v_3(v_3 + p_3)]_x + \mu_{32}(k_2)(v_3 - v_2) + \mu_{31}(k_1)(v_3 - v_1) = a_3[v_{3,e}(k_1, k_2, k_3) - v_1],$$

where $\mu_{ij}(k_j)$ is the relative drag function over the i^{th} class caused due to the existence and proportion of vehicles from class j, and p is a pressure term. But, [200] proposed a new macroscopic model aimed at addressing traffic emissions and the presumption of drivers based on these emissions. This model seeks to mitigate traffic congestion and reduce pollution levels. The formulation of the model is presented as follows:

$$v_t + vv_x + \left(\frac{-0.03599 + 0.000314k}{-0.579 + 0.0134v(k)}\right)\left(\frac{1}{h_s + \tau v}\right)v_x = \frac{-v}{\tau}.$$

This equation accounts for various factors affecting traffic flow and emissions, integrating the dynamics of driver behaviour in response to emission levels.

Furthermore, [201] introduced a macroscopic second-order traffic flow model designed to evaluate traffic dynamics in the context of Connected and Autonomous Vehicles (CAVs). The model is defined by the equation:

$$v_t + (v - \vartheta)v_x = a(v_e - v).$$

In this context, ϑ represents the reaction velocity, which is determined by the safe longitudinal gap and density-dependent transition times, providing a comprehensive framework for analysing CAV interactions within traffic systems.

Additionally, [202] derived a macro continuum model from a micro car-following model that incorporates the effects of both upslope and downslope driving conditions. The model is articulated as:

$$v_t + vv_x = a(\gamma v_e - v) + a\gamma v_e'\left(\frac{k_x}{2k} + \frac{k_{xx}}{6k^2}\right).$$

This equation highlights the interplay between vehicle velocity, environmental factors, and driver reactions to varying road gradients, offering insights into how topography impacts traffic flow dynamics.

Furthermore, [203] devised a novel macroscopic traffic system that observes the transition distance between vehicles and the sensitivity of drivers during traffic evolution. The model is formulated as:

$$(kv)_t + \left(kv^2 + \left(\frac{(d_t - d_s)\tau}{t^2}\right)^2 k\right)_x = a(v_e - v),$$

where d_s represents the safe distance headway, reflecting the importance of maintaining appropriate spacing between vehicles for safe driving.

Adding to these, [204] proposed a macroscopic model aimed at understanding non-homogeneous traffic conditions, particularly during transitions. The model is expressed as:

$$(kv)_t + \left(\frac{(kv)^2}{k} + f\zeta k\right)_x = k\left(\frac{v(k) - v}{\tau}\right),$$

where ζ signifies the lateral distance headway, which plays a crucial role in harmonizing the movement of vehicles within a traffic stream.

Additionally, [205] introduced a macroscopic traffic flow model that integrates traffic alignment behaviour at transitions. This model posits that vehicle velocity is influenced by the distance headway and driver response time. The formulation is given by:

$$v_t + \left(v - \frac{(\beta h_f + (a - \beta)h_r)\tau_a}{\tau^2}\right)v_x + \frac{\left(\frac{(\beta h_f + (a-\beta)h_r)}{\tau}\right)^2}{k}k_x = a(v_e - v),$$

where h_r denotes the rearward distance, while β represents the driver memory factor, emphasizing the role of cognitive processes in driving behaviour.

These advancements illustrate the ongoing efforts to refine traffic flow models to better reflect real-world driving conditions, particularly in light of technological developments and the growing emphasis on environmental considerations.

3. Mathematical properties

A fundamental property of second-order macroscopic traffic flow models is hyperbolicity, which determines whether the system supports wave-like solutions. For classical models such as the Payne-Whitham formulation, the system may exhibit non-physical characteristic speeds, allowing information to propagate faster than vehicles. This issue motivated the development of anisotropic models such as the ARZ system, which ensures that characteristic speeds remain physically consistent with traffic flow direction. In general, the characteristic speeds depend on both the velocity and density gradients, and their structure determines the propagation of traffic waves and disturbances.

Stability analysis plays a crucial role in understanding the emergence of stop-and-go waves in traffic flow. Linear stability analysis typically examines small perturbations around a uniform flow state and identifies conditions under which these perturbations grow or decay. For many second-order models, stability depends on the slope of the equilibrium velocity function $v_e(k)$, the relaxation time, and anticipation or pressure terms. In particular, large relaxation times or steep equilibrium gradients can lead to instability, resulting in the formation of congestion waves. Conversely, the inclusion of viscosity or nonlocal effects can stabilize traffic flow by damping perturbations.

Well-posedness and admissibility of solutions are also central considerations in macroscopic traffic flow modelling. Hyperbolic systems may develop discontinuities (shock waves), corresponding to abrupt changes in traffic density and velocity. Proper entropy conditions and boundary conditions are required to ensure physically meaningful solutions. In network and multi-lane models, additional challenges arise due to coupling conditions at junctions and lane-changing interactions. These factors influence both analytical properties and numerical implementations of the models.

The parameters appearing in second-order models have clear physical interpretations but may be difficult to calibrate in practice. For example, the relaxation time represents driver response to equilibrium conditions, while pressure and anticipation terms reflect interactions among vehicles. Viscosity coefficients are often introduced to stabilize solutions but may not correspond directly to measurable quantities. Parameter identifiability remains a key challenge, as multiple parameter combinations can produce similar traffic behaviour. This highlights the importance of combining theoretical modelling with empirical calibration.

4. Empirical validation and data-driven assessment

Empirical validation of macroscopic traffic flow models relies on a variety of data sources that capture traffic dynamics at different spatial and temporal scales. Common data sources include loop detector measurements, which provide aggregated flow, density, and speed data; vehicle trajectory datasets obtained from video tracking or GPS devices; and probe vehicle data from connected navigation systems. More recently, connected and autonomous vehicle technologies have enabled the collection of high-resolution traffic data, offering new opportunities for calibrating and validating macroscopic models.

Model validation is typically performed by comparing simulated outputs with observed traffic data using quantitative metrics such as root mean square error (RMSE) in speed or density, travel time discrepancies, and the accuracy of shock wave propagation. Additional measures include the ability of a model to reproduce stop-and-go waves and congestion patterns observed in real traffic conditions.

While many second-order models provide valuable theoretical insights, not all have been extensively validated against empirical data. Classical models such as the Payne-Whitham formulation are often used for theoretical analysis, whereas more recent models incorporating anisotropy, nonlocal effects, and multi-class dynamics have demonstrated improved agreement with observed traffic behaviour. However, challenges remain in model calibration due to parameter uncertainty, data noise, and variability in driver behaviour across different traffic environments.

5. Notable trends

The development of second-order macroscopic traffic flow models has significantly advanced the understanding of traffic dynamics.

The 1970s marked a foundational period for these models, primarily characterized by systems of partial differential equations. During this time, Payne introduced the dynamic velocity equation, incorporating anticipation and relaxation effects, illustrating how drivers adjust their speeds in response to changing traffic conditions [49]. [50] expanded on these concepts by discussing linear and nonlinear wave propagation in traffic flow. However, limited empirical validation constrained the depth of research, resulting in relatively few publications.

The late 1970s and 1980s witnessed further advancements in traffic flow models, integrating more realistic representations of driver behaviour. Warren Phillips introduced a continuum traffic model derived from kinetic theory, emphasizing the importance of driver anticipation and reaction times [44]. Building on this, [51] proposed a macroscopic freeway model that addressed dense traffic conditions and stop-start waves.

By the 1990s, traffic flow modelling shifted towards refining previous models to better represent complex conditions. Researchers began investigating factors like freeway congestion and traffic clusters. For instance, [53] introduced a continuum model focusing on congested freeways, and [54] explored the structure of traffic clusters, highlighting the dynamic nature of traffic as a system with complex flow patterns.

The decade from 2000 to 2010 saw a significant evolution in second-order macroscopic traffic flow models, with increased emphasis on incorporating complex variables, stochastic processes, and real-world applications. A key advancement was the renewed interest in models that accounted for driver behaviour and reaction times. [41] revitalized second-order traffic models by introducing frameworks that incorporated both density and velocity gradients, ensuring anisotropic behaviour. Simultaneously, [42] critiqued existing higher-order continuum traffic models for exhibiting gas-like behaviour and proposed a non-equilibrium traffic model that ensured disturbances propagated slower than traffic. Around the same time, [58] introduced a continuum model integrating microscopic and macroscopic elements, addressing the issue of backward-traveling waves.

A key trend during this period was the integration of time delays into traffic models. Researchers increasingly examined how these delays influenced driving behaviour [61,64,74,94,102,192]. In parallel, models incorporating viscosity effects gained prominence. Key contributions included models from [52,54], which introduced viscosity terms to account for the dissipation of disturbances in dense traffic conditions. Subsequent models [64,75,79,80,83,87,110,144,157,165,187,195] further emphasized the role of viscosity in smoothing traffic flow and reducing oscillations caused by stop-start waves.

Following 2010, the study of traffic flow modelling underwent a remarkable transformation, driven partly by the development of intelligent transportation systems (ITS) [126]. A significant milestone was the introduction of ACC and CACC models. [95,185] highlighted the ability of CACC systems to improve coordination between vehicles, thereby enhancing traffic flow efficiency and reducing congestion. Moreover, researchers focused on incorporating nonlinear interactions between traffic flow variables. Lane-changing behaviour saw considerable progress [72,75,84,88,100,114,156], as did research on multi-lane traffic dynamics [65,67,74,157], multi-anticipative behaviour [76,105,182], and multi-class scenarios [45,46,184].

During this era, researchers also focused on how road topography affects traffic flow. Road gradient was extensively explored [98,168,189], road curvature was shown to significantly impact vehicle speed and acceleration [135,153,163,164,190,202], and static bottlenecks were found to affect vehicle interactions and flow rates [89,92]. Ramp flow dynamics also received attention [72,111], and the impact of potholes on vehicular dynamics was quantified [186].

Early models primarily focused on fundamental density-flow relationships. However, as research advanced, newer models began to incorporate the effects of jerking, throttle dynamics, signal lights, and taillights. The foundational work by [115] established a basis for further studies on jerk dynamics. Recent studies [122,131,132,183] built upon these models. A significant development has been the refinement of models integrating electronic throttle dynamics [136,152,168,169,183,190,191,195,197]. The incorporation of signal lights [78,150] and the role of taillights [167,175] further highlighted how visual and control cues impact traffic dynamics.

In the early 2020s, macroscopic traffic flow models expanded significantly, increasingly integrating driver behavioural aspects. Models were introduced that differentiated between timid and aggressive drivers [117, 119,163], providing more realistic simulations of driver behaviour. Models also began incorporating driver memory [133,134,145,148,165,167,179,203–205], and bounded rationality adjustments based on a threshold [118,163,176].

These shifts prompted researchers to integrate sustainability into traffic flow modelling. Studies by [194, 200] introduced macroscopic models that incorporated emissions as a central variable, reflecting the growing recognition of the need for traffic models addressing environmental concerns. Recent years have also seen an emphasis on smart city technologies and connected and autonomous vehicles [165,166,191,195,201].

The evolution of macroscopic traffic flow models illustrates a trajectory of increasing complexity and the integration of technology, behaviour, and environmental considerations.

6. Future directions

As the field of traffic flow modelling continues to advance, the integration of macroscopic models with emerging technologies and evolving societal needs will be crucial in addressing contemporary challenges in urban mobility. One promising direction is the incorporation of advanced data analytics. The increasing availability of real-time data from connected vehicles, smart traffic systems, and various sensors offers an unprecedented opportunity to refine existing models. By utilizing big data analytics and machine learning techniques, researchers can enhance the predictive accuracy of these models, allowing them to respond dynamically to real-time traffic conditions.

A deeper understanding of human factors influencing driver behaviour is another critical area for research inquiry. Traditional macroscopic models often simplify driver interactions, failing to account for the complexities of human decision-making. By integrating more psychological and physiological insights into traffic flow models, researchers can create more realistic simulations that capture how various factors, such as stress levels, distractions, and risk perceptions, affect driver behaviour.

With increasing concerns about climate change and environmental sustainability, second-order macroscopic models must emphasize sustainability metrics. Researchers could explore how different traffic management strategies impact emissions, energy consumption, and urban air quality. By incorporating sustainability considerations, macroscopic models can aid policymakers in making informed decisions that balance the need for efficient transportation with environmental stewardship.

The rise of CAVs presents new challenges and opportunities for macroscopic traffic modelling. Further research could focus on developing new modelling frameworks that account for the unique dynamics introduced by CAVs, such as their ability to communicate with each other and with infrastructure. These models should explore how the presence of CAVs alters traffic flow patterns, potentially leading to reduced congestion and improved safety.

Additionally, the development of integrated multimodal transportation systems will be critical for future research. Macroscopic models could be modified to account for the interactions among various transportation modes, such as public transit, cycling, and pedestrian traffic, allowing for a comprehensive understanding of how each mode impacts traffic flow.

Real-time traffic management solutions are essential in the context of smart cities. Second-order models could be designed to support real-time applications, enabling traffic managers to dynamically adjust traffic signals and implement adaptive control strategies based on current conditions. The integration of Internet of Things (IoT) technologies into macroscopic models will further facilitate the development of smart cities that prioritize efficient and sustainable urban mobility.

Fostering cross-disciplinary collaboration among traffic engineers, urban planners, computer scientists, and social scientists will lead to innovative modelling approaches that better reflect the complexities of traffic systems. The application of network theory in traffic flow modelling also presents an exciting avenue, providing insights into how traffic flows through complex urban environments and identifying critical nodes and links within the transportation network.

The integration of artificial intelligence (AI) and optimization algorithms into macroscopic models can revolutionize traffic management. AI can facilitate the development of intelligent systems that learn from traffic patterns and optimize signal timings and route choices in real-time. The exploration of socio-economic factors and the role of policy and regulatory frameworks in shaping traffic dynamics are further directions that can lead to more effective transportation planning and policy design.

7. Conclusion

This review has presented a comprehensive examination of second-order macroscopic traffic flow models, tracing their evolution from early relaxation-based formulations of the 1970s to modern extensions incorporating anisotropy, nonlocal interactions, multi-class dynamics, and intelligent transportation systems. By organizing models within a chronological and thematic framework, the review illustrates how successive formulations address the limitations of earlier approaches.

A key observation from the literature is that model development is largely driven by the need to resolve specific deficiencies. Classical Payne-Whitham formulations introduced relaxation and anticipation mechanisms but suffered from non-physical wave propagation, which was later resolved through anisotropic formulations. The incorporation of viscosity, time delays, and multi-class interactions in subsequent decades reflects continued efforts to improve realism and stability. More recently, models have incorporated electronic throttle dynamics, driver memory, bounded rationality, road geometry effects, and environmental factors such as emissions.

Despite these advances, no single model fully captures all aspects of real-world traffic dynamics. Each formulation represents a trade-off between physical realism, mathematical tractability, and computational efficiency. Several open challenges remain in the field, including the rigorous calibration of model parameters using high-resolution data, the integration of connected and autonomous vehicle effects into continuum frameworks, and the development of models that simultaneously capture multi-scale interactions and network-level dynamics.

Furthermore, while many models exhibit desirable mathematical properties such as stability and well-posedness under certain conditions, these properties are often sensitive to parameter choices and may not hold universally across traffic regimes. Strengthening the connection between theoretical modelling and empirical validation remains an essential priority for the field.

This review is subject to certain limitations. The focus has been restricted to second-order macroscopic continuum models, and related approaches such as microscopic, mesoscopic, and hybrid models have not been covered in detail. While efforts were made to include representative and influential models across the coverage period, some specialized or emerging contributions may not have been exhaustively captured.

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