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From Boolean algebras to effect algebras: Sharp and unsharp logic in classical and quantum systems

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Abstract: This paper examines how logical structures represent sharp and unsharp propositions in classical and quantum systems. Classical sharp logic is modeled by Boolean algebras, where propositions have bivalent truth values, complements are globally defined, and distributivity holds. Quantum sharp logic is modeled by orthomodular lattices of closed Hilbert-space subspaces or projection operators; it retains definite yes–no outcomes for ideal measurements but rejects the Boolean assumptions of global truth valuation and distributivity. Classical unsharp logic uses fuzzy membership or probabilistic truth values to describe vagueness and incomplete information, whereas quantum unsharp logic is expressed through effects and effect algebras, which capture noisy, inefficient, and generalized measurements. The paper's central question is which algebraic structure is appropriate for each combination of classical or quantum behavior and sharp or unsharp measurement. The analysis shows that the four cases form a coherent hierarchy: Boolean algebras describe deterministic classical events, orthomodular lattices describe ideal quantum propositions, fuzzy or probabilistic models describe classical partial truth, and effect algebras describe realistic quantum effects. This classification clarifies the mathematical and physical role of unsharpness and explains why effect algebras are needed beyond projection-based quantum logic.

Keywords: Boolean algebra, orthomodular lattice, effect algebra, quantum logic, unsharp logic, fuzzy sets, projection operator, positive operator-valued measure

1. Introduction

Logical systems provide the mathematical language used to assign truth values to propositions about physical events. In classical reasoning, propositions are usually assumed to be sharp: an event either occurs or it does not, and its truth value is either 1 or 0. This view is represented by Boolean algebra, the algebraic structure introduced by Boole for classical logic and set-theoretic reasoning [1]. Boolean logic is therefore suitable for deterministic classical events such as the outcome of a coin toss or the membership of an element in a well-defined set.

Quantum theory changes this picture in a fundamental way. Ideal quantum measurements still yield sharp outcomes at the moment of observation, but the propositions associated with those outcomes do not form a Boolean algebra. Birkhoff and von Neumann showed that quantum propositions are naturally represented by the lattice of closed subspaces of a Hilbert space, or equivalently by projection operators, producing an orthomodular rather than a Boolean structure [2]. The difference is not merely formal. Superposition, noncommutativity, and contextuality prevent the simultaneous assignment of context-independent truth values to all quantum propositions [3,4].

A second distinction arises from measurement precision. Sharp logic, whether classical or quantum, concerns ideal yes–no propositions. Many practical situations, however, involve graded truth, uncertainty, noise, or inefficient detection. Classical unsharp logic is commonly modeled by fuzzy sets, where membership can take any value in the interval $[0,1]$ [5]. Quantum unsharp logic requires a different mathematical object: the effect. Effects are positive operators E satisfying $0 \leq E \leq I$, and they occur naturally in positive operator-valued measures used to describe generalized quantum measurements [4,6]. The abstract algebraic treatment of these objects is provided by effect algebras [7–9].

The research question addressed here is: which algebraic structure correctly represents propositions when the system is classical or quantum and the measurement is sharp or unsharp? The contribution of the paper is a unified comparison of the four resulting cases: classical sharp, quantum sharp, classical unsharp, and quantum unsharp logic. The comparison clarifies how Boolean algebras, orthomodular lattices, fuzzy or probabilistic models, and effect algebras differ in their treatment of events, operations, complements, truth values, and physical interpretation.

The remainder of the paper follows this order. §2 discusses classical sharp logic and its Boolean representation. §3 explains quantum sharp logic through Hilbert-space subspaces and orthomodular lattices. §4 presents classical unsharp logic in terms of fuzzy and probabilistic truth values. §5 develops quantum unsharp logic through effects, POVMs, and effect algebras. §6 summarizes the findings and states their implications for the logical modeling of realistic measurements.

2. Classical sharp logic

2.1. Meaning of "sharp"

A proposition is sharp when it has a definite truth value. In classical logic, the only allowed truth values are

$$0 = \text{false}, \qquad 1 = \text{true}.$$

Sharpness therefore excludes intermediate values. A proposition is not partly true or partly false; it is evaluated exactly as true or false once the outcome under consideration is known.

2.2. Classical sharp logic

In the classical deterministic setting, all elementary outcomes are assumed to be well defined. Consider a coin toss with two elementary outcomes: heads and tails. Let

$$H : \text{"the coin shows heads"}, \qquad T : \text{"the coin shows tails"}.$$

When the coin lands, exactly one of these propositions is true. If the outcome is heads, then $H = 1$ and $T = 0$; if the outcome is tails, then $H = 0$ and $T = 1$. This example illustrates the essential feature of classical sharp logic: a single realized outcome determines the truth value of every proposition about the experiment.

This type of reasoning is appropriate when events are determinate and mutually exclusive at the level being modeled. Any uncertainty in such a setting is epistemic: it reflects incomplete knowledge of the realized outcome rather than indeterminacy in the logical structure.

2.3. Mathematical representation: Boolean algebra

Classical sharp logic is represented by Boolean algebra [1]. Let Ω be the sample space of an experiment. Events are subsets of Ω, and logical operations are identified with set-theoretic operations:

$$A \wedge B = A \cap B, \qquad A \vee B = A \cup B, \qquad A' = \Omega \setminus A.$$

For a coin toss,

$$\Omega = \{\text{Heads}, \text{Tails}\},$$

and the events "Heads" and "Tails" are represented by the subsets $\{\text{Heads}\}$ and $\{\text{Tails}\}$, respectively.

The truth value of an event $A \subseteq \Omega$ at an outcome $\omega \in \Omega$ is given by the characteristic function

$$\chi_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

This function formalizes bivalence: an outcome either belongs to an event or it does not. Boolean algebra is therefore the natural algebraic structure for classical sharp propositions.

2.4. Why Boolean algebras are suitable

Boolean algebras are suitable for classical sharp logic because they capture the main assumptions of deterministic classical reasoning [1]. First, they support bivalence: every proposition receives one of two truth values. Second, they provide a set-like structure in which conjunction, disjunction, and negation correspond to intersection, union, and complement. Third, they satisfy distributivity:

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C).$$

This law expresses the compatibility of all classical propositions within a single event space.

Boolean algebras also provide exact complements. For every event A, its complement A' satisfies

$$A \wedge A' = \emptyset, \qquad A \vee A' = \Omega.$$

Thus, each proposition has a unique negation, and one of A or A' must hold at every outcome. Finally, Boolean algebras allow global truth assignments: a single outcome $\omega \in \Omega$ determines truth values for all events simultaneously. These properties explain why Boolean algebra describes classical sharp logic but also indicate why it becomes inadequate for quantum propositions.

2.5. Key properties of classical sharp logic

The defining properties of classical sharp logic can be summarized as determinism, bivalence, distributivity, complementarity, and global valuation. Determinism means that each realized outcome fixes the truth values of all propositions. Bivalence restricts truth to the values 0 and 1. Distributivity ensures that logical combinations can be regrouped without changing meaning. Complementarity assigns each proposition a unique opposite, and global valuation permits all propositions to be evaluated within the same sample space. These features make classical sharp logic internally complete for deterministic classical systems.

2.6. Summary

Classical sharp logic is the logic of determinate classical events. Its propositions are modeled as subsets of a sample space, and its operations are Boolean operations. The resulting structure is powerful precisely because it assumes that all propositions can be jointly evaluated. The later sections show that this assumption fails in quantum theory and must be modified further when measurements become unsharp.

3. Quantum sharp logic

3.1. Meaning of "sharp" in the quantum setting

A sharp quantum measurement is an ideal measurement associated with projection operators. It produces a definite outcome from a specified set of mutually exclusive alternatives. For example, a measurement of electron spin along the z-axis yields either spin-up or spin-down. In that restricted sense, the outcome is sharp.

The important difference from classical sharp logic is that quantum propositions need not have predetermined truth values before measurement. A system may be prepared in a superposition of possible outcomes, and incompatible observables cannot all be assigned simultaneous classical values. Sharpness in quantum theory therefore refers to the definiteness of the measurement outcome, not to the existence of a global Boolean truth assignment.

3.2. Example: Spin measurement

The spin of a spin-$\frac{1}{2}$ particle measured along the z-axis has two possible sharp outcomes:

$$|\uparrow_z\rangle \quad \text{and} \quad |\downarrow_z\rangle.$$

The proposition "spin is up along the z-axis" is represented by the projection operator

$$P_{\uparrow z} = |\uparrow_z\rangle\langle\uparrow_z|,$$

while the opposite outcome is represented by

$$P_{\downarrow z} = |\downarrow_z\rangle\langle\downarrow_z|.$$

These projections are orthogonal and exhaustive for the z-spin measurement. A system prepared in $|\uparrow_z\rangle$ yields spin-up with probability one. A system prepared in a superposition, however, yields outcomes according to the Born rule rather than according to a pre-existing Boolean truth value.

3.3. Mathematical representation: Orthomodular lattice

In quantum sharp logic, propositions are represented by closed subspaces of a Hilbert space, or equivalently by projection operators [2,3]. Logical operations are defined geometrically:

$$A \wedge B = A \cap B, \qquad A \vee B = \overline{\mathrm{span}(A \cup B)}, \qquad A^{\perp} = \text{orthogonal complement of } A.$$

Here, the meet $\wedge$ corresponds to intersection, the join $\vee$ corresponds to the smallest closed subspace containing both subspaces, and orthocomplementation corresponds to taking all vectors orthogonal to a given subspace.

These operations form an orthomodular lattice rather than a Boolean algebra. The difference arises because Hilbert-space subspaces do not generally satisfy distributivity. Orthomodularity preserves enough structure to support a coherent logic of sharp quantum propositions while allowing the non-Boolean behavior imposed by superposition and incompatible measurements.

3.4. Why not Boolean algebra?

Boolean algebra assumes that all propositions belong to a common distributive structure and can be assigned simultaneous truth values. Quantum theory violates both assumptions. The events associated with incompatible observables, such as spin along different axes, cannot be treated as subsets of a single classical sample space. Instead, they are represented by subspaces whose lattice operations depend on Hilbert-space geometry.

Three features explain the failure of Boolean representation. First, superposition permits states such as

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1,$$

where no classical assignment of a definite spin-up or spin-down truth value is available before measurement. Second, entanglement shows that the properties of a composite system cannot always be reduced to independent properties of its parts. For the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big),$$

the correlations are definite even though neither subsystem has its own determinate classical value. Third, contextuality shows that the value assigned to a quantum proposition cannot be independent of the compatible observables measured with it. This incompatibility with global truth valuation is one of the central lessons of quantum logic [4].

3.5. Why quantum logic is not Boolean

The failure of distributivity gives a direct algebraic demonstration of the non-Boolean character of quantum logic. Let the Hilbert space be $\mathbb{C}^2$, with basis vectors

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and define

$$|+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big).$$

Consider the one-dimensional subspaces

$$A = \text{span}\{|0\rangle\}, \qquad B = \text{span}\{|+\rangle\}, \qquad C = \text{span}\{|1\rangle\}.$$

Since $|+\rangle$ and $|1\rangle$ are linearly independent,

$$B \vee C = \mathbb{C}^2.$$

Therefore,

$$A \wedge (B \vee C) = A.$$

On the other hand, $A \cap B = \{0\}$ and $A \cap C = \{0\}$, so

$$(A \wedge B) \vee (A \wedge C) = \{0\}.$$

Consequently,

$$A \wedge (B \vee C) \neq (A \wedge B) \vee (A \wedge C).$$

This example shows that the Hilbert-space lattice of quantum propositions is not distributive. The loss of distributivity is not a technical defect; it is the algebraic expression of the fact that quantum alternatives combine through linear superposition rather than through ordinary set union.

3.6. Orthomodular law

Although distributivity fails, quantum propositions obey the orthomodular law. If $A \leq B$, meaning that the subspace A is contained in the subspace B, then

$$B = A \vee (B \wedge A^{\perp}).$$

The law states that a larger event B can be decomposed into the part contained in A and the part orthogonal to A within B. In Hilbert-space terms, this decomposition preserves a disciplined relation between implication and orthocomplementation. Orthomodularity is therefore the structural principle that replaces distributivity in sharp quantum logic.

3.7. Comparison: Classical vs. quantum sharp logic

Table 1. Comparison of classical and quantum sharp logic

Aspect	Classical Sharp Logic	Quantum Sharp Logic
Typical example	Coin toss: heads or tails	Spin measurement: spin-up or spin-down along a chosen axis
Events	Subsets of a sample space Ω	Closed subspaces of a Hilbert space or projection operators
Mathematical structure	Boolean algebra [1]	Orthomodular lattice [2,3]
Operations	Intersection, union, complement	Meet, join, orthocomplement
Distributivity	Holds for all events	Fails in general; orthomodularity holds
Truth values	Global bivalence: 0 or 1	Sharp outcome values within a measurement context
Measurement outcome	Determinate once the classical outcome is known	Definite after measurement; probabilistic before measurement for non-eigenstates
Complements	Exact set complements, $A \wedge A' = \emptyset$ and $A \vee A' = \Omega$	Orthogonal complements, $P \wedge P^{\perp} = 0$ and $P \vee P^{\perp} = I$
Global valuation	Available	Not available for all observables in general

The comparison in Table 1 shows that the word sharp has different implications in the two domains. In classical logic, sharpness is tied to an underlying global event space. In quantum logic, sharpness is tied to ideal projective measurement, but the collection of all sharp propositions remains non-Boolean. The main interpretive consequence is that quantum sharp logic preserves definite outcomes without preserving classical distributive reasoning.

4. Classical unsharp logic

4.1. Meaning of "Unsharp"

Unsharpness refers to truth values or event descriptions that are not restricted to the binary set $\{0, 1\}$. In classical reasoning, unsharpness arises when propositions involve vagueness, incomplete information, or probabilistic assessment. A statement such as "it will rain tomorrow" may be represented not as simply true or false but by a probability, for example 0.7, or by a fuzzy membership value representing the degree to which the situation belongs to the class of rainy conditions [5].

This kind of unsharpness does not require quantum phenomena. It already appears in ordinary reasoning about risk, diagnosis, classification, and decision-making. Its purpose is to represent partial truth or graded belief when binary truth values are too rigid.

4.2. Mathematical representation: Fuzzy set theory

Fuzzy set theory replaces the characteristic function of an ordinary set with a membership function

$$\mu_A : X \to [0, 1].$$

For an element $x \in X$, the value $\mu_A(x)$ expresses the degree to which x belongs to the fuzzy set A [5]. The sharp case is recovered when membership values are only 0 or 1.

Standard fuzzy operations generalize Boolean operations as follows:

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\},$$

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\},$$

$$\mu_{A'}(x) = 1 - \mu_A(x).$$

These formulas preserve the intuitive meanings of conjunction, disjunction, and negation while allowing graded membership. Probabilistic models provide a related but conceptually distinct account: a probability measures uncertainty about occurrence, whereas a fuzzy value measures degree of membership or vague truth.

4.3. Why Boolean algebra is not sufficient?

Boolean algebra is not sufficient for classical unsharp logic because bivalence is too restrictive for many empirical and linguistic propositions. Weather, medical risk, approximate classification, and preference statements often require values between 0 and 1. Treating such propositions as strictly true or false loses information about uncertainty and degree.

The shift from Boolean values to fuzzy or probabilistic values changes the interpretation of logical operations. Instead of asking only whether an event occurs, one asks how strongly a predicate applies or how likely an event is. This shift is essential for modeling reasoning in situations where measurements are imprecise or the boundary of a concept is gradual.

4.4. Connection with sharp logic

Classical unsharp logic includes classical sharp logic as a limiting case. When every membership value is either 0 or 1, fuzzy membership reduces to the characteristic function of an ordinary set, and the classical Boolean interpretation is recovered. Thus, unsharp classical logic does not reject sharp logic; it broadens the range of admissible truth values so that deterministic, probabilistic, and vague propositions can be treated in a common mathematical language.

This relation is important for comparison with quantum theory. In the classical case, unsharpness can often be introduced by replacing a two-valued truth function with a many-valued or probabilistic one. In the quantum case, unsharpness requires not only graded probabilities but also noncommutative operator structure, because the events themselves are not generally Boolean.

5. Quantum unsharp logic

5.1. Sharp measurements in theory

In the ideal formulation of quantum mechanics, a sharp measurement is represented by a family of mutually orthogonal projection operators. A projection P satisfies

$$P^2 = P, \qquad P^\dagger = P.$$

The complement of the proposition represented by P is represented by $I - P$. For spin measurement along the z-axis, the projectors

$$P_{\uparrow z} = |\uparrow_z\rangle\langle\uparrow_z|, \qquad P_{\downarrow z} = |\downarrow_z\rangle\langle\downarrow_z|,$$

represent the two mutually exclusive outcomes. If the system is prepared in $|\uparrow_z\rangle$, the first outcome has probability 1 and the second has probability 0.

The collection of such projections forms the orthomodular lattice of sharp quantum propositions [2,3]. This lattice is appropriate for ideal projective measurements but does not fully describe measurements affected by inefficiency, background noise, or coarse-graining.

5.2. Real-world limitations

Physical measurements are rarely perfectly sharp. Detectors can miss particles, signals can be corrupted by electronic or environmental noise, and measurement devices can have limited efficiency. Even when the quantum state is well prepared, the apparatus may fail to register the ideal outcome. A detector with efficiency η may produce a positive signal only with probability

$$P(\text{yes}) = \eta\, P_{\text{ideal}}(\text{yes}),$$

where P_{ideal} is the probability predicted by the corresponding ideal measurement.

These limitations require a measurement formalism that treats outcomes as effects rather than only as projections. This change is not merely experimental bookkeeping. It alters the logical interpretation of propositions because a measurement outcome can now correspond to an operator lying strictly between 0 and I.

5.3. From sharp to unsharp measurements

Generalized quantum measurements are described by positive operator-valued measures (POVMs) [4,6]. A POVM is a family of positive operators $\{E_i\}$ satisfying

$$0 \le E_i \le I, \qquad \sum_i E_i = I.$$

Each E_i is called an effect. If the system is in the density state ρ, the probability of outcome i is

$$p(i) = \mathrm{Tr}(\rho E_i).$$

Projective measurements are recovered when each E_i is an orthogonal projection and the projectors sum to I.

POVMs capture detector inefficiency and noise because the effects need not be idempotent or mutually orthogonal. Inefficient detection can be represented by scaling an ideal projection and adding a no-detection effect. Noise can be represented by replacing sharp projectors with effects that overlap. In this setting, a proposition does not simply correspond to a subspace; it corresponds to a positive operator whose expectation value gives the probability of a measurement outcome.

5.4. Intuition

The difference between projection operators and effects can be understood through a photon detector. In the sharp ideal case, the detector always clicks when a photon is present and never clicks when it is absent. The yes–no measurement is represented by projectors such as

$$P_{\text{yes}} = |1\rangle\langle 1|, \qquad P_{\text{no}} = |0\rangle\langle 0|,$$

with $P_{\text{yes}} + P_{\text{no}} = I$.

If the detector has efficiency 70%, a photon in state $|1\rangle$ produces a click only with probability 0.7. The corresponding unsharp yes-effect can be written as

$$E_{\text{yes}} = 0.7|1\rangle\langle 1|,$$

and the complementary no-effect is

$$E_{\text{no}} = I - E_{\text{yes}} = 0.3|1\rangle\langle 1| + |0\rangle\langle 0|.$$

Here, E_{yes} is not a projection because $E_{\text{yes}}^2 \neq E_{\text{yes}}$. It is nevertheless a valid effect because $0 \leq E_{\text{yes}} \leq I$. The example shows why quantum unsharp logic is naturally probabilistic: even a definite input state can yield a non-deterministic detector response.

Effect algebras provide the abstract algebraic setting for such propositions [7,8]. In a Hilbert space H, the set

$$E(H) = \{E \in B(H) : 0 \leq E \leq I\},$$

of all effects contains the projection lattice as a special subset. The partial addition $E \oplus F$ is defined precisely when $E + F \leq I$, in which case $E \oplus F = E + F$. This partiality reflects the probabilistic constraint that mutually exclusive alternatives cannot have total effect greater than the identity.

Projection operators satisfy $P^2 = P$ and correspond to sharp propositions. General effects satisfy only $0 \leq E \leq I$ and correspond to unsharp propositions. This distinction explains why orthomodular lattices are not enough for realistic quantum measurements: they describe ideal projective events but omit the broader family of effects used in POVMs [6,9].

5.5. Comparison: Quantum sharp vs. quantum unsharp logic

The comparison in Table 2 identifies the key mathematical transition from projections to effects. In the sharp case, the logical structure is governed by subspace geometry. In the unsharp case, the central object is a positive operator whose expectation value gives an operational probability. This transition is essential for describing laboratory measurements, because real apparatuses normally introduce efficiency loss, noise, or coarse-graining.

Table 2. Comparison of quantum sharp and quantum unsharp logic

Aspect	Quantum Sharp Logic	Quantum Unsharp Logic
Measurement type	Ideal, projective, repeatable	Generalized, noisy, coarse-grained, or inefficient
Mathematical object	Projection P with $P^2 = P$	Effect E with $0 \leq E \leq I$
Algebraic structure	Orthomodular lattice [2,3]	Effect algebra [7,8]
Outcome interpretation	Definite yes–no outcome in a measurement context	Probability-valued response determined by $\mathrm{Tr}(\rho E)$
Addition rule	Lattice meet, join, and orthocomplement	Partial addition $E \oplus F$ defined when $E + F \leq I$
Physical example	Ideal spin measurement along a fixed axis	Photon detection with finite efficiency
Limitation	Does not model detector inefficiency or unsharp effects	Includes sharp projections as special cases while modeling imperfections

5.6. Master comparison: Classical and quantum, sharp and unsharp logic

The master comparison in Table 3 answers the paper's central question by matching each logical setting with its appropriate mathematical structure. The vertical distinction separates classical from quantum behavior; the horizontal distinction separates sharp from unsharp measurement. Classical sharp logic is Boolean because all propositions can be evaluated in one global event space. Quantum sharp logic is orthomodular because propositions are tied to Hilbert-space geometry and measurement context. Classical unsharp logic uses graded values to represent vagueness or uncertainty. Quantum unsharp logic requires effects because uncertainty is encoded in positive operators rather than merely in scalar truth values.

Table 3. Comparison of classical sharp, quantum sharp, classical unsharp, and quantum unsharp logic

Aspect	Classical Sharp	Quantum Sharp	Classical Unsharp	Quantum Unsharp
Typical measurement	Ideal deterministic classical observation	Ideal projective quantum measurement	Probabilistic or vague classical assessment	Generalized quantum measurement
Outcome	Binary, such as heads or tails	Binary within a chosen context, such as spin-up or spin-down	Degree or probability, such as 0.7	Probability from an effect, such as detector response 0.7
Mathematical structure	Boolean algebra [1]	Orthomodular lattice [2,3]	Fuzzy set or probability model [5]	Effect algebra [7–9]
Events represented as	Subsets of a sample space Ω	Closed subspaces or projectors	Membership functions or probability events	Effects $0 \leq E \leq I$
Truth values	0 or 1	Sharp measurement outcomes; no general global valuation	Values in $[0,1]$	Probabilities $\mathrm{Tr}(\rho E)$ in $[0,1]$
Operations	Intersection, union, complement	Meet, join, orthocomplement	Generalized conjunction, disjunction, negation	Partial addition and orthosupplement
Core property	Distributive and globally valued	Orthomodular and contextual	Graded or probabilistic	Operator-valued and probabilistic
Physical example	Coin toss	Spin along a fixed axis	Weather-risk assessment	Inefficient photon detection
Main limitation	Cannot express graded uncertainty	Cannot express unsharp detector effects	Cannot express noncommutative quantum events	Requires operator structure and partial operations

This classification also clarifies the role of effect algebras. They are not simply a many-valued version of Boolean logic. They are algebraic structures designed for the partial, operator-valued composition of quantum effects. As a result, they provide the correct language for describing realistic quantum measurements while retaining projection-based logic as the sharp limiting case.

6. Conclusion

This paper addressed the question of which algebraic structures represent sharp and unsharp propositions in classical and quantum systems. The analysis shows that no single classical logic can cover all four cases. Classical sharp logic is correctly modeled by Boolean algebras because deterministic classical events admit bivalent truth values, exact complements, distributivity, and global valuations. Quantum sharp logic is modeled by orthomodular lattices because ideal quantum propositions are sharp at measurement but do not support a global Boolean assignment of truth values.

The treatment of unsharpness further separates the classical and quantum cases. In classical reasoning, unsharpness can be modeled by fuzzy membership values or probabilities, allowing propositions to have degrees in $[0,1]$. In quantum measurement theory, unsharpness is represented by effects E satisfying $0 \leq E \leq I$, and the appropriate algebraic structure is the effect algebra. This difference is significant because quantum unsharpness is not merely scalar vagueness; it is tied to the operator structure of generalized measurements.

The main contribution of the study is a coherent fourfold classification linking Boolean algebras, orthomodular lattices, fuzzy or probabilistic models, and effect algebras to their corresponding measurement settings. The comparison demonstrates that effect algebras are necessary whenever realistic quantum measurements involve inefficiency, noise, or coarse-graining. The implication is that logical modeling in physics must be selected according to both the nature of the system and the precision of measurement. Sharp logics describe idealized propositions, whereas unsharp logics provide the mathematical tools needed for empirical uncertainty and realistic quantum experiments.

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