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DEGREE SUBTRACTION ADJACENCY EIGENVALUES AND ENERGY OF GRAPHS OBTAINED FROM REGULAR GRAPHS

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Abstract. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G and let $d_G(v_i)$ be the degree of a vertex v_i in G. The degree subtraction adjacency matrix of G is a square matrix $DSA(G) = [d_{ij}]$, in which $d_{ij} = d_G(v_i) - d_G(v_j)$, if v_i is adjacent to v_j and $d_{ij} = 0$, otherwise. In this paper we express the eigenvalues of the degree subtraction adjacency matrix of subdivision graph, semitotal point graph, semitotal line graph and total graph of a regular graph in terms of the adjacency eigenvalues of G. Further we obtain the degree subtraction adjacency energy of these graphs.

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1. Introduction

Let G be a simple, undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The degree of a vertex v_i denoted by $d_G(v_i)$ is the number of edges incident to it. If all vertices have same degree equal to r then G is called an r-regular graph.
The adjacency matrix of G is a square matrix of order n, defined as $A = A(G) = [a_{ij}]$, where $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The characteristic polynomial of $A(G)$ is denoted by $\phi(G : \lambda)$, that is, $\phi(G : \lambda) = \det|\lambda I - A(G)|$, where I is an identity matrix. The characteristic polynomial of the adjacency

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matrix of a complete graph K_n is $\phi(K_n : \lambda) = (\lambda - n + 1)(\lambda + 1)^{n-1}$. The roots of the equation $\phi(G : \lambda) = 0$ are called the adjacency eigenvalues of G [1] and they are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. Two non-isomorphic graphs are said to be cospectral if they have same eigenvalues. For any graph G, $-\Delta \leq \lambda_i \leq \Delta$, where Δ is the maximum degree. Thus for an r-regular graph, $\lambda_i + r \geq 0$ for $i = 1, 2, \ldots, n$.
The vertex-edge incidence matrix of G is defined as $B = B(G) = [b_{ij}]$, where $b_{ij} = 1$ if the vertex v_i is incident to an edge e_j and $b_{ij} = 0$, otherwise.
It is easy to observe that [1]

$$BB^T = A + D,$$

where $D = \mathrm{diag}[d_G(v_1), d_G(v_2), \ldots, d_G(v_n)]$ is a diagonal degree matrix of G and B^T is the transpose of B.
If G is an r-regular graph, then

$$BB^T = A + rI. \tag{1}$$

The other matrices of a graph exists in the literature such as distance matrix [2], Laplacian matrix [3], Laplacian distance matrix [4], sum-eccentricity matrix [5, 6], degree sum matrix [7, 8], degree sum adjacency matrix [9], Zagreb matrix [10], degree subtraction matrix [11], degree product matrix [12], degree square sum matrix [13] and average-degree eccentricity matrix [14].
In [15] the degree subtraction adjacency (DSA) matrix is defined as $DSA(G) = [d_{ij}]$, where

$$d_{ij} = \begin{cases} d_G(v_i) - d_G(v_j), & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $DSA(G)$ is called the DSA-polynomial and is denoted by $\psi(G : \xi)$. Thus $\psi(G : \xi) = \det(\xi I - DSA(G))$, where I is an identity matrix of order n.
For any regular graph of order n, $\psi(G : \xi) = \xi^n$. The line graph $L(G)$ of a regular graph is regular. Hence $\psi(L(G) : \xi) = \xi^m$, where m is the number of edges of G.
The eigenvalues of $DSA(G)$, denoted by $\xi_1, \xi_2, \ldots, \xi_n$ are called DSA-eigenvalues of G. Two non-isomorphic graphs are said to be DSA-cospectral if they have same DSA-eigenvalues. Since $DSA(G)$ is a skew-symmetric matrix, its eigenvalues are purely imaginary or zero.
The DSA-energy of a graph G is defined as

$$DSAE(G) = \sum_{i=1}^{n} |\xi_i|. \tag{2}$$

The Eq. (2) is analogous to the ordinary graph energy defined as [16]

$$E_A(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of G. The ordinary graph energy is well studied by many researchers [17].
In [15] the DSA-polynomial and DSA-energy of a path, complete bipartite graph, wheel, windmill graph and corona graph have been obtained. In this paper we obtain the DSA-eigenvalues and DSA-energy of subdivision graph, semitotal point graph, semitotal line graph and of toal graph of regular graphs.

2. DSA-eigenvalues

A subdivision graph of G is a graph $S(G)$ obtained from G by inserting a new vertex on each edge of G [18]. Thus if G has n vertices and m edges, then $S(G)$ has $n+m$ vertices and $2m$ edges. If $u \in V(G)$ then $d_{S(G)}(u) = d_G(u)$ and if v is subdivided vertex then $d_{S(G)}(v) = 2$.

Lemma 2.1. [1] *If M is a non-singular matrix, then we have*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|.$$

Theorem 2.2. *Let G be an r-regular graph on n vertices and m edges. Then*

$$\psi(S(G):\xi) = \begin{cases} \xi^{\frac{n}{2}}(\xi^2+2)^{\frac{n}{2}}, & \text{if} \quad r=1 \\ \xi^{2n}, & \text{if} \quad r=2 \\ (-1)^n (r-2)^{2n} \xi^{m-n} \phi\left(G : \frac{-\xi^2 - r(r-2)^2}{(r-2)^2}\right), & \text{if} \quad r \geq 3. \end{cases}$$

Proof. (i) If $r=1$, then G is a union of $k \geq 1$ edges. Thus G has $n = 2k$ vertices and k edges. The vertices of $S(G)$ can be labeled in such a way that

$$DSA(S(G)) = \begin{bmatrix} O & B^T \\ -B & O \end{bmatrix},$$

where B is vertex-edge incidence matrix of G and O is zero matrix. Therefore by Lemma 2.1 and Eq. (1)

$$\begin{aligned}
\psi(S(G):\xi) &= \begin{vmatrix} \xi I_m & -B^T \\ B & \xi I_n \end{vmatrix} \\
&= \xi^m \left| \xi I_n + B \frac{I_m}{\xi} B^T \right| \\
&= \xi^{m-n} \left| \xi^2 I_n + BB^T \right| \\
&= \xi^{m-n} \left| \xi^2 I_n + A + rI_n \right| \\
&= (-1)^n \xi^{m-n} \left| -(\xi^2 + r) I_n - A \right| \\
&= (-1)^n \xi^{m-n} \phi(G : -(\xi^2 + r)) \\
&= (-1)^{2k} \xi^{k-2k} \left((\xi^2+1)^2 - 1 \right)^k \\
&= \xi^k (\xi^2+2)^k
\end{aligned}$$

$$= \quad \xi^{\frac{n}{2}}(\xi^2+2)^{\frac{n}{2}}.$$

(ii) If $r=2$, then each component of G is cycle. Therefore $S(G)$ is 2-regular graph on $2n$ vertices. Hence

$$\psi(S(G):\xi)=\xi^{2n}.$$

(iii) Let $r\geq 3$. The vertices of $S(G)$ can be labeled in such a way that

$$DSA(S(G))=\begin{bmatrix} O & (2-r)B^T \\ -(2-r)B & O \end{bmatrix},$$

where B is vertex-edge incidence matrix of G and O is zero matrix. Therefore by Lemma 2.1 and Eq. (1)

$$\begin{aligned}
\psi(S(G):\xi) &= \begin{vmatrix} \xi I_m & -(2-r)B^T \\ (2-r)B & \xi I_n \end{vmatrix} \\
&= \xi^m \left| \xi I_n + (2-r)^2 B\frac{I_m}{\xi}B^T \right| \\
&= \xi^{m-n}\left|\xi^2 I_n + (r-2)^2(A+rI)\right| \\
&= (-1)^n (r-2)^{2n}\xi^{m-n}\left|\left(\frac{-\xi^2-r(r-2)^2}{(r-2)^2}\right)I_n - A\right| \\
&= (-1)^n (r-2)^{2n}\xi^{m-n}\phi\left(G:\frac{-\xi^2-r(r-2)^2}{(r-2)^2}\right).
\end{aligned}$$

□

By Theorem 2.2, we have following corollary.

Corollary 2.3. *Let G be an r-regular graph on n vertices and m edges. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the adjacency eigenvalues of G and $\mathbf{i}=\sqrt{-1}$.*
(i) *If $r=1$ then DSA-eigenvalues of $S(G)$ are 0 ($\frac{n}{2}$ times) and $\pm\mathbf{i}\sqrt{2}$ ($\frac{n}{2}$ times).*
(ii) *If $r=2$ then DSA-eigenvalues of $S(G)$ are all zeros.*
(iii) *If $r\geq 3$ then DSA-eigenvalues of $S(G)$ are 0 ($m-n$ times) and $\pm\mathbf{i}(r-2)\sqrt{r+\lambda_i}$, $i=1,2,\ldots,n$.*

The semitotal pont graph of G, denoted by $T_1(G)$, is a graph with vertex set $V(G)\cup E(G)$ and two vertices in $T_1(G)$ are adjacent if they are adjacent vertices in G or one is a vertex and other is an edge incident to it in G [19].
Note that if $u\in V(G)$ then $d_{T_1(G)}(u)=2d_G(u)$ and if $e\in E(G)$ then $d_{T_1(G)}(e)=2$.

Theorem 2.4. *Let G be an r-regular graph on n vertices and m edges. Then*

$$\psi(T_1(G):\xi)=\begin{cases} \xi^{m+n}, & \text{if} \quad r=1 \\ (-1)^n(2r-2)^{2n}\xi^{m-n}\phi\left(G:\frac{-\xi^2-r(2r-2)^2}{(2r-2)^2}\right), & \text{if} \quad r\geq 2. \end{cases}$$

Proof. (i) If $r=1$, then $T_1(G)$ is a regular graph of degree two on $m+n$ vertices. Hence
$$\psi(T_1(G):\xi)=\xi^{m+n}.$$
(ii) Let $r\geq 2$. The vertices of $T_1(G)$ can be labeled in such a way that
$$DSA(T_1(G))=\begin{bmatrix} O & (2-2r)B^T \\ -(2-2r)B & O \end{bmatrix},$$
where B is vertex-edge incidence matrix of G and O is a zero matrix.
Therefore by Lemma 2.1, and Eq. (1)
$$\begin{aligned}
\psi(T_1(G):\xi) &= \begin{vmatrix} \xi I_m & -(2-2r)B^T \\ (2-2r)B & \xi I_n \end{vmatrix} \\
&= \xi^m \left|\xi I_n+(2r-2)^2 B\frac{I_m}{\xi}B^T\right| \\
&= \xi^{m-n}\left|\xi^2 I_n+(2r-2)^2(A+rI)\right| \\
&= (-1)^n(2r-2)^{2n}\xi^{m-n}\left|\left(\frac{-\xi^2-r(2r-2)^2}{(2r-2)^2}\right)I_n-A\right| \\
&= (-1)^n(2r-2)^{2n}\xi^{m-n}\phi\left(G:\frac{-\xi^2-r(2r-2)^2}{(2r-2)^2}\right).
\end{aligned}$$
□

By Theorem 2.4, we have following corollary.

Corollary 2.5. *Let G be an r-regular graph on n vertices and m edges. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the adjacency eigenvalues of G and $\mathbf{i}=\sqrt{-1}$.*
(i) *If $r=1$ then DSA-eigenvalues of $T_1(G)$ are all zeros.*
(ii) *If $r\geq 2$ then DSA-eigenvalues of $T_1(G)$ are 0 $(m-n$ times) and $\pm\mathbf{i}(2r-2)\sqrt{r+\lambda_i}$, $i=1,2,\ldots,n$.*

Semitotal line graph of G, denoted by $T_2(G)$, is a graph with vertex set $V(G)\cup E(G)$ and two vertices in $T_2(G)$ are adjacent if one is a vertex and other is an edge incident to it in G or both are edges adjacent in G [1]. Note that if $u\in V(G)$ then $d_{T_2(G)}(u)=d_G(u)$ and if $e=uv\in E(G)$ then $d_{T_2(G)}(e)=d_G(u)+d_G(v)$.

Theorem 2.6. *Let G be an r-regular graph $(r\geq 1)$ on n vertices and m edges. Then*
$$\psi(T_2(G):\xi)=(-1)^n r^{2n}\xi^{m-n}\phi\left(G:\frac{-\xi^2-r^3}{r^2}\right).$$

Proof. The vertices of $T_2(G)$ can be labeled in such a way that
$$DSA(T_2(G))=\begin{bmatrix} O & rB^T \\ -rB & O \end{bmatrix},$$
where B is vertex-edge incidence matrix of G and O is a zero matrix.

Therefore by Lemma 2.1 and Eq. (1)

$$\begin{aligned}
\psi(T_2(G):\xi) &= \begin{vmatrix} \xi I_m & -rB^T \\ rB & \xi I_n \end{vmatrix} \\
&= \xi^m \left| \xi I_n + r^2 B \frac{I_m}{\xi} B^T \right| \\
&= \xi^{m-n} \left| \xi^2 I_n + r^2 (A + rI) \right| \\
&= (-1)^n r^{2n} \xi^{m-n} \left| \left(\frac{-\xi^2 - r^3}{r^2} \right) I_n - A \right| \\
&= (-1)^n r^{2n} \xi^{m-n} \phi \left(G : \frac{-\xi^2 - r^3}{r^2} \right).
\end{aligned}$$

□

By Theorem 2.6, we have following corollary.

Corollary 2.7. *Let G be an r-regular graph ($r \geq 1$) on n verties and m edges. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of G and $\mathbf{i} = \sqrt{-1}$. Then DSA-eigenvalues of $T_2(G)$ are 0 ($m - n$ times) and $\pm \mathbf{i} r \sqrt{r + \lambda_i}$, $i = 1, 2, \ldots, n$.*

Total graph of G, denoted by $T(G)$, is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $T(G)$ are adjacent if and only if they are adjacent vertics of G or adjacent edges of G or one is a vertex and other is an edge incident to it in G [18]. Total graph of a regular graph is regular. Hence if G is regular, then

$$\psi(T(G):\xi) = \xi^{m+n}.$$

Two different graphs having same eigenvalues are called cospectral. If G_1 and G_2 are adjacency cospectral graphs with same regularity, then by Corollaries 2.3, 2.5 and 2.7, the graphs $S(G_1)$ and $S(G_2)$; $T_1(G_1)$ and $T_1(G_2)$; $T_2(G_1)$ and $T_2(G_2)$ form a pair of DSA-cospectral graphs.

3. DSA-energy

By Corollaries 2.3, 2.5 and 2.7 and by Eq. (2) we get the following proposition.

Proposition 3.1. *Let G be an r-regular graph on n vertices and m edges. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of G. Then*
(i)

$$DSAE(S(G)) = \begin{cases} n\sqrt{2}, & \text{if} \quad r = 1 \\ 0, & \text{if} \quad r = 2 \\ 2(r-2) \sum\limits_{i=1}^{n} \sqrt{r + \lambda_i}, & \text{if} \quad r \geq 3. \end{cases}$$

(ii)

$$DSAE(T_1(G)) = \begin{cases} 0, & \text{if} \quad r = 1 \\ 2(2r-2)\sum_{i=1}^{n} \sqrt{r+\lambda_i}, & \text{if} \quad r \geq 2. \end{cases}$$

(iii) $DSAE(T_2(G)) = 2r \sum_{i=1}^{n} \sqrt{r+\lambda_i}$ *for* $r \geq 1$.

(iv) $DSAE(T(G)) = 0$ *for* $r \geq 1$.

By Proposition 3.1 we have following result.

Proposition 3.2. *If G is an r-regular graph $(r \geq 3)$ on n vertices, then*
(i) $(2r-2)DSAE(S(G)) = (r-2)DSAE(T_1(G))$;
(ii) $rDSAE(S(G)) = (r-2)DSAE(T_2(G))$;
(iii) $rDSAE(T_1(G)) = (2r-2)DSAE(T_2(G))$.

Proposition 3.3. *Let G be an r-regular graph $(r \geq 3)$ on n vertices. Then*

$$DSAE(S(G)) < DSAE(T_2(G)) < DSAE(T_1(G)).$$

Proof. For $r \geq 3$, we see that

$$r-2 < r < 2r-2.$$

Hence by Proposition 3.2, this implies

$$DSAE(S(G)) < DSAE(T_2(G)) < DSAE(T_1(G)).$$

□

Competing Interests

The authors declare that they have no competing interests.

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