DEGREE SUBTRACTION ADJACENCY EIGENVALUES AND ENERGY OF GRAPHS OBTAINED FROM REGULAR GRAPHS

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Abstract. Let \( V(G) = \{v_1, v_2, \ldots, v_n \} \) be the vertex set of \( G \) and let \( d_G(v_i) \) be the degree of a vertex \( v_i \) in \( G \). The degree subtraction adjacency matrix of \( G \) is a square matrix \( DSA(G) = [d_{ij}] \), in which \( d_{ij} = d_G(v_i) - d_G(v_j) \), if \( v_i \) is adjacent to \( v_j \) and \( d_{ij} = 0 \), otherwise. In this paper we express the eigenvalues of the degree subtraction adjacency matrix of subdivision graph, semitotal point graph, semitotal line graph and total graph of a regular graph in terms of the adjacency eigenvalues of \( G \). Further we obtain the degree subtraction adjacency energy of these graphs.

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1. Introduction

Let \( G \) be a simple, undirected graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n \} \) and edge set \( E(G) = \{e_1, e_2, \ldots, e_m \} \). The degree of a vertex \( v_i \) denoted by \( d_G(v_i) \) is the number of edges incident to it. If all vertices have same degree equal to \( r \) then \( G \) is called an \( r \)-regular graph.

The adjacency matrix of \( G \) is a square matrix of order \( n \), defined as \( A = A(G) = [a_{ij}] \), where \( a_{ij} = 1 \) if \( v_i \) is adjacent to \( v_j \) and \( a_{ij} = 0 \), otherwise. The characteristic polynomial of \( A(G) \) is denoted by \( \phi(G : \lambda) \), that is, \( \phi(G : \lambda) = \det |\lambda I - A(G)| \), where \( I \) is an identity matrix. The characteristic polynomial of the adjacency matrix of a graph is a fundamental tool in graph theory.
matrix of a complete graph $K_n$ is $φ(K_n : λ) = (λ - n + 1)(λ + 1)^{n-1}$. The roots of the equation $φ(G : λ) = 0$ are called the adjacency eigenvalues of $G$ \[1\] and they are denoted by $λ_1, λ_2, \ldots, λ_n$. Two non-isomorphic graphs are said to be cospectral if they have same eigenvalues. For any graph $G$, $−Δ ≤ λ_i ≤ Δ$, where $Δ$ is the maximum degree. Thus for an $r$-regular graph, $λ_i + r ≥ 0$ for $i = 1, 2, \ldots, n$.

The vertex-edge incidence matrix of $G$ is defined as $B = B(G) = [b_{ij}]$, where $b_{ij} = 1$ if the vertex $v_i$ is incident to an edge $e_j$ and $b_{ij} = 0$, otherwise. It is easy to observe that \[1\]

\[BB^T = A + D,\]

where $D = \text{diag}[d_G(v_1), d_G(v_2), \ldots, d_G(v_n)]$ is a diagonal degree matrix of $G$ and $B^T$ is the transpose of $B$.

If $G$ is an $r$-regular graph, then

\[BB^T = A + rI.\]  \((1)\)

The other matrices of a graph exists in the literature such as distance matrix \[2\], Laplacian matrix \[3\], Laplacian distance matrix \[4\], sum-eccentricity matrix \[5, 6\], degree sum matrix \[7, 8\], degree sum adjacency matrix \[9\], Zagreb matrix \[10\], degree subtraction matrix \[11\], degree product matrix \[12\], degree square sum matrix \[13\] and average-degree eccentricity matrix \[14\].

In \[15\] the degree subtraction adjacency (DSA) matrix is defined as $DSA(G) = [d_{ij}]$, where

\[d_{ij} = \begin{cases} d_G(v_i) - d_G(v_j), & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise.} \end{cases}\]

The characteristic polynomial of $DSA(G)$ is called the DSA-polynomial and is denoted by $ψ(G : ξ)$. Thus $ψ(G : ξ) = \det(ξI - DSA(G))$, where $I$ is an identity matrix of order $n$.

For any regular graph of order $n$, $ψ(G : ξ) = ξ^n$. The line graph $L(G)$ of a regular graph is regular. Hence $ψ(L(G) : ξ) = ξ^m$, where $m$ is the number of edges of $G$.

The eigenvalues of $DSA(G)$, denoted by $ξ_1, ξ_2, \ldots, ξ_n$ are called DSA-eigenvalues of $G$. Two non-isomorphic graphs are said to be DSA-cospectral if they have same DSA-eigenvalues. Since $DSA(G)$ is a skew-symmetric matrix, its eigenvalues are purely imaginary or zero.

The DSA-energy of a graph $G$ is defined as

\[DSAE(G) = \sum_{i=1}^{n} |ξ_i|. \]  \((2)\)

The Eq. \((2)\) is analogous to the ordinary graph energy defined as \[16\]

\[E_A(G) = \sum_{i=1}^{n} |λ_i|.\]
where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of $G$. The ordinary graph energy is well studied by many researchers \[17\]. In \[15\] the DSA-polynomial and DSA-energy of a path, complete bipartite graph, wheel, windmill graph and corona graph have been obtained. In this paper we obtain the DSA-eigenvalues and DSA-energy of subdivision graph, semitotal point graph, semitotal line graph and of toal graph of regular graphs.

2. DSA-eigenvalues

A subdivision graph of $G$ is a graph $S(G)$ obtained from $G$ by inserting a new vertex on each edge of $G$ \[18\]. Thus if $G$ has $n$ vertices and $m$ edges, then $S(G)$ has $n + m$ vertices and $2m$ edges. If $u \in V(G)$ then $d_{S(G)}(u) = d_G(u)$ and if $v$ is subdivided vertex then $d_{S(G)}(v) = 2$.

Lemma 2.1. \[1\] If $M$ is a non-singular matrix, then we have

$$
\begin{vmatrix}
M & N \\
P & Q
\end{vmatrix} = |M||Q - PM^{-1}N|.
$$

Theorem 2.2. Let $G$ be an $r$-regular graph on $n$ vertices and $m$ edges. Then

$$
\psi(S(G) : \xi) = \begin{cases} 
\xi^{\frac{n}{2}}(\xi^2 + 2)^{\frac{n}{2}}, & \text{if } r = 1 \\
\xi^{2n}, & \text{if } r = 2 \\
(-1)^{n}(r-2)^{2n}\xi^{m-n}\phi(G : \frac{-\xi^2-(r-2)^2}{(r-2)^2}), & \text{if } r \geq 3.
\end{cases}
$$

Proof. (i) If $r = 1$, then $G$ is a union of $k \geq 1$ edges. Thus $G$ has $n = 2k$ vertices and $k$ edges. The vertices of $S(G)$ can be labeled in such a way that

$$
DSA(S(G)) = \begin{bmatrix}
O & B^T \\
-B & O
\end{bmatrix},
$$

where $B$ is vertex-edge incidence matrix of $G$ and $O$ is zero matrix. Therefore by Lemma 2.1 and Eq. (1)
= \frac{1}{\xi^2}(\xi^2 + 2)^2.

(ii) If \( r = 2 \), then each component of \( G \) is cycle. Therefore \( S(G) \) is 2-regular graph on \( 2n \) vertices. Hence

\[ \psi(S(G) : \xi) = \xi^{2n}. \]

(iii) Let \( r \geq 3 \). The vertices of \( S(G) \) can be labeled in such a way that

\[ DSA(S(G)) = \begin{bmatrix} O & (2-r)B^T \\ -(2-r)B & O \end{bmatrix}, \]

where \( B \) is vertex-edge incidence matrix of \( G \) and \( O \) is zero matrix. Therefore by Lemma 2.1 and Eq. (1)

\[
\begin{align*}
\psi(S(G) : \xi) &= \left| \begin{array}{cc}
\xi I_m & -(2-r)B^T \\
(2-r)B & \xi I_n
\end{array} \right| \\
&= \xi^m \left| \begin{array}{cc}
\xi I_n + (2-r)^2 B^T & -B I_m \\
-B & \xi I_n
\end{array} \right| \\
&= \xi^m \left| \xi I_n + (2-r)^2 B^T \right| \\
&= (-1)^n (r-2)2n \xi^{m-n} \left( G : \frac{-\xi^2 - r(r-2)^2}{(r-2)^2} \right) I_n - A \\
&= (-1)^n (r-2)2n \xi^{m-n} \phi \left( G : \frac{-\xi^2 - r(r-2)^2}{(r-2)^2} \right) \\
&= (-1)^n (r-2)2n \xi^{m-n} \phi \left( G : \frac{-\xi^2 - r(r-2)^2}{(r-2)^2} \right).
\]

By Theorem 2.2, we have following corollary.

**Corollary 2.3.** Let \( G \) be an \( r \)-regular graph on \( n \) vertices and \( m \) edges. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the adjacency eigenvalues of \( G \) and \( i = \sqrt{-1} \).

(i) If \( r = 1 \) then DSA-eigenvalues of \( S(G) \) are 0 (\( \frac{1}{\xi} \) times) and \( \pm i \sqrt{2} \) (\( \frac{1}{\xi} \) times).

(ii) If \( r = 2 \) then DSA-eigenvalues of \( S(G) \) are all zeros.

(iii) If \( r \geq 3 \) then DSA-eigenvalues of \( S(G) \) are 0 (\( m - n \) times) and \( \pm i(r-2)\sqrt{r+\lambda_i}, \ i = 1, 2, \ldots, n \).

The semitotal pont graph of \( G \), denoted by \( T_1(G) \), is a graph with vertex set \( V(G) \cup E(G) \) and two vertices in \( T_1(G) \) are adjacent if they are adjacent vertices in \( G \) or one is a vertex and other is an edge incident to it in \( G \).\[19]. Note that if \( u \in V(G) \) then \( d_{T_1(G)}(u) = 2d_G(u) \) and if \( e \in E(G) \) then \( d_{T_1(G)}(e) = 2 \).

**Theorem 2.4.** Let \( G \) be an \( r \)-regular graph on \( n \) vertices and \( m \) edges. Then

\[
\psi(T_1(G) : \xi) = \begin{cases} 
\xi^{m+n}, & \text{if } r = 1 \\
(-1)^n (2r-2)^2n \xi^{m-n} \phi \left( G : \frac{-\xi^2 - r(r-2)^2}{(2r-2)^2} \right), & \text{if } r \geq 2.
\end{cases}
\]
Proof. (i) If $r = 1$, then $T_1(G)$ is a regular graph of degree two on $m+n$ vertices. Hence

$$\psi(T_1(G) : \xi) = \xi^{m+n}.$$  

(ii) Let $r \geq 2$. The vertices of $T_1(G)$ can be labeled in such a way that

$$DSA(T_1(G)) = \begin{bmatrix} O & (2-2r)B^T \\ -(2-2r)B & O \end{bmatrix},$$

where $B$ is vertex-edge incidence matrix of $G$ and $O$ is a zero matrix.

Therefore by Lemma 2.1 and Eq. [1]

$$\psi(T_1(G) : \xi) = \begin{vmatrix} \xi I_m & -(2-2r)B^T \\ (2-2r)B & \xi I_n \end{vmatrix}$$

$$= \xi^m \begin{vmatrix} \xi I_n + (2r - 2)^2 B I_m B^T \xi \end{vmatrix}$$

$$= \xi^{m-n} \begin{vmatrix} \xi I_n + (2r - 2)^2 (A + rI) \end{vmatrix}$$

$$= (-1)^n (2r - 2)^{2n} \xi^{m-n} \begin{vmatrix} \frac{-\xi^2 - r(2r - 2)^2}{(2r - 2)^2} I_n - A \end{vmatrix}$$

$$= (-1)^n (2r - 2)^{2n} \xi^{m-n} \phi \left( G : \frac{-\xi^2 - r(2r - 2)^2}{(2r - 2)^2} \right).$$  

$\square$

By Theorem 2.4, we have following corollary.

**Corollary 2.5.** Let $G$ be an $r$-regular graph on $n$ vertices and $m$ edges. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of $G$ and $i = \sqrt{-1}$.

(i) If $r = 1$ then DSA-eigenvalues of $T_1(G)$ are all zeros.

(ii) If $r \geq 2$ then DSA-eigenvalues of $T_1(G)$ are $0$ ($m-n$ times) and $\pm i (2r - 2)^{2n} \lambda_i$, $i = 1, 2, \ldots, n$.

Semitotal line graph of $G$, denoted by $T_2(G)$, is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $T_2(G)$ are adjacent if one is a vertex and other is an edge incident to it in $G$ or both are edges adjacent in $G$ $[\Box]$. Note that if $u \in V(G)$ then $d_{T_2(G)}(u) = d_G(u)$ and if $e = uv \in E(G)$ then $d_{T_2(G)}(e) = d_G(u) + d_G(v)$.

**Theorem 2.6.** Let $G$ be an $r$-regular graph ($r \geq 1$) on $n$ vertices and $m$ edges. Then

$$\psi(T_2(G) : \xi) = (-1)^n r^{2n} \xi^{m-n} \phi \left( G : \frac{-\xi^2 - r^3}{r^2} \right).$$

Proof. The vertices of $T_2(G)$ can be labeled in such a way that

$$DSA(T_2(G)) = \begin{bmatrix} O & rB^T \\ -rB & O \end{bmatrix},$$

where $B$ is vertex-edge incidence matrix of $G$ and $O$ is a zero matrix.
Therefore by Lemma 2.1 and Eq. (1)

\[
\psi(T_2(G) : \xi) = \begin{vmatrix}
\xi I_m & -rB^T \\
rB & \xi I_n
\end{vmatrix} = \xi^m \begin{vmatrix}
\xi I_n + r^2 B I_n B^T \\
\xi
\end{vmatrix} = \xi^{m-n} \left| \xi^2 I_n + r^2(A + rI) \right| = (-1)^n r^{2n} \xi^{m-n} \left( \frac{-\xi^2 - r^3}{r^2} \right) I_n - A \\
= (-1)^n r^{2n} \xi^{m-n} \phi \left( G : \frac{-\xi^2 - r^3}{r^2} \right). \]

\[\square\]

By Theorem 2.6 we have following corollary.

**Corollary 2.7.** Let \( G \) be an \( r \)-regular graph \( (r \geq 1) \) on \( n \) vertices and \( m \) edges. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the adjacency eigenvalues of \( G \) and \( i = \sqrt{-1} \). Then DSA-eigenvalues of \( T_2(G) \) are 0 \( (m-n \) times) and \( \pm i \sqrt{r^2 + \lambda_i} \), \( i = 1, 2, \ldots, n \).

Total graph of \( G \), denoted by \( T(G) \), is a graph with vertex set \( V(G) \cup E(G) \) and two vertices in \( T(G) \) are adjacent if and only if they are adjacent vertices of \( G \) or adjacent edges of \( G \) or one is a vertex and other is an edge incident to it in \( G \) [18]. Total graph of a regular graph is regular. Hence if \( G \) is regular, then

\[
\psi(T(G) : \xi) = \xi^{m+n}.
\]

Two different graphs having same eigenvalues are called cospectral. If \( G_1 \) and \( G_2 \) are adjacency cospectral graphs with same regularity, then by Corollaries 2.3, 2.5 and 2.7 the graphs \( S(G_1) \) and \( S(G_2) \); \( T_1(G_1) \) and \( T_1(G_2) \); \( T_2(G_1) \) and \( T_2(G_2) \) form a pair of DSA-cospectral graphs.

3. DSA-energy

By Corollaries 2.3, 2.5 and 2.7 and by Eq. (2) we get the following proposition.

**Proposition 3.1.** Let \( G \) be an \( r \)-regular graph on \( n \) vertices and \( m \) edges. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the adjacency eigenvalues of \( G \). Then

(i)

\[
\text{DSA}(S(G)) = \begin{cases} 
 n \sqrt{2}, & \text{if } r = 1 \\
 0, & \text{if } r = 2 \\
 2(r - 2) \sum_{i=1}^{n} \sqrt{r + \lambda_i}, & \text{if } r \geq 3.
\end{cases}
\]
DSAE(T_1(G)) = \begin{cases} 0, & \text{if } r = 1 \\ 2(2r - 2) \sum_{i=1}^{n} \sqrt{r + \lambda_i}, & \text{if } r \geq 2. \end{cases}

(ii) DSAE(T_2(G)) = 2r \sum_{i=1}^{n} \sqrt{r + \lambda_i} \text{ for } r \geq 1.

(iv) DSAE(T(G)) = 0 \text{ for } r \geq 1.

By Proposition 3.1, we have the following result.

**Proposition 3.2.** If \( G \) is an \( r \)-regular graph \( (r \geq 3) \) on \( n \) vertices, then

(i) \( (2r - 2) \text{DSAE}(S(G)) = (r - 2) \text{DSAE}(T_1(G)) \);

(ii) \( r \text{DSAE}(S(G)) = (r - 2) \text{DSAE}(T_2(G)) \);

(iii) \( r \text{DSAE}(T_1(G)) = (2r - 2) \text{DSAE}(T_2(G)) \).

**Proposition 3.3.** Let \( G \) be an \( r \)-regular graph \( (r \geq 3) \) on \( n \) vertices. Then

\[ \text{DSAE}(S(G)) < \text{DSAE}(T_2(G)) < \text{DSAE}(T_1(G)). \]

**Proof.** For \( r \geq 3 \), we see that

\[ r - 2 < r < 2r - 2. \]

Hence by Proposition 3.2, this implies

\[ \text{DSAE}(S(G)) < \text{DSAE}(T_2(G)) < \text{DSAE}(T_1(G)). \]

\[ \square \]

**Competing Interests**

The authors declare that they have no competing interests.

**References**


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