PATH DECOMPOSITION NUMBER OF CERTAIN GRAPHS

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Abstract. Let $G$ be a simple, finite and connected graph. A graph is said to be decomposed into subgraphs $H_1$ and $H_2$ which is denoted by $G = H_1 \oplus H_2$, if $G$ is the edge disjoint union of $H_1$ and $H_2$. Assume that $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$ and if each $H_i$, $1 \leq i \leq k$, is a path or cycle in $G$, then the collection of edge-disjoint subgraphs of $G$ denoted by $\psi$ is called a path decomposition of $G$. If each $H_i$ is a path in $G$ then $\psi$ is called an acyclic path decomposition of $G$. The minimum cardinality of a path decomposition of $G$, denoted by $\pi(G)$, is called the path decomposition number and the minimum cardinality of an acyclic path decomposition of $G$, denoted by $\pi_a(G)$, is called the acyclic path decomposition number of $G$. In this paper, we determine path decomposition number for a number of graphs in particular, the Cartesian product of graphs. We also provided bounds for $\pi(G)$ and $\pi_a(G)$ for these graphs.

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1. Introduction

Let $P_m$, $C_m$, $K_m$, $K_m - I$, $K_{m,m} - I$ denote path of length $m$, cycle of length $m$, complete graph on $m$ vertices, complete graph on $m$ vertices minus a 1-factor and complete bipartite graph on $2m$ vertices minus a 1-factor respectively. All graphs considered in this paper are simple, finite and connected. We refer to the book [1] for graph theoretic terminology used in this article. A graph is said to be decomposed into subgraphs $H_1$ and $H_2$ which is denoted by $G = H_1 \oplus H_2$, if $G$ is the edge disjoint union of $H_1$ and $H_2$. Assume that $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$

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and if each $H_i$, $1 \leq i \leq k$, is a path or cycle in $G$, then the collection of edge-disjoint subgraphs of $G$ denoted by $\psi$ is called a path decomposition of $G$. If each $H_i$ is a path in $G$ then $\psi$ is called an acyclic path decomposition of $G$. The minimum cardinality of a path decomposition of $G$, denoted by $\pi(G)$, is called the path decomposition number and the minimum cardinality of an acyclic path decomposition of $G$, denoted by $\pi_a(G)$, is called the acyclic path decomposition number of $G$.

If $P = (x_1, x_2, ..., x_m)$ is a path in a graph $G$, then the vertices $x_2, x_3, ..., x_{m-1}$ are called the internal vertices of $P$ and $x_1, x_m$ are called external vertices of $P$. Here, by a first vertex and end vertex of path $P$ we mean the vertices $x_1$ and $x_m$ respectively. Let $P = (x_1, x_2, ..., x_m)$ and $Q = (y_1, y_2, ..., y_m)$ be two paths in $G$, by joining $x_1$ to $y_1$ ($x_m$ to $y_m$, respectively) we obtain the path $R = (y_m, y_{m-1}, ..., y_1, x_1, x_2, ..., x_m)$ ($R = (x_1, x_2, ..., x_m, y_m, y_{m-1}, ..., y_1)$, respectively).

1.1. Definition. The Cartesian product $G \square H$ of two graphs $G$ and $H$ is a graph with vertex set $V(G) \times V(H)$ in which $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if one of the following condition holds:

(i) $x_1 = x_2$ and $\{y_1, y_2\} \in E(H)$,
(ii) $y_1 = y_2$ and $\{x_1, x_2\} \in E(G)$.

The graphs $G$ and $H$ are known as the factors of $G \square H$.

Suppose we are dealing with $m$-copies of a graph $G$ we denote these $m$-copies of $G$ by $G^i$, where $i = 1, 2, 3, ..., m$.

The Cartesian product graph $G \square H$ may also be viewed as the graph obtained from $G$ by replacing each vertex $i \in V(G)$ by a copy $H^i$ (say) of $H$ and each of its edges $\{i, k\}$ with $|V(H)|$ edges joining corresponding vertices of $H^i$ and $H^k$. Henceforth, for any vertex $i \in V(G)$ we refer the copy of $H$, denoted by $H^i$, in $G \square H$ corresponding to the vertex $i$ as the $i^{th}$ copy of $H$ in $G \square H$.

The problem of finding $C_k$-decomposition of $K_{2n+1}$ or $K_{2n} - I$ where $I$ is a 1-factor of $K_{2n}$, is completely settled by Alspach, Gavlas and Sajna in two different papers (see [2, 3]). Obviously, every graph admits a decomposition in which each subgraph $H_i$ is either a path or a cycle. Gallai conjectured that the minimum number of paths into which every connected graph on $n$ vertices can be decomposed into is not less than $\left\lceil \frac{n}{2} \right\rceil$ (see [4]). A significant contribution to the parameter $\pi$ was by Lovasz [4] when he proved that a graph on $n$ vertices can be decomposed into $\left\lfloor \frac{n}{2} \right\rfloor$ paths and cycles. Harary introduced the parameter $\pi_a$, this was further studied by Harary and Schwenk in [5] when they considered the evolution number of the path number of a given graph. Staton et al. in [6, 7] provided further results on path numbers and considered the case of the tripartite graphs. Péroche [8] gave some results on the path numbers of certain multipartite graphs. Arumugam and Suseela [9] shed some lights on the acyclic path decomposition of unicyclic graphs. A recent work by Arumugam et al. [10] studied the parameter $\pi$ and further determined the value of $\pi$ for some graphs. They also provided some bounds for $\pi$ and characterize graphs attaining the
bounds. Furthermore, they proved that the difference between the parameter $\pi$ and $\pi_n$ can be arbitrary large.

In this paper, we determine the value of $\pi$ for the graph $K_n - I$, $K_{n,n} - I$ and the Cartesian products $P_m \square C_n$ and $C_m \square C_n$. In addition, we classify the graphs that attain some of the bounds mentioned in [10].

2. Path decomposition number of $K_n - I$ and $K_{n,n} - I$

Theorem 2.1. [2] For even integers $m$ and $n$ with $4 \leq m \leq n$, the graph $K_n - I$ decomposes cycles of length $m$ if and only if the number of edges in $K_n - I$ is a multiple of $m$.

Lemma 2.2. [11] Let $m \equiv 2(\text{mod} \ 4)$, $n \equiv 1(\text{mod} \ 2)$ and $6 \leq m \leq 2n$. Then $C_m|K_{n,n} - I$ if and only if $m|n(n-1)$.

Theorem 2.3. Given the graph $K_n - I$, where $n$ is even, the minimum path decomposition number for $K_n - I$ is $\frac{n-2}{2}$.

Proof. The graph $K_n - I$ has $n$ vertices and $\frac{n(n-2)}{2}$ edges. The largest cycle which is a subgraph of $K_n - I$ is a cycle of order $n$. Now, by Theorem 2.1, $C_n|K_n - I$. We only need to know the number of copies $C_n$ that can be gotten from $K_n - I$, which is $\frac{n-2}{2}$. Thus, we have $\frac{n-2}{2}$ copies of $C_n$ in $K_n - I$. Therefore, $\pi(K_n - I) = \frac{n-2}{2}$.

Lemma 2.4. If $n \geq 4$ and an even integer, then $K_{n,n} - I$ is a decomposable.

Proof. Let $X = \{1^1, 2^1, 3^1, ..., n^1\}$ and $Y = \{1^2, 2^2, 3^2, ..., n^2\}$ form the column set of vertices in $K_{n,n} - I$. Also, two vertices $a^i$ and $b^j$, has an edge in $K_{n,n} - I$, if $a \neq b$ and $i \neq j$, $i < j = 2$. Since $n$ is even, the degree of each vertex in $K_{n,n} - I$ is odd.

Next, remove the edges

$$E(a^i, b^j) = \begin{cases} (a, a - 1), & a = 1, 2 \\ (a, a - 2), & a = 3, 4, 5, ..., n \end{cases}, a^i \in X, b^j \in Y$$

which are exactly $n$ number of $P_2$’s. By removal of these edges, each vertex in $K_{n,n} - I$ would be of even degree. In total, we have $n(n-2)$ edges. At this point, we need to show that the subgraph $(K_{n,n} - I) \setminus E(a^i, b^j)$ admits a $C_{2n}$ decomposition.

Now, by $C^r_{2n}$, $r \leq 1$, we mean the $r^{th}$ copy of $C_{2n}$ in $(K_{n,n} - I) \setminus E(a^i, b^j)$. With exception of $C^1_{2n}$, all other $C^r_{2n}$, $r > 1$, follow a similar pattern. The construction of these cycles of order $2n$ is given below.

$$C^1_{2n} = 1^1, 2^1, 3^1, 4^1, ..., (n-1)^1, n^2, (n-2)^1, (n-3)^2, (n-4)^1, (n-5)^2, 2^1, 1^2, n^1, (n-1)^2, 1^1.$$

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For \( r = 2, 3, 4, \ldots, \frac{n-2}{2} \) we have that
\[
C_r^{n} = 1^1, (2r-1)^2, n^1, (2r-2)^2, (n-1^1, (2r-3)^2, (n-2)^1, \ldots, 1^2,
(n-2r+2)^1, (n-1)^2, (n-2r+1)^1, n^2, (n-2r)^1, (n-2)^2,
(n-2r-1)^1, (n-3)^2, (n-2r-2)^1, (n-4)^2, \ldots, (2r)^2, 1^1.
\]

From the above construction, we conclude that the graph \((K_{n,n} - I) \setminus E(a^i, b^j)\) admits a \(C_{2n}\) decomposition. Clearly, \( r = \frac{n-2}{2} \) and thus \( C_{2n} \{ K_{n,n} - I \setminus E(a^i, b^j) \} = (C_{2n} \oplus C_{2n} \oplus C_{2n} \oplus \cdots \oplus \frac{n-2}{2} C_{2n}). \) Finally, we have that \( K_{n,n} - I \) is \((\frac{n-2}{2} C_{2n}, nP_2)\)-decomposable. Hence the proof. \( \Box \)

**Theorem 2.5.** For the complete bipartite graph \( K_{n,n} - I \), we have that
\[
\pi(K_{n,n} - I) = \begin{cases} \frac{n-1}{3n-2}, & \text{if } n \text{ is odd} \\ \frac{n-2}{2}, & \text{otherwise} \end{cases}
\]

**Proof.** The graph \( K_{n,n} - I \) has \( 2n \) vertices and \( n(n-1) \) edges. The largest cycle which is a subgraph of \( K_{n,n} - I \) is a cycle of order \( 2n \). We now prove this theorem in two cases.

**Case 1:** when \( n \) is odd
By Lemma 2.2, \( C_{2n} \{ K_{n,n} - I \} \). We only need to know the number of copies of \( C_{2n} \) that can be obtained from \( K_{n,n} - I \), which is \( \frac{n-1}{2} \). Therefore, \( \pi(K_{n,n} - I) = \frac{n-1}{2} \).

**Case 2:** when \( n \) is even
By Lemma 2.4, the graph \( K_{n,n} - I \) can be decomposed into \( \frac{n-2}{2} \) copies of \( C_{2n} \) and \( n \) copies of \( P_2 \). Since no vertex is repeated in these \( n \) copies of \( P_2 \), we have that \( \pi(K_{n,n} - I) = \frac{3n-2}{2} \). The proof of this theorem is complete. \( \Box \)

To end this section we now give the following remark. This remark is immediate from Theorem 2.3 and Theorem 2.5

**Remark 2.6.** In [10], it was mentioned that every graph \( G \) which is Hamiltonian cycle decomposable attains the bound that \( \pi(G) \geq \lceil \frac{n}{2} \rceil \). This is true as we see from Theorem 2.3 and Theorem 2.5 that the complete graph minus a one-factor and the complete bipartite graph \( K_{n,n} - I \), where \( n \) is odd, attains this bound. Now, when \( n \) is even in \( K_{n,n} - I \) we have \( \pi(K_{n,n} - I) = \frac{3n+1}{2} \).

3. Path decomposition number of \( P_m \Box C_n \) and \( C_m \Box C_n \)

**Theorem 3.1.** Let \( m \) and \( n \) be positive integers then
\[
\pi(P_m \Box C_n) = \pi_n(P_m \Box C_n) = n.
\]

**Proof.** First we give the construction of \( P_m \) paths by constructing Hamilton paths of order \( n \) in each copy of \( C_n \) in \( P_m \Box C_n \). Let \( i \) be an odd number, in each copy of \( C_n^i \), join the end vertex of the Hamilton path in the \( i^{th} \) copy with the end vertex of the \( C_n^{i+1} \) copy of \( P_m \Box C_n \). Similarly, suppose \( i \) is even, in each copy of \( C_n^i \), join the first vertex of the Hamilton path in the \( i^{th} \) copy with the first vertex of the \( C_n^{i+1} \) copy of \( P_m \Box C_n \).
Lastly, the left out edges which has not been covered by the path $\pi C$. Now, notice that the left out edges which has not been covered by the cycle $n C$. Since the Cartesian product of graph is commutative, the result

\[ \pi(P_m \square C_n) = \pi_a(P_m \square C_n) = n. \]

**Remark 3.2.** Since the Cartesian product of graph is commutative, the result in Theorem 3.1 holds for the graph $C_m \square P_n$ where $m$ and $n$ are positive integers.

**Theorem 3.3.** Let $m$ and $n$ be positive integers such that $3 \leq n \leq m$, then $\pi(C_m \square C_n) = n$.

**Proof.** Since both $m$ and $n$ are positive integers, the proof of this theorem is split in two cases.

**Case 1:** when $m$ is even and $n \geq 3$.

First we give the construction of $C_{mn}$ cycles by constructing Hamilton paths of order $n$ in each copy of $C_n$ in $C_m \square C_n$. Let $i$ be an odd number, in each copy of $C_n^i$, join the end vertex of the Hamilton path in the $i^{th}$ copy with the end vertex of the $C_n^{i+1}$ copy of $C_m \square C_n$. Similarly, suppose $i$ is even, in each copy of $C_n^i$, join the first vertex of the Hamilton path in the $i^{th}$ copy with the first vertex of the $C_n^{i+1}$ copy of $C_m \square C_n$.

Next, for each internal vertex in the Hamilton path, join the vertices $x_j^i$ and $x_j^{i+1}$, $1 \leq i \leq m$, $i$ is calculated in modulo $m$ and $2 \leq j \leq n - 1$. By this, we have $n - 2$ copies of $C_m \square C_n$.

Now, notice that the left out edges which has not been covered by the cycle $C_{mn}$ and the $n - 2$ copies of $C_m$ form a cycle of order $2m$. So we have that $\pi(C_m \square C_n) = n$.

**Case 2:** when $m$ is odd and $n \geq 3$.

Here, we first give the construction of $C_{mn-1}$ cycles. For $1 \leq i \leq m-2$, construct Hamilton paths of order $n$ in each $C_n^i$ copy in $C_m \square C_n$. Suppose $i$ is odd, in each copy of $C_n^i$, join the end vertex of the Hamilton path in the $i^{th}$ copy with the end vertex of the $C_n^{i+1}$ copy of $C_m \square C_n$. In the same way, if $i$ is even, in each copy of $C_n^i$, join the first vertex of the Hamilton path in the $i^{th}$ copy of $C_n^i$ with the first vertex of the $C_n^{i+1}$ copy of $C_m \square C_n$. This gives a path of order $n(m-2)$.

Next, let $x$ be the first vertex in the $C_{n-1}^m$ copy of $C_m \square C_n$. Now, construct a path $P_{n-1}$ from $C_{n-1}^m \setminus x$. Join the end vertex of $C_{n-1}^m$ copy to the end vertex of $C_{n-1}^{m-2}$ copy of $C_m \square C_n$. Since $x$ is removed from $C_{n-1}^m$, join the second vertex $x_2^{m-1}$ of $C_{n-1}^{m-1}$ to the second vertex $x_1^m$ of $C_n^m$ and then move in a clockwise direction to the first vertex in the $m^{th}$ copy of $C_m \square C_n$. To get the desired $C_{mn-1}$ cycle, join the first vertex $x_1^m$ of $C_n^m$ to $x_1^1$ of $C_n^1$ in the graph $C_m \square C_n$.

Furthermore, aside the second vertex, each internal vertex $x_j^i$ and $x_j^{i+1}$, $1 \leq i \leq m$, $i$ is calculated in modulo $m$ and $3 \leq j \leq n - 1$ when joined in all other copies of $C_n$ results to $n - 3$ copies of $C_m$ in $C_m \square C_n$. 


The left out edges which have not been covered by the cycle $C_{mn-1}$ and the $n-3$ copies of $C_m$ form cycles $C_{m+2}$ and $C_{2m-1}$. We now give the construction of cycles $C_{m+2}$ and $C_{2m-1}$ as follows. By $x_j^i$ we mean the $j^{th}$ vertex of $C_n$ in copy $i$ of the graph $C_m \square C_n$.

$$C_{m+2} = x_1^1, x_2^1, x_2^2, ..., x_{m-1}^m, x_1^m, x_2, x_2^1.$$  
$$C_{2m-1} = x_1^n, x_1^1, x_2^n, x_2^1, x_3^n, x_3^1, ..., x_{m-1}^m, x_{m-1}^n, x_m, x_1^m.$$  

Therefore we have that $\pi(C_m \square C_n) = n$. This completes the proof. $\square$

We now conclude this section with the following remark.

**Remark 3.4.** We note here in this section that although Arumugam *et al.* in [10] gave a relationship between the path decomposition number (or acyclic path decomposition number, as the case maybe) and the maximum degree $\Delta$ of some graphs, we note that for the product $G \square H$ there is no such relationship since the parameters $\pi(G \square H)$ and $\pi_a(G \square H)$ do not depend on $\Delta(G \square H)$.

### 4. Conclusion and future work

So far in this work we have provided the path decomposition number for $K_n-I$, $K_{n,n}-I$ and the product $P_m \square C_n$ and $C_m \square C_n$. The question for determining the acyclic path decomposition number for these graphs certainly deserves attention. As a future work, we intend to provide the acyclic path decomposition number for these graphs and possibly look into other types of product graphs, e.g. lexicographic and tensor products.

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### Competing Interests

The authors declare that they have no competing interests.

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