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PATH DECOMPOSITION NUMBER OF CERTAIN GRAPHS

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Abstract. Let G be a simple, finite and connected graph. A graph is said to be *decomposed* into subgraphs H_1 and H_2 which is denoted by $G = H_1 \oplus H_2$, if G is the edge disjoint union of H_1 and H_2. Assume that $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$ and if each H_i, $1 \leq i \leq k$, is a path or cycle in G, then the collection of edge-disjoint subgraphs of G denoted by ψ is called a *path decomposition* of G. If each H_i is a path in G then ψ is called an *acyclic path decomposition* of G. The minimum cardinality of a path decomposition of G, denoted by $\pi(G)$, is called the *path decomposition number* and the minimum cardinality of an acyclic path decomposition of G, denoted by $\pi_a(G)$, is called the *acyclic path decomposition number* of G. In this paper, we determine path decomposition number for a number of graphs in particular, the Cartesian product of graphs. We also provided bounds for $\pi(G)$ and $\pi_a(G)$ for these graphs.

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1. Introduction

Let P_m, C_m, K_m, $K_m - I$, $K_{m,m} - I$ denote path of length m, cycle of length m, complete graph on m vertices, complete graph on m vertices minus a 1-factor and complete bipartite graph on $2m$ vertices minus a 1-factor respectively. All graphs considered in this paper are simple, finite and connected. We refer to the book [1] for graph theoretic terminology used in this article. A graph is said to be *decomposed* into subgraphs H_1 and H_2 which is denoted by $G = H_1 \oplus H_2$, if G is the edge disjoint union of H_1 and H_2. Assume that $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$

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and if each H_i, $1 \leq i \leq k$, is a path or cycle in G, then the collection of edge-disjoint subgraphs of G denoted by ψ is called a *path decomposition* of G. If each H_i is a path in G then ψ is called an *acyclic path decomposition* of G. The minimum cardinality of a path decomposition of G, denoted by $\pi(G)$, is called the *path decomposition number* and the minimum cardinality of an acyclic path decomposition of G, denoted by $\pi_a(G)$, is called the *acyclic path decomposition number* of G. If $P = (x_1, x_2, ..., x_m)$ is a path in a graph G, then the vertices $x_2, x_3, ..., x_{m-1}$ are called the *internal vertices* of P and x_1, x_m are called *external vertices* of P. Here, by a first vertex and end vertex of path P we mean the vertices x_1 and x_m respectively. Let $P = (x_1, x_2, ..., x_m)$ and $Q = (y_1, y_2, ..., y_m)$ be two paths in G, by *joining* x_1 to y_1 (x_m to y_m, respectively) we obtain the path $R = (y_m, y_{m-1}, ..., y_1, x_1, x_2, ..., x_m)$ $\big(R = (x_1, x_2, ..., x_m, y_m, y_{m-1}, ..., y_1)$, respectively$\big)$.

1.1. Definition. The Cartesian product $G \square H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$ in which (x_1, y_1) and (x_2, y_2) are adjacent if one of the following condition holds:
(i) $x_1 = x_2$ and $\{y_1, y_2\} \in E(H)$,
(ii) $y_1 = y_2$ and $\{x_1, x_2\} \in E(G)$.

The graphs G and H are known as the factors of $G \square H$.
Suppose we are dealing with m-copies of a graph G we denote these m-copies of G by G^i, where $i = 1, 2, 3, ..., m$.
The Cartesian product graph $G \square H$ may also be viewed as the graph obtained from G by replacing each vertex $i \in V(G)$ by a copy H^i (say) of H and each of its edges $\{i, k\}$ with $|V(H)|$ edges joining corresponding vertices of H^i and H^k. Henceforth, for any vertex $i \in V(G)$ we refer the copy of H, denoted by H^i, in $G \square H$ corresponding to the vertex i as the i^{th} copy of H in $G \square H$.
The problem of finding C_k-decomposition of K_{2n+1} or $K_{2n} - I$ where I is a 1-factor of K_{2n}, is completely settled by Alspach, Gavlas and Sajna in two different papers (see [2, 3]). Obviously, every graph admits a decomposition in which each subgraph H_i is either a path or a cycle. Gallai conjectured that the minimum number of paths into which every connected graph on n vertices can be decomposed into is not less than $\lceil \frac{n}{2} \rceil$ (see [4]). A significant contribution to the parameter π was by Lovasz [4] when he proved that a graph on n vertices can be decomposed into $\lfloor \frac{n}{2} \rfloor$ paths and cycles. Harary introduced the parameter π_a, this was further studied by Harary and Schwenk in [5] when they considered the evolution number of the path number of a given graph. Staton *et al.* in [6, 7] provided further results on path numbers and considered the case of the tripartite graphs. Péroche [8] gave some results on the path numbers of certain multipartite graphs. Arumugam and Suseela [9] shed some lights on the acyclic path decomposition of unicyclic graphs. A recent work by Arumugam *et al.* [10] studied the parameter π and further determined the value of π for some graphs. They also provided some bounds for π and characterize graphs attaining the

bounds. Furthermore, they proved that the difference between the parameter π and π_a can be arbitrary large.
In this paper, we determine the value of π for the graph $K_n - I$, $K_{n,n} - I$ and the Cartesian products $P_m \,\square\, C_n$ and $C_m \,\square\, C_n$. In addition, we classify the graphs that attain some of the bounds mentioned in [10].

2. Path decomposition number of $K_n - I$ and $K_{n,n} - I$

Theorem 2.1. [2] *For even integers m and n with $4 \leq m \leq n$, the graph $K_n - I$ decomposes cycles of length m if and only if the number of edges in $K_n - I$ is a multiple of m*

Lemma 2.2. [11] *Let $m \equiv 2(mod\ 4)$, $n \equiv 1(mod\ 2)$ and $6 \leq m \leq 2n$. Then $C_m | K_{n,n} - I$ if and only if $m|n(n-1)$.*

Theorem 2.3. *Given the graph $K_n - I$, where n is even, the minimum path decomposition number for $K_n - I$ is $\frac{n-2}{2}$.*

Proof. The graph $K_n - I$ has n vertices and $\frac{n(n-2)}{2}$ edges. The largest cycle which is a subgraph of $K_n - I$ is a cycle of order n. Now, by Theorem 2.1, $C_n | K_n - I$. We only need to know the number of copies C_n that can be gotten from $K_n - I$, which is $\frac{n-2}{2}$. Thus, we have $\frac{n-2}{2}$ copies of C_n in $K_n - I$. Therefore, $\pi(K_n - I) = \frac{n-2}{2}$. □

Lemma 2.4. *If $n \geq 4$ and an even integer, then $K_{n,n} - I$ is $\left(\frac{n-2}{2} C_{2n}, nP_2\right)$-decomposable.*

Proof. Let $X = \{1^1, 2^1, 3^1, ..., n^1\}$ and $Y = \{1^2, 2^2, 3^2, ..., n^2\}$ form the column set of vertices in $K_{n,n} - I$. Also, two vertices a^i and b^j, has an edge in $K_{n,n} - I$, if $a \neq b$ and $i \neq j$, $i < j = 2$. Since n is even, the degree of each vertex in $K_{n,n} - I$ is odd.
Next, remove the edges

$$E(a^i, b^j) = \begin{cases} (a, n-a+1), & a = 1, 2 \\ (a, a-2), & a = 3, 4, 5, ..., n \end{cases}, a^i \in X, b^j \in Y$$

which are exactly n number of P_2's. By removal of these edges, each vertex in $K_{n,n} - I$ would be of even degree. In total, we have $n(n-2)$ edges. At this point, we need to show that the subgraph $(K_{n,n} - I) \setminus E(a^i, b^j)$ admits a C_{2n} decomposition.
Now, by C_{2n}^r, $r \leq 1$, we mean the r^{th} copy of C_{2n} in $(K_{n,n} - I) \setminus E(a^i, b^j)$. With exception of C_{2n}^1, all other C_{2n}^r, $r > 1$, follow a similar pattern. The construction of these cycles of order $2n$ is given below.

$$C_{2n}^1 = 1^1, 2^2, 3^1, 4^2, ..., (n-1)^1, n^2, (n-2)^1, (n-3)^2, (n-4)^1, (n-5)^2, 2^1, 1^2, n^1, (n-1)^2, 1^1.$$

For $r = 2, 3, 4, ..., \frac{n-2}{2}$ we have that

$$\begin{aligned} C_{2n}^{r} = &1^1, (2r-1)^2, n^1, (2r-2)^2, (n-1)^1, (2r-3)^2, (n-2)^1, ..., 1^2, \\ &(n-2r+2)^1, (n-1)^2, (n-2r+1)^1, n^2, (n-2r)^1, (n-2)^2, \\ &(n-2r-1)^1, (n-3)^2, (n-2r-2)^1, (n-4)^2, ..., (2r)^2, 1^1. \end{aligned}$$

From the above construction, we conclude that the graph $(K_{n,n} - I) \setminus E(a^i, b^j)$ admits a C_{2n} decomposition. Clearly, $r = \frac{n-2}{2}$ and thus $C_{2n}|\{K_{n,n} - I \setminus E(a^i, b^j)\} = (C_{2n} \oplus C_{2n} \oplus C_{2n} \oplus \cdots \oplus \frac{n-2}{2}C_{2n})$. Finally, we have that $K_{n,n} - I$ is $\left(\frac{n-2}{2}C_{2n}, nP_2\right)$-decomposable. Hence the proof. □

Theorem 2.5. *For the complete bipartite graph $K_{n,n} - I$, we have that*

$$\pi(K_{n,n} - I) = \begin{cases} \frac{n-1}{2}, & if\ n\ is\ odd \\ \frac{3n-2}{2}, & otherwise \end{cases}$$

Proof. The graph $K_{n,n} - I$ has $2n$ vertices and $n(n-1)$ edges. The largest cycle which is a subgraph of $K_{n,n} - I$ is a cycle of order $2n$. We now prove this theorem in two cases.

Case 1: when n is odd

By Lemma 2.2, $C_{2n}|K_{n,n} - I$. We only need to know the number of copies of C_{2n} that can be obtained from $K_{n,n} - I$, which is $\frac{n-1}{2}$. Therefore, $\pi(K_{n,n} - I) = \frac{n-1}{2}$.

Case 2: when n is even

By Lemma 2.4, the graph $K_{n,n} - I$ can be decomposed into $\frac{n-2}{2}$ copies of C_{2n} and n copies of P_2. Since no vertex is repeated in these n copies of P_2, we have that $\pi(K_{n,n} - I) = \frac{3n-2}{2}$. The proof of this theorem is complete. □

To end this section we now give the following remark. This remark is immediate from Theorem 2.3 and Theorem 2.5.

Remark 2.6. In [10], it was mentioned that every graph G which is Hamiltonian cycle decomposable attains the bound that $\pi(G) \geq \lceil \frac{\Delta}{2} \rceil$. This is true as we see from Theorem 2.3 and Theorem 2.5 that the complete graph minus a one-factor and the complete bipartite graph $K_{n,n} - I$, where n is odd, attains this bound. Now, when n is even in $K_{n,n} - I$ we have $\pi(K_{n,n} - I) = \frac{3\Delta+1}{2}$.

3. Path decomposition number of $P_m \square C_n$ and $C_m \square C_n$

Theorem 3.1. *Let m and n be positive integers then*
$\pi(P_m \square C_n) = \pi_a(P_m \square C_n) = n$.

Proof. First we give the construction of P_{mn} paths by constructing Hamilton paths of order n in each copy of C_n in $P_m \square C_n$. Let i be an odd number, in each copy of C_n^i, join the end vertex of the Hamilton path in the i^{th} copy with the end vertex of the C_n^{i+1} copy of $P_m \square C_n$. Similarly, suppose i is even, in each copy of C_n^i, join the first vertex of the Hamilton path in the i^{th} copy with the first vertex of the C_n^{i+1} copy of $P_m \square C_n$.

Next, for each internal vertex in the Hamilton path, join the vertices x_j^i and x_j^{i+1}, $1 \leq i \leq m$, i is calculated in modulo m and $2 \leq j \leq n-1$. By this, we have $n-2$ copies of P_m in $P_m \,\square\, C_n$.
Lastly, the left out edges which has not been covered by the path P_{mn} and the $n-2$ copies of P_m form a path of order $2m$. So we have that $\pi(P_m \,\square\, C_n) = \pi_a(P_m \,\square\, C_n) = n$. □

Remark 3.2. Since the Cartesian product of graph is commutative, the result in Theorem 3.1 holds for the graph $C_m \,\square\, P_n$ where m and n are positive integers.

Theorem 3.3. *Let m and n be positive integers such that $3 \leq n \leq m$, then $\pi(C_m \,\square\, C_n) = n$.*

Proof. Since both m and n are positive integers, the proof of this theorem is split in two cases.
Case 1: when m is even and $n \geq 3$.
First we give the construction of C_{mn} cycles by constructing Hamilton paths of order n in each copy of C_n in $C_m \,\square\, C_n$. Let i be an odd number, in each copy of C_n^i, join the end vertex of the Hamilton path in the i^{th} copy with the end vertex of the C_n^{i+1} copy of $C_m \,\square\, C_n$. Similarly, suppose i is even, in each copy of C_n^i, join the first vertex of the Hamilton path in the i^{th} copy with the first vertex of the C_n^{i+1} copy of $C_m \,\square\, C_n$.
Next, for each internal vertex in the Hamilton path, join the vertices x_j^i and x_j^{i+1}, $1 \leq i \leq m$, i is calculated in modulo m and $2 \leq j \leq n-1$. By this, we have $n-2$ copies of $C_m \,\square\, C_n$.
Now, notice that the left out edges which has not been covered by the cycle C_{mn} and the $n-2$ copies of C_m form a cycle of order $2m$. So we have that $\pi(C_m \,\square\, C_n) = n$.
Case 2: when m is odd and $n \geq 3$.
Here, we first give the construction of C_{mn-1} cycles. For $1 \leq i \leq m-2$, construct Hamilton paths of order n in each C_n^i copy in $C_m \,\square\, C_n$. Suppose i is odd, in each copy of C_n^i join the end vertex of the Hamilton path in the i^{th} copy with the end vertex of the C_n^{i+1} copy of $C_m \,\square\, C_n$. In the same way, if i is even, in each copy of C_n^i join the first vertex of the Hamilton path in the i^{th} copy of C_n^i with the first vertex of the C_n^{i+1} copy of $C_m \,\square\, C_n$. This gives a path of order $n(m-2)$.
Next, let x be the first vertex in the C_n^{m-1} copy of $C_m \,\square\, C_n$. Now, construct a path P_{n-1} from $C_n^{m-1} \backslash x$. Join the end vertex of C_n^{m-1} copy to the end vertex of C_n^{m-2} copy of $C_m \,\square\, C_n$. Since x is removed from C_n^{m-1}, join the second vertex x_2^{m-1} of C_n^{m-1} to the second vertex x_2^m of C_n^m and then move in a clockwise direction to the first vertex in the m^{th} copy of $C_m \,\square\, C_n$. To get the desired C_{mn-1} cycle, join the first vertex x_1^m of C_n^m to x_1^1 of C_n^1 in the graph $C_m \,\square\, C_n$.
Furthermore, aside the second vertex, each internal vertex x_j^i and x_j^{i+1}, $1 \leq i \leq m$, i is calculated in modulo m and $3 \leq j \leq n-1$ when joined in all other copies of C_n results to $n-3$ copies of C_m in $C_m \,\square\, C_n$.

The left out edges which have not been covered by the cycle C_{mn-1} and the $n-3$ copies of C_m form cycles C_{m+2} and C_{2m-1}. We now give the construction of cycles C_{m+2} and C_{2m-1} as follows. By x_j^i we mean the j^{th} vertex of C_n in copy i of the graph $C_m \square C_n$.

$$C_{m+2} = x_2^1, x_2^2, x_2^3, ..., x_2^{m-1}, x_1^{m-1}, x_1^m, x_2^m, x_2^1.$$
$$C_{2m-1} = x_n^1, x_1^1, x_1^2, x_n^2, x_n^3, x_1^3, ..., x_1^{m-1}, x_n^{m-1}, x_n^m, x_n^1.$$

Therefore we have that $\pi(C_m \square C_n) = n$. This completes the proof. □

We now conclude this section with the following remark.

Remark 3.4. We note here in this section that although Arumugam *et al.* in [10] gave a relationship between the path decomposition number (or acyclic path decomposition number, as the case maybe) and the maximum degree Δ of some graphs, we note that for the product $G \square H$ there is no such relationship since the parameters $\pi(G \square H)$ and $\pi_a(G \square H)$ do not depend on $\Delta(G \square H)$.

4. Conclusion and future work

So far in this work we have provided the path decomposition number for $K_n - I$, $K_{n,n} - I$ and the product $P_m \square C_n$ and $C_m \square C_n$. The question for determining the acyclic path decomposition number for these graphs certainly deserves attention. As a future work, we intend to provide the acyclic path decomposition number for these graphs and possibly look into other types of product graphs, e.g. lexicographic and tensor products.

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Competing Interests

The authors declare that they have no competing interests.

References

1. Chartrand, G., Lesniak, L., & Zhang, P. (2010). *Graphs & digraphs*. Chapman and Hall/CRC.
2. Alspach, B., & Gavlas, H. (2001). Cycle decompositions of K_n and K_{n-I}. *Journal of Combinatorial Theory, Series B*, 81(1), 77-99.
3. Šajna, M. (2002). Cycle decompositions III: complete graphs and fixed length cycles. *Journal of Combinatorial Designs*, 10(1), 27-78.
4. Lovasz, L. (1968). On covering of graphs. *In Theory of Graphs, P. Erdős and G. Katona, Eds., Procedure Collage, Academic Press, Tihany, Hungary*, 231-236.
5. Harary, F., & Schwenk, A. J. (1972). Evolution of the path number of a graph: Covering and packing in graphs, II. *In Graph theory and computing*, 39-45.

6. Stanton, R. G., Cowan, D. D. & James, L. O. (1970). Some results on path numbers. *In Proceedings of the Louisiana Conference on Combinatorics, Graph Theory and computing*, 112-135.
7. Stanton, R. G., James, L. O., & Cowan, D. D. (1972). Tripartite path numbers. *In Graph theory and computing*, 285-294.
8. Peroche, B. (2011). The path-numbers of some multipartite graphs. *Combinatorics,* 79, 195-197.
9. Arumugam, S., & Suseela, J. S. (1998). Acyclic graphoidal covers and path partitions in a graph. *Discrete Mathematics*, 190(1-3), 67-77.
10. Arumugam, S., Hamid, I. S. & Abraham, V. M. (2013). Decomposition of graphs into path and cycles. *Journal of Discrete Mathematics*, ID 721051, 1-6.
11. Archdeacon, D., Debowsky, M., Dinitz, J. & Gavlas, H. (2004). Cycle system in the complete bipartite graph minus a one-factor. *Discrete Mathematics*, 284, 37-43.

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