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Reinterpreting the middle-levels theorem via natural enumeration of ordered trees

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Abstract: Let $0 < k \in \mathbb{Z}$. A reinterpretation of the proof of existence of Hamilton cycles in the middle-levels graph M_k induced by the vertices of the $(2k+1)$-cube representing the k- and $(k+1)$-subsets of $\{0,\ldots,2k\}$ is given via an associated dihedral quotient graph of M_k whose vertices represent the ordered (rooted) trees of order $k+1$ and size k.

Keywords: Middle-levels graph, Hamilton cycle, dihedral quotient graph, ordered tree.

MSC: 05C62, 05C75, 06A05.

1. Introduction

Let $0 < k \in \mathbb{Z}$. The *middle-levels graph* M_k [1] is the subgraph induced by the k-th and $(k+1)$-th levels (formed by the k- and $(k+1)$-subsets of $[2k+1] = \{0,\ldots,2k\}$) in the Hasse diagram [2] of the Boolean lattice $2^{[2k+1]}$. The dihedral group D_{4k+2} acts on M_k via translations mod $2k+1$ (Section 4) and complemented reversals (Section 5). The sequence $\mathcal{S}$ [3] A239903 of *restricted-growth strings* or *RGS's* ([4] page 325, [5] page 224 item (u)) will be shown to unify the presentation of all M_k's. In fact, the first C_k terms of $\mathcal{S}$ will stand for the orbits of $V(M_k)$ under natural D_{4k+2}-action, where $C_k = \frac{(2k)!}{k!(k+1)!}$ is the k-th Catalan number [3] A000108. This will provide a reinterpretation (Section 10) of the Middle-Levels Theorem on the existence of Hamilton cycles in M_k [6,7] via *k-germs* (Section 2) of RGS's. For the history of this theorem, [7] may be consulted, but the conjecture answered by it was learned from B. Bollobás on January 23, 1983, who mentioned it as a P. Erdös conjecture.

In Section 6, the cited D_{4k+2}-action on $M_k = (V(M_k), E(M_k))$ allows to project M_k onto a quotient graph R_k whose vertices stand for the first C_k terms of $\mathcal{S}$ via the lexical-matching colors [1] (or *lexical colors*) $0,1,\ldots,k$ on the $k+1$ edges incident to each vertex (Section 12). In preparation, RGS's α are converted in Section 3 into $(2k+1)$-strings $F(\alpha)$, composed by the $k+1$ lexical colors and k asterisks, representing all ordered (rooted) k-edge trees (Remark 1) via a "Castling" procedure that facilitates enumeration. These trees (encoded as $F(\alpha)$) are shown to represent the vertices of R_k via "Uncastling" procedure (Section 8).

In Section 9, the 2-factor W^k_{01} of R_k determined by the colors 0 and 1 is analyzed from the RGS-dihedral action viewpoint. From this, W^k_{01} is seen in Section 10 to morph into Hamilton cycles of M_k via symmetric differences with 6-cycles whose presentation is alternate to that of [6]. In particular, an integer sequence $\mathcal{S}_0$ is shown to exist such that, for each integer $k > 0$, the neighbors via color k of the RGS's in R_k ordered as in $\mathcal{S}$ correspond to an idempotent permutation on the first C_k terms of $\mathcal{S}_0$. This and related properties hold for colors $0,1,\ldots,k$ (Theorem 10 and Remarks 2-4) in part reflecting and extending from Observation 2 in Section 9 properties of plane trees (i.e., classes of ordered trees under root rotation). Moreover, Section 11 considers suggestive symmetry properties by reversing the designation of the roots in the ordered trees so that each lexical-color i adjacency ($0 \le i \le k$) can be seen from the lexical-color $(k-i)$ viewpoint.

2. Restricted-Growth Strings and k-Germs

Let $0 < k \in \mathbb{Z}$. The sequence of (pairwise different) RGS's $\mathcal{S} = (\beta(0),\ldots,\beta(17),\ldots) = (0,1,10,11,12,100,101,110,111,112,120,121,122,123,1000,1001,1010,1011,\ldots)$ has the lengths of its contiguous pairs $(\beta(i-1),\beta(i))$ constant unless $i = C_k$ for $0 < k \in \mathbb{Z}$, in which case $\beta(i-1) = \beta(C_k - 1) = 12\cdots k$ and $\beta(i) = \beta(C_k) = 10^k = 10\cdots 0$.

To view the continuation of the sequence $\mathcal{S}$, each RGS $\beta = \beta(m)$ will be transformed, for every $k \in \mathbb{Z}$ such that $k \geq$ length(β), into a $(k-1)$-string $\alpha = a_{k-1}a_{k-2}\cdots a_2a_1$ by prefixing $k-$ length(β) zeros to β. We say such α is a *k-germ*. More generally, a *k-germ* α $(1 < k \in \mathbb{Z})$ is formally defined to be a $(k-1)$-string $\alpha = a_{k-1}a_{k-2}\cdots a_2a_1$ such that:

(1) the leftmost position (called position $k-1$) of α contains the entry $a_{k-1} \in \{0,1\}$;
(2) given $1 < i < k$, the entry a_{i-1} (at position $i-1$) satisfies $0 \leq a_{i-1} \leq a_i + 1$.

Every k-germ $a_{k-1}a_{k-2}\cdots a_2a_1$ yields the $(k+1)$-germ $0a_{k-1}a_{k-2}\cdots\ a_2a_1$. A *non-null RGS* is obtained by stripping a k-germ $\alpha = a_{k-1}a_{k-2}\cdots a_1 \neq 00\cdots 0$ off all the null entries to the left of its leftmost position containing a 1. Such a non-null RGS is again denoted α. We also say that the *null RGS* $\alpha = 0$ corresponds to every null k-germ α, for $0 < k \in \mathbb{Z}$. (We use the same notations $\alpha = \alpha(m)$ and $\beta = \beta(m)$ to denote both a k-germ and its associated RGS).

The k-germs are ordered as follows. Given 2 k-germs, say $\alpha = a_{k-1}\cdots a_2a_1$ and $\beta = b_{k-1}\cdots b_2b_1$, where $\alpha \neq \beta$, we say that α precedes β, written $\alpha < \beta$, whenever either:

(i) $a_{k-1} < b_{k-1}$ or
(ii) $a_j = b_j$, for $k-1 \leq j \leq i+1$, and $a_i < b_i$, for some $k-1 > i \geq 1$.

The resulting order on k-germs $\alpha(m)$, $(m \leq C_k)$, corresponding biunivocally (via the assignment $m\ \ \alpha(m)$) with the natural order on m, yields a listing that we call the *natural (k-germ) enumeration*. Note that there are exactly C_k k-germs $\alpha = \alpha(m) < 10^k$, $\forall k > 0$.

3. Castling of ordered trees

Table 1. The Castling Procedure for $k = 2, 3, 4$.

<table>
<tr><th>m</th><th>α</th><th>β</th><th>$F(\beta)$</th><th>i</th><th>$W^i \mid X \mid Y \mid Z^i$</th><th>$W^i \mid Y \mid X \mid Z^i$</th><th>$F(\alpha)$</th><th>$\alpha$</th></tr>
<tr><td>0</td><td>0</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>012**</td><td>0</td></tr>
<tr><td>1</td><td>1</td><td>0</td><td>012**</td><td>1</td><td>0 | 1 | 2* | *</td><td>0 | 2* | 1 | *</td><td>02*1*</td><td>1</td></tr>
<tr><td>0</td><td>00</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>012 3** *</td><td>00</td></tr>
<tr><td>1</td><td>01</td><td>00</td><td>0123***</td><td>1</td><td>0 | 1 | 23** | *</td><td>0 | 23** | 1 | *</td><td>023**1*</td><td>01</td></tr>
<tr><td>2</td><td>10</td><td>00</td><td>0123***</td><td>2</td><td>01 | 2 | 3* | **</td><td>01 | 3* | 2 | **</td><td>013*2**</td><td>10</td></tr>
<tr><td>3</td><td>11</td><td>10</td><td>013*2**</td><td>1</td><td>0 | 13* | 2* | *</td><td>0 | 2* | 13* | *</td><td>02*13**</td><td>11</td></tr>
<tr><td>4</td><td>12</td><td>11</td><td>02*13**</td><td>1</td><td>0 | 2*1 | 3* | *</td><td>0 | 3* | 2*3 | *</td><td>03*2*1*</td><td>12</td></tr>
<tr><td>0</td><td>000</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>01234****</td><td>000</td></tr>
<tr><td>1</td><td>001</td><td>000</td><td>01234****</td><td>1</td><td>0 | 1 | 234*** | *</td><td>0 | 234*** | 1 | *</td><td>0234***1*</td><td>001</td></tr>
<tr><td>2</td><td>010</td><td>000</td><td>01234****</td><td>2</td><td>01 | 2 | 34** | **</td><td>01 | 34** | 2 | **</td><td>0134**2**</td><td>010</td></tr>
<tr><td>3</td><td>011</td><td>010</td><td>0134**2**</td><td>1</td><td>0 | 134** | 2* | *</td><td>0 | 2* | 134** | *</td><td>02*134***</td><td>011</td></tr>
<tr><td>4</td><td>012</td><td>011</td><td>02*134***</td><td>1</td><td>0 | 2*1 | 34** | *</td><td>0 | 34** | 2*1 | *</td><td>034**2*1*</td><td>012</td></tr>
<tr><td>5</td><td>100</td><td>000</td><td>01234****</td><td>3</td><td>012 | 3 | 4* | ***</td><td>012 | 4* | 3 | ***</td><td>0124*3***</td><td>100</td></tr>
<tr><td>6</td><td>101</td><td>100</td><td>0124*3***</td><td>1</td><td>0 | 1 | 24*3** | *</td><td>0 | 24*3** | 1 | *</td><td>024*3**1*</td><td>101</td></tr>
<tr><td>7</td><td>110</td><td>100</td><td>0124*3***</td><td>2</td><td>01 | 24* | 3* | **</td><td>01 | 3* | 24* | **</td><td>013*24***</td><td>110</td></tr>
<tr><td>8</td><td>111</td><td>110</td><td>013*24***</td><td>1</td><td>0 | 13* | 24** | *</td><td>0 | 24** | 13* | *</td><td>024**13**</td><td>111</td></tr>
<tr><td>9</td><td>112</td><td>111</td><td>024**13**</td><td>1</td><td>0 | 24**1 | 3* | *</td><td>0 | 3* | 24**1 | *</td><td>03*24**1*</td><td>112</td></tr>
<tr><td>10</td><td>120</td><td>110</td><td>013*24***</td><td>2</td><td>01 | 3*2 | 4* | **</td><td>01 | 4* | 3*2 | **</td><td>014*3*2**</td><td>120</td></tr>
<tr><td>11</td><td>121</td><td>120</td><td>014*3*2**</td><td>1</td><td>0 | 14*3* | 2* | *</td><td>0 | 2* | 14*3* | *</td><td>02*14*3**</td><td>121</td></tr>
<tr><td>12</td><td>122</td><td>121</td><td>02*14*3**</td><td>1</td><td>0 | 2*34* | 3* | *</td><td>0 | 3* | 2*14* | *</td><td>03*2*14**</td><td>122</td></tr>
<tr><td>13</td><td>123</td><td>122</td><td>03*2*14**</td><td>1</td><td>0 | 3*2*1 | 4* | *</td><td>0 | 4* | 3*2*1 | *</td><td>04*3*2*1*</td><td>123</td></tr>
</table>

Theorem 1. *To each k-germ* $\alpha = a_{k-1}\cdots a_1 \neq 0^{k-1}$ *with rightmost entry* $a_i \neq 0$ $(k > i \geq 1)$ *corresponds a k-germ* $\beta(\alpha) = b_{k-1}\cdots b_1 < \alpha$ *with* $b_i = a_i - 1$ *and* $a_j = b_j$, $(j \neq i)$. *All k-germs form an ordered tree* $\mathcal{T}_k$ *rooted at* 0^{k-1}, *each k-germ* $\alpha \neq 0^{k-1}$ *with* $\beta(\alpha)$ *as parent.*

Proof. The statement, illustrated for $k = 2, 3, 4$ by means of the first 3 columns of Table 1, is straightforward. Table 1 also serves as illustration to the proof of Theorem 2, below. □

By representing $\mathcal{T}_k$ with each node β having its children α enclosed between parentheses following β and separating siblings with commas, we can write:

$$\mathcal{T}_4 = 000(001, 010(011(012)), 100(101, 110(111(121)), 120(121(122(123))))).$$

Theorem 2. *To each k-germ $\alpha = a_{k-1}\cdots a_1$ corresponds a $(2k+1)$-string $F(\alpha) = f_0 f_1 \cdots f_{2k}$ whose entries are both the numbers $0, 1, \ldots, k$ (once each) and k asterisks $(*)$ and such that:*

(A) $F(0^{k-1}) = 012\cdots(k-2)(k-1)k * \cdots *$;
(B) if $\alpha \neq 0^{k-1}$, then $F(\alpha)$ is obtained from $F(\beta) = F(\beta(\alpha)) = h_0 h_1 \cdots h_{2k}$ by means of the following "Castling Procedure" steps:

1. *let $W^i = h_0 h_1 \cdots h_{i-1} = f_0 f_1 \cdots f_{i-1}$ and $Z^i = h_{2k-i+1} \cdots h_{2k-1} h_{2k} = f_{2k-i+1} \cdots f_{2k-1} f_{2k}$ be respectively the initial and terminal substrings of length i in $F(\beta)$;*
2. *let $\Omega > 0$ be the leftmost entry of the substring $U = F(\beta) \setminus (W^i \cup Z^i)$ and consider the concatenation $U = X|Y$, with Y starting at entry $\Omega + 1$; then, $F(\beta) = W^i|X|Y|Z^i$;*
3. *set $F(\alpha) = W^i|Y|X|Z^i$.*

In particular:

(a) the leftmost entry of each $F(\alpha)$ is 0; $k*$ is a substring of $F(\alpha)$, but $*k$ is not;
(b) a number to the immediate right of any $b \in [0, k)$ in $F(\alpha)$ is larger than b;
(c) W^i is an (ascending) number i-substring and Z^i is formed by i of the k asterisks.

Proof. Let $\alpha = a_{k-1}\cdots a_1 \neq 0^{k-1}$ be a k-germ. In the sequence of applications of 1-3 along the path from root 0^{k-1} to α in $\mathcal{T}_k$, unit augmentation of a_i for larger values of i, $(0 < i < k)$, must occur earlier, and then in strictly descending order of the entries i of the intermediate k-germs. As a result, the length of the inner substring $X|Y$ is kept non-decreasing after each application. This is illustrated in Table 1, where the order of presentation of X and Y is reversed in successively decreasing steps. In the process, (a)-(c) are seen to be fulfilled.

The 3 successive subtables in Table 1 have C_k rows each, where $C_2 = 2$, $C_3 = 5$ and $C_4 = 14$; in the subtables, the k-germs α are shown both on the second and last columns via natural enumeration in the first column; the images $F(\alpha)$ of those α are on the penultimate column; the remaining columns in the table are filled, from the second row on, as follows:

(i) $\beta = \beta(\alpha)$, arising in Theorem 1;
(ii) $F(\beta)$, taken from the penultimate column in the previous row;
(iii) the length i of W^i and Z^i $(k-1 \geq i \geq 1)$;
(iv) the decomposition $W^i|Y|X|Z^i$ of $F(\beta)$;
(v) the decomposition $W^i|X|Y|Z^i$ of $F(\alpha)$, re-concatenated in the following, penultimate, column as $F(\alpha)$, with $\alpha = F^{-1}(F(\alpha))$ in the last column.

□

Remark 1. As in the case of $\mathcal{T}_k$ in Theorem 1, an *ordered (rooted) tree* [6] is a tree T with:

(a) a specified node v_0 as the root of T;
(b) an embedding of T into the plane with v_0 on top;
(c) the edges between the j- and $(j+1)$-levels of T (formed by the nodes at distance j and $(j+1)$ from v_0, where $0 \leq j < \text{height}(T)$) having (parent) nodes at the j-level above their children at the $(j+1)$-level;
(d) the children in (c) ordered in a left-to-right fashion.

Each k-edge ordered tree T is both represented by a k-germ α and by its associated $(2k+1)$-string $F(\alpha)$, so we write $T = T_\alpha$. In fact, we perform a depth first search (DFS, or DFS) on T with its vertices from v_0 downward denoted as v_i $(i = 0, 1, \ldots, k)$ in a right-to-left breadth-first search (BFS) way. Such DFS yields the claimed $F(\alpha)$ by writing successively:

(i) the subindex i of each v_i in the DFS downward appearance and
(ii) an asterisk for the edge e_i ending at each child v_i in the DFS upward appearance.

Then, we write: $F(T_\alpha) = F(\alpha)$. Now, Theorem 1 can be taken as a *tree-surgery transformation* from T_β onto T_α for each k-germ $\alpha \neq 0^{k-1}$ via the vertices v_i and edges e_i (whose parent vertices are generally reattached in different ways). This remark is used in Sections 9-11 in reinterpreting the Middle-Levels Theorem. (In Section 11, an alternate viewpoint on ordered trees taking a_k as the root instead of a_0 is considered).

To each $F(\alpha)$ corresponds a binary n-string $\theta(\alpha)$ of weight k obtained by replacing each number in $[k] = \{0, 1, \ldots, k-1\}$ by 0 and each asterisk $*$ by 1. By attaching the entries of $F(\alpha)$ as subscripts to the corresponding entries of $\theta(\alpha)$, a subscripted binary n-string $\hat{\theta}(\alpha)$ is obtained, as shown for $k = 2, 3$ in the 4th column of Table 2. Let $\aleph(\theta(\alpha))$ be given by the *complemented reversal* of $\theta(\alpha)$, that is:

$$\text{if } \theta(\alpha) = a_0 a_1 \cdots a_{2k}, \text{ then } \aleph(\theta(\alpha)) = \bar{a}_{2k} \cdots \bar{a}_1 \bar{a}_0, \tag{1}$$

where $\bar{0} = 1$ and $\bar{1} = 0$. A subscripted version $\hat{\aleph}$ of $\aleph$ is obtained for $\hat{\theta}(\alpha)$, as shown in the fifth column of Table 2, with the subscripts of $\hat{\aleph}$ reversed with respect to those of $\aleph$. Each image of a k-germ α under $\aleph$ is an n-string of weight $k+1$ and has the 1's indexed with numeric subscripts and the 0's indexed with the asterisk subscript. The number subscripts reappear from Section 7 on as lexical colors [1] for the graphs M_k.

Table 2. Subscripted binary n-strings $\hat{\theta}(\alpha)$ and complemented reversals via $\aleph$ for $k = 2, 3$.

m	α	$\theta(\alpha)$	$\hat{\theta}(\alpha)$	$\hat{\aleph}(\theta(\alpha)) = \aleph(\hat{\theta}(\alpha))$	$\aleph(\theta(\alpha))$
0	0	00011	$0_0 0_1 0_2 1_* 1_*$	$0_* 0_* 1_2 1_1 1_0$	00111
1	1	00101	$0_0 0_2 1_* 0_1 1_*$	$0_* 1_1 0_* 1_2 1_0$	01011
0	00	0000111	$0_0 0_1 0_2 0_3 1_* 1_* 1_*$	$0_* 0_* 0_* 1_3 1_2 1_1 1_0$	0001111
1	01	0001101	$0_0 0_2 0_3 1_* 1_* 0_1 1_*$	$0_* 1_1 0_* 0_* 1_3 1_2 1_0$	0100111
2	10	0001011	$0_0 0_1 0_3 2_* 0_1 1_* 1_*$	$0_* 0_* 1_2 0_* 1_3 1_1 1_0$	0010111
3	11	0010011	$0_0 0_2 1_* 0_1 0_3 1_* 1_*$	$0_* 0_* 1_3 1_1 0_* 1_2 1_0$	0011011
4	12	0010101	$0_0 0_3 1_* 0_2 1_* 0_1 1_*$	$0_* 1_1 0_* 1_2 0_* 1_3 1_0$	0101011

4. Translations mod $n = 2k + 1$

Let $n = 2k + 1$. The *n-cube graph* H_n is the Hasse diagram of the Boolean lattice $2^{[n]}$ on the set $[n] = \{0, \ldots, n-1\}$. It is convenient to express each vertex v of H_n in 3 different equivalent ways:

(a) ordered set $A = \{a_0, a_1, \ldots, a_{j-1}\} = a_0 a_1 \cdots a_{j-1} \subseteq [n]$ that v represents, $(0 < j \leq n)$;
(b) characteristic binary n-vector $B_A = (b_0, b_1, \ldots, b_{n-1})$ of ordered set A in (a) above, where $b_i = 1$ if and only if $i \in A$, $(i \in [n])$;
(c) polynomial $\epsilon_A(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}$ associated to B_A in (b) above.

Ordered set A and vector B_A in (a) and (b) respectively are written for short as $a_0 a_1 \cdots a_{j-1}$ and $b_0 b_1 \cdots b_{n-1}$. A is said to be the *support* of B_A.

For each $j \in [n]$, let $L_j = \{A \subseteq [n] \text{ with } |A| = j\}$ be the *j-level* of H_n. Then, M_k is the subgraph of H_n induced by $L_k \cup L_{k+1}$, for $1 \leq k \in \mathbb{Z}$. By viewing the elements of $V(M_k) = L_k \cup L_{k+1}$ as polynomials, as in (c) above, a regular (i.e., free and transitive) *translation mod n* action Υ' of $\mathbb{Z}_n$ on $V(M_k)$ is seen to exist, given by:

$$\Upsilon' : \mathbb{Z}_n \times V(M_k) \quad V(M_k), \text{ with } \Upsilon'(i, v) = v(x) x^i \ (\text{mod } 1 + x^n), \tag{2}$$

where $v \in V(M_k)$ and $i \in \mathbb{Z}_n$. Now, Υ' yields a quotient graph M_k/π of M_k, where π stands for the equivalence relation on $V(M_k)$ given by:

$$\epsilon_A(x) \pi \epsilon_{A'}(x) \Leftarrow \exists i \in \mathbb{Z} \text{ with } \epsilon_{A'}(x) \equiv x^i \epsilon_A(x) \ (\text{mod } 1 + x^n),$$

with $A, A' \in V(M_k)$. This is to be used in the proofs of Theorems 4 and 8. Clearly, M_k/π is the graph whose vertices are the equivalence classes of $V(M_k)$ under π. Also, π induces a partition of $E(M_k)$ into equivalence classes, to be taken as the edges of M_k/π.

5. Complemented reversals

Let $(b_0b_1\cdots b_{n-1})$ denote the class of $b_0b_1\cdots b_{n-1} \in L_i$ in L_i/π. Let $\rho_i : L_i \quad L_i/\pi$ be the canonical projection given by assigning $b_0b_1\cdots b_{n-1}$ to $(b_0b_1\cdots b_{n-1})$, for $i \in \{k, k+1\}$. The definition of $\aleph$ in display (1) is easily extended to a bijection, again denoted $\aleph$, from L_k onto L_{k+1}. Let $\aleph_\pi : L_k/\pi \quad L_{k+1}/\pi$ be given by $\aleph_\pi((b_0b_1\cdots b_{n-1})) = (\bar{b}_{n-1}\cdots\bar{b}_1\bar{b}_0)$. Observe $\aleph_\pi$ is a bijection. Notice the commutative identities $\rho_{k+1}\aleph = \aleph_\pi\rho_k$ and $\rho_k\aleph^{-1} = \aleph_\pi^{-1}\rho_{k+1}$.

The following geometric representations will be handy. List vertically the vertex parts L_k and L_{k+1} of M_k (resp. L_k/π and L_{k+1}/π of M_k/π) so as to display a splitting of $V(M_k) = L_k \cup L_{k+1}$ (resp. $V(M_k)/\pi = L_k/\pi \cup L_{k+1}/\pi$) into pairs, each pair contained in a horizontal line, the 2 composing vertices of such pair equidistant from a vertical line ϕ (resp. ϕ/π, depicted through M_2/π on the left of Figure 1, Section 6 below). In addition, we impose that each resulting horizontal vertex pair in M_k (resp. M_k/π) be of the form $(B_A, \aleph(B_A))$ (resp. $((B_A), (\aleph(B_A)) = \aleph_\pi((B_A))))$, disposed from left to right at both sides of ϕ. A non-horizontal edge of M_k/π will be said to be a *skew edge*.

Theorem 3. *To each skew edge $e = (B_A)(B_{A'})$ of M_k/π corresponds another skew edge $\aleph_\pi((B_A))\aleph_\pi^{-1}((B_{A'}))$ obtained from e by reflection on the line ϕ/π. Moreover:*

(i) the skew edges of M_k/π appear in pairs, with the endpoints of the edges in each pair forming 2 horizontal pairs of vertices equidistant from ϕ/π;

(ii) each horizontal edge of M_k/π has multiplicity equal either to 1 or to 2.

Proof. The skew edges $B_AB_{A'}$ and $\aleph^{-1}(B_{A'})\aleph(B_A)$ of M_k are reflection of each other about ϕ. Their endopoints form 2 horizontal pairs $(B_A, \aleph(B_{A'}))$ and $(\aleph^{-1}(B_A), B_{A'})$ of vertices. Now, ρ_k and ρ_{k+1} extend together to a covering graph map $\rho : M_k \quad M_k/\pi$, since the edges accompany the projections correspondingly, exemplified for $k = 2$ as follows:

$$\begin{aligned}
\aleph((B_A)) &= \aleph((00011)) \\
&= \aleph(\{00011, 10001, 11000, 01100, 00110\}) \\
&= \{00111, 01110, 11100, 11001, 10011\} \\
&= (00111), \\
\aleph^{-1}((B'_A)) &= \aleph^{-1}((01011)) \\
&= \aleph^{-1}(\{01011, 10110, 10110, 11010, 10101\}) \\
&= \{00101, 10010, 01001, 10100, 01010\} \\
&= (00101).
\end{aligned}$$

Here, the order of the elements in the image of class (00011) (resp. (01011)) mod π under $\aleph$ (resp. $\aleph^{-1}$) are shown reversed, from right to left (cyclically between braces, continuing on the right once one reaches the leftmost brace). Such reversal holds for every $k > 2$:

$$\begin{aligned}
\aleph((B_A)) &= \aleph((b_0\cdots b_{2k})) \\
&= \aleph(\{b_0\cdots b_{2k},\ b_{2k}\ldots b_{2k-1},\ \ldots,\ b_1\cdots b_0\}) \\
&= \{\bar{b}_{2k}\cdots\bar{b}_0,\ \bar{b}_{2k-1}\cdots\bar{b}_{2k},\ \ldots,\ \bar{b}_1\cdots\bar{b}_0\} = \\
&(\bar{b}_{2k}\cdots\bar{b}_0), \\
\aleph^{-1}((B'_A)) &= \aleph^{-1}((\bar{b}'_{2k}\cdots\bar{b}'_0)) \\
&= \aleph^{-1}(\{\bar{b}'_{2k}\cdots\bar{b}'_0,\ \bar{b}'_{2k-1}\cdots\bar{b}'_{2k},\ \ldots,\ \bar{b}'_1\cdots\bar{b}'_0\}) \\
&= \{b'_0\cdots b'_{2k},\ b'_{2k}\cdots b'_{2k-1},\ \ldots,\ b'_1\cdots b'_0\} \\
&= (b'_0\cdots b'_{2k}),
\end{aligned}$$

where $(b_0\cdots b_{2k}) \in L_k/\pi$ and $(b'_0\cdots b'_{2k}) \in L_{k+1}/\pi$. This establishes (i).

Every horizontal edge $v\aleph_\pi(v)$ of M_k/π has $v \in L_k/\pi$ represented by $\bar{b}_k\cdots\bar{b}_1 0 b_1\cdots b_k$ in L_k, (so $v = (\bar{b}_k\cdots\bar{b}_1 0 b_1\cdots b_k)$). There are 2^k such vertices in L_k and at most 2^k corresponding vertices in L_k/π.

For example, $(0^{k+1}1^k)$ and $(0(01)^k)$ are endpoints in L_k/π of 2 horizontal edges of M_k/π, each. To prove that this implies (ii), we have to see that there cannot be more than 2 representatives $\bar{b}_k \cdots \bar{b}_1 b_0 b_1 \cdots b_k$ and $\bar{c}_k \cdots \bar{c}_1 c_0 c_1 \cdots c_k$ of a vertex $v \in L_k/\pi$, with $b_0 = 0 = c_0$. Such a v is expressible as $v = (d_0 \cdots b_0 d_{i+1} \cdots d_{j-1} c_0 \cdots d_{2k})$, with $b_0 = d_i$, $c_0 = d_j$ and $0 < j - i \leq k$. Let the substring $\sigma = d_{i+1} \cdots d_{j-1}$ be said $(j-i)$-*feasible*. Let us see that every $(j-i)$-feasible substring σ forces in L_k/π only vertices ω leading to 2 different (parallel) horizontal edges in M_k/π incident to v. In fact, periodic continuation mod n of $d_0 \cdots d_{2k}$ both to the right of $d_j = c_0$ with minimal cyclic substring $\bar{d}_{j-1} \cdots \bar{d}_{i+1} 1 d_{i+1} \cdots d_{j-1} 0 = P_r$ and to the left of $d_i = b_0$ with minimal cyclic substring $0 d_{i+1} \cdots d_{j-1} 1 \bar{d}_{j-1} \cdots \bar{d}_{i+1} = P_\phi$ yields a 2-way infinite string that winds up onto a class $(d_0 \cdots d_{2k})$ containing such an ω. For example, some pairs of feasible substrings σ and resulting vertices ω are:

$$(\sigma,\omega) = (\emptyset,(oo1)),\ (0,(o0o11)),\ (1,(o1o)),\ (0^2,(o00o111)),\ (01,(o01o011)),\ (1^2,o11o0)),$$
$$(0^3,o000o1111)),\ (010,(o010o101101)),\ (01^2,(o011o)),\ (101,(o101o)),\ (1^3,(o111o00)),$$

with 'o' replacing $b_0 = 0$ and $c_0 = 0$, and where $k = \lfloor \frac{n}{2} \rfloor$ has successive values $k = 1,2,1,3,3,2,4,5,2,2,3$. If σ is a feasible substring and $\bar{\sigma}$ is its complemented reversal via $\aleph$, then the possible symmetric substrings $P_\phi \sigma P_r$ about $o\sigma o = 0\sigma 0$ in a vertex v of L_k/π are in order of ascending length:

$$\begin{gathered}
0\sigma 0,\\
\bar{\sigma}0\sigma 0\bar{\sigma},\\
1\bar{\sigma}0\sigma 0\bar{\sigma}1,\\
\sigma 1\bar{\sigma}0\sigma 0\bar{\sigma}1\sigma,\\
0\sigma 1\bar{\sigma}0\sigma 0\bar{\sigma}1\sigma 0,\\
\bar{\sigma}0\sigma 1\bar{\sigma}0\sigma 0\bar{\sigma}1\sigma 0\bar{\sigma},\\
1\bar{\sigma}0\sigma 1\bar{\sigma}0\sigma 0\bar{\sigma}1\sigma 0\bar{\sigma}1,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,
\end{gathered}$$

where we use again '0' instead of 'o' for the entries immediately preceding and following the shown central copy of σ. The lateral periods of P_r and P_ϕ determine each one horizontal edge at v in M_k/π up to returning to b_0 or c_0, so no entry $e_0 = 0$ of $(d_0 \cdots d_{2k})$ other than b_0 or c_0 happens such that $(d_0 \cdots d_{2k})$ has a third representative $\bar{e}_k \cdots \bar{e}_1 0 e_1 \cdots e_k$ (besides $\bar{b}_k \cdots \bar{b}_1 0 b_1 \cdots b_k$ and $\bar{c}_k \cdots \bar{c}_1 0 c_1 \cdots c_k$). Thus, those 2 horizontal edges are produced solely from the feasible substrings $d_{i+1} \cdots d_{j-1}$ characterized above. □

To illustrate Theorem 3, let $1 < h < n$ in $\mathbb{Z}$ be such that $\gcd(h,n) = 1$ and let $\lambda_h : L_k/\pi \quad L_k/\pi$ be given by $\lambda_h((a_0 a_1 \cdots a_n)) \quad (a_0 a_h a_{2h} \cdots a_{n-2h} a_{n-h})$. For each such $h \leq k$, there is at least one h-feasible substring σ and a resulting associated vertex $v \in L_k/\pi$ as in the proof of Theorem 3. For example, starting at $v = (0^{k+1}1^k) \in L_k/\pi$ and applying λ_h repeatedly produces a number of such vertices $v \in L_k/\pi$. If we assume $h = 2h'$ with $h' \in \mathbb{Z}$, then an h-feasible substring σ has the form $\sigma = \bar{a}_1 \cdots \bar{a}_{h'} a_{h'} \cdots a_1$, so there are at least $2^{h'} = 2^{\frac{h}{2}}$ such h-feasible substrings.

6. Dihedral quotients

An *involution* of a graph G is a graph map $\aleph : G \quad G$ such that $\aleph^2$ is the identity. If G has an involution, an $\aleph$-*folding* of G is a graph H, possibly with loops, whose vertices v' and edges or loops e' are respectively of the form $v' = \{v, \aleph(v)\}$ and $e' = \{e, \aleph(e)\}$, where $v \in V(G)$ and $e \in E(G)$; e has endvertices v and $\aleph(v)$ if and only if $\{e, \aleph(e)\}$ is a loop of G.

Note that both maps $\aleph : M_k \quad M_k$ and $\aleph_\pi : M_k/\pi \quad M_k/\pi$ in Section 5 are involutions. Let $\langle B_A \rangle$ denote each horizontal pair $\{(B_A), \aleph_\pi((B_A))\}$ (as in Theorem 3) of M_k/π, where $|A| = k$. An $\aleph$-folding R_k of M_k/π is obtained whose vertices are the pairs $\langle B_A \rangle$ and having:

(1) an edge $\langle B_A \rangle \langle B_{A'} \rangle$ per skew-edge pair $\{(B_A)\aleph_\pi((B_{A'})), (B_{A'})\aleph_\pi((B_A))\}$;
(2) a loop at $\langle B_A \rangle$ per horizontal edge $(B_A)\aleph_\pi((B_A))$; because of Theorem 3, there may be up to 2 loops at each vertex of R_k.

Theorem 4. *R_k is a quotient graph of M_k under an action $\Upsilon : D_{2n} \times M_k \quad M_k$.*

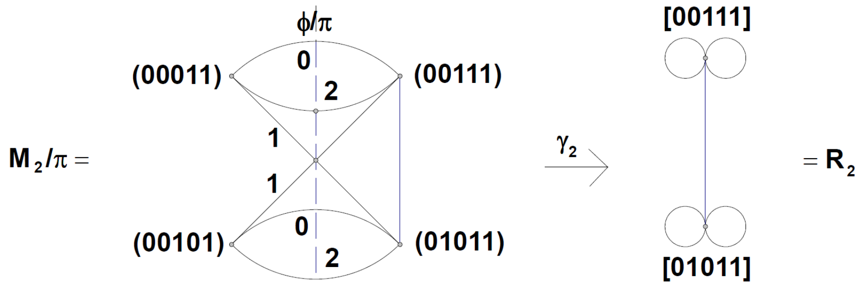

Figure 1. Reflection symmetry of M_2/π about a line ϕ/π and resulting graph map γ_2

Proof. D_{2n} is the semidirect product $\mathbb{Z}_n \rtimes_\varrho \mathbb{Z}_2$ via the group homomorphism $\varrho : \mathbb{Z}_2 \quad \mathrm{Aut}(\mathbb{Z}_n)$, where $\varrho(0)$ is the identity and $\varrho(1)$ is the automorphism $i \quad (n-i)$, $\forall i \in \mathbb{Z}_n$. If $* : D_{2n} \times D_{2n} \quad D_{2n}$ indicates group multiplication and $i_1, i_2 \in \mathbb{Z}_n$, then $(i_1, 0) * (i_2, j) = (i_1 + i_2, j)$ and $(i_1, 1) * (i_2, j) = (i_1 - i_2, \bar{j})$, for $j \in \mathbb{Z}_2$. Set $\Upsilon((i,j), v) = \Upsilon'(i, \aleph^j(v))$, $\forall i \in \mathbb{Z}_n, \forall j \in \mathbb{Z}_2$, where Υ' is as in display (2). Then, Υ is a well-defined D_{2n}-action on M_k. By writing $(i,j) \cdot v = \Upsilon((i,j), v)$ and $v = a_0 \cdots a_{2k}$, we have $(i,0) \cdot v = a_{n-i+1} \cdots a_{2k} a_0 \cdots a_{n-i} = v'$ and $(0,1) \cdot v' = \bar{a}_{i-1} \cdots \bar{a}_0 \bar{a}_{2k} \cdots \bar{a}_i = (n-i, 1) \cdot v = ((0,1) * (i,0)) \cdot v$, leading to the compatibility condition $((i,j) * (i',j')) \cdot v = (i,j) \cdot ((i',j') \cdot v)$. □

Theorem 4 yields a graph projection $\gamma_k : M_k/\pi \quad R_k$ for the action Υ, given for $k = 2$ in Figure 1. In fact, γ_2 is associated with reflection of M_2/π about the dashed vertical symmetry axis ϕ/π so that R_2 (containing 2 vertices and one edge between them, with each vertex incident to 2 loops) is given as its image. Both the representations of M_2/π and R_2 in the figure have their edges indicated with colors 0,1,2, as arising inSection 7.

7. Lexical procedure

Let P_{k+1} be the subgraph of the unit-distance graph of $\mathbb{R}$ (the real line) induced by the set $[k+1] = \{0, \ldots, k\}$. We draw the grid $\Gamma = P_{k+1} \square P_{k+1}$ in the plane $\mathbb{R}^2$ with a diagonal ∂ traced from the lower-left vertex $(0,0)$ to the upper-right vertex (k,k). For each $v \in L_k/\pi$, there are $k+1$ n-tuples of the form $b_0 b_1 \cdots b_{n-1} = 0 b_1 \cdots b_{n-1}$ that represent v with $b_0 = 0$. For each such n-tuple, we construct a $2k$-path D in Γ from $(0,0)$ to (k,k) in $2k$ steps indexed from $i = 0$ to $i = 2k-1$. This leads to a lexical edge-coloring implicit in [1]; see the following statement and Figure 2 (Section 8), containing examples of such a $2k$-path D in thick trace.

Theorem 5. *[1] Each $v \in L_k/\pi$ has its $k+1$ incident edges assigned colors $0, 1, \ldots, k$ by means of the following "Lexical Procedure", where $0 \le i \in \mathbb{Z}$, $w \in V(\Gamma)$ and D is a path in Γ. Initially, let $i = 0$, $w = (0,0)$ and D contain solely the vertex w. Repeat $2k$ times the following sequence of steps (1)-(3), and then perform once the final steps (4)-(5):*

(1) If $b_i = 0$, then set $w' := w + (1,0)$; otherwise, set $w' := w + (0,1)$.
(2) Reset $V(D) := v(D) \cup \{w'\}$, $E(D) := E(D) \cup \{ww'\}$, $i := i+1$ and $w := w'$.
(3) If $w \neq (k,k)$, or equivalently, if $i < 2k$, then go back to step (1).
(4) Set $\check{v} \in L_{k+1}/\pi$ to be the vertex of M_k/π adjacent to v and obtained from its representative n-tuple $b_0 b_1 \cdots b_{n-1} = 0 b_1 \cdots b_{n-1}$ by replacing the entry b_0 by $\bar{b}_0 = 1$ in $\check{v}$, keeping the entries b_i of v unchanged in $\check{v}$ for $i > 0$.
(5) Set the color of the edge $v\check{v}$ to be the number c of horizontal (alternatively, vertical) arcs of D above ∂.

Proof. If addition and subtraction in $[n]$ are taken modulo n and we write $[y, x) = \{y, y+1, y+2, \ldots, x-1\}$, for $x, y \in [n]$, and $S^c = [n] \setminus S$, for $S = \{i \in [n] : b_i = 1\} \subseteq [n]$, then the cardinalities of the sets $\{y \in S^c \setminus x : |[y,x) \cap S| < |[y,x) \cap S^c|\}$ yield all the edge colors, where $x \in S^c$ varies. □

The Lexical Procedure of Theorem 5 yields a 1-factorization not only for M_k/π but also for R_k and M_k. This is clarified by the end of Section 8.

8. Uncastling and lexical 1-factorization

A notation $\delta(v)$ is assigned to each pair $\{v, \aleph_\pi(v)\} \in R_k$, where $v \in L_k/\pi$, so that there is a unique k-germ $\alpha = \alpha(v)$ with $\langle F(\alpha) \rangle = \delta(v)$, (where the notation $\langle \cdot \rangle$ is as in $\langle B_A \rangle$ in Section 6). We exemplify $\delta(v)$ for $k = 2$ in Figure 2, with the Lexical Procedure (indicated by arrows "") departing from $v = (00011)$ (top) and

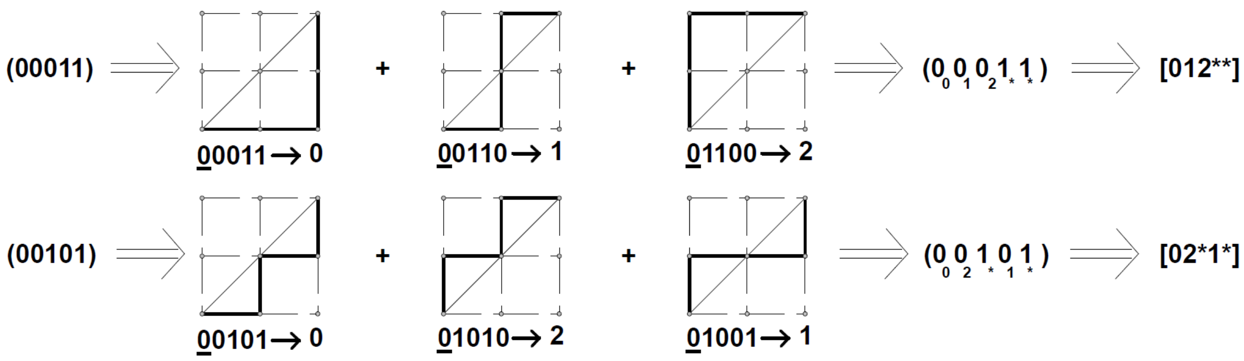

Figure 2. Representing lexical-color assignment for $k = 2$

$v = (00101)$ (bottom), passing to sketches of Γ (separated by symbols "+"), one sketch (in which to trace the edges of $D \subset \Gamma$ as in Theorem 5) per representative $b_0 b_1 \cdots b_{n-1} = 0 b_1 \cdots b_{n-1}$ of v shown under the sketch (where $b_0 = 0$ is underscored) and pointing via an arrow "" to the corresponding color $c \in [k+1]$. Recall this c is the number of horizontal arcs of D below ∂.

In each of the 2 cases in Figure 2 (top, bottom), an arrow "" to the right of the sketches points to a modification $\hat{v}$ of $b_0 b_1 \cdots b_{n-1} = 0 b_1 \cdots b_{n-1}$ obtained by setting as a subindex of each 0 (resp. 1) its associated color c (resp. an asterisk "$*$"). Further to the right, a third arrow "" points to the n-tuple $\delta(v)$ formed by the string of subindexes of entries of $\hat{v}$ in the order they appear from left to right.

Theorem 6. *Let $\alpha(v^0) = a_{k-1} \cdots a_1 = 00 \cdots 0$. To each $\delta(v)$ corresponds a sole k-germ $\alpha = \alpha(v)$ with $\langle F(\alpha) \rangle = \delta(v)$ by means of the following "Uncastling Procedure": Given $v \in L_k/\pi$, let $W^i = 01 \cdots i$ be the maximal initial numeric (i.e., colored) substring of $\delta(v)$, so the length of W^i is $i+1$ $(0 \le i \le k)$. If $i = k$, let $\alpha(v) = \alpha(v^0)$; else, set $m = 0$ and:*

1. *set $\delta(v^m) = \langle W^i | X | Y | Z^i \rangle$, where Z^i is the terminal j_m-substring of $\delta(v^m)$, with $j_m = i+1$, and let X, Y (in that order) start at contiguous numbers Ω and $\Omega - 1 \ge i$;*
2. *set $\delta(v^{m+1}) = \langle W^i | Y | X | Z^i \rangle$;*
3. *obtain $\alpha(v^{m+1})$ from $\alpha(v^m)$ by increasing its entry a_{j_m} by 1;*
4. *if $\delta(v^{m+1}) = [01 \cdots k * \cdots *]$, then stop; else, increase m by 1 and go to step 1.*

Proof. This is a procedure inverse to that of Castling (Section 3), so 1-4 follow. □

Theorem 6 allows to produce a finite sequence $\delta(v^0), \delta(v^1), \ldots, \delta(v^m), \ldots, \delta(v^s)$ of n-strings with $j_0 \ge j_1 \ge \cdots \ge j_m \cdots \ge j_{s-1}$ as in steps 1-4, and k-germs $\alpha(v^0), \alpha(v^1), \ldots, \alpha(v^m), \ldots, \alpha(v^s)$, taking from $\alpha(v^0)$ through the k-germs $\alpha(v^m)$, $(m = 1, \ldots, s-1)$, up to $\alpha(v) = \alpha(v^s)$ via unit incrementation of a_{j_m}, for $0 \le m < s$, where each incrementation yields the corresponding $\alpha(v^{m+1})$. Recall F is a bijection from the set $V(\mathcal{T}_k)$ of k-germs onto $V(R_k)$, both sets being of cardinality C_k. Thus, to deal with $V(R_k)$ it is enough to deal with $V(\mathcal{T}_k)$, a fact useful in interpreting Theorem 7. For example $\delta(v^0) = \langle 04*3*2*1*\rangle = \langle 0|4*|3*2*1|*\rangle = \langle W^0|X|Y|Z^0\rangle$ with $m = 0$ and $\alpha(v^0) = 123$, continued in Table 3 with $\delta(v^1) = \langle W^0|Y|X|Z^0\rangle$, finally arriving to $\alpha(v^s) = \alpha(v^6) = 000$.

Table 3. Continuation of the Uncastling Procedure started at $\alpha(v^0) = 123$.

$j_0{=}0$	$\delta(v^1)$	$=$	$\langle 0\|3*2*1\|4*\|*\rangle$	$=$	$\langle 03*2*14**\rangle$	$=$	$\langle 0\|3*\|2*14*\|*\rangle$	$\alpha(v^1){=}122$	$\langle F(122)\rangle{=}\delta(v^1)$
$j_1{=}0$	$\delta(v^2)$	$=$	$\langle 0\|2*14*\|3*\|*\rangle$	$=$	$\langle 02*14*3**\rangle$	$=$	$\langle 0\|2*\|14*3*\|*\rangle$	$\alpha(v^2){=}121$	$\langle F(121)\rangle{=}\delta(v^2)$
$j_2{=}0$	$\delta(v^3)$	$=$	$\langle 0\|14*3*\|2*\|*\rangle$	$=$	$\langle 014*3*2**\rangle$	$=$	$\langle 01\|4*\|3*2\|**\rangle$	$\alpha(v^3){=}120$	$\langle F(120)\rangle{=}\delta(v^3)$
$j_3{=}1$	$\delta(v^4)$	$=$	$\langle 01\|3*2\|4*\|**\rangle$	$=$	$\langle 013*24***\rangle$	$=$	$\langle 01\|3*\|24*\|**\rangle$	$\alpha(v^4){=}110$	$\langle F(110)\rangle{=}\delta(v^4)$
$j_4{=}1$	$\delta(v^5)$	$=$	$\langle 01\|24*\|3*\|**\rangle$	$=$	$\langle 0124*3***\rangle$	$=$	$\langle 012\|4*\|3\|**\rangle$	$\alpha(v^5){=}100$	$\langle F(100)\rangle{=}\delta(v^5)$
$j_5{=}2$	$\delta(v^6)$	$=$	$\langle 012\|3\|4*\|***\rangle$	$=$	$\langle 01234****\rangle$			$\alpha(v^6){=}000$	$\langle F(000)\rangle{=}\delta(v^6)$

A pair of skew edges $(B_A)\aleph_\pi((B_{A'}))$ and $(B_{A'})\aleph((B_A))$ in M_k/π, to be called a *skew reflection edge pair* (*SREP*), provides a color notation for any $v \in L_{k+1}/\pi$ such that in each particular edge class mod π:

(I) all edges receive a common color in $[k+1]$ regardless of the endpoint on which the Lexical Procedure (or its modification immediately below) for $v \in L_{k+1}/\pi$ is applied;

(II) the 2 edges in each SREP in M_k/π are assigned a common color in $[k+1]$.

The modification in step (I) consists in replacing in Figure 2 each v by $\aleph_\pi(v)$ so that on the left we have instead now (00111) (top) and (01011) (bottom) with respective sketch subtitles

$$\begin{array}{lll} 0011\underline{1}0, & 1001\underline{1}1, & 1100\underline{1}2, \\ 0101\underline{1}0, & 1010\underline{1}2, & 0110\underline{1}1, \end{array}$$

resulting in similar sketches when the steps (1)-(5) of the Lexical Procedure are taken with right-to-left reading and processing of the entries on the left side of the subtitles (before the arrows “”), where the values of each b_i must be taken complemented, (i.e., as $\bar{b}_i$).

Since an SREP in M_k determines a unique edge ϵ of R_k (and vice versa), the color received by the SREP can be attributed to ϵ, too. Clearly, each vertex of either M_k or M_k/π or R_k defines a bijection from its incident edges onto the color set $[k+1]$. The edges obtained via $\aleph$ or $\aleph_\pi$ from these edges have the same corresponding colors.

Theorem 7. *A 1-factorization of M_k/π by the colors $0,1,\dots,k$ is obtained via the Lexical Procedure that can be lifted to a covering 1-factorization of M_k and subsequently collapsed onto a folding 1-factorization of R_k. This insures the notation $\delta(v)$ for each $v \in V(R_k)$ so that there is a unique k-germ $\alpha = \alpha(v)$ with $\langle F(\alpha)\rangle = \delta(v)$.*

Proof. As pointed out in (II) above, each SREP in M_k/π has its edges with a common color in $[k+1]$. Thus, the $[k+1]$-coloring of M_k/π induces a well-defined $[k+1]$-coloring of R_k. This yields the claimed collapsing to a folding 1-factorization of R_k. The lifting to a covering 1-factorization in M_k is immediate. The arguments above determine that the collapsing 1-factorization in R_k induces the claimed k-germs $\alpha(v)$. □

9. Union of lexical 1-factors of colors 0 and 1

Given a k-germ α, let (α) represent the dihedral class $\delta(v) = \langle F(\alpha)\rangle$ with $v \in L_k/\pi$. Let W_{01}^k be the 2-factor given by the union of the 1-factors of colors 0 and 1 in M_k (namely those formed by lifting the edges $\alpha\alpha^0$ and $\alpha\alpha^1$ of R_k in the notation of Section 12, instead of those of colors k and $k-1$, as in [6]). The cycles of W_{01}^k are grown in this section from specific paths, as suggested in Figure 3 for $k = 2,3,4$ (say: cycle C_0 that starts with vertically expressed path $X(0)$, for $k = 2$; cycles C_0, C_1 that start with vertically expressed paths $X(0), X(1)$, for $k = 3$; and cycles C_0, C_1, C_2 that start with vertically expressed paths $X(0), X(1), X(2)$, for $k = 4$). Here, vertices $v \in L_k$ (resp. $v \in L_{k+1}$) will be represented with:

(a) 0- (resp. 1-) entries replaced by their respective colors $0,1,\dots,k$ (resp. asterisks);
(b) 1- (resp. 0-) entries replaced by asterisks (resp. their respective colors $0,1,\dots,k$);
(c) delimiting chevron symbols ">" or “$\rangle$” (resp. "<" or “$\langle$”), instead of parentheses or brackets, indicating the reading direction of *“forward”* (resp. *“backward”*) n-tuples.

Each such vertex v is showen to belong (via the set membership symbol expressed by “ε” in Figure 3) to (α_v), where α_v is the k-germ of v, also expressed as its (underlined decimal) natural order. In each case, Figure 3 shows a vertically presented path $X(i)$ of length $4k-1$ in the corresponding cycle C_i starting at the vertex $w = b_0b_1\cdots b_{2k} = 01\cdots *$ of smallest natural order and proceeding by traversing the edges colored 1 and 0, alternatively. The terminal vertex of such subpath is $b_{2k}b_0b_1\cdots b_{2k-1} = *01\cdots b_{2k-1}$, obtained by translation mod n from w.

Observation 1. *The initial entries of the vertices in each C_i are presented downward, first in the 0-column of $X(i)$, then in the $(2k-j)$-column of $X(i)$, $(j \in [2k]$, only up to $|C_i|$.*

In Figure 3, initial entries are red if they are in $\{0,1\}$ and each cycle C_i is encoded on its top right by a vertical sequence of expressions $(0\cdots 1)(1\cdots 0)$ that allows to get the sequence of initial entries of the succeeding vertices of C_i by interspersing asterisks between each 2 terms inside parentheses (), then removing those ().

An ordered tree $T = T_v$ (Remark 1) for each v in the exemplified $C_i(= C$ in Proposition 2(v) [6]) is shown at the lower right of its case in Figure 3. Each of these T_v for a specific C_i corresponds to the k-germ α_v (so we write $F(\alpha_v) = F(T_{\alpha_v})$) and is headed in the figure by its (underlined decimal) natural order. In the figure,

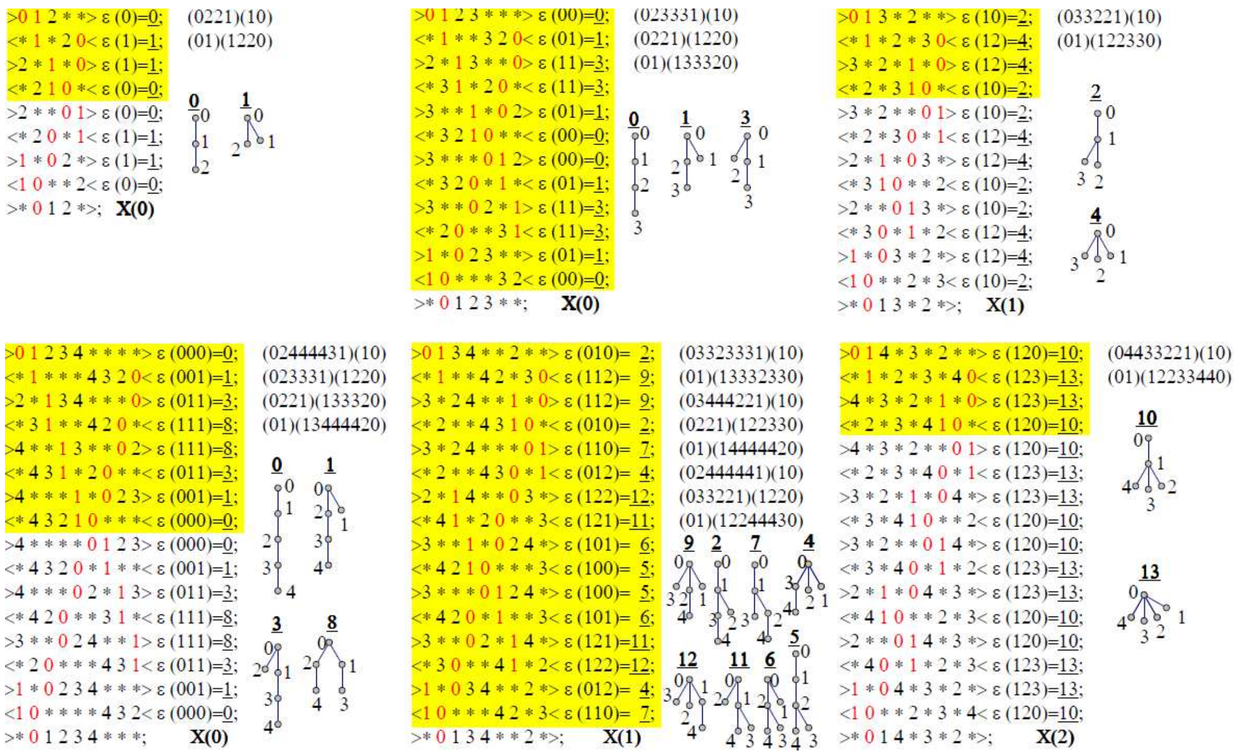

Figure 3. Cycles of W_{01}^k in M_k, $(k = 2, 3, 4)$

vertices of each T_v are denoted i, instead of v_i ($i \in [k+1]$). The trees corresponding to the k-germs in each case are obtained by applying *root rotation* [6], consisting in replacing the tree root by its leftmost child and redrawing the ordered tree accordingly.

A *plane tree* is an equivalence relation of ordered trees under root rotations. In the notation of Section 12, applying one root rotation has the same effect as traversing first an edge $\alpha\alpha_0$ in C_i and then edge $\beta\beta_1$, also in C_i, where $\beta = \alpha_0$. In each case, a yellow box shows a subpath of $X(i)$ with $\frac{1}{n}|V(C_i)|$ vertices of C_i that takes into account the rotation symmetry of the associated plane tree $\mathcal{T}_i$.

In Figure 3 for $k = 4$, successive application of root rotations on the second cycle, C_1, produces the cycle $(\underline{9}, \underline{2}, \underline{4}, \underline{11}, \underline{5}, \underline{6}, \underline{12}, \underline{7})$, the square graph of C_1, that starts downward from the second row or upward from the third row, thus covering respectively the vertices of L_4 or L_5 in the class.

Let D_i ($i \geq 0$) be the set of substrings of length $2i$ in an $F(\alpha)$ with exactly i color-entries such that in every prefix (i.e. initial substring), the number of asterisk-entries is at least as large as the number of color-entries. The elements of $D = \cup_{i \geq 0} D_i$ are known as *Dyck words*. Each $F(\alpha)$ is of the form $0v1u*$, where u and v are Dyck words and 0 and 1 are colors in $[k+1]$ [6].

The "forward" n-tuple $F(\alpha)$ in $(\underline{9}, \underline{2}, \underline{4}, \underline{11}, \underline{5}, \underline{6}, \underline{12}, \underline{7})$ can be written with parentheses enclosing such Dyck words v and u, namely $0()1(34**2*)*$, for $\underline{2}$; $0(3*24**)1()*$, for $\underline{9}$; $0()1(3*24**)*$, for $\underline{7}$; $0(3*2*)1(4*)*$, for $\underline{12}$; $0(24*3**)1()*$, for $\underline{6}$; $0()1(24*3**)*$, for $\underline{5}$; $0(2*)1(4*3*)*$, for $\underline{11}$; and $0(34**2*)1()*$, for $\underline{4}$. Similar treatment holds for "backward" n-tuples.

As in Figure 3, each pair $(0\cdots1)(1\cdots0)$ represents 2 paths in the corresponding cycle, of lengths $2|(0\cdots1)| - 1$ and $2|(1\cdots0)|$, adding up to $4k+2$. If these 2 paths are of the form $0v1u*$ and $0v'1u'*$ (this one read in reverse), then $|u| + 2 = |(0\cdots1)|$ and $|u'| + 2 = |(1\cdots0)|$. Reading these paths starts at a 0-entry and ends at a 1-entry. In reality, the collections of paths obtained from the 1-factors here have the leftmost entry of the n-tuples representing their vertices constantly equal to $1 \in \mathbb{Z}_2$ before taking into consideration (items (a) and (b) above, but with the reading orientations given in item (c).

The 1-factor of color 0 makes the endvertices of each of its edges to have their representative plane trees obtained from each other by *horizontal reflection* $\Phi = F\alpha^0F^{-1}$. For example, Figure 3 shows that for $k = 2$: both $\underline{0}, \underline{1}$ in $X(0)$ are fixed via Φ; for $k = 3$: $\underline{0}$ in $X(0)$ is fixed via Φ and $\underline{1}, \underline{3}$ in $X(0)$ correspond to each other via Φ; and $\underline{2}, \underline{4}$ in $X(1)$ are fixed via Φ; for $k = 4$: $\underline{0}, \underline{8}$ in $X(0)$ are fixed via Φ and $\underline{1}, \underline{3}$ in $X(0)$ correspond to each other via Φ; $\underline{5}, \underline{9}$ in $X(1)$ are fixed via Φ and the pairs $(\underline{5}, \underline{9})$, $(\underline{2}, \underline{7})$, $(\underline{4}, \underline{12})$ and $(\underline{6}, \underline{11})$ in $X(1)$ are pairs of correspondent plane trees via Φ; and $\underline{10}, \underline{13}$ in $X(2)$ are fixed via Φ. This horizontal reflection symmetry arises from Theorem 3. It accounts for each pair of contiguous rows in any $X(i)$ corresponding to a 0-colored edge.

For $k = 5$, this symmetry via Φ occurs in all cycles C_i $(i \in [6])$. But we also have $F(\underline{22}) = \rangle 024 * *135 * **\rangle$, where $\underline{22} = (1111)$ in C_0 and $F(\underline{39}) = \rangle 03 * 2 * 15 * 4 * *\rangle$, where $\underline{39} = (1232)$ in C_3, both having their 1-colored edges leading to reversed reading between L_5 and L_6, again by Theorem 3. Moreover, $F((11 \cdots 1))$ has a similar property only if k is odd; but if k is even, a 0-colored edge takes place, instead of the 1-colored edge for k odd. These cases reflect the following lemma (which can alternatively be implied from Theorem 10 (B)-(C)) via the correspondence $i \leftrightarrow k - i$, $(i \in [k+1])$.

Observation 2. *(A) Every 0-colored edge is represented via $\Phi = F\alpha^0 F^{-1}$.*
(B) Every 1-colored edge is represented via the composition Ψ of Φ (first) and root rotation (second).

By Theorem 3(ii), the number ξ of contiguous pairs of vertices of M_k in each C_i with a common k-germ happens in pairs. The first cases for which this ξ is null happens for $k = 6$, namely for the 2 reflection pairs $(\rangle 012356 * *4 * * * *\rangle, \rangle 01235 * 46 * * * **\rangle)$ and $(\rangle 01246 * 5 * *3 * **\rangle, \rangle 0124 * 36 * 5 * * * *\rangle)$ whose respective ordered trees are enantiomorphic, i.e they are reflection via Φ of each other. We say that these 2 cases are *enantiomorphic*. In fact, the presence of a pair of enantiomorphic ordered trees in a case of an $X(i)$ will be distinguished by saying that the case is *enantiomorphic*. For example, $k = 4$ offers $(\underline{1}, \underline{3})$ as the sole enantiomorphic pair in $X(0)$, and $(\underline{2}, \underline{7})$, $(\underline{4}, \underline{12})$, $(\underline{11}, \underline{6})$ as all the enantiomorphic pairs in $X(1)$. Each enantiomorphic cycle C_i or each cycle C_i with $\xi = 2$ has $|C_i| = 2k(4k+2)$. If $\xi = 2\zeta$ with $\zeta > 1$, then $|C_i| = \frac{2k(4k+1)}{\zeta}$. On account of these facts, we have the following:

Observation 3. *For each integer $k > 1$, there is a natural bijection Λ from the k-edge plane trees onto the cycles of W_{01}^k, as well as a partition $\mathcal{P}_k$ of the k-germs (or the ordered trees they represent via F), with each class of $\mathcal{P}_k$ in natural correspondence either to a k-edge plane tree or to a pair of enantiomorphic k-edge plane trees disconnecting in W_{01}^k the forward (in L_k) and reversed (in L_{k+1}) readings of each vertex v of their associated cycles via Λ.*

10. Reinterpretation of the middle-levels theorem

For each cycle C_i of W_{01}^k, the ordered trees of its plane tree $\mathcal{T}_i$ with leftmost subpath of length 1 from v_0 to a vertex v_h determine 6-cycles touching C_i in 2 nonadjacent edges as follows:

Let $t_i < k$ be the number of degree 1 vertices of $\mathcal{T}_i$. Let τ_i be the number of rotation symmetries of $\mathcal{T}_i$. Then, there are $\frac{t_i}{\tau_i}$ classes mod n of pairs of vertices u, v at distance 5 in C_i with $u \in L_{k+1}$ ahead of $v \in L_k$ in $X(i)$ and associated color $h \in \{2, \ldots, k\}$ such that:

(i) u and v are adjacent via h;
(ii) u (resp. v) has cyclic backward (resp. forward) reading $\langle \cdots * h0 * \cdots \langle$ (resp. $\rangle \cdots * 0h * \cdots \rangle$);
(iii) the column in which the occurrences of h in (ii) happen at distance 5 looks between u and v (both included) as the transpose of $(h, *, 0, 0, *, h)$. Recall there are n such columns.

In each case, the vertices u' and v' in C_i preceding respectively u and v in X_i are end vertices of a 3-path $u'u''v''v'$ in M_k with the edge $u''v''$ in a cycle $C_j \neq C_i$ in W_{01}^k.

The 6-cycle $U_i^j = (uu'u''v''v'v)$ has as symmetric difference $U_i^j \Delta (C_i \cup C_j)$ a cycle in M_k whose vertex set is $V(C_i \cup C_j)$. With $u', u'', v'', v', v, u, u'$ shown vertically, Figure 4 illustrates U_i^j, twice each for $k = 3, 4$.

That symmetric difference replaces respectively the edges $u''v''$, $v'v$, uu' in $C_i \cup C_j$ by $u'u''$, $v''v''$, vu. In the figure, vertically contiguous positions holding a common number g (meaning adjacency via color g) are presented in red if $g \in \{0, 1\}$, e.g. $u''v''$ with $g = 1$, in column say r_1 exactly at the position where u and v differ, (however having common color h as in (i) above) and in orange, otherwise. The column r_2 (resp r_3) in each instance of Figure 3 containing color 1 in u' (resp. 0 in v') and a color $c \in \{2, \ldots, k\}$ in v' (resp. color $d \in \{2, \ldots, k\}$ in u') starts with $1, *, c, c$, (resp. $d, d, *, 0$), where $c, d \in \{2, \ldots, k\}$. Then, r_1, r_2, r_3 are the only columns having changes in the binary version of U_i^j. All other columns have their first 4 entries alternating asterisks and colors. In the first disposition in item (ii) above, we have that: **(ii')** u'' (resp. v'') has cyclic backward (resp. forward) reading $\langle \cdots d10 * \cdots \langle$ (resp. $\rangle \cdots * 1 * 0 \cdots \rangle$).

In the previous paragraph, "ahead" can be replaced by "behind", yielding additional 6-cycles U_i^j by modifying adequately the accompanying text.

>2 * 1 3 * * 0> ε (11)= 3 ε X(0);	>0 1 3 * 2 * *> ε (10)= 2 ε X(1);	>0 1 3 4 * * 2 * *> ε (010)= 2 ε X(1);	>0 1 4 * 3 * 2 * *> ε (120)=10 ε X(2);
<* 2 * 3 1 0 *< ε (10)= 2 ε X(1);	<* * * 3 2 1 0< ε (00)= 0 ε X(0);	<* * * * 4 3 2 1 0< ε (000)= 0 ε X(0);	<* * * 3 * 4 2 1 0< ε (100)= 5 ε X(1);
>3 * 2 * 1 * 0> ε (12)= 4 ε X(1);	>0 2 3 * * 1 *> ε (01)= 1 ε X(0);	>0 2 3 4 * * * 1 *> ε (001)= 1 ε X(0);	>0 2 4 * 3 * * 1 *> ε (101)= 6 ε X(1);
<* 3 2 0 * 1 *< ε (01)= 1 ε X(0);	<* 2 * 3 0 * 1< ε (12)= 4 ε X(1);	<* 2 * * 4 3 0 * 1< ε (012)= 4 ε X(1);	<* 2 * 3 * 4 0 * 1< ε (123)=13 ε X(2);
>3 * * 0 2 * 1> ε (11)= 3 ε X(0);	>2 * 1 * 0 3 *> ε (12)= 4 ε X(1);	>2 * 1 4 * * 0 3 *> ε (122)=12 ε X(1);	>3 * 2 * 1 * 0 4 *> ε (123)=13 ε X(2);
<* 3 1 * 2 0 *< ε (11)= 3 ε X(0);	<* 1 * 2 * 3 0< ε (12)= 4 ε X(1);	<* 1 * * 4 2 * 3 0< ε (112)= 9 ε X(1);	<* 1 * 2 * 3 * 4 0< ε (123)=13 ε X(2);
>2 * 1 3 * * 0> ε (11)= 3 ε X(0);	>0 1 3 * 2 * *> ε (10)= 2 ε X(1);	>0 1 3 4 * * 2 * *> ε (010)= 2 ε X(1);	>0 1 4 * 3 * 2 * *> ε (120)=10 ε X(2);

Figure 4. Examples of 6-cycles U_i^j for $k = 4, 5$

Theorem 8. *[6,7] Let $0 < k \in \mathbb{Z}$. A Hamilton cycle in M_k is obtained by means of the symmetric differences of W_{01}^k with the members of a set of pairwise edge-disjoint 6-cycles U_i^j.*

Proof. Clearly, the statement holds for $k = 1$. Assume $k > 1$. Let $\mathcal{D}$ be the digraph whose vertices are the cycles C_i of W_{01}^k, with an arc from C_i to C_j, for each 6-cycle U_i^j, where $C_i, C_j \in V(\mathcal{D})$ with $i \neq j$. Since M_k is connected, then $\mathcal{D}$ is connected.

Moreover, the outdegree and indegree of every C_i in $\mathcal{D}$ is $2n\frac{t_i}{\tau_i}$ (see items (ii) and (ii') above), in proportion with the length of C_i, a $\frac{1}{n}$-th of which is illustrated in each yellow box of Figure 3.

Consider a spanning tree $\mathcal{D}'$ of $\mathcal{D}$. Since all vertices of $\mathcal{D}$ have outdegree > 0, there is a $\mathcal{D}'$ in which the outdegree of each vertex is 1. This way, we avoid any pair of 6-cycles U_i^j with common C_i in which the associated distance-6 subpaths from u' to v in C_i do not have edges in common. For each $a \in A(\mathcal{D}')$, let $\nabla(a)$ be its associated 6-cycle U_i^j. Then, $\{\nabla(a); a \in A(\mathcal{D}')\}$ can be selected as a collection of edge-disjoint 6-cycles. By performing all symmetric differences $\nabla(a)\Delta(C_i \cup C_j)$ corresponding to these 6-cycles, a Hamilton cycle is obtained. □

11. Alternate viewpoint on ordered trees

To have the viewpoint of [6], replace v_0 by v_k as root of the ordered trees. We start with examples. Figure 5 shows on its left-hand side the 14 ordered trees for $k = 4$ encoded at the bottom of Table 1. Each such tree $T = T(\alpha)$ is headed on top by its k-germ α, in which the entry i producing T via Castling is in red. Such T has its vertices denoted on their left and its edges denoted on their right, with their notation v_i and e_j given in Section 9. Castling here is indicated in any particular tree $T = T(\alpha) \neq T(00\cdots0)$ by distinguishing in red the largest subtree common with that of the parent tree of T (as in Theorem 1) whose Castling reattachment produces T. This subtree corresponds with substring X in Theorem 2. In each case of such parent tree, the vertex in which the corresponding tree-surgery transformation leads to such a child tree T is additionally labeled (on its right) with the expression of its k-germ, in which the entry to be modified in the case is set in red color.

On the other hand, the 14 trees in the right-hand side of Figure 5 have their labels set by making the root to be v_k (instead of v_0), then going downward to v_0 (instead of v_k) while gradually increasing (instead of decreasing) a unit in the subindex j of the denomination v_j, sibling by sibling from left to right at each level. The associated k-germ headers on this right-hand side of the figure correspond to the new root viewpoint. This determines a bijection Θ established by correspondence between the old and the new header k-germs. In our example, it yields an involution formed by the pairs $(001, 100)$, $(011, 110)$, $(120, 012)$ and $(112, 121)$, with fixed $000, 010, 101, 111, 122$ and 123.

The function Θ seen from the k-germ viewpoint, namely as the composition function $F^{-1}\Theta F$, behaves as follows. Let $\alpha = a_{k-1}a_{k-2}\cdots a_2a_1$ be a k-germ and let a_i be the rightmost occurrence of its largest integer value. A substring β of α is said to be an *atom* if it is either formed by a sole 0 or is a maximal strictly increasing substring of α not starting with 0. For example, consider the 17-germ $\alpha' = 0123223442310121$. By enclosing the successive atoms between parentheses, α' can be written as $\alpha' = (0)123(2)(234)4(23)(1)(0)(12)(1)$, obtained by inserting in a *base string* $\gamma' = 1\cdots a_i = 1234$ all those atoms according to their order, where γ' appears partitioned into subsequent un-parenthesized atoms distributed and interspered from left to right in α' just once each as further to the right as possible.

This atom-parenthesizing procedure works for every k-germ α and determines a corresponding base string γ, like the γ' in our example.

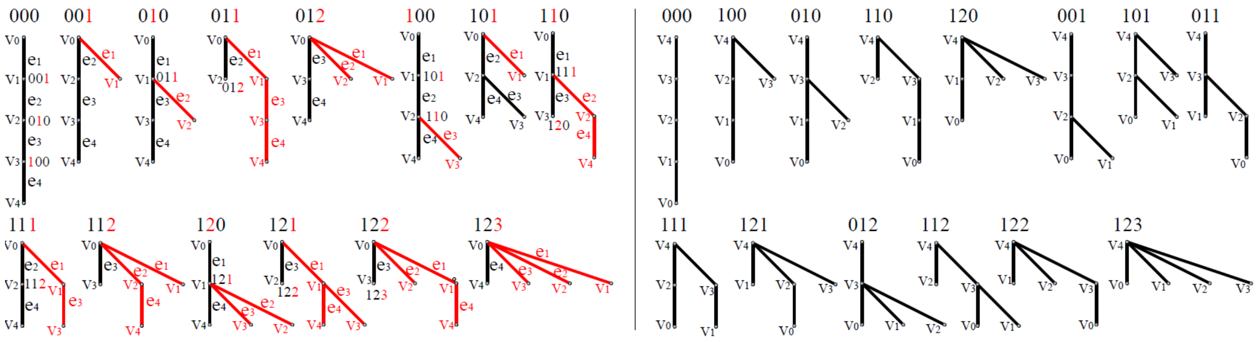

Figure 5. Generation of ordered trees for $k = 4$

Theorem 9. *Given a k-germ $\alpha = a_{k-1}a_{k-2}\cdots a_2a_1$, let a_i be the leftmost occurrence of its largest integer value. Then, α is obtained from a base string $\gamma = 1\cdots a_i$ by inserting in γ all atoms of $\alpha \setminus \gamma$ in their left-to-right order. Moreover, $F^{-1}\Theta F(\alpha)$ is obtained by reversing the insertion of those atoms in γ, in right-to-left fashion. For both insertions, γ is partitioned into subsequent atoms distributed in α and $F^{-1}\Theta F(\alpha)$ as further to the right as possible.*

Proof. $F^{-1}\Theta F(\alpha)$ is obtained by reversing the position of the parenthesized atoms, inserting them between the substrings of a partition of γ as the one above but for this reversing situation. In the above example of α', it is $F^{-1}\Theta F(\alpha') = (1)(12)(0)(1)12(234)34(23)(2)(0)$. □

12. Germ structure of 1-factorizations

Table 4. Presentations $\delta(v) = \{v, \aleph_\pi(v)\} \in R_k$, for $k = 2, 3$, $(v \in L_k/\pi)$.

m	α	$F(\alpha)$	$F^3(\alpha)$	$F^2(\alpha)$	$F^1(\alpha)$	$F^0(\alpha)$	α^3	α^2	α^1	α^0
0	0	$012**$	–	$012**$	$02*1*$	$12**0$	–	0	1	0
1	1	$02*1*$	–	$1*02*$	$012**$	$2*1*0$	–	1	0	1
0	00	$0123***$	$0123***$	$013*2**$	$023**1*$	$123***0$	00	10	01	00
1	01	$023**1*$	$1*023**$	$1*03*2*$	$0123***$	$2*13**0$	01	12	00	11
2	10	$013*2**$	$02*20**$	$0123***$	$03*2*1*$	$13*2**0$	11	00	12	10
3	11	$02*13**$	$013*2**$	$13**02*$	$02*13**$	$10**2*3$	10	11	11	01
4	12	$03*2*1*$	$2*1*03*$	$1*023**$	$013*2**$	$3*2*1*0$	12	01	10	12

We present each vertex of R_k via the pair $\delta(v) = \{v, \aleph_\pi(v)\} \in V(R_k)$ $(v \in L_k/\pi)$ of Section 8 and via the k-germ α for which $\delta(v) = \langle F(\alpha)\rangle$, and view R_k as the graph whose vertices are the k-germs α, with adjacency inherited from that of their δ-notation via F^{-1} (i.e. Uncastling). So, $V(R_k)$ is presented as in the natural (k-germ) enumeration (see Section 2 and Subsection 1.3 [8]).

To start with, examples of such presentation are shown in Table 4 for $k = 2$ and 3, where m, $\alpha = \alpha(m)$ and $F(\alpha)$ are shown in the first 3 columns, for $0 \leq m < C_k$. The neighbors of $F(\alpha)$ are presented in the central columns of the table as $F^k(\alpha)$, $F^{k-1}(\alpha)$, …, $F^0(\alpha)$ respectively for the edge colors $k, k-1, \ldots, 0$, with notation given via the effect of function $\aleph$. The last columns yield the k-germs α^k, α^{k-1}, …, α^0 associated via F^{-1} respectively to the listed neighbors $F^k(\alpha)$, $F^{k-1}(\alpha)$, …, $F^0(\alpha)$ of $F(\alpha)$ in R_k.

Table 5. Colored Adjacency Table CAT(4).

m	α	α^4	α^3	α^2	α^1	α^0	m	α	α^4	α^3	α^2	α^1	α^0
–	– –	– –	– –	– –	– –	– –	–	– –	– –	– –	– –	– –	– –
0	000	000	100	010	001	000	7	110	100	111	110	012	010
1	001	001	101	012	000	011	8	111	111	110	122	011	111
2	010	011	121	000	112	110	9	112	101	122	112	010	112
3	011	010	120	011	111	001	10	120	122	011	100	123	120
4	012	012	123	001	110	122	11	121	121	010	121	122	101
5	100	110	000	120	101	100	12	122	120	112	111	121	012
6	101	112	001	123	100	121	13	123	123	012	101	120	123
–	– –	– –	– –	– –	– –	– –	–	– –	– –	– –	– –	– –	– –
		$3**$	$***$	$3**$	$*2*$	$**1$			$3**$	$***$	$3**$	$*2*$	$**1$

For $k = 4$ and 5, Tables 5 and 6 have a similar respective natural enumeration adjacency disposition. We can generalize these tables directly to *Colored Adjacency Tables* denoted $CAT(k)$, for $k > 1$. This way, Theorem 10(A) below is obtained as indicated in the aggregated last row upending Tables 5 and 6 citing the only non-asterisk entry, for each of $i = k, k-2, \ldots, 0$, as a number $j = (k-1), \ldots, 1$ that leads to entry equality in both columns $\alpha = a_{k-1} \cdots a_j \cdots a_1$ and $\alpha^i = a^i_{k-1} \cdots a^i_j \cdots a^i_1$, that is $a_j = a^i_j$. Other important properties are contained in the remaining items of Theorem 10, including (B), that the columns α^0 in all CAT(k), $(k > 1)$, yield an (infinte) integer sequence.

Table 6. Colored Adjacency Table CAT(5).

m	α	α^5	α^4	α^3	α^2	α^1	α^0	m	α	α^5	α^4	α^3	α^2	α^1	α^0
–	– –	– –	– –	– –	– –	– –	– –	–	– –	– –	– –	– –	– –	– –	– –
0	0000	0000	1000	0100	0010	0001	0000	21	1110	1111	1100	1221	0110	1112	1110
1	0001	0001	1001	0101	0012	0000	0011	22	1111	1110	1111	1220	0122	1111	0111
2	0010	0011	1011	0121	0000	0112	0110	23	1112	1122	1101	1233	0112	1110	1222
3	0011	0010	1010	0120	0011	0111	0001	24	1120	1011	1222	1121	0100	1123	1120
4	0012	0012	1012	0123	0001	0110	0122	25	1121	1010	1221	1120	0121	1122	0101
5	0100	0110	1210	0000	1120	1101	1100	26	1122	1112	1220	1223	0111	1121	1122
6	0101	0112	1212	0001	1123	1100	1121	27	1123	1012	1233	1123	0101	1120	1223
7	0110	0100	1200	0111	1110	0012	0010	28	1200	1220	0110	1000	1230	1201	1200
8	0111	0111	1211	0110	1122	0011	1111	29	1201	1223	0112	1001	1234	1200	1231
9	0112	0101	1201	0122	1112	0010	0112	30	1210	1210	0100	1211	1220	1012	1011
10	0120	0122	1232	0011	1100	1223	1220	31	1211	1222	0111	1210	1233	1011	1221
11	0121	0121	1231	0010	1121	1222	1101	32	1212	1212	0101	1232	1223	1010	1212
12	0122	0120	1230	0112	1111	1221	0012	33	1220	1200	1122	1111	1210	0123	0120
13	0123	0123	1234	0012	1101	1220	1233	34	1221	1221	1121	1110	1232	0122	1211
14	1000	1100	0000	1200	1010	1001	1000	35	1222	1211	1120	1222	1222	0121	1112
15	1001	1101	0001	1201	1012	1000	1011	36	1223	1201	1223	1122	1212	0120	1123
16	1010	1121	0011	1231	1000	1212	1210	37	1230	1233	0122	1011	1200	1234	1230
17	1011	1120	0010	1230	1011	1211	1001	38	1231	1232	0121	1010	1231	1233	1201
18	1012	1123	0012	1234	1001	1210	1232	39	1232	1231	0120	1212	1221	1232	1012
19	1100	1000	1110	1100	0120	0101	0100	40	1233	1230	1123	1112	1211	1231	0123
20	1101	1001	1112	1101	0123	0100	0121	41	1234	1234	0123	1012	1201	1230	1234
–	– –	– –	– –	– –	– –	– –	– –	–	– –	– –	– –	– –	– –	– –	– –
		4***	****	4***	*3**	**2*	***1			4***	****	4***	*3**	**2*	***1

Theorem 10. *Let: $k > 1$, $j(\alpha^k) = k-1$ and $j(\alpha^{i-1}) = i$, $(i = k-1, \ldots, 1)$. Then:*

(A) each column α^{i-1} in $CAT(k)$, for $i \in [k] \cup \{k+1\}$, preserves the respective $j(\alpha^{i-1})$-th entry of α;
(B) the columns α^k of all $CAT(k)$'s for $k > 1$ coincide into an RGS sequence and thus into an integer sequence $\mathcal{S}_0$, the first C_k terms of which form an idempotent permutation for each k;
(C) the integer sequence $\mathcal{S}_1$ given by concatenating the m-indexed intervals $[0,2), [2,5), \ldots, [C_{k-1}, C_k)$, etc. in column α^{k-1} of the corresponding tables $CAT(2)$, $CAT(3), \ldots$, $CAT(k)$, etc. allows to encode all columns α^{k-1}'s;
(D) for each $k > 1$, there is an idempotent permutation given in the m-indexed interval $[0, C_k)$ of the column α^{k-1} of $CAT(k)$; such permutation equals the one given in the interval $[0, C_k)$ of the column α^{k-2} of $CAT(k+1)$.

Proof. (A) holds as a continuation of the observation made above with respect to the last aggregated row in Figure 5. Let α be a k-germ. Then α shares with α^k (e.g. the leftmost column α^i in Tables 4, 5 and 6, for $0 \le i \le k$) all the entries to the left of the leftmost entry 1, which yields (B). Note that if $k = 3$ then $m = 2, 3, 4$ yield for α^{k-1} the idempotent permutation $(2,0)(4,1)$, illustrating (C). (D) can be proved similarly. □

The sequences in Theorem 10 (B)-(C) start as follows, with intervals ended in ";":

$\{0\}\cup\mathbb{Z}^+=$	0,	1;	2,	3,	4;	5,	6,	7,	8,	9,	10,	11,	12,	13;	14	15,	16,...
$(B)=$	0,	1;	3,	2,	4;	7,	9,	5,	8,	6,	12,	11,	10,	13;	19,	20,	25,...
$(C)=$	1,	0;	0,	3,	1;	0,	1,	8,	7,	12,	3,	2,	9,	4;	0,	1,	3,...

Remark 2. With the notation of Section 11 and Theorem 9, for each of the involutions α^i $(0 < i < k)$, it holds that $\alpha^i\Theta = \Theta\alpha^{k-i}$. This implies that

(A) every k-colored edge represents an adjacency via $\Phi' = F\alpha_k F^{-1}$ and
(B) every $(k-1)$-colored edge represents an adjacency via $\Psi' = F\Psi F^{-1}$.

In addition, the reflection symmetry of Φ' yields the sequence S_0 cited in Theorem 10(B). A similar observation yields from Ψ' the sequence S_1 cited in Theorem 10(C).

Given a k-germ $\alpha = a_{k-1} \cdots a_1$, we want to express $\alpha^k, \alpha^{k-1}, \ldots, \alpha^0$ as functions of α. Given a substring $\alpha' = a_{k-j} \cdots a_{k-i}$ of α $(0 < j \le i < k)$, let:

(a) the *reverse string* off α' be $\psi(\alpha') = a_{k-i} \cdots a_{k-j}$;
(b) the *ascent* of α' be

  (i) its maximal initial ascending substring, if $a_{k-j} = 0$, and
  (ii) its maximal initial non-descending substring with at most 2 equal nonzero terms, if $a_{k-j} > 0$.

Then, the following remarks allow to express the k-germs $\alpha^p = \beta = b_{k-1} \cdots b_1$ via the colors $p = k, k-1, \dots, 0$, independently of F^{-1} and F.

Remark 3. Assume $p = k$. If $a_{k-1} = 1$, take $0|\alpha$ instead of $\alpha = a_{k-1} \cdots a_1$, with $k-1$ instead of k, removing afterwards from the resulting β the added leftmost 0. Now, let $\alpha_1 = a_{k-1} \cdots a_{k-i_1}$ be the ascent of α. Let $B_1 = i_1 - 1$, where $i_1 = ||\alpha_1||$ is the length of α_1. It can be seen that β has ascent $\beta_1 = b_{k-1} \cdots b_{k-i_1}$ with $\alpha_1 + \psi(\beta_1) = B_1 \cdots B_1$. If $\alpha \neq \alpha_1$, let α_2 be the ascent of $\alpha \setminus \alpha_1$. Then there is a $||\alpha_2||$-germ β_2 with $\alpha_2 + \psi(\beta_2) = B_2 \cdots B_2$ and $B_2 = ||\alpha_1|| + ||\alpha_2|| - 2$. Inductively when feasible for $j > 2$, let α_j be the ascent of $\alpha \setminus (\alpha_1|\alpha_2| \cdots |\alpha_{j-1})$. Then there is a $||\alpha_j||$-germ β_j with $\alpha_j + \psi(\beta_j) = B_j \cdots B_j$ and $B_j = ||\alpha_{j-1}|| + ||\alpha_j|| - 2$. This way, $\beta = \beta_1|\beta_2| \cdots |\beta_j| \cdots$.

Remark 4. Assume $k > p > 0$. By Theorem 10 (A), if $p < k-1$, then $b_{p+1} = a_{p+1}$; in this case, let $\alpha' = \alpha \setminus \{a_{k-1} \cdots a_q\}$ with $q = p+1$. If $p = k-1$, let $q = k$ and let $\alpha' = \alpha$. In both cases (either $p < k-1$ or $p = k-1$) let $\alpha'_1 = a_{q-1} \cdots a_{k-i_1}$ be the ascent of α'. It can be seen that $\beta' = \beta \setminus \{b_{k-1} \cdots b_q\}$ has ascent $\beta'_1 = b_{k-1} \cdots b_{k-i_1}$ where $\alpha'_1 + \psi(\beta'_1) = B'_1 \cdots B'_1$ with $B'_1 = i_1 + a_q$. If $\alpha' \neq \alpha'_1$ then let α'_2 be the ascent of $\alpha' \setminus \alpha'_1$. Then there is a $||\alpha'_2||$-germ β'_2 where $\alpha'_2 + \psi(\beta'_2) = B'_2 \cdots B'_2$ with $B'_2 = ||\alpha'_1|| + ||\alpha'_2|| - 2$. Inductively when feasible for $j > 2$, let α_j be the ascent of $\alpha' \setminus (\alpha'_1|\alpha'_2| \cdots |\alpha'_{j-1})$. Then there is a $||\alpha'_j||$-germ β'_j where $\alpha'_j + \psi(\beta'_j) = B'_j \cdots B'_j$ with $B'_j = ||\alpha'_{j-1}|| + ||\alpha'_j|| - 2$. This way, $\beta' = \beta'_1|\beta'_2| \cdots |\beta'_j| \cdots$.

We process the left-hand side from position q. If $p > 1$, we set $a_{a_q+2} \cdots a_q + \psi(b_{b_q+2} \cdots b_q)$ to equal a constant string $B \cdots B$, where $a_{a_q+2} \cdots a_q$ is an ascent and $a_{a_q+2} = b_{b_q+2}$. Expressing all those numbers a_i, b_i as a_i^0, b_i^0, respectively, in order to keep an inductive approach, let $a_q^1 = a_{a_q+2}$. While feasible, let $a_{q+1}^1 = a_{a_q+1}$, $a_{q+2}^1 = a_{a_q}$ and so on. In this case, let $b_q^1 = b_{b_q+2}$, $b_{q+1}^1 = b_{b_q+1}$, $b_{q+2}^1 = b_{b_q}$ and so on. Now, $a_{a_q^1+2}^1 \cdots a_q^1 + \psi(b_{b_q^1+2}^1 \cdots b_q^1)$ equals a constant string, where $a_{a_q^1+2}^1 \cdots a_q^1$ is an ascent and $a_{a_q^1+2}^1 = b_{b_q^1+2}^1$. The continuation of this procedure produces a subsequent string a_q^2 and so on, until what remains to reach the leftmost entry of α is smaller than the needed space for the procedure itself to continue, in which case, a remaining initial ascent is shared by both α and β. This allows to form the left-hand side of $\alpha^p = \beta$ by concatenation.

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