Article

Degree-based topological indices of product graphs

Xiaojing Wang¹, Zhen Lin²,* and Lianying Miao¹

¹ School of Mathematics, China University of Mining and Technology, Xuzhou, 221116, Jiangsu, P.R. China.
² School of Mathematics and Statistics, Qinghai Normal University, Xining, 810008, Qinghai, P.R. China.
* Correspondence: lnlinzhen@163.com

Academic Editor: Aisha Javed
Received: 21 August 2021; Accepted: 12 October 2021; Published: 1 November 2021.

Abstract: In this paper, we obtain the quantitative calculation formula of the degree-based topological indices of four standard product for the path and regular graphs, which unify to solve the question on product of these basic graphs without having to deal with it one by one separately. As applications, we give corresponding calculation formula of the general Randić index, the first general Zagreb index and the general sum-connectivity index.

Keywords: Degree-based topological indices; Cartesian product; Direct product; Strong product; Lexicographic product.

MSC: 05C07; 05C09; 05C76; 05C90; 05C92.

1. Introduction

Throughout the article, G is a simple undirected connected graph with vertex set \( V(G) \) and edge set \( E(G) \). The number of vertices and edges of G is called order and size, respectively. If the vertices \( u \) and \( v \) are adjacent, then we write \( u \sim v \). For \( v \in V(G) \), \( d_v = d_G(v) \) denotes the degree of vertex \( v \) in G. Denote by \( P_n \) and \( K_{1,n-1} \) the path and star with \( n \) vertices, respectively.

Cheminformatics is a new interdisciplinary composed of chemistry, mathematics and information science, which contributes a major role in the field of chemical sciences by implementing graph theory to mathematical modeling of chemical occurrence. In cheminformatics, the topological indices play a significant role in predicting the biological activities and properties of chemical compounds due to the fact that the numerical characteristics of topological indices reflect certain physico-chemical properties of chemical compounds, such as boiling point, stability, strain energy etc. A large number of topological indices have been studied in the models of Quantitative structure-activity relationships (QSAR) and structure-property relationships (QSPR), such as Wiener index, Randić index, Zagreb index, ABC index and so on.

The study on degree-based topological indices has been one of the hotspots in cheminformatics [1]. Let \( K = \{ (i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq j \leq n - 1 \} \) and \( m_{i,j} = m_{i,j}(G) \) be the number of edges in G joining vertices of degree \( i \) and \( j \). For any set of numbers \( \{ \varphi_{i,j} \}_{(i, j) \in K} \), the general formula of degree-based topological indices is

\[
DTI(G) = \sum_{(i, j) \in K} m_{i,j}(G) \varphi_{i,j}.
\]

In particular, we obtain the first Zagreb index and the second Zagreb index when \( \varphi_{i,j} = i + j \) and \( \varphi_{i,j} = ij \), respectively.

In 1998, the general Randić index of a graph G, introduced by Bollobás and Erdős [2], is defined as

\[
R^t = R^t(G) = \sum_{v \in V(G)} \left( \frac{d_v}{d_{G}} \right)^t, \quad t \in \mathbb{R}.
\]

Clearly, we have that \( R^0 \) is the number of edges, \( R^{-\frac{1}{2}} \) is the Randić index [3], \( R^{-1} \) is the modified second Zagreb index [3], \( R^2 \) is the reciprocal Randić index [4], \( R^2 \) is the second Hyper-Zagreb index [4], \( R^3 \) is the second Zagreb index [5], etc.
In 2005, the first general Zagreb index of a graph $G$ was introduced by Li and Zheng [6] and is defined as

$$Z^t = Z^t(G) = \sum_{v \in V(G)} d_i^t = \sum_{v \in V(G)} \left(d_i^{t-1} + d_j^{t-1}\right), \quad t \in \mathbb{R}.$$ 

It is easy to see that $Z^0$ is the number of vertices, $Z^1$ is twice the number of edges, $Z^2$ is the first Zagreb index [5], $Z^3$ is the forgotten topological index [7], etc.

In 2010, Zhou and Trinajstić [8] proposed the general sum-connectivity index of a graph $G$ as follows:

$$\chi^t = \chi^t(G) = \sum_{v \in V(G)} (d_i + d_j)^t, \quad t \in \mathbb{R}.$$ 

It is not difficult to find that $2\chi^{-1}$ is the harmonic index [9], $\chi^{-\frac{1}{2}}$ is the sum-connectivity index [10], $\chi^{\frac{1}{2}}$ is the reciprocal sum-connectivity index [11], etc.

The product graphs are useful in constructing many important structural models with regularities [12], especially the following four standard product graphs which are widely used in network design [13], multiprocessor system [14], automata theory [15] and other fields. Let $G_1$ and $G_2$ be two graphs with disjoint vertex sets $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$, respectively. The Cartesian product of $G_1$ and $G_2$, denoted by $G_1 \square G_2$ is the graph, where $(u_i, v_j) \sim (u_r, v_s)$ if either $(u_i = u_r$ and $v_j = v_s)$ or $(u_i \sim u_r$ and $v_j = v_s).$ The direct product or Kronecker product of $G_1$ and $G_2$, denoted by $G_1 \otimes G_2$, is the graph where $(u_i, v_j) \sim (u_r, v_s)$ if $u_i \sim u_r$ in $G_1$ and $v_j \sim v_s$ in $G_2.$ The strong product of $G_1$ and $G_2$, denoted by $G_1 \boxtimes G_2$, is graph where $(u_i, v_j) \sim (u_r, v_s)$ if either $(u_i = u_r$ and $u_j = u_s)$ or $(u_i = u_r$ in $G_1$ and $u_j = u_s)$ or $(u_i \sim u_r$ in $G_1$ and $u_j \sim u_s$ in $G_2).$ The lexicographic product of $G_1$ and $G_2$, denoted by $G_1[G_2]$, is the graph where $(u_i, v_j) \sim (u_r, v_s)$ if either $(u_i \sim u_r$ in $G_1$) or $(u_i = u_r$ and $v_j \sim v_s$ in $G_2).$

In this paper, we give a unified approach to solve the computational problems of degree-based topological indices of standard product graphs for the path and regular graphs, which is generalization of many specific degree-based topological indices. As applications, the corresponding calculation formulas of the general Randić index, the first general Zagreb index and the general sum-connectivity index are obtained.

2. Cartesian product

Theorem 1. Let $P_{n_1}$ and $P_{n_2}$ be two path graphs of order $n_1$ and $n_2$, respectively. Then

$$DTI\left(P_{n_1} \square P_{n_2}\right) = 8\varphi_{2,3} + 2(n_1 + n_2 - 6)\varphi_{3,3} + 2(n_1 + n_2 - 4)\varphi_{3,4} + (2n_1n_2 - 5n_1 - 5n_2 + 12)\varphi_{4,4}$$

for $n_1 \geq n_2 \geq 3$.

Proof. By the definition of Cartesian product, we obtain the basic information on $P_{n_1} \square P_{n_2}$ in the Table 1.

<table>
<thead>
<tr>
<th>$m_{2,3}$</th>
<th>$m_{3,3}$</th>
<th>$m_{3,4}$</th>
<th>$m_{4,4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$2(n_1 + n_2 - 6)$</td>
<td>$2(n_1 + n_2 - 4)$</td>
<td>$2n_1n_2 - 5n_1 - 5n_2 + 12$</td>
</tr>
</tbody>
</table>

Thus we have

$$DTI\left(P_{n_1} \square P_{n_2}\right) = \sum_{(i,j) \in K} m_{i,j}(G) \varphi_{i,j} = 8\varphi_{2,3} + 2(n_1 + n_2 - 6)\varphi_{3,3} + 2(n_1 + n_2 - 4)\varphi_{3,4} + (2n_1n_2 - 5n_1 - 5n_2 + 12)\varphi_{4,4}.$$ 

This completes the proof.

Corollary 1. Let $P_{n_1}$ and $P_{n_2}$ be two path graphs of order $n_1$ and $n_2$, respectively. Then

$$R^t\left(P_{n_1} \square P_{n_2}\right) = 8 \cdot 2^t \cdot 3^t(n_1 + n_2 - 6) + 2 \cdot 12^t(n_1 + n_2 - 4) + 16^t(2n_1n_2 - 5n_1 - 5n_2 + 12),$$

$$Z^t\left(P_{n_1} \square P_{n_2}\right) = 8(2^{t-1} + 3^{t-1}) + 4 \cdot 3^{t-1}(n_1 + n_2 - 6) + 2(n_1 + n_2 - 4)(3^{t-1} + 4^{t-1}) + 2 \cdot 4^{t-1}(2n_1n_2 - 5n_1 - 5n_2 + 12),$$
χ^t(P_{n_1} □ P_{n_2}) = 8 \cdot 5^t + 2 \cdot 6^t (n_1 + n_2 - 6) + 2 \cdot 7^t (n_1 + n_2 - 4) + 8^t (2n_1n_2 - 5n_1 - 5n_2 + 12)

for n_1 \geq n_2 \geq 3.

**Theorem 2.** Let P_{n_1} and G_r be a path and a r-regular graph of order n_1 and n_2, respectively. Then

\[ DTI(P_{n_1} □ G_r) = r n_2 \varphi_{r+1,r+1} + 2 n_2 \varphi_{r+1,r+2} + \frac{1}{2} [r n_2 (n_1 - 2) + 2n_1n_2 - 6n_2] \varphi_{r+2,r+2} \]

for n_1 \geq n_2 \geq 2.

**Proof.** By the definition of Cartesian product, we obtain the basic information on P_{n_1} □ G_r in the following Table 2.

<table>
<thead>
<tr>
<th>m_{r+1,r+1}</th>
<th>m_{r+1,r+2}</th>
<th>m_{r+2,r+2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>rn_2</td>
<td>2n_2</td>
<td>\frac{rn_2(n_1 - 2)}{2} + n_1n_2 - 3n_2</td>
</tr>
</tbody>
</table>

Thus we have

\[ DTI(P_{n_1} □ G_r) = \sum_{(i,j) \in K} m_{i,j} (G) \varphi_{i,j} = r n_2 \varphi_{r+1,r+1} + 2 n_2 \varphi_{r+1,r+2} + \frac{1}{2} [r n_2 (n_1 - 2) + 2n_1n_2 - 6n_2] \varphi_{r+2,r+2}. \]

This completes the proof. □

**Corollary 2.** Let P_{n_1} and G_r be a path and a r-regular graph of order n_1 and n_2, respectively. Then

\[ R^t(P_{n_1} □ G_r) = r n_2 (r + 1)^{2t} + 2 n_2 (r + 1)^t (r + 2)^t + \left[ \frac{rn_2(n_1 - 2)}{2} + n_2(n_1 - 3) \right] (r + 2)^t, \]

\[ Z^t(P_{n_1} □ G_r) = 2rn_2 (r + 1)^{t-1} + 2n_2 [(r + 1)^{t-1} + (r + 2)^{t-1}] + 2 (r + 2)^{t-1} \left[ \frac{rn_2(n_1 - 2)}{2} + n_2(n_1 - 3) \right], \]

\[ \chi^t(P_{n_1} □ G_r) = 2^t r n_2 (r + 1)^t + 2 n_2 (2r + 3)^t + 2^t (r + 2)^t \left[ \frac{rn_2(n_1 - 2)}{2} + n_2(n_1 - 3) \right] \]

for n_1 \geq n_2 \geq 2.

**Theorem 3.** Let G_r and P_{n_2} be a r-regular and a path of order n_1 and n_2, respectively. Then

\[ DTI(G_r □ P_{n_2}) = r n_1 \varphi_{r+1,r+1} + 2 n_1 \varphi_{r+1,r+2} + \frac{1}{2} [r n_1 (n_2 - 2) + 2n_1n_2 - 6n_1] \varphi_{r+2,r+2} \]

for n_1 \geq n_2 \geq 2.

**Proof.** By the definition of Cartesian product, we obtain the basic information on G_r □ P_{n_2} in the following Table 3.

<table>
<thead>
<tr>
<th>m_{r+1,r+1}</th>
<th>m_{r+1,r+2}</th>
<th>m_{r+2,r+2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>rn_1</td>
<td>2n_1</td>
<td>\frac{rn_1(n_2 - 2)}{2} + n_1n_2 - 3n_1</td>
</tr>
</tbody>
</table>

Thus we have

\[ DTI(G_r □ P_{n_2}) = \sum_{(i,j) \in K} m_{i,j} (G) \varphi_{i,j} = r n_1 \varphi_{r+1,r+1} + 2 n_1 \varphi_{r+1,r+2} + \frac{1}{2} [r n_1 (n_2 - 2) + 2n_1n_2 - 6n_1] \varphi_{r+2,r+2}. \]
This completes the proof. □

**Corollary 3.** Let $G_r$ and $P_{n_2}$ be a $r$-regular and a path of order $n_1$ and $n_2$, respectively. Then

\[
R^j(G_r \square P_{n_2}) = rn_1(r + 1)^{2j} + 2n_1(r + 1)^j(r + 2)^{j+1} + \left[\frac{2r_1(n_2 - 2)}{2} + n_1(n_2 - 3)\right](r + 2)^{2j},
\]

\[
Z^j(G_r \square P_{n_2}) = 2rn_1(r + 1)^{2j-1} + 2n_1\left[(r + 1)^{j+1} + (r + 2)^{j-1}\right] + 2(r + 2)^{j+1}\left[\frac{2r_1(n_2 - 2)}{2} + n_1(n_2 - 3)\right],
\]

\[
\chi^j(G_r \square P_{n_2}) = 2^jrn_1(r + 1)^j + 2n_1(2r + 3)^j + 2^j(r + 2)^j\left[\frac{2r_1(n_2 - 2)}{2} + n_1(n_2 - 3)\right]
\]

for $n_1 \geq n_2 \geq 2$.

**Theorem 4.** Let $G_1$ and $G_2$ be a $r_1$-regular graph and a $r_2$-regular graph with order $n_1$ and $n_2$, respectively. Then

\[
DTI(G_1 \square G_2) = \frac{n_1n_2(r_1 + r_2)}{2} \varphi_{r_1+r_2,r_1+r_2}
\]

for $n_1 \geq n_2 \geq 2$.

**Proof.** By the definition of Cartesian product, we have $G_1 \square G_2$ is a $(r_1 + r_2)$-regular graph with $\frac{n_1n_2(r_1 + r_2)}{2}$ edges. Thus

\[
DTI(G_1 \square G_2) = \sum_{(i,j) \in K} m_{i,j}(G) \varphi_{i,j} = \frac{n_1n_2(r_1 + r_2)}{2} \varphi_{r_1+r_2,r_1+r_2}.
\]

This completes the proof. □

**Corollary 4.** Let $G_1$ and $G_2$ be a $r_1$-regular graph and a $r_2$-regular graph with order $n_1$ and $n_2$, respectively. Then

\[
R^j(G_1 \square G_2) = \frac{n_1n_2(r_1 + r_2)^{2j+1}}{2},
\]

\[
Z^j(G_1 \square G_2) = n_1n_2(r_1 + r_2)^j,
\]

\[
\chi^j(G_1 \square G_2) = 2^{j-1}n_1n_2(r_1 + r_2)^{j+1}
\]

for $n_1 \geq n_2 \geq 2$.

3. Direct product

**Theorem 5.** Let $P_{n_1}$ and $P_{n_2}$ be two path graphs of order $n_1$ and $n_2$, respectively. Then

\[
DTI(P_{n_1} \square P_{n_2}) = 4\varphi_{1,4} + 4\varphi_{2,2} + 4(n_1 + n_2 - 6) \varphi_{2,4} + 2(n_1 - 3)(n_2 - 3) \varphi_{4,4}
\]

for $n_1 \geq n_2 \geq 3$.

**Proof.** By the definition of direct product, we obtain the basic information on $P_{n_1} \square P_{n_2}$ in the following Table 4.

<table>
<thead>
<tr>
<th>$m_{1,4}$</th>
<th>$m_{2,2}$</th>
<th>$m_{2,4}$</th>
<th>$m_{4,4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>$4(n_1 + n_2 - 6)$</td>
<td>$2(n_1 - 3)(n_2 - 3)$</td>
</tr>
</tbody>
</table>

Thus we have

\[
DTI(P_{n_1} \square P_{n_2}) = \sum_{(i,j) \in K} m_{i,j}(G) \varphi_{i,j} = 4\varphi_{1,4} + 4\varphi_{2,2} + 4(n_1 + n_2 - 6) \varphi_{2,4} + 2(n_1 - 3)(n_2 - 3) \varphi_{4,4}.
\]
This completes the proof.

**Corollary 5.** Let $P_{n_1}$ and $P_{n_2}$ be two path graphs of order $n_1$ and $n_2$, respectively. Then

- $R^t(P_{n_1} \otimes P_{n_2}) = 8 \cdot 4^t + 2 \cdot 8^t \cdot (n_1 n_2 - n_1 - n_2 - 3)$,
- $Z^t(P_{n_1} \otimes P_{n_2}) = 4 \left[ 1 + 2^t + (n_1 + n_2 - 6) \left( 2^{t-1} + 4^{t-1} \right) \right] + 4^t \left[ 1 + (n_1 - 3) (n_2 - 3) \right]$,
- $\chi^t(P_{n_1} \otimes P_{n_2}) = 4 \left[ 4^t + 5^t + 6^t (n_1 + n_2 - 6) \right] + 2^{3t+1} (n_1 - 3) (n_2 - 3)$

for $n_1 \geq n_2 \geq 3$.

**Theorem 6.** Let $P_{n_1}$ and $G_r$ be a path and a $r$-regular of order $n_1$ and $n_2$, respectively. Then

$$\text{DTI} \left( P_{n_1} \otimes G_r \right) = 2rn_2 \phi_{r,2r} + rn_2 (n_1 - 3) \phi_{2r,2r}$$

for $n_1 \geq n_2 \geq 3$.

**Proof.** By the definition of direct product, we obtain the basic information on $P_{n_1} \otimes G_r$ in the following Table 5.

<table>
<thead>
<tr>
<th>$m_{r,2r}$</th>
<th>$m_{2r,2r}$</th>
<th>$rn_2 (n_1 - 3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2rn_2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus we have

$$\text{DTI} \left( P_{n_1} \otimes G_r \right) = \sum_{(i,j) \in K} m_{i,j} (G) \phi_{i,j} = 2rn_2 \phi_{r,2r} + rn_2 (n_1 - 3) \phi_{2r,2r}$$

This completes the proof.

**Corollary 6.** Let $P_{n_1}$ and $G_r$ be a path and a $r$-regular of order $n_1$ and $n_2$, respectively. Then

- $R^t(P_{n_1} \otimes G_r) = 2^{t+1} \cdot 2^{t+1} n_2 + 2^{t+1} \cdot 2^{t+1} n_2 (n_1 - 3)$,
- $Z^t(P_{n_1} \otimes G_r) = (2 + 2^t) (r^t n_2 + 2^t r^t n_2 (n_1 - 3))$,
- $\chi^t(P_{n_1} \otimes G_r) = 2n_2 \cdot 3^t, \cdot 2^{t+1} + 4^t 2^{t+1} n_2 (n_1 - 3)$

for $n_1 \geq n_2 \geq 3$.

**Theorem 7.** Let $G_r$ and $P_{n_2}$ be a $r$-regular and a path of order $n_1$ and $n_2$, respectively. Then

$$\text{DTI} \left( G_r \otimes P_{n_2} \right) = 2rn_1 \phi_{r,2r} + rn_1 (n_2 - 3) \phi_{2r,2r}$$

for $n_1 \geq n_2 \geq 3$.

**Proof.** By the definition of direct product, we obtain the basic information on $G_r \otimes P_{n_2}$ in the following Table 6.

<table>
<thead>
<tr>
<th>$m_{r,2r}$</th>
<th>$m_{2r,2r}$</th>
<th>$rn_1 (n_2 - 3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2rn_1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus we have

$$\text{DTI} \left( G_r \otimes P_{n_2} \right) = \sum_{(i,j) \in K} m_{i,j} (G) \phi_{i,j} = 2rn_1 \phi_{r,2r} + rn_1 (n_2 - 3) \phi_{2r,2r}$$

This completes the proof.
Corollary 7. Let $G_r$ and $P_{n_2}$ be a $r$-regular and a path of order $n_1$ and $n_2$, respectively. Then

\[
R^t(G_r \otimes P_{n_2}) = 2^t + 1 \cdot 2^{t+1} n_1 + 2^{t+1} n_1 (n_2 - 3),
\]

\[
Z^t(G_r \otimes P_{n_2}) = r^t n_1 (2 + 2^t) + 2^t r^t n_1 (n_2 - 3),
\]

\[
\chi^t(G_r \otimes P_{n_2}) = 2 \cdot 3^t \cdot r^t + 1 + 4^t r^t n_1 (n_2 - 3)
\]

for $n_1 \geq n_2 \geq 3$.

Theorem 8. Let $G_1$ and $G_2$ be a $r_1$-regular graph and a $r_2$-regular graph with order $n_1$ and $n_2$, respectively. Then

\[
DTI(G_1 \otimes G_2) = \frac{r_1 r_2 n_1 n_2}{2} \varphi_{r_1 r_2, r_1 r_2}
\]

for $n_1 \geq n_2 \geq 2$.

Proof. By the definition of direct product, we have $G_1 \otimes G_2$ is a $r_1 r_2$-regular graph with $\frac{r_1 r_2 n_1 n_2}{2}$ edges. Thus

\[
DTI(G_1 \otimes G_2) = \sum_{(i,j) \in k} m_{i,j}(G) \varphi_{i,j} = \frac{r_1 r_2 n_1 n_2}{2} \varphi_{r_1 r_2, r_1 r_2}.
\]

This completes the proof. \(\square\)

Corollary 8. Let $G_1$ and $G_2$ be a $r_1$-regular graph and a $r_2$-regular graph with order $n_1$ and $n_2$, respectively. Then

\[
R^t(G_1 \otimes G_2) = \frac{n_1 n_2 (r_1 r_2)^{t+1}}{2},
\]

\[
Z^t(G_1 \otimes G_2) = n_1 n_2 (r_1 r_2)^t,
\]

\[
\chi^t(G_1 \otimes G_2) = 2^{t-1} n_1 n_2 (r_1 r_2)^{t+1}
\]

for $n_1 \geq n_2 \geq 2$.

4. Strong product

Theorem 9. Let $P_{n_1}$ and $P_{n_2}$ be two path graphs of order $n_1$ and $n_2$, respectively. Then

\[
DTI(P_{n_1} \otimes P_{n_2}) = 8 \varphi_{3,5} + 4 \varphi_{3,8} + 2 (n_1 + n_2 - 4) \varphi_{5,5} + (6n_1 + 6n_2 - 32) \varphi_{5,8} + [4n_1 n_2 - 11 (n_1 + n_2) + 30] \varphi_{8,8}
\]

for $n_1 \geq n_2 \geq 3$.

Proof. By the definition of strong product, we obtain the basic information on $P_{n_1} \otimes P_{n_2}$ in the following Table 7.

<table>
<thead>
<tr>
<th>$m_{3,5}$</th>
<th>$m_{3,8}$</th>
<th>$m_{5,5}$</th>
<th>$m_{5,8}$</th>
<th>$m_{8,8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4</td>
<td>$2(n_1 + n_2) - 8$</td>
<td>$6(n_1 + n_2) - 32$</td>
<td>$4n_1 n_2 - 11(n_1 + n_2) + 30$</td>
</tr>
</tbody>
</table>

Thus we have

\[
DTI(P_{n_1} \otimes P_{n_2}) = \sum_{(i,j) \in k} m_{i,j}(G) \varphi_{i,j}
\]

\[
= 8 \varphi_{3,5} + 4 \varphi_{3,8} + 2 (n_1 + n_2 - 4) \varphi_{5,5} + (6n_1 + 6n_2 - 32) \varphi_{5,8} + [4n_1 n_2 - 11 (n_1 + n_2) + 30] \varphi_{8,8}.
\]

This completes the proof. \(\square\)
Corollary 9. Let $P_{n_1}$ and $P_{n_2}$ be two path graphs of order $n_1$ and $n_2$, respectively. Then
\[
R^t(P_{n_1} \otimes P_{n_2}) = 8 \cdot 15^t + 4 \cdot 24^t + 25^t \cdot [2(n_1 + n_2 - 4)] + 40^t \cdot (6n_1 + 6n_2 - 32)
\]
\[
+ 64^t \cdot [4n_1n_2 - 11(n_1 + n_2) + 30],
\]
\[
Z^t(P_{n_1} \otimes P_{n_2}) = 8 \cdot (3^{t-1} + 3^{t-1}) + 4 \cdot (3^{t-1} + 8^{t-1}) + 4 \cdot 5^{t-1} \cdot (n_1 + n_2) - 4
\]
\[
+ 4(n_1 + 6n_2 - 32) \cdot (5^{t-1} + 8^{t-1}) + 2 \cdot 8^{t-1} \cdot [4n_1n_2 - 11(n_1 + n_2) + 30],
\]
\[
\chi^t(P_{n_1} \otimes P_{n_2}) = 8^{t+1} + 4 \cdot 11^t + 10^t \cdot 2(n_1 + n_2) - 8] + 13^t \cdot (6n_1 + 6n_2 - 32)
\]
\[
+ 16^t \cdot [4n_1n_2 - 11(n_1 + n_2) + 30]
\]
for $n_1 \geq n_2 \geq 3$.

Theorem 10. Let $P_{n_1}$ and $G_r$ be a path and a $r$-regular of order $n_1$ and $n_2$, respectively. Then
\[
DTI(P_{n_1} \otimes G_r) = rn_2 \varphi_{2r+1,2r+1} + 2(n_1 + n_2) \varphi_{2r+1,3r+2} + \frac{1}{2} [n_1n_2(3r+2) - 2n_2(4r + 3)] \varphi_{3r+2,3r+2}
\]
for $n_1 > n_2 \geq 2$.

Proof. By the definition of strong product, we obtain the basic information on $P_{n_1} \otimes G_r$ in the following Table 8.

<table>
<thead>
<tr>
<th>$m_{2r+1,2r+1}$</th>
<th>$m_{2r+1,3r+2}$</th>
<th>$m_{3r+2,3r+2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$rn_2$</td>
<td>$2(n_1 + n_2)$</td>
<td>$n_1n_2(\frac{3r}{2} + 1) - n_2(4r + 3)$</td>
</tr>
</tbody>
</table>

Thus we have
\[
DTI(P_{n_1} \otimes G_r) = \sum_{(i,j) \in K} m_{i,j}(G) \varphi_{i,j}
\]
\[
=rn_2 \varphi_{2r+1,2r+1} + 2(n_1 + n_2) \varphi_{2r+1,3r+2} + \frac{1}{2} [n_1n_2(3r+2) - 2n_2(4r + 3)] \varphi_{3r+2,3r+2}.
\]
This completes the proof. □

Corollary 10. Let $P_{n_1}$ and $G_r$ be a path and a $r$-regular of order $n_1$ and $n_2$, respectively. Then
\[
R^t(P_{n_1} \otimes G_r) = rn_2(2r + 1)^{2i} + 2n_2(2r + 1)(2r + 1)^{i} + n_2 \varphi_{2r+1} + \frac{3r}{2} + 1 - n_2(4r + 3)(3r + 2)^{2i},
\]
\[
Z^t(P_{n_1} \otimes G_r) = 2rn_2(2r + 1)^{t-1} + 2n_2(2r + 1)(2r + 1)^{t-1} + 2n_2(2r + 1)^{t-1} + 2(3r + 2)^{t-1} + n_2 \varphi_{2r+1} + \frac{3r}{2} + 1 - n_2(4r + 3),
\]
\[
\chi^t(P_{n_1} \otimes G_r) = 2r^2n_2(2r + 1)^{t} + 2n_2(2r + 1)(5r + 3)^{t} + 2n_2(2r + 1)^{t} + n_2 \varphi_{2r+1} + \frac{3r}{2} + 1 - n_2(4r + 3)
\]
for $n_1 > n_2 \geq 2$.

Theorem 11. Let $G_r$ and $P_{n_2}$ be a $r$-regular and a path of order $n_1$ and $n_2$, respectively. Then
\[
DTI(G_r \otimes P_{n_2}) = rn_1 \varphi_{2r+1,2r+1} + 2(n_1 + n_2) \varphi_{2r+1,3r+2} + \frac{1}{2} [n_1n_2(3r+2) - 2n_1(4r + 3)] \varphi_{3r+2,3r+2}
\]
for $n_1 \geq n_2 \geq 3$.

Proof. By the definition of strong product, we obtain the basic information on $G_r \otimes P_{n_2}$ in the following Table 9.
Corollary 11. Let $G_r$ and $P_{n_2}$ be a $r$-regular and a path of order $n_1$ and $n_2$, respectively. Then

$$R^t(G_r \otimes P_{n_2}) = r n_1 (2r + 1)^{2t} + 2 n_1 (r + 1) (2r + 1)^t (3r + 2)^t + (3r + 2)^t = n_1 n_2 \left( \frac{3r}{2} + 1 \right) - n_1 \left( 4r + 3 \right),$$

$$Z^t(G_r \otimes P_{n_2}) = 2 r n_1 (2r + 1)^{t-1} + 2 n_1 (r + 1) \left[ (2r + 1)^{t-1} (3r + 2)^{t-1} + 2 (3r + 2)^{t-1} \right] + 2 (3r + 2)^{t-1} = n_1 n_2 \left( \frac{3r}{2} + 1 \right) - n_1 \left( 4r + 3 \right),$$

$$\chi^t(G_r \otimes P_{n_2}) = 2^t r n_1 (2r + 1)^t + 2 n_1 (r + 1) (5r + 3)^t + 2^t (3r + 2)^t = n_1 n_2 \left( \frac{3r}{2} + 1 \right) - n_1 \left( 4r + 3 \right)$$

for $n_1 \geq n_2 \geq 3$.

Theorem 12. Let $G_1$ and $G_2$ be a $r_1$-regular graph and a $r_2$-regular graph with order $n_1$ and $n_2$, respectively. Then

$$DTI(G_1 \otimes G_2) = \frac{n_1 n_2 (r_1 r_2 + r_1 + r_2)}{2} \varphi_{r_1 r_2 + r_1 + r_2}$$

for $n_1 \geq n_2 \geq 2$.

Proof. By the definition of strong product, we have $G_1 \otimes G_2$ is a $(r_1 r_2 + r_1 + r_2)$-regular graph with $n_1 n_2 (r_1 r_2 + r_1 + r_2)$ edges. Thus

$$DTI(G_1 \otimes G_2) = \sum_{(i,j) \in E} m_{i,j}(G) \varphi_{i,j} = \frac{n_1 n_2 (r_1 r_2 + r_1 + r_2)}{2} \varphi_{r_1 r_2 + r_1 + r_2}.$$

This completes the proof.

Corollary 12. Let $G_1$ and $G_2$ be a $r_1$-regular graph and a $r_2$-regular graph with order $n_1$ and $n_2$, respectively. Then

$$R^t(G_1 \otimes G_2) = \frac{n_1 n_2 (r_1 r_2 + r_1 + r_2)^{2t+1}}{2},$$

$$Z^t(G_1 \otimes G_2) = n_1 n_2 (r_1 r_2 + r_1 + r_2)^t,$$

$$\chi^t(G_1 \otimes G_2) = 2^{t-1} n_1 n_2 (r_1 r_2 + r_1 + r_2)^{t+1}$$

for $n_1 \geq n_2 \geq 2$.

5. Lexicographic product

Theorem 13. Let $P_{n_1}$ and $P_{n_2}$ be two path graphs of order $n_1$ and $n_2$, respectively. Then

$$DTI(P_{n_1}[P_{n_2}]) = 4 \varphi_{n_2+1,2n_2+2} + 8 \varphi_{n_2+1,2n_2+1} + 4 (n_2 - 2) \varphi_{n_2+1,2n_2+2} + 2 (n_2 - 3) \varphi_{n_2+1,2n_2+2} + 4 (n_2 - 2) \varphi_{n_2+2,2n_2+1} + 2 (n_2 - 2)^2 \varphi_{n_2+2,2n_2+2} + 4 (n_2 - 3) \varphi_{n_2+1,2n_2+1} + [2 (n_1 - 2) + 4 (n_1 - 3) (n_2 - 2)] \varphi_{2n_2+1,2n_2+2} + [(n_1 - 2) (n_2 - 3) + (n_1 - 3) (n_2 - 2)^2] \varphi_{2n_2+2,2n_2+2}.$$
for $n_1 \geq n_2 \geq 3$.

**Proof.** By the definition of lexicographic product, we obtain the basic information on $P_{n_1}[P_{n_2}]$ in the following Table 10.

<table>
<thead>
<tr>
<th>$m_{n_1, n_2+1, n_2+2}$</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{n_1+1,2n_2+1}$</td>
<td>8</td>
</tr>
<tr>
<td>$m_{n_1+1,2n_2+2}$</td>
<td>4 $(n_2-2)$</td>
</tr>
<tr>
<td>$m_{n_1+2,2n_2+2}$</td>
<td>2 $(n_2-3)$</td>
</tr>
<tr>
<td>$m_{n_1+2,2n_2+1}$</td>
<td>4 $(n_2-2)$</td>
</tr>
<tr>
<td>$m_{n_1+2,2n_2+2}$</td>
<td>2 $(n_2-2)^2$</td>
</tr>
<tr>
<td>$m_{2n_1+1,2n_2+2}$</td>
<td>4 $(n_1-3)$</td>
</tr>
<tr>
<td>$m_{2n_1+1,2n_2+2}$</td>
<td>2 $(n_1-2) + 4(n_1-3)(n_2-2)$</td>
</tr>
<tr>
<td>$m_{2n_1+2,2n_2+2}$</td>
<td>$(n_1-2)(n_2-3) + (n_1-3)(n_2-2)^2$</td>
</tr>
</tbody>
</table>

Thus we have

$$DTI(P_{n_1}[P_{n_2}]) = \sum_{(i,j) \in K} m_{i,j} (G) \varphi_{i,j}$$

$$= 4\varphi_{n_1+1, n_2+2} + 8\varphi_{n_1+2,2n_2+1} + 4(n_2-2)\varphi_{n_1+2,2n_2+2} + 2(n_2-3)\varphi_{n_1+2,2n_2+2} + 4(n_2-2)\varphi_{n_2+2,2n_2+1} + 2(n_2-2)^2\varphi_{n_2+2,2n_2+2} + 4(n_1-3)\varphi_{2n_1+1,2n_2+1} + 2(n_1-2) + 4(n_1-3)(n_2-2)]\varphi_{2n_1+1,2n_2+2} + [(n_1-2)(n_2-3) + (n_1-3)(n_2-2)^2]\varphi_{2n_1+2,2n_2+2}.$$

This completes the proof.  

**Corollary 13.** Let $P_{n_1}$ and $P_{n_2}$ be two path graphs of order $n_1$ and $n_2$, respectively. Then

$$R^i(P_{n_1}[P_{n_2}]) = 4[(n_2+1)(n_2+2)]^i + 8[(n_2+1)(2n_2+1)]^i + 4(n_2-2)[(n_2+1)(2n_2+2)]^i + 2(n_2-3)(n_2+2)^{i-1} + 4(n_2-2)[(n_2+2)(2n_2+1)]^{i-1} + 2(n_2-3)(2n_2+1)^{i-1} + 2(n_2-2)[(n_2+2)(2n_2+2)]^{i-1} + 8(n_1-3)(2n_2+2)^{i-1} + 2(n_2-2)[(n_2+1)(2n_2+2)]^i + 8(n_1-3)(2n_2+1)^{i-1} + 2(n_1-2)(n_2-3)(n_2-2)^2 + [(n_1-2)(n_2-3) + (n_1-3)(n_2-2)^2] + 4(n_2+3)^i + 8(3n_2+2)^i + 8\cdot 3^i\cdot(n_2-2)\cdot(n_2-3) + 2(n_2-3)(2n_2+4)^i + 2(n_2-2)^2(3n_2+2)^i + 4(n_1-3)(4n_2+2)^i + 2(n_1-2) + 4(n_1-3)(n_2-2)(4n_2+3)^i + 4n_1+1)^i[(n_1-2)(n_2-3) + (n_1-3)(n_2-2)^2]$$

for $n_1 \geq n_2 \geq 3$.  

Theorem 15. Let \( P_{n_1} \) and \( G_r \) be a path and a \( r \)-regular of order \( n_1 \) and \( n_2 \), respectively. Then

\[
DTI(P_{n_1}[G_r]) = m_{n_1+1,n_2+1} + 2n_1[1 + r(n_2 - 2)] \varphi_{n_1+1,n_2+2} + \frac{1}{2} \left[ m_{n_1+2,n_2+2} + m_{n_2+2,r} \right] \varphi_{2,n_2+2}
\]

for \( n_1 > n_2 \geq 2 \).

Proof. By the definition of lexicographic product, we obtain the basic information on \( P_{n_1}[G_r] \) in the following Table 11.

Table 11. The basic information on \( P_{n_1}[G_r] \).

<table>
<thead>
<tr>
<th>( m_{r+n_2,n_2+r} )</th>
<th>( m_{r+n_2,2n_2+r} )</th>
<th>( m_{2n_2+r,2n_2+r} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_2 )</td>
<td>( 2n_2^2 )</td>
<td>( \frac{m_{2n_2+r,2n_2+r}}{2} + n_2^2(n_1-3) )</td>
</tr>
</tbody>
</table>

Thus we have

\[
DTI(P_{n_1}[G_r]) = \sum_{(i,j) \in K} m_{i,j}(G) \varphi_{i,j}
\]

= \( m_{n_1+1,n_2+1} + 2n_1[1 + r(n_2 - 2)] \varphi_{n_1+1,n_2+2} + \frac{1}{2} \left[ m_{n_1+2,n_2+2} + m_{n_2+2,r} \right] \varphi_{2,n_2+2}
\]

This completes the proof. \(\square\)

Corollary 14. Let \( P_{n_1} \) and \( G_r \) be a path and a \( r \)-regular of order \( n_1 \) and \( n_2 \), respectively. Then

\[
R^t(P_{n_1}[G_r]) = r n_2 (r + n_2)^{2^t} + 2 n_2^2 (r + n_2)^{2^t} (2 n_2 + r)^t + \left[ \frac{r n_2 (n_1 - 2)}{2} + \left( n_1 - 3 \right) n_2^2 \right] (2 n_2 + r)^{2^t},
\]

\[
Z^t(P_{n_1}[G_r]) = 2 r n_2 (r + n_2)^{t-1} + 2 n_2^2 \left[ (r + n_2)^{t-1} + (2 n_2 + r)^{t-1} \right] + \left[ r n_2 (n_1 - 2) + \left( n_1 - 3 \right) n_2^2 \right] (2 n_2 + r)^{t-1},
\]

\[
\chi^t(P_{n_1}[G_r]) = 2^t r n_2 (r + n_2)^{t-1} + 2 n_2^2 (2 r + 3 n_2)^{t-1} + 2 \left[ \frac{r n_2 (n_1 - 2)}{2} + \left( n_1 - 3 \right) n_2^2 \right] (2 n_2 + r)^t
\]

for \( n_1 > n_2 \geq 2 \).

Theorem 15. Let \( G_r \) and \( P_{n_2} \) be a \( r \)-regular and a path of order \( n_1 \) and \( n_2 \), respectively. Then

\[
DTI(G_r[P_{n_2}]) = 2 r n_1 \varphi_{n_1+1,n_1+2} + 2 n_1[1 + r(n_2 - 2)] \varphi_{n_1+1,n_1+2} + \frac{1}{2} \left[ 2 n_1 (n_2 - 3) + n_1 (n_2 - 2)^2 \right] \varphi_{n_1+1,n_1+2}
\]

for \( n_1 \geq n_2 \geq 3 \).

Proof. By the definition of lexicographic product, we obtain the basic information on \( G_r[P_{n_2}] \) in the following Table 12. Thus we have

\[
DTI(G_r[P_{n_2}]) = \sum_{(i,j) \in K} m_{i,j}(G) \varphi_{i,j}
\]

= \( 2 r n_1 \varphi_{n_1+1,n_1+2} + 2 n_1[1 + r(n_2 - 2)] \varphi_{n_1+1,n_1+2} + \frac{1}{2} \left[ 2 n_1 (n_2 - 3) + n_1 (n_2 - 2)^2 \right] \varphi_{n_1+1,n_1+2}
\]

This completes the proof. \(\square\)
Corollary 15. Let $G_r$ and $P_{n_2}$ be a $r$-regular and a path of order $n_1$ and $n_2$, respectively. Then
\begin{align*}
R'(G_r[P_{n_2}]) &= 2rn_1(n_2+1)^{2i} + 2n_1[1 + r(n_2 - 2)][(rn_2 + 1)(rn_2 + 2)^i] \\
&+ \left[n_1(n_2 - 3) + \frac{rn_1(n_2 - 2)^2}{2}\right](rn_2 + 2)^{i-1}, \\
Z'(G_r[P_{n_2}]) &= 4rn_1(n_2+1)^{2i-1} + 2n_1[1 + r(n_2 - 2)][(rn_2 + 1)^{i-1} + (rn_2 + 2)^{i-1}] \\
&+ \left[2n_1(n_2 - 3) + rn_1(n_2 - 2)^2\right](rn_2 + 2)^{i-1}, \\
\chi'(G_r[P_{n_2}]) &= 2^{i-1}rn_1(n_2+1)^{i} + 2n_1[1 + r(n_2 - 2)](2rn_2 + 3)^i \\
&+ 2i\left[n_1(n_2 - 3) + \frac{rn_1(n_2 - 2)^2}{2}\right](rn_2 + 2)^i
\end{align*}
for $n_1 \geq n_2 \geq 3$.

Theorem 16. Let $G_1$ and $G_2$ be a $r_1$-regular graph and a $r_2$-regular graph with order $n_1$ and $n_2$, respectively. Then
\[
DTI(G_1[G_2]) = \frac{1}{2}n_1n_2(r_2 + r_1n_2) \varphi_{r_1,r_2,r_1,n_2+2}
\]
for $n_1 \geq n_2 \geq 2$.

Proof. By the definition of lexicographic product, we have $G_1[G_2]$ is a $(r_1n_2 + r_2)$-regular graph. Thus
\[
DTI(G_1[G_2]) = \sum_{(i,j) \in K} m_{i,j}(G) \varphi_{i,j} = \frac{1}{2}n_1n_2(r_2 + r_1n_2) \varphi_{r_1,r_2,r_1,n_2+2}.
\]
This completes the proof.

Corollary 16. Let $G_1$ and $G_2$ be a $r_1$-regular graph and a $r_2$-regular graph with order $n_1$ and $n_2$, respectively. Then
\begin{align*}
R'(G_1[G_2]) &= \frac{n_1n_2(r_2 + r_1n_2)^{2i+1}}{2}, \\
Z'(G_1[G_2]) &= n_1n_2(r_2 + r_1n_2)^i, \\
\chi'(G_1[G_2]) &= 2^{i-1}n_1n_2(r_2 + r_1n_2)^{i+1}
\end{align*}
for $n_1 \geq n_2 \geq 2$.

6. Conclusion

In this paper, we give a unified approach to solve the computational problems of degree-based topological indices of standard product graphs for the path, star and regular graphs. It is imaginable to use other graph operations to calculate degree-based topological indices uniformly in the future.

Acknowledgments: This work was supported by the Qinghai science and technology plan project (No. 2021-ZJ-703) and the National Natural Science Foundation of China (No. 11771443).

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: “The authors declare no conflict of interest.”

References


© 2021 by the authors; licensee PSRP, Lahore, Pakistan. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (http://creativecommons.org/licenses/by/4.0/).