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A fuzzy solution of nonlinear partial differential equations of fractional order

Mawia Osman[1,2,*], Ahmed Hamoud[3,*], Altyeb Mohammed Mustafa[4], Zengtai Gong [2], Bchir Mahamat Acyl[2], Abdoulaye Ali[2] and Bakry Musa[2]

1 College of Mathematics and Computer Science, Zhejiang Normal University, Jinhua, P.R. China.
2 College of Mathematics and Statistics, Northwest Normal University, Lanzhou, P.R. China.
3 Department of Mathematics, Taiz University, Taiz, Yemen.
4 Department of Applied Mathematics, Faculty of Mathematical Science, University of Khartoum, Khartoum, Sudan.
* Correspondence: mawiaosman@yahoo.com(M.O); drahmed985@yahoo.com(A.H); Tel.: +8615682896302

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Abstract: In this paper, the fuzzy nonlinear partial differential equations of fractional order are considered. The generalization differential transform method (DTM) and fuzzy variational iteration method (VIM) were applied to solve fuzzy nonlinear partial differential equations of fractional order. The above methods are investigated based on Taylor's formula, and fuzzy Caputo's fractional derivative. The proposed methods are also illustrated by some examples. The results reveal the methods are a highly effective scheme for obtaining the fuzzy fractional partial differential equations.

Keywords: Fuzzy numbers; Fuzzy-valued functions; Fuzzy fractional partial differential equations; DTM; VIM; Taylor's formula; Fuzzy Caputo's fractional derivative.

MSC: 42B10; 42C40.

1. Introduction

Fuzzy partial differential equations (FPDEs) have garnered significant attention from scientists and engineers due to their frequent use in modeling various industrial applications such as electromagnetic fields, heat and mass transfer, meteorology, static and dynamic of structures, biomechanics, and other fields. Consequently, numerous authors have studied FPDEs, as evidenced by the existing literature on the subject, [1,2].

The study of fractional calculus has gained significant importance in the last three decades, mainly due to its diverse applications in physics, engineering, and other fields. In recent years, several researchers have focused on introducing and exploring fractional calculus, as evidenced by a growing body of literature; for instance, see the following references [3–8].

The differential transform method (DTM) was first introduced by Zhou [9] for solving linear-nonlinear initial value problems in electric circuit analysis. Unlike the traditional higher-order Taylor series, which requires the symbolic computation of the necessary derivatives of the data functions, DTM constructs an analytic solution in the form of a polynomial. Recently, several researchers have used DTM to solve partial differential equations, as reported in [3,6,10–12].

Another popular method for solving differential equations is the variational iteration method (VIM), which was initially introduced by He in [13]. Many researchers have since used VIM to solve differential-integral equations. For instance, Osman *et al.*, [14] compared the fuzzy Adomian decomposition method with fuzzy VIM for solving fuzzy heat-like and wave-like equations with variable coefficients under gH-differentiability, while Hamoud *et al.*, [15] investigated the use of the modified variational iteration technique to solve Fredholm integro-differential equations. In this work, we compare the performance of DTM and VIM for solving fuzzy nonlinear partial differential equations of fractional order using Taylor's formula and fuzzy Caputo's fractional derivative.

The paper is organized as follows: In §2, we provide a brief overview of some fundamental concepts and definitions. In §3, we present the generalized two-dimensional differential transform method and fuzzy variational iteration method, which are the main focus of this work. In §4, we demonstrate the effectiveness and simplicity of the proposed methods through various examples. Finally, we summarize our findings and draw conclusions in §5.

2. Basic concepts

In this section, we give some necessary definitions of fuzzy theory and fuzzy fractional calculus which are used in the work.

2.1. The results about fuzzy number space E^1

We recall that $E^1 = \{\tilde{u} : R \to [0,1] : \tilde{u}$ satisfies $(1)-(4)$ below$\}$

1. $\tilde{u}$ is normal, i.e., there exists $x_0 \in R$ such that $u(x_0) = 1$;
2. $\tilde{u}$ is convex, i.e., for all and $r \in [0,1], x, y \in R$,

$$\tilde{u}(rx + (1-r)y) \geq \min\{\tilde{u}(x), \tilde{u}(y)\},$$

holds;

3. $\tilde{u}$ is upper semicontinuous, i.e., for any $x_0 \in R$,

$$\tilde{u}(x_0) \geq \lim_{x \longrightarrow x_0^{\pm}} \tilde{u}(x);$$

4. supp $\tilde{u} = \{x \in R | \tilde{u}(x) > 0\}$ is the support of $\tilde{u}$, and its closure cl (supp $\tilde{u}$) is compact.

For $0 < \lambda \leq 1$ is a closed interval for all $\lambda \in [0,1]$.
For $\tilde{u}, \tilde{v} \in E^1, k \in R$, the addition and scalar multiplication are defined by the equations

$$[\tilde{u} + \tilde{v}]_\lambda = [\tilde{u}]_\lambda + [\tilde{v}]_\lambda,$$

$$[k\tilde{u}]_\lambda = k[\tilde{u}]_\lambda.$$

Define $D : E^1 \times E^1 \to R^+ \cup \{0\}$ by the equation

$$D(\tilde{u}, \tilde{v}) = \sup_{\lambda \in [0,1]} d([\tilde{u}]_\lambda [\tilde{v}]_\lambda),$$

where d is Hausdorff metric space,

$$d([\tilde{u}]_\lambda, [\tilde{v}]_\lambda) = \inf\{\varepsilon : [\tilde{u}]_\lambda \subset N([\tilde{v}]_\lambda, \varepsilon), [\tilde{v}]_\lambda \subset N([\tilde{u}]_\lambda, \varepsilon)\}$$
$$\max\{|\underline{u}_\lambda - \underline{v}_\lambda|, |\overline{u}_\lambda - \overline{v}_\lambda|\},$$

where $N([\tilde{v}]_\lambda, \varepsilon), N([\tilde{u}]_\lambda, \varepsilon)$ is the ε-neighborhood of $[u]_\lambda, [v]_\lambda$, respectively, and $\underline{u}_\lambda, \underline{v}_\lambda, \overline{u}_\lambda, \overline{v}_\lambda$ are endpoints of $[u]_\lambda, [v]_\lambda$, respectively. Using the result of [16],

- (E^1, D) is complete metric space,
- $D(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}) = D(\tilde{u}, \tilde{v})$ for all $\tilde{u}, \tilde{v}, \tilde{w} \in E^1$
- $D(k\tilde{u}, k\tilde{v}) = |k| D(\tilde{u}, \tilde{v})$.

In addition, we can introduce a partial order in E^1 by $\tilde{u} \leq \tilde{v}$ if and only if $[\tilde{u}]_\lambda \leq [v]_\lambda$, $\lambda \in [0,1]$ if and only if $\underline{u}_\lambda \leq \underline{v}_\lambda, \overline{u}_\lambda \leq \overline{v}_\lambda, \lambda \in [0,1]$.

Definition 1. [17]. A fuzzy number u in parametric forms is a pair $(\underline{u}_\lambda, \overline{u}_\lambda)$ of functions $\underline{u}_\lambda, \overline{u}_\lambda, 0 \leq \lambda \leq 1$, which satisfy the following requirements:

- $\underline{u}_\lambda$ is a bounded non-decreasing left continuous function in (0,1], and right continuous at 0,
- $\overline{u}_\lambda$ is a bounded non-increasing left continuous function in (0,1], and right continuous at 0,

- $\underline{u}_\lambda \le \overline{u}_\lambda$, $0 \le \lambda \le 1$.

The λ-level set of fuzzy numbers is a closed and bounded interval $[\underline{u}(x;\lambda), \overline{u}(x;\lambda)]$, where $\underline{u}(x;\lambda)$ denotes the left-hand endpoint of $[u(x)]_\lambda$ and $\overline{u}(x;\lambda)$ the right-hand endpoint $[u(x)]_\lambda$.

Definition 2. [18]. A fuzzy-valued function f of two variable is a rule that assigns to each ordered pair of real numbers, (x,t), in a set D, a unique fuzzy number denoted by $f(x,t)$. The set D is the domain of f and its range is the set of values taken by f, i.e., $\{f(x,t)|(x,t) \in D\}$.

The parametric representation of the fuzzy valued function $f : D \to E^1$ is expressed by $f(x,t;\lambda) = [\underline{f}(x,t;\lambda), \overline{f}(x,t;\lambda)]$, for all $(x,t) \in D$ and $\lambda \in [0,1]$.

Definition 3. [18]. Let $f : D \to E^1$ be a fuzzy-valued function of two variable. Then, we say that the fuzzy limit of $f(x,t)$ as (x,t) approaches to (a,b) is $L \in E^1$,and we write $\lim_{(x,t)\to(a,b)} f(x,t) = L$ if for every number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that if $(x,t) \in D, \| (x,t) - (a,b) \| < \delta \Rightarrow D(f(x,t), L) < \varepsilon$, where $\| \cdot \|$ denotes the Euclidean norm in R^n.

Definition 4. [17]. Let $\tilde{u}(x,t) : D \to E^1$ and $(x_0,t) \in D$. We say that $\tilde{u}$ is strongly generalized Hukuhara differentiability on (x_0,t) (GH-differentiability for short) if there exists an element $\frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)} \in E^1$ such that

(i) for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0+h,t) \ominus_H \tilde{u}(x_0,t), \tilde{u}(x_0,t) \ominus_H \tilde{u}(x_0-h,t)$ and the limits (in the metric D)

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0+h,t) \ominus_H \tilde{u}(x_0,t)}{h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0,t) \ominus_H \tilde{u}(x_0-h,t)}{h} = \left.\frac{\partial \tilde{u}}{\partial x}\right|_{(x_0,t)},$$

or

(ii) for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0,t) \ominus_H \tilde{u}(x_0+h,t), \tilde{u}(x_0-h,t) \ominus_H \tilde{u}(x_0,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0,t) \ominus_H \tilde{u}(x_0+h,t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0-h,t) \ominus_H \tilde{u}(x_0,t)}{-h} = \left.\frac{\partial \tilde{u}}{\partial x}\right|_{(x_0,t)},$$

or

(iii) for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0+h,t) \ominus_H \tilde{u}(x_0,t), \tilde{u}(x_0-h,t) \ominus_H \tilde{u}(x_0,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0+h,t) \ominus_H \tilde{u}(x_0,t)}{h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0-h,t) \ominus_H \tilde{u}(x_0,t)}{-h} = \left.\frac{\partial \tilde{u}}{\partial x}\right|_{(x_0,t)},$$

or

(iv) for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0,t) \ominus_H \tilde{u}(x_0+h,t), \tilde{u}(x_0,t) \ominus_H \tilde{u}(x_0-h,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0,t) \ominus_H \tilde{u}(x_0+h,t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0,t) \ominus_H \tilde{u}(x_0-h,t)}{h} = \left.\frac{\partial \tilde{u}}{\partial x}\right|_{(x_0,t)}.$$

2.2. Fuzzy fractional calculus

We denote $C^F[a,b]$ as a space of all fuzzy-valued functions which are continuous on $[a,b]$, and the space of all Lebesgue integrable fuzzy-valued functions on the bounded interval $[a,b] \subset \mathbb{R}$ by $L^F[a,b]$, we denote the space of fuzzy-valued functions $f(x)$ which have continuous H-derivative up to order $n-1$ on $[a,b]$ such that $f^{(n-1)}(x) \in AC^F([a,b])byAC^{(n)F}([a,b])$, where $AC^F([a,b])$ denote the set of all fuzzy-valued functions which are absolutely continuous.

Definition 5. [19]. Suppose $f(x) \in C^F[a,b] \cap L^F[a,b]$, the fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as:

$$(I^\alpha_{a+} f)(x;\lambda) = [(I^\alpha_{a+} \underline{f})(x;\lambda), (I^\alpha_{a+} \overline{f})(x;\lambda)],$$

where $0 \le \lambda \le 1$

$$(I^\alpha_{a+} \underline{f})(x;\lambda) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\underline{f}(t;\lambda)\mathrm{d}t}{(x-t)^{1-\alpha}}, 0 \le \lambda \le 1,$$

$$(I^{\alpha}_{a+}\overline{f})(x;\lambda) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\overline{f}(t;\lambda)\mathrm{d}t}{(x-t)^{1-\alpha}}, 0 \le \lambda \le 1.$$

Let $f(x) \in C^F((0,a]) \cap L^F(0,a)$, be a given function such that $f(t;\lambda) = [\underline{f}(t;\lambda), \overline{f}(t;\lambda)]$ for all $t \in (0,a]$ and $0 \le m \le 1$. We define $D^{\alpha}_a f(t;\lambda)$ the fuzzy fractional Riemann-Liouville derivative of order $0 < \alpha < 1$ of f in the parametric from,

$$D^{\alpha}_a f(t;\lambda) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{d}{dt}\int_0^t (t-s)^{-\alpha}\underline{f}(s;\lambda)ds, \frac{d}{dt}\int_0^t (t-s)^{-\alpha}\overline{f}(s;\lambda)ds\right],$$

provided that equation defines a fuzzy number $D^{\alpha}_a f(t) \in E^1$. In fact,

$$D^{\alpha}_a f(t;\lambda) = [D^{\alpha}_a \underline{f}(t;\lambda), D^{\alpha}_a \overline{f}(t;\lambda)].$$

Obviously, $D^{\alpha}_a f(t) = \frac{d}{dt} I^{1-\alpha} f(t)$ *for* $t \in (0,a]$.

Definition 6. The one-parameter Mittag-Leffler function $E_\alpha(p)$ with $\alpha > 0$ is expressed as follows:

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0. \tag{1}$$

We present the generalized Taylor's formula that involves Caputo fractional derivatives. This generalization is introduced in [20].

Theorem 7. [21] *Suppose that* $f(x) \in C[a,b]$ *and* $D^{\alpha}_a f(x) \in C(a,b]$, *for* $0 < \alpha \le 1$ *then we obtain*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)}(D^{\alpha}_a f)(\vartheta)\cdot(x-a)^{\alpha}, \tag{2}$$

with $a \le \vartheta \le x, \forall_x \in (a,b]$ *and* D^{α} *is the Caputo fractional derivative of order* $\alpha > 0$.

Theorem 8. [21] *Let* $(D^{\alpha}_a)^n f(x), (D^{\alpha}_a)^{n+1} f(x) \in C(a,b]$, *for* $0 < \alpha \le 1$, *then we obtain [24]*

$$(J^{n\alpha}_a (D^{\alpha}_a)^n f)(x) - (J^{(n+1)\alpha}_a (D^{\alpha}_a)^{n+1} f)(x) = \frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)}((D^{\alpha}_a)^n f)(a+), \tag{3}$$

where $(D^{\alpha}_a)^n = D^{\alpha}_a \cdot D^{\alpha}_a \cdots D^{\alpha}_a, \quad (n-times)$.

Theorem 9. [20]. *Let* $f(x) = x^{\mu} g(x)$, *where* $\mu > -1$ *and* $g(x)$ *has the generalized power series expansion* $g(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n\alpha}$ *with radius of convergence* $R > 0$, *where* $0 < \alpha \le 1$. *Then*

$$D^{\gamma}_a D^{\beta}_a f(x) = D^{\gamma+\beta}_a f(x), \tag{4}$$

for all $x \in (0,R)$ *if one of the following conditions is satisfied:*

- $\beta < \mu + 1$, *and* α *arbitrary; or*
- $\beta \ge \mu + 1, \gamma$ *arbitrary, and* $a_\mu = 0$ *for* $j = 0, 1, \cdots, m-1$,*where* $m - 1 < \beta \le m$.

Theorem 10. [20] *Let that* $(D^{\alpha}_a)^{\mu} f(x) \in C(a,b]$ *for* $j = 0, 1 \cdots, n+1$, *where* $0 \le \alpha \le 1$, *then we obtain*

$$f(x) = \sum_{\varsigma=0}^{n} \frac{(x-a)^{\varsigma\alpha}}{\Gamma(i\alpha+1)}((D^{\alpha}_a)^{\varsigma} f)(a+) + \frac{((D^{\alpha}_a)^{n+1} f)(\vartheta)}{\Gamma((n+1)\alpha+1)}\cdot(x-a)^{(n+1)\alpha}, \tag{5}$$

with $a \le \vartheta \le x, \ \forall x \in (a,b]$.

Theorem 11. [20] *Let that* $(D_a^\alpha)^\mu f(x) \in C(a,b]$ *for* $j = 0, 1 \cdots, n+1$, *where* $0 < \alpha \le 1$. *If* $x \in [a,b]$, *then*

$$f(x) \cong P_N^\alpha(x) = \sum_{\varsigma=0}^{N} \frac{(x-a)^{\varsigma\alpha}}{\Gamma(i\alpha+1)}((D_a^\alpha)^\varsigma f)(a). \tag{6}$$

Moreover, there is a value ϑ *with* $a \le \vartheta \le x$ *so that the error term* R_N^α *has the form:*

$$R_N^\alpha(x) = \frac{((D_a^\alpha)^{N+1} f)(\vartheta)}{\Gamma((N+1)\alpha+1)} \cdot (x-a)^{(N+1)\alpha}. \tag{7}$$

The accuracy of $P_N^\alpha(x)$ *increases when we choose large* N *and decreases as the value of* x *moves away from the center* a.

3. Analysis of the methods

In this section, we present the generalized two-dimensional differential transform method with fuzzy variational iteration method.

3.1. Generalized two-dimensional differential transform method

We consider the real-valued function of two variable $u(x,t)$, and let that it can be represented as a product of two single-variable real-valued function, i.e., $u(x,t) = f(x)g(t)$. Based on the properties of generalized one-dimensional DTM, the function $u(x,t)$ can be represented as

$$\begin{aligned} u(x,t) &= \sum_{\mu=0}^{\infty} F_\alpha(j) \cdot (x-x_0)^{j\alpha} \sum_{h=0}^{\infty} G_\beta(h) \cdot (t-t_0)^{h\beta} \\ &= \sum_{\mu=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(j,h)(x-x_0)^{j\alpha}(t-t_0)^{h\beta} \end{aligned} \tag{8}$$

where $0 < \alpha, \beta \le 1$, $U_{\alpha,\beta}(j,h) = F_\alpha(j)G_\beta(h)$ is called the spectrum of $u(x,t)$. If real-valued function $u(x,t)$ is analytic and differentiated continuously with respect to time t in the domain of interest, then we define the generalized two-dimensional DTM of the function $u(x,t)$ is given by

$$U_{\alpha,\beta}(j,h) = \frac{1}{\Gamma(\alpha j+1)\Gamma(\beta h+1)}[(D_{x_0}^\alpha)^\mu (D_{t_0}^\beta)^h u(x,t)]_{(x_0,t_0)}, \tag{9}$$

where $(D_{x_0}^\alpha)^\mu = D_{x_0}^\alpha D_{x_0}^\alpha \cdots D_{x_0}^\alpha$.

The lowercase $u(x,t)$ represent the original real-valued function while the uppercase $U_{\alpha,\beta}(j,h)$ stands for the transformed function. The generalized inverse DTM of $U_{\alpha,\beta}(j,h)$ as follows

$$u(x,t) = \sum_{\mu=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(j,h) \cdot (x-x_0)^{j\alpha}(t-t_0)^{h\beta} \tag{10}$$

In case of $\alpha = \beta = 1$, the generalized two-dimensional DTM (9) reduces to the classical two-dimensional DTM [22–24].

By using (9) and (10), we can propose some fundamental properties of the generalized two-dimensional DTM.

Theorem 12. *If* $u(x,t) = v(x,t) \pm w(x,t)$, *then* $U_{\alpha,\beta}(j,h) = V_{\alpha,\beta}(j,h) \pm W_{\alpha,\beta}(j,h)$.

Theorem 13. *If* $u(x,t) = \lambda v(x,t)$, *then* $U_{\alpha,\beta}(j,h) = \lambda V_{\alpha,\beta}(j,h)$.

Theorem 14. *If* $u(x,t) = v(x,t)w(x,t)$, *then*

$$U_{\alpha,\beta}(j,h) = \sum_{r=0}^{\mu} \sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(j-r,s). \tag{11}$$

Theorem 15. *If* $u(x,t) = (x-x_0)^{n\alpha}(t-t_0)^{m\alpha}$, *then* $U_{\alpha,\beta}(j,h) = \delta(j-n)(h-m)$.

Theorem 16. *If* $u(x,t) = D^{\alpha}_{x_0}v(x,t)$ *and* $0 < \alpha \le 1$, *then we obtain*

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha(j+1)+1)}{\Gamma(\alpha j+1)}V_{\alpha,\beta}(j+1,h). \tag{12}$$

Theorem 17. *If* $u(x,t) = D^{\gamma}_{x_0}v(x,t), m-1 < \gamma \le m$ *and* $v(x,t) = f(x)g(t)$, *where* $f(x)$ *satisfies the conditions in Theorem 14, then*

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha j+\gamma+1)}{\Gamma(\alpha j+1)}V_{\alpha,\beta}(j+\gamma/\alpha,h). \tag{13}$$

Theorem 18. *If* $u(x,t) = D^{\alpha}_{x_0}D^{\beta}_{t_0}v(x,t)$, $0 < \alpha,\ \beta \le 1$, *then we get*

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha(j+1)+1)\Gamma(\beta(h+1)+1)}{\Gamma(\alpha j+1)\Gamma(\beta h+1)}V_{\alpha,\beta}(j+1,h+1). \tag{14}$$

Theorem 19. *If* $u(x,t) = D^{\gamma}_{x_0}D^{\eta}_{t_0}v(x,t)$, *where* $m-1 < \gamma \le m, n-1 < \eta \le n$ *and* $v(x,t) = f(x)g(t)$, *where real-valued functions* $f(x)$ *and* $g(x)$ *satisfies the conditions in Theorem 14, then*

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha j+\gamma+1)}{\Gamma(\alpha j+1)}\frac{\Gamma(\beta h+\eta+1)}{\Gamma(\beta h+1)}U_{\alpha,\beta}(j+\gamma/\alpha,h+\eta/\beta). \tag{15}$$

3.2. Fuzzy variational iteration method

We consider the following a fuzzy fractional differential equation as:

$$\mathcal{L}\tilde{u}(x,t) = \tilde{g}(x,t), \quad t > 0, \tag{16}$$

where

$$\mathcal{L}\tilde{u}(x,t) = D^{\alpha}_{t}\tilde{u}(x,t) + \mathcal{L}\tilde{u}(x,t) + \mathcal{N}u(x,t), \quad t > 0,$$

$m-1 < \alpha \le m, m \in \mathcal{N}, \mathcal{L}$ is a linear and $\mathcal{N}$ nonlinear operator, $\tilde{g}(x,t;\lambda) = \left[\underline{g}(x,t;\lambda), \overline{g}(x,t;\lambda)\right]$ is a known analytic fuzzy-valued function and D^{α}_{t} is the fuzzy Caputo's fractional derivative of order α.

The initial conditions (16) are given in terms of the field variables and their integer order as follows:

$$\frac{\partial^{\mu}\underline{u}(x,0)(\lambda)}{\partial t^{\mu}} = \underline{f}_{\mu}(x;\lambda), \quad \nu = 0,1,2,\cdots,m-1,$$
$$\frac{\partial^{\mu}\overline{u}(x,0)(\lambda)}{\partial t^{\mu}} = \overline{f}_{\mu}(x;\lambda), \quad \nu = 0,1,2,\cdots,m-1.$$

where $\tilde{f}_{\mu}(x;\lambda) = \left[\underline{f}_{\mu}(x;\lambda), \overline{f}_{\mu}(x;\lambda)\right]$ fuzzy-valued functions. The solution of

$$\underline{u}(x,t;\lambda) = \lim_{j\to\infty}\underline{u}_n(x,t;\lambda) \qquad \overline{u}(x,t;\lambda) = \lim_{j\to\infty}\overline{u}_n(x,t;\lambda).$$

The Eq. (16) can be derived from the iteration formula,

$$\begin{cases} \underline{u}_{\mu+1}(x,t;\lambda) = \underline{u}_{\mu}(x,t;\lambda) - J^{\alpha}\left[\mathcal{L}u(x,t;\lambda) - g(x,t;\lambda)\right], & 0 < \alpha \le 1, \\ \underline{u}_{\mu+1}(x,t;\lambda) = \underline{u}_{\mu}(x,t;\lambda) - (\alpha-1)J^{\alpha}\left[\mathcal{L}u(x,t;\lambda) - g(x,t;\lambda)\right], & 1 < \alpha \le 2, \\ \underline{u}_{\mu+1}(x,t;\lambda) = \underline{u}_{\mu}(x,t;\lambda) - (\alpha-2)J^{\alpha}\left[\mathcal{L}u(x,t;\lambda) - g(x,t;\lambda)\right], & 2 < \alpha \le 3, \\ \qquad\vdots \\ \underline{u}_{\mu+1}(x,t;\lambda) = u_{\mu}(x,t;\lambda) - \dfrac{(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)}{(m-1)!}J^{\alpha}\left[\mathcal{L}u(x,t;\lambda) - g(x,t;\lambda)\right], & m-1 < \alpha \le m, \end{cases} \tag{17}$$

and

$$\begin{cases} \overline{u}_{\mu+1}(x,t;\lambda) = \overline{u}_{\mu}(x,t;\lambda) - J^{\alpha}\left[\mathcal{L}u(x,t;\lambda) - g(x,t;\lambda)\right], & 0 < \alpha \leq 1, \\ \overline{u}_{\mu+1}(x,t;\lambda) = \overline{u}_{\mu}(x,t;\lambda) - (\alpha-1)J^{\alpha}\left[\mathcal{L}u(x,t;\lambda) - g(x,t;\lambda)\right], & 1 < \alpha \leq 2, \\ \overline{u}_{\mu+1}(x,t;\lambda) = \overline{u}_{\mu}(x,t;\lambda) - (\alpha-2)J^{\alpha}\left[\mathcal{L}u(x,t;\lambda) - g(x,t;\lambda)\right], & 2 < \alpha \leq 3, \\ \qquad\vdots \\ \overline{u}_{\mu+1}(x,t;\lambda) = u_{\mu}(x,t;\lambda) - \dfrac{(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)}{(m-1)!}J^{\alpha}\left[\mathcal{L}u(x,t;\lambda) - g(x,t;\lambda)\right], & m-1 < \alpha \leq m, \end{cases} \tag{18}$$

where J^{α} is Riemann-Liouville fuzzy fractional integral operator of order $\alpha > 0$, and D_t^{α} the fuzzy Caputo fractional derivative, of order $\alpha > 0$. Applying the alternative approach of fuzzy VIM as follows

$$\underline{u}(x,t;\lambda) = \sum_{\mu=0}^{\infty} \underline{v}(x,t;\lambda), \tag{19}$$

$$\overline{u}(x,t;\lambda) = \sum_{\mu=0}^{\infty} \overline{v}(x,t;\lambda). \tag{20}$$

Applying the iteration formula

$$\begin{cases} \underline{v}_0(x,t;\lambda) = \displaystyle\sum_{\mu=0}^{m-1} \frac{f_k(x;\lambda)}{\mu!} t^k \\ \underline{v}_{\mu+1}(x,t;\lambda) = -\dfrac{(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-m+1)}{(m-1)!} J^{\alpha}\left(\mathcal{L}\left[\displaystyle\sum_{\varsigma=0}^{\mu} v_{\varsigma}(x,t;\lambda)\right] - g(x,t;\lambda)\right) \end{cases} \tag{21}$$

and

$$\begin{cases} \overline{v}_0(x,t;\lambda) = \displaystyle\sum_{\mu=0}^{m-1} \frac{f_{\mu}(x;\lambda)}{\mu!} t^{\mu} \\ \overline{v}_{\mu+1}(x,t;\lambda) = -\dfrac{(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-m+1)}{(m-1)!} J^{\alpha}\left(\mathcal{L}\left[\displaystyle\sum_{\varsigma=0}^{\mu} v_{\varsigma}(x,t;\lambda)\right] - g(x,t;\lambda)\right) \end{cases}. \tag{22}$$

3.3. Convergence analysis of fuzzy VIM

In this part, we propose the convergence analysis of the fuzzy VIM as the following theorem.

Theorem 20. *Assume that A an operator from a Hilbert space H to H. The series solution of (19) and (20), convergence if $\exists\ 0 < \gamma < 1$ such that $\|A[v_0(x,t;\lambda) + v_1(x,t;\lambda) + \cdots + v_{\mu+1}(x,t;\lambda)]\| \leq \gamma\|A[v_0(x,t;\lambda) + v_1(x,t;\lambda) + \cdots + v_{\mu}(x,t;\lambda)]\|$, that is*

$$\|v_{\mu+1}(x,t;\lambda)\| \leq \gamma\|v_{\mu}(x,t;\lambda)\|, \quad \forall j \in \mathbb{N} \cup \{0\},$$
$$\|v_{\mu+1}(x,t;\lambda)\| \leq \gamma\|v_{\mu}(x,t;\lambda)\|, \quad \forall j \in \mathbb{N} \cup \{0\}.$$

Proof. Define the sequence $\{S_n(x,t;\lambda)\}_{n=0}^{\infty}$, where $\lambda \in [0,1]$ we obtain

$$\begin{cases} \underline{S}_0(x,t;\lambda) = \underline{v}_0(x,t;\lambda), \\ \underline{S}_1(x,t;\lambda) = \underline{v}_0(x,t;\lambda) + \underline{v}_1(x,t;\lambda), \\ \qquad\vdots \\ \underline{S}_n(x,t;\lambda) = \underline{v}_0(x,t;\lambda) + \underline{v}_1(x,t;\lambda) + \cdots + \underline{v}_n(x,t;\lambda), \end{cases} \tag{23}$$

and

$$\begin{cases} \overline{S}_0(x,t;\lambda) = \overline{v}_0(x,t;\lambda), \\ \overline{S}_1(x,t;\lambda) = \overline{v}_0(x,t;\lambda) + \overline{v}_1(x,t;\lambda), \\ \qquad\vdots \\ \overline{S}_n(x,t;\lambda) = \overline{v}_0(x,t;\lambda) + \overline{v}_1(x,t;\lambda) + \cdots + \overline{v}_n(x,t;\lambda), \end{cases} \tag{24}$$

and show that $\{S_n(x,t;\lambda)\}_{n=0}^{\infty}$, is a Cauchy sequence in the Hilbert space H

$$\begin{aligned} \|\underline{S}_{n+1}(x,t;\lambda) - \underline{S}_n(x,t;\lambda)\| &= \|\underline{v}_{n+1}(x,t;\lambda)\| \le \gamma\|\underline{v}_n(x,t;\lambda)\| \\ &\le \gamma^2\|\underline{v}_{n-1}(x,t;\lambda)\| \le \cdots \le \gamma^{n+1}\|\underline{v}_0(x,t;\lambda)\| \end{aligned} \tag{25}$$

$$\begin{aligned} \|\overline{S}_{n+1}(x,t;\lambda) - \overline{S}_n(x,t;\lambda)\| &= \|\overline{v}_{n+1}(x,t;\lambda)\| \le \gamma\|\overline{v}_n(x,t;\lambda)\| \\ &\le \gamma^2\|\overline{v}_{n-1}(x,t;\lambda)\| \le \cdots \le \gamma^{n+1}\|\overline{v}_0(x,t;\lambda)\|. \end{aligned} \tag{26}$$

For $n, \mu \in \mathbb{N}, n \ge \mu$, we obtain

$$\begin{aligned} \|\underline{S}_n(x,t;\lambda) - \underline{S}_\mu(x,t;\lambda)\| &= \| \left(\underline{S}_n(x,t;\lambda) - \underline{S}_{n-1}(x,t;\lambda)\right) + \left(\underline{S}_{n-1}(x,t;\lambda) - \underline{S}_{n-2}(x,t;\lambda)\right) \\ &\quad + \cdots + \left(S_{\mu+1}(x,t;\lambda) - \underline{S}_\mu(x,t;\lambda)\right) \| \\ &\le \| \left(\underline{S}_n(x,t;\lambda) - \underline{S}_{n-1}(x,t;\lambda)\right) \| + \| \left(\underline{S}_{n-1}(x,t;\lambda) - \underline{S}_{n-2}(x,t;\lambda)\right) \| \\ &\quad + \cdots + \| \left(\underline{S}_{\mu+1}(x,t;\lambda) - \underline{S}_\mu(x,t;\lambda)\right) \| \\ &\le \gamma^n\|\underline{v}_0(x,t;\lambda)\| + \gamma^{n-1}\|\underline{v}_0(x,t;\lambda)\| + \cdots + \gamma^{\mu+1}\|\underline{v}_0(x,t;\lambda)\| \\ &= \frac{1-\gamma^{n-\mu}}{1-\gamma}\gamma^{\mu+1}\|\underline{v}_0(x,t;\lambda)\|, \end{aligned} \tag{27}$$

and

$$\begin{aligned} \|\overline{S}_n(x,t;\lambda) - \overline{S}_\mu(x,t;\lambda)\| &= \| \left(\overline{S}_n(x,t;\lambda) - \overline{S}_{n-1}(x,t;\lambda)\right) + \left(\overline{S}_{n-1}(x,t;\lambda) - S_{n-2}(x,t;\lambda)\right) \\ &\quad + \cdots + \left(\overline{S}_{\mu+1}(x,t;\lambda) - \overline{S}_\mu(x,t;\lambda)\right) \| \\ &\le \| \left(\overline{S}_n(x,t;\lambda) - \overline{S}_{n-1}(x,t;\lambda)\right) \| + \| \left(\overline{S}_{n-1}(x,t;\lambda) - \overline{S}_{n-2}(x,t;\lambda)\right) \| \\ &\quad + \cdots + \| \left(\overline{S}_{\mu+1}(x,t;\lambda) - \overline{S}_\mu(x,t;\lambda)\right) \| \\ &\le \gamma^n\|\overline{v}_0(x,t;\lambda)\| + \gamma^{n-1}\|\overline{v}_0(x,t;\lambda)\| + \cdots + \gamma^{\mu+1}\|\overline{v}_0(x,t;\lambda)\| \\ &= \frac{1-\gamma^{n-\mu}}{1-\gamma}\gamma^{\mu+1}\|\overline{v}_0(x,t;\lambda)\|, \end{aligned} \tag{28}$$

and since $0 \le \gamma \le 1$, we obtain

$$\lim n, \mu \to \infty \|\underline{S}_n(x,t;\lambda) - \underline{S}_\mu(x,t;\lambda)\| = \tilde{0}, \tag{29}$$

$$\lim n, \mu \to \infty \|\overline{S}_n(x,t;\lambda) - \overline{S}_\mu(x,t;\lambda)\| = \tilde{0}. \tag{30}$$

Thus, $\{S_n(x,t;\lambda)\}_{n=0}^{\infty}$, is a Cauchy sequence in the Hilbert space H and it implies that the series solution of (19) and (20), convergence. □

4. Examples

In this section, we have utilized the differential transform method and fuzzy variational iteration method to solve fuzzy partial differential equations. To demonstrate the effectiveness and versatility of the proposed methods, we have presented three illustrative examples.

Example 1. We consider the following fuzzy fractional KdV equation

$$D_t^\alpha \tilde{u} \ominus_{gH} 3(\tilde{u}^2)_x \oplus \tilde{u}_{xxx} = \tilde{0}, \tag{31}$$

subject to

$$\tilde{u}(x,0) = [(1+2\lambda)^n, (5-2\lambda)^n] \odot 6x \tag{32}$$

where $(n = 1,2,3,\cdots)$, and $0 < \alpha \leq 1$. The parametric from of (31) is

$$D_t^\alpha \underline{u} - 3(\underline{u}^2)_x + \underline{u}_{xxx} = \tilde{0}, \tag{33}$$

$$D_t^\alpha \overline{u} - 3(\overline{u}^2)_x + \overline{u}_{xxx} = \tilde{0}, \tag{34}$$

for $\lambda \in [0,1]$, where $\underline{u}$ stands for $\underline{u}(x,t;\lambda)$, and similarly for $\overline{u}$.

Case [A]. The differential transform method: Applying the two-dimensional DTM to both sides of (33) and (34), we obtain

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}\underline{U}(j,h+1;\lambda) = 3(j+1)\sum_{r=0}^{\mu+1}\sum_{s=0}^{h}\underline{U}(r,h-s;\lambda)\underline{U}(j-r+1,s;\lambda) - (j+1)(j+2)(j+3)\underline{U}(j+3,h;\lambda) \tag{35}$$

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}\overline{U}(j,h+1;\lambda) = 3(j+1)\sum_{r=0}^{\mu+1}\sum_{s=0}^{h}\overline{U}(r,h-s;\lambda)\overline{U}(j-r+1,s;\lambda) - (j+1)(j+2)(j+3)\overline{U}(j+3,h;\lambda). \tag{36}$$

Taking the initial condition (32), we obtain

$$\begin{cases} \underline{U}(j,0)(\lambda) = \tilde{0}, & j = 0,2,3,\cdots \\ \underline{U}(1,0)(\lambda) = 6(1+2\lambda)^n, \end{cases} \tag{37}$$

and

$$\begin{cases} \overline{U}(j,0)(\lambda) = \tilde{0}, & j = 0,2,3,\cdots \\ \overline{U}(1,0)(\lambda) = 6(5-2\lambda)^n. \end{cases} \tag{38}$$

Using (38) into (35), we can get some value of $\tilde{U}(k,h;\lambda) = \left[\underline{U}(k,h;\lambda), \overline{U}(k,h;\lambda)\right]$, we get

$$\begin{cases} \underline{U}(j,1)(\lambda) = \tilde{0}, \\ \underline{U}(1,1)(\lambda) = (1+2\lambda)^n\left[\dfrac{6^3}{\Gamma(\alpha+1)}\right], \\ \underline{U}(j,2)(\lambda) = \tilde{0}, \\ \underline{U}(1,2)(\lambda) = (1+2\lambda)^n\left[\dfrac{2\cdot 6^5}{\Gamma(2\alpha+1)}\right], \\ \underline{U}(j,3)(\lambda) = \tilde{0}, \\ \underline{U}(1,3)(\lambda) = (1+2\lambda)^n\left[\dfrac{4\cdot 6^7}{\Gamma(3\alpha+1)} + \dfrac{6^7(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right], \end{cases} \tag{39}$$

and

$$\begin{cases} \overline{U}(j,1)(\lambda) = \tilde{0}, \\ \overline{U}(1,1)(\lambda) = (5-2\lambda)^n\left[\dfrac{6^3}{\Gamma(\alpha+1)}\right], \\ \overline{U}(j,2)(\lambda) = \tilde{0}, \\ \overline{U}(1,2)(\lambda) = (5-2\lambda)^n\left[\dfrac{2\cdot 6^5}{\Gamma(2\alpha+1)}\right], \\ \overline{U}(j,3)(\lambda) = \tilde{0}, \\ \overline{U}(1,3)(\lambda) = (5-2\lambda)^n\left[\dfrac{4\cdot 6^7}{\Gamma(3\alpha+1)} + \dfrac{6^7(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right]. \end{cases} \tag{40}$$

The solution for $\tilde{U}(j,h;\lambda)$ is

$$\underline{u}(x,t;\lambda) = (1+2\lambda)^n \left[6x + \frac{6^3}{\Gamma(\alpha+1)} x t^{\alpha} + \frac{2\cdot 6^5}{\Gamma(2\alpha+1)} x t^{2\alpha} + \left[\frac{4\cdot 6^7}{\Gamma(3\alpha+1)} + \frac{6^7(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right] x t^{3\alpha} + \cdots\right], \tag{41}$$

$$\overline{u}(x,t;\lambda) = (5-2\lambda)^n \left[6x + \frac{6^3}{\Gamma(\alpha+1)} x t^{\alpha} + \frac{2\cdot 6^5}{\Gamma(2\alpha+1)} x t^{2\alpha} + \left[\frac{4\cdot 6^7}{\Gamma(3\alpha+1)} + \frac{6^7(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right] x t^{3\alpha} + \cdots\right]. \tag{42}$$

We can obtain the solution for $\alpha = 1$, as follows:

$$\underline{u}(x,t;\lambda) = (1+2\lambda)^n \left(1 + 36t + 36^2 t^2 + 36^3 t^3 + \cdots\right);$$
$$\overline{u}(x,t;\lambda) = (5-2\lambda)^n \left(1 + 36t + 36^2 t^2 + 36^3 t^3 + \cdots\right),$$

we get

$$\tilde{u}(x,t;\lambda) = \left[(1+2\lambda)^n, (5-2\lambda)^n\right] \odot \left[\frac{6x}{1-36t}\right], \quad 0 \le \lambda \le 1.$$

Case [B]. Fuzzy variational iteration method: By utilizing (21) and (22), we can derive the iteration formula for solving problem (31):

$$\begin{aligned} &\underline{v}_0(x,t;\lambda) = (1+2\lambda)^n 6x, \\ &\underline{v}_{k+1}(x,t;\lambda) = -J\left\{D_t^{\alpha} v(x,t;\lambda) - 3(v^2)_x(x,t;\lambda) + v_{xxx}(x,t;\lambda)\right\}, \end{aligned} \tag{43}$$

and

$$\begin{aligned} &\overline{v}_0(x,t;\lambda) = (5-2\lambda)^n 6x, \\ &\overline{v}_{k+1}(x,t;\lambda) = -J\left\{D_t^{\alpha} v(x,t;\lambda) - 3(v^2)_x(x,t;\lambda) + v_{xxx}(x,t;\lambda)\right\}. \end{aligned} \tag{44}$$

Taking the mentioned iteration formula, we get

$$\begin{cases} \underline{v}_1(x,t;\lambda) = (1+2\lambda)^n \left[\dfrac{6^3 x}{\Gamma(\alpha+1)} t^{\alpha}\right], \\ \underline{v}_2(x,t;\lambda) = (1+2\lambda)^n \left[\dfrac{2\cdot 6^7 x}{\Gamma(2\alpha+1)} t^{2\alpha} + \dfrac{6^7 x}{\Gamma(\alpha+1)^2} \dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} t^{3\alpha}\right], \\ \qquad \vdots \end{cases} \tag{45}$$

and

$$\begin{cases} \overline{v}_1(x,t;\lambda) = (5-2\lambda)^n \left[\dfrac{6^3 x}{\Gamma(\alpha+1)} t^{\alpha}\right], \\ \overline{v}_2(x,t;\lambda) = (5-2\lambda)^n \left[\dfrac{2\cdot 6^7 x}{\Gamma(2\alpha+1)} t^{2\alpha} + \dfrac{6^7 x}{\Gamma(\alpha+1)^2} \dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} t^{3\alpha}\right], \\ \qquad \vdots. \end{cases} \tag{46}$$

For $\alpha = 1$, we obtain the approximate solution as follows:

$$u(x,t;\lambda) \approx \sum_{\mu=0}^{3} v_{\mu}(x,t;\lambda) = (1+2\lambda)^n \left[6x\left(1 + 36t + 36^2 t^2 + \cdots\right)\right], \tag{47}$$

$$u(x,t;\lambda) \approx \sum_{\mu=0}^{3} v_{\mu}(x,t;\lambda) = (5-2\lambda)^n \left[6x\left(1 + 36t + 36^2 t^2 + \cdots\right)\right], \tag{48}$$

after some steps and the exact solution obtained as follows:

$$u(x,t;\lambda) = \sum_{\mu=0}^{3} v_\mu(x,t;\lambda) = (1+2\lambda)^n \left[6x\left(1+36t+36^2t^2+\cdots\right)\right], \tag{49}$$

$$u(x,t;\lambda) = \sum_{\mu=0}^{3} v_\mu(x,t;\lambda) = (5-2\lambda)^n \left[6x\left(1+36t+36^2t^2+\cdots\right)\right], \tag{50}$$

we obtain

$$\tilde{u}(x,t;\lambda) = [(1+2\lambda)^n, (5-2\lambda)^n] \odot \left[\frac{6x}{1-36t}\right], \quad 0 \le \lambda \le 1.$$

Example 2. We consider the following fuzzy fractional $K(2,2)$ equation

$$D_t^\alpha \tilde{u} \oplus (\tilde{u}^2)_x \oplus (\tilde{u}^2)_{xxx} = \tilde{0}, \quad 0 < \alpha \le 1, \tag{51}$$

subject to

$$\tilde{u}(x,0) = [(0.1+0.1\alpha)^n, (0.3-0.1\alpha)^n] \oplus x, \tag{52}$$

where $(n = 1,2,3,\cdots)$. The parametric from of (51) is

$$D_t^\alpha \underline{u} + (\underline{u}^2)_x + (\underline{u}^2)_{xxx} = \tilde{0}, \quad 0 < \alpha \le 1, \tag{53}$$

$$D_t^\alpha \overline{u} + (\overline{u}^2)_x + (\overline{u}^2)_{xxx} = \tilde{0}, \quad 0 < \alpha \le 1, \tag{54}$$

for $\lambda \in [0,1]$, where $\underline{u}$ stands for $\underline{u}(x,t;\lambda)$, and similarly for $\overline{u}$.

Case [A]. The differential transform method: Applying the two-dimensional DTM to both sides of (53) and (54), we obtain

$$\begin{aligned}\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}\underline{U}(j,h+1;\lambda) = &-(j+1)\sum_{r=0}^{\mu+1}\sum_{s=0}^{h}\underline{U}(r,h-s;\lambda)\underline{U}(j-r+1,s;\lambda),\\ &-(j+1)(j+2)(j+3)\sum_{r=0}^{\mu+3}\sum_{s=0}^{h}\underline{U}(r,h-s;\lambda)\underline{U}(j-r+3,s;\lambda),\end{aligned} \tag{55}$$

$$\begin{aligned}\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}\overline{U}(j,h+1;\lambda) = &-(j+1)\sum_{r=0}^{\mu+1}\sum_{s=0}^{h}\underline{U}(r,h-s;\lambda)\overline{U}(j-r+1,s;\lambda),\\ &-(j+1)(j+2)(j+3)\sum_{r=0}^{\mu+3}\sum_{s=0}^{h}\overline{U}(r,h-s;\lambda)\overline{U}(j-r+3,s;\lambda).\end{aligned} \tag{56}$$

Taking the initial condition (52), we obtain

$$\begin{cases}\underline{U}(j,0)(\lambda) = (0.1+0.1\alpha)^n, & j = 0,2,3,\cdots\\ \underline{U}(1,0)(\lambda) = 1+(0.1+0.1\alpha)^n,\end{cases} \tag{57}$$

and

$$\begin{cases}\overline{U}(j,0)(\lambda) = (0.3-0.1\alpha)^n, & j = 0,2,3,\cdots\\ \overline{U}(1,0)(\lambda) = 1+(0.3-0.1\alpha)^n.\end{cases} \tag{58}$$

Using (58) into (55) with we can get some value of $\tilde{U}(k,h;\lambda) = \left[\underline{U}(k,h;\lambda), \overline{U}(k,h;\lambda)\right]$, we get

$$\begin{cases} \underline{U}(j,1)(\lambda) = (0.1+0.1\lambda)^n, \\ \underline{U}(1,1)(\lambda) = (0.1+0.1\lambda)^n + \left[\dfrac{-2}{\Gamma(\alpha+1)}\right], \\ \underline{U}(j,2)(\lambda) = (0.1+0.1\lambda)^n, \\ \underline{U}(1,2)(\lambda) = (0.1+0.1\lambda)^n + \left[\dfrac{2^3}{\Gamma(2\alpha+1)}\right], \\ \underline{U}(j,3)(\lambda) = (0.1+0.1\lambda)^n, \\ \underline{U}(1,3)(\lambda) = (0.1+0.1\lambda)^n + \left[-\dfrac{2^5}{\Gamma(3\alpha+1)} - \dfrac{2^3\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right], \end{cases} \tag{59}$$

and

$$\begin{cases} \overline{U}(j,1)(\lambda) = (0.3-0.1\lambda)^n, \\ \overline{U}(1,1)(\lambda) = (0.3-0.1\lambda)^n + \left[\dfrac{-2}{\Gamma(\alpha+1)}\right], \\ \overline{U}(j,2)(\lambda) = (0.3-0.1\lambda)^n, \\ \overline{U}(1,2)(\lambda) = (0.3-0.1\lambda)^n + \left[\dfrac{2^3}{\Gamma(2\alpha+1)}\right], \\ \overline{U}(j,3)(\lambda) = (0.3-0.1\lambda)^n, \\ \overline{U}(1,3)(\lambda) = (0.3-0.1\lambda)^n + \left[-\dfrac{2^5}{\Gamma(3\alpha+1)} - \dfrac{2^3\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right], \end{cases} \tag{60}$$

The solution for $\tilde{U}(j,h;\lambda)$ is

$$\underline{u}(x,t;\lambda) = (0.1+0.1\alpha)^n + \left[x - \frac{2xt^\alpha}{\Gamma(\alpha+1)} + \frac{2^3xt^{2\alpha}}{\Gamma(2\alpha+1)} - \left[\frac{2^5}{\Gamma(3\alpha+1)} + \frac{2^3\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right]xt^{3\alpha} + \cdots\right], \tag{61}$$

$$\overline{u}(x,t;\lambda) = (0.3-0.1\alpha)^n + \left[x - \frac{2xt^\alpha}{\Gamma(\alpha+1)} + \frac{2^3xt^{2\alpha}}{\Gamma(2\alpha+1)} - \left[\frac{2^5}{\Gamma(3\alpha+1)} + \frac{2^3\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right]xt^{3\alpha} + \cdots\right]. \tag{62}$$

We can obtained the solution for $\alpha = 1$, as follows:

$$\underline{u}(x,t;\lambda) = (0.1+0.1\lambda)^n + \left(x - 2xt + 4xt^2 + 8xt^3 + \cdots\right),$$
$$\overline{u}(x,t;\lambda) = (0.3-0.1\lambda)^n + \left(x - 2xt + 4xt^2 + 8xt^3 + \cdots\right),$$

we obtain

$$\tilde{u}(x,t;\lambda) = [(0.1+0.1\lambda)^n, (0.3-0.1\lambda)^n] \oplus \left[\frac{x}{1+2t}\right], \quad 0 \le \lambda \le 1.$$

Case [B]. Fuzzy variational iteration method: The fuzzy iteration formula for problem (53) and (54), we obtain

$$\begin{aligned} \underline{v}_0(x,t;\lambda) &= (0.1+0.1\lambda)^n + x, \\ \underline{v}_{k+1}(x,t;\lambda) &= -J\left\{D_t^\alpha v(x,t;\lambda) + (v^2)_x(x,t;\lambda) + (v^2)_{xxx}(x,t;\lambda)\right\}, \end{aligned} \tag{63}$$

and

$$\begin{aligned} \overline{v}_0(x,t;\lambda) &= (0.3-0.1\lambda)^n + x, \\ \overline{v}_{k+1}(x,t;\lambda) &= -J\left\{D_t^\alpha v(x,t;\lambda) + (v^2)_x(x,t;\lambda) + (v^2)_{xxx}(x,t;\lambda)\right\}. \end{aligned} \tag{64}$$

Taking the mentioned iteration formula, we obtain

$$\begin{cases} \underline{v}_1(x,t;\lambda) = (0.1+0.1\lambda)^n - \left[\dfrac{2x}{\Gamma(\alpha+1)}t^{\alpha}\right], \\ \underline{v}_2(x,t;\lambda) = (0.1+0.1\lambda)^n + \left[\dfrac{2^3x}{\Gamma(2\alpha+1)}t^{2\alpha} - \dfrac{2^3x}{\Gamma(\alpha+1)^2}\dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}\right], \\ \qquad\vdots \end{cases} \tag{65}$$

and

$$\begin{cases} \overline{v}_1(x,t;\lambda) = (0.3-0.1\lambda)^n - \left[\dfrac{2x}{\Gamma(\alpha+1)}t^{\alpha}\right], \\ \overline{v}_2(x,t;\lambda) = (0.3-0.1\lambda)^n + \left[\dfrac{2^3x}{\Gamma(2\alpha+1)}t^{2\alpha} - \dfrac{2^3x}{\Gamma(\alpha+1)^2}\dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}\right], \\ \qquad\vdots. \end{cases} \tag{66}$$

For $\alpha = 1$, we obtain the approximate solution as follows:

$$\underline{u}(x,t;\lambda) \approx \sum_{\mu=0}^{3} \underline{v}_{\mu}(x,t;\lambda) = (0.1+0.1\lambda)^n + \left[x\left(1-2t+4t^2+\cdots\right)\right], \tag{67}$$

$$\overline{u}(x,t;\lambda) \approx \sum_{\mu=0}^{3} \overline{v}_{\mu}(x,t;\lambda) = (0.3-0.1\lambda)^n + \left[x\left(1-2t+4t^2+\cdots\right)\right], \tag{68}$$

after some steps and the exact solution obtained as:

$$\underline{u}(x,t;\lambda) = \sum_{\mu=0}^{\infty} \underline{v}_{\mu}(x,t;\lambda) = (0.1+0.1\lambda)^n + \left[x\left(1-2t+4t^2+\cdots\right)\right], \tag{69}$$

$$\overline{u}(x,t;\lambda) = \sum_{\mu=0}^{3} \overline{v}_{\mu}(x,t;\lambda) = (0.3-0.1\lambda)^n + \left[x\left(1-2t+4t^2+\cdots\right)\right], \tag{70}$$

we get

$$\tilde{u}(x,t;\lambda) = \left[(0.1+0.1\alpha)^n, (0.3-0.1\alpha)^n\right] \oplus \left[\frac{x}{1+2t}\right], \quad 0 \le \lambda \le 1.$$

Example 3. We consider the following fuzzy modified fractional KdV (mKdV) equation

$$D_t^{\alpha}\tilde{u} \oplus \frac{1}{2} \odot (\tilde{u}^2)_{\mathrm{x}} \ominus_{gH} \tilde{u}_{\mathrm{xx}} = \tilde{0}, \quad 0 < \alpha \le 1, \tag{71}$$

subject to

$$\tilde{u}(x,0) = \left[(0.2+0.2r)^n, (0.6-0.2r)^n\right] \ominus_{gH} x, \tag{72}$$

where $(n = 1,2,3,\cdots)$. The parametric from of (71) is

$$D_t^{\alpha}\underline{u} + \frac{1}{2}(\underline{u}^2)_{\mathrm{x}} - \underline{u}_{\mathrm{xx}} = \tilde{0}, \quad 0 < \alpha \le 1, \tag{73}$$

$$D_t^{\alpha}\overline{u} + \frac{1}{2}(\overline{u}^2)_{\mathrm{x}} - \overline{u}_{\mathrm{xx}} = \tilde{0}, \quad 0 < \alpha \le 1, \tag{74}$$

for $\lambda \in [0,1]$, where $\underline{u}$ stands for $\underline{u}(x,t;\lambda)$, and similarly for $\overline{u}$.
Case [A]. The differential transform method: Applying the two-dimensional DTM to (73) and (74), we get

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}\underline{U}(j,h+1;\lambda) = -\frac{1}{2}(j+1)\sum_{r=0}^{\mu+1}\sum_{s=0}^{h}\underline{U}(r,h-s;\lambda)\underline{U}(j-r+1,s;\lambda) + (j+1)(j+2)U(j+2,h;\lambda) \tag{75}$$

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}\overline{U}(j,h+1;\lambda) = -\frac{1}{2}(j+1)\sum_{r=0}^{\mu+1}\sum_{s=0}^{h}\overline{U}(r,h-s;\lambda)\overline{U}(j-r+1,s;\lambda) + (j+1)(j+2)U(j+2,h;\lambda). \tag{76}$$

Using the initial condition (72), we get

$$\begin{cases} \underline{U}(j,0)(\lambda) = (0.2+0.2r)^n, & j = 0,2,3,\cdots \\ \underline{U}(1,0)(\lambda) = (0.2+0.2r)^n - 1, \end{cases} \tag{77}$$

and

$$\begin{cases} \overline{U}(j,0)(\lambda) = (0.6-0.2r)^n, & j = 0,2,3,\cdots \\ \overline{U}(1,0)(\lambda) = (0.6-0.2r)^n - 1. \end{cases} \tag{78}$$

From (78) into (75) with we can get some value of $\tilde{U}(k,h;\lambda)$, we obtain

$$\begin{cases} \underline{U}(j,1)(\lambda) = (0.2+0.2\lambda)^n, \\ \underline{U}(1,1)(\lambda) = (0.2+0.2\lambda)^n - \left[\dfrac{-1}{\Gamma(\alpha+1)}\right], \\ \underline{U}(j,2)(\lambda) = (0.2+0.2\lambda)^n, \\ \underline{U}(1,2)(\lambda) = (0.2+0.2\lambda)^n - \left[\dfrac{2}{\Gamma(2\alpha+1)}\right], \\ \underline{U}(j,3)(\lambda) = (0.2+0.2\lambda)^n, \\ \underline{U}(1,3)(\lambda) = (0.2+0.2\lambda)^n - \left[-\dfrac{4}{\Gamma(3\alpha+1)} - \dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right], \end{cases} \tag{79}$$

and

$$\begin{cases} \overline{U}(j,1)(\lambda) = (0.6-0.2\lambda)^n, \\ \overline{U}(1,1)(\lambda) = (0.6-0.2\lambda)^n - \left[\dfrac{-1}{\Gamma(\alpha+1)}\right], \\ \overline{U}(j,2)(\lambda) = (0.6-0.2\lambda)^n, \\ \overline{U}(1,2)(\lambda) = (0.6-0.2\lambda)^n - \left[\dfrac{2}{\Gamma(2\alpha+1)}\right], \\ \overline{U}(j,3)(\lambda) = (0.6-0.2\lambda)^n, \\ \overline{U}(1,3)(\lambda) = (0.6-0.2\lambda)^n - \left[-\dfrac{4}{\Gamma(3\alpha+1)} - \dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right]. \end{cases} \tag{80}$$

The solution for $\tilde{U}(j,h;\lambda)$ is

$$\underline{u}(x,t;\lambda) = (0.2+0.2\lambda)^n - \left[x - \frac{xt^{\alpha}}{\Gamma(\alpha+1)} + \frac{2xt^{2\alpha}}{\Gamma(2\alpha+1)} - \left[\frac{4}{\Gamma(3\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right]xt^{3\alpha} + \cdots\right], \tag{81}$$

$$\overline{u}(x,t;\lambda) = (0.6-0.2\lambda)^n - \left[x - \frac{xt^{\alpha}}{\Gamma(\alpha+1)} + \frac{2xt^{2\alpha}}{\Gamma(2\alpha+1)} - \left[\frac{4}{\Gamma(3\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right]xt^{3\alpha} + \cdots\right]. \tag{82}$$

We can obtained the solution for $\alpha = 1$, as follows:

$$\underline{u}(x,t;\lambda) = (0.2+0.2\lambda)^n - \left(x - xt + xt^2 - xt^3 + \cdots\right),$$
$$\overline{u}(x,t;\lambda) = (0.6-0.2\lambda)^n - \left(x - xt + xt^2 - xt^3 + \cdots\right),$$

we obtain

$$\tilde{u}(x,t;\lambda) = [(0.2+0.2\lambda)^n, (0.6-0.2\lambda)^n] \ominus_{gH} \left[\frac{x}{1+t}\right], \quad 0 \le \lambda \le 1.$$

Case [B]. Fuzzy variational iteration method: The fuzzy iteration formula for problem (73) and (74), we obtain

$$\begin{aligned} &\underline{v}_0(x,t;\lambda) = (0.2+0.2\lambda)^n - x, \\ &\underline{v}_{k+1}(x,t;\lambda) = -J^\alpha \left\{ D_t^\alpha \underline{v}(x,t;\lambda) + \frac{1}{2}(\underline{v}^2)_x(x,t;\lambda) + \underline{v}_{xx}(x,t;\lambda) \right\}, \end{aligned} \tag{83}$$

and

$$\begin{aligned} &\overline{v}_0(x,t;\lambda) = (0.6-0.2\lambda)^n - x, \\ &\overline{v}_{k+1}(x,t;\lambda) = -J^\alpha \left\{ D_t^\alpha \overline{v}(x,t;\lambda) + \frac{1}{2}(\overline{v}^2)_x(x,t;\lambda) + \overline{v}_{xx}(x,t;\lambda) \right\}. \end{aligned} \tag{84}$$

According to the mentioned iteration formula, we obtain

$$\begin{cases} \underline{v}_1(x,t;\lambda) = (0.2+0.2\lambda)^n + \left[\dfrac{x}{\Gamma(\alpha+1)} t^\alpha\right], \\ \underline{v}_2(x,t;\lambda) = (0.2+0.2\lambda)^n - \left[\dfrac{2x}{\Gamma(2\alpha+1)} t^{2\alpha} - \dfrac{x}{\Gamma(\alpha+1)^2} \dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} t^{3\alpha}\right], \\ \underline{v}_3(x,t;\lambda) = (0.2+0.2\lambda)^n - \left[-\dfrac{2^2 x}{\Gamma(3\alpha+1)} t^{3\alpha} + \dfrac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)} \left[\dfrac{2x}{\Gamma(\alpha+1)^2} \dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} + \dfrac{2^2 x}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right] t^{4\alpha} \right. \\ \qquad -\dfrac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)} \left[\dfrac{2^2 x}{\Gamma(2\alpha+1)^2} + \dfrac{2x}{\Gamma(\alpha+1)^3} \dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right] t^{5\alpha} + \dfrac{2^2 x}{\Gamma(\alpha+1)^2} \dfrac{\Gamma(5\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)} t^{6\alpha} \\ \qquad \left. -\dfrac{x}{\Gamma(\alpha+1)^4} \dfrac{\Gamma(2\alpha+1)^2\Gamma(6\alpha+1)}{\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)} t^{7\alpha}\right], \\ \qquad \vdots \end{cases}$$

and

$$\begin{cases} \overline{v}_1(x,t;\lambda) = (0.6-0.2\lambda)^n + \left[\dfrac{x}{\Gamma(\alpha+1)} t^\alpha\right], \\ \overline{v}_2(x,t;\lambda) = (0.6-0.2\lambda)^n - \left[\dfrac{2x}{\Gamma(2\alpha+1)} t^{2\alpha} - \dfrac{x}{\Gamma(\alpha+1)^2} \dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} t^{3\alpha}\right], \\ \overline{v}_3(x,t;\lambda) = (0.6-0.2\lambda)^n - \left[-\dfrac{2^2 x}{\Gamma(3\alpha+1)} t^{3\alpha} + \dfrac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)} \left[\dfrac{2x}{\Gamma(\alpha+1)^2} \dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} + \dfrac{2^2 x}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right] t^{4\alpha} \right. \\ \qquad -\dfrac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)} \left[\dfrac{2^2 x}{\Gamma(2\alpha+1)^2} + \dfrac{2x}{\Gamma(\alpha+1)^3} \dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right] t^{5\alpha} + \dfrac{2^2 x}{\Gamma(\alpha+1)^2} \dfrac{\Gamma(5\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)} t^{6\alpha} \\ \qquad \left. -\dfrac{x}{\Gamma(\alpha+1)^4} \dfrac{\Gamma(2\alpha+1)^2\Gamma(6\alpha+1)}{\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)} t^{7\alpha}\right], \\ \qquad \vdots \end{cases}$$

For $\alpha = 1$, we obtain the approximate solution as follows:

$$\underline{u}(x,t;\lambda) \approx \sum_{\mu=0}^{3} \underline{v}_{\mu}(x,t;\lambda) = (0.2+0.2\lambda)^n - \left[x\left(1-t+t^2-t^3+\cdots\right)\right], \tag{85}$$

$$\overline{u}(x,t;\lambda) \approx \sum_{\mu=0}^{3} \overline{v}_{\mu}(x,t;\lambda) = (0.6-0.2\lambda)^n - \left[x\left(1-t+t^2-t^3+\cdots\right)\right], \tag{86}$$

after some steps and the exact solution obtained as follows:

$$\underline{u}(x,t;\lambda) = \sum_{\mu=0}^{\infty} \underline{v}_{\mu}(x,t;\lambda) = (0.2+0.2\lambda)^n - \left[x\left(1-t+t^2-t^3+\cdots\right)\right], \tag{87}$$

$$\overline{u}(x,t;\lambda) = \sum_{\mu=0}^{3} \overline{v}_{\mu}(x,t;\lambda) = (0.6-0.2\lambda)^n - \left[x\left(1-t+t^2-t^3+\cdots\right)\right], \tag{88}$$

we obtain

$$\tilde{u}(x,t;\lambda) = \left[(0.2+0.2r)^n, (0.6-0.2r)^n\right] \ominus_{gH} \frac{x}{1+t}, \quad 0 \le \lambda \le 1.$$

5. Conclusion

This paper successfully applied the generalized differential transform method (DTM) and fuzzy variational iteration method (VIM) to solve fuzzy nonlinear fractional partial differential equations (FPDEs) of fractional order using Taylor's formula and fuzzy Caputo's fractional derivative. The results of the illustrative examples demonstrate that these methods are effective and user-friendly mathematical tools to solve fuzzy FPDEs. In future research, we plan to investigate novel approaches towards bipolar complex fuzzy sets and their applications in generalized similarity measures, as well as a novel approach towards bipolar soft sets and their applications in T-spherical fuzzy sets and spherical fuzzy sets.

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References

[1] Alikhani, R., & Bahrami, F. (2019). Fuzzy partial differential equations under the cross product of fuzzy numbers. *Information Sciences, 494*, 80-99.

[2] Harir, A., Melliani, S., & Chadli, L. S. (2020). Solving fuzzy Burgers equation by variational iteration method. *Journal of Mathematics and Computer Science, 21*, 136-149.

[3] Osman, M., Gong, Z., & Mustafa, A. M. (2021). Application to the fuzzy fractional diffusion equation by using fuzzy fractional variational Homotopy perturbation iteration method. *Advance Research Journal of Multidisciplinary Discoveries, 51*(3), 15-26.

[4] Osman, M., Gong, Z. T., & Mohammed, A. (2021). Differential transform method for solving fuzzy fractional wave equation. *Journal of Computational Analysis and Applications, 29*(3), 431-453.

[5] Rahman, N. A. A., & Ahmad, M. Z. (2017). Solving fuzzy fractional differential equations using fuzzy sumudu transform. *Journal of Nonlinear Sciences and Applications, 10*(11), 2620-2632.

[6] Rivaz, A., Fard, O. S., & Bidgoli, T. A. (2015). Solving fuzzy fractional differential equations by generalized differential transform method. *SeMA Springer*, 1-22.

[7] Salahshour, S., Allahviranloo, T., & Abbasbandy, S. (2012). Solving fuzzy fractional differential equations by fuzzy Laplace transforms. *Communications in Nonlinear Science and Numerical Simulation, 17*(7), 1372-1381.

[8] Salah, A., Khan, M., & Gondal, M. A. (2013). A novel solution procedure for fuzzy fractional heat equations by homotopy analysis transform method. *Neural Computing and Applications, 23*(2), 269-271.

[9] Zhou, J.-K. (1986). *Differential Transformation and its Applications for Electrical Circuits* (in Chinese). Wuhan, China: Huazhong University Press.

[10] Osman, M., Gong, Z., Mustafa, A. M., & Yang, H. (2021). Solving fuzzy $(1+n)$−dimensional Burgers equation. *Advances in Difference Equations, 2021*, 219. https://doi.org/10.1186/s13662-021-03376-y

[11] Mirzaee, F., & Yari, M. K. (2016). A novel computing three-dimensional differential transform method for solving fuzzy partial differential equations. *Ain Shams Engineering Journal, 7*, 695-708.

[12] Ünal, E., & Gökdoğan, A. (2017). Solution of conformable fractional ordinary differential equations via differential transform method. *Optik, 128*, 264-273.

[13] He, J.-H. (1998). Approximate analytical solution for seepage flow with fractional derivatives in porous media. *Computer Methods in Applied Mechanics and Engineering, 167*(1-2), 57-68.

[14] Osman, M., Gong, Z., & Mustafa, A. M. (2020). Comparison of fuzzy Adomian decomposition method with fuzzy VIM for solving fuzzy heat-like and wave-like equations with variable coefficients. *Advances in Difference Equations, 2020*, 327. https://doi.org/10.1186/s13662-020-02784-w

[15] Hamoud, A., Dawood, L., Ghadle, K., & Atshan, S. (2019). Usage of the modified variational iteration technique for solving Fredholm integro-differential equations. *International Journal of Mechanical and Production Engineering Research and Development, 9*(2), 895-902.

[16] Kaleva, O. (1987). Fuzzy differential equations. *Fuzzy Sets and Systems, 24*(3), 301-317.

[17] Gong, Z., & Hao, Y. (2018). Fuzzy Laplace transform based on the Henstock integral and its applications in discontinuous fuzzy systems. *Fuzzy Sets and Systems, 283*, 1-28.

[18] Allahviranloo, T., Gouyandeha, Z., Armanda, A., & Hasanoglub, A. (2015). On fuzzy solutions for heat equation based on generalized Hukuhara differentiability. *Fuzzy Sets and Systems, 265*, 1-23.

[19] Ghazanfari, B., & Ebrahimi, P. (2015). Differential transformation method for solving fuzzy fractional heat equations. *International Journal of Mathematical Modeling and Computations, 5*(1), 81-89.

[20] Odibat, Z., & Shawagfeh, N. (2007). Generalized Taylor's formula. *Applied Mathematics and Computation, 186*(1), 286-293.

[21] Caputo, M. (1969). Linear models of dissipation whose Q is almost frequency independent. Part II. *Journal of the Royal Australian Mathematical Society, 13*(4), 529-539.

[22] Odibat, Z., & Momani, S. (2008). Generalized differential transform method for linear partial differential equations of fractional order. *Applied Mathematics Letters, 21*(2), 194-199.

[23] Momani, S., Odibat, Z., & Ertürk, V. S. (2007). Generalized differential transform method for solving a space-and time-fractional diffusion-wave equation. *Physics Letters A, 370*(5-6), 379-387.

[24] Li, J.-C., & Huang, G.-L. (2010). New approximate solution for time-fractional coupled KdV equations by generalized differential transform method. *Chinese Physics B, 19*(11), 110203-110305.