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Compatible maps of type (β) in intuitionistic generalized fuzzy metric spaces

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Abstract: This paper presents several fixed point theorems for intuitionistic generalized fuzzy metric spaces with an implicit relation. Specifically, we utilize compatible maps of type (β) in intuitionistic generalized fuzzy metric spaces to derive our fixed point theorems. Our results not only extend but also generalize some fixed point theorems that were previously established in complete fuzzy metric spaces. This is achieved by introducing a novel technique, which enhances the applicability and scope of the existing fixed point theorems.

Keywords: Fixed point theorem; Fuzzy metric space; Implicit relations; Intuitionistic fuzzy metric space.

MSC: 47H10; 54H25.

1. Introduction

The concept of fuzzy sets was first introduced by Zadeh in 1965 [1]. Later in 1975, Kramosil and Michalek proposed fuzzy metric spaces as a generalization of metric spaces [2]. The notion of fuzzy metric spaces was further refined by George and Veeramani in 1994, who used continuous t-norms to modify the original definition [3]. In fuzzy metric spaces, Mishra *et al.*, introduced the idea of asymptotically commuting maps in 2005, which plays an essential role in several applications [4].

Another significant contribution to the field is the study of weak compatibility in fuzzy metric spaces, which was introduced by Singh and Jain in 2006 [5]. In 2006, Sedghi and Shobe introduced the concept of $\mathcal{M}$-fuzzy metric space and proved a common fixed point theorem in it [7]. Veerapandi *et al.*, defined generalized $\mathcal{M}$-fuzzy metric spaces and proved several fixed point and coincident point theorems in 2007 [8].

In 1998, Cho *et al.*, introduced the notion of compatible maps of type (β) [9]. In 2008, Altun and Turkoglu developed the concept of compatible maps of type (β) on complete fuzzy metric spaces and proved common fixed point theorems with the help of an implicit relation [10].

Meanwhile, the notion of intuitionistic fuzzy sets was introduced and studied by Atanassov in 1986, as a generalization of fuzzy sets [11]. Park defined the notion of intuitionistic fuzzy metric spaces in 1998 as a further generalization of fuzzy metric spaces due to George and Veeramani [12]. Alaca *et al.*, introduced the concept of intuitionistic fuzzy metric spaces using continuous t-norm and t-conorm in 2005 as another generalization of fuzzy metric spaces in the sense of Kramosil and Michalek [13].

Overall, these developments have significantly expanded the theory of fuzzy metric spaces and provided several generalizations of the original concept. The study of fuzzy metric spaces continues to be an active area of research, with ongoing efforts to explore new applications and extensions of these concepts.

In this paper, we focus on intuitionistic generalized fuzzy metric spaces. Our main contributions are the extension and generalization of the fixed point theorems for compatible maps of type (β) and weakly compatible maps in intuitionistic generalized fuzzy metric spaces with an implicit relation. Our results can be

applied to many existing metric spaces as well as to the new class of intuitionistic generalized fuzzy metric spaces.

We believe that our results provide useful tools for further research in fixed point theory and related areas. Additionally, our approach and techniques may inspire researchers to investigate the fixed point theory in other generalizations of metric spaces, leading to new and interesting results.

2. Preliminaries

Let us start with some essential definitions and known results that will help us to establish our main results.

Definition 1. A 5-tuple $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is said to be an intuitionistic Generalized Fuzzy Metric Space (shortly IGFM-Space), if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t- conorm and $\mathcal{M}$ and $\mathcal{N}$ are fuzzy sets on $X^3 \times (0, \infty)$ satisfying the following conditions: For all $x, y, z, a \in X$ and $s, t > 0$.

(IGFM 1) $\mathcal{M}(x, y, z, t) + \mathcal{N}(x, y, z, t) \leq 1$,
(IGFM 2) $\mathcal{M}(x, y, z, t) > 0$,
(IGFM 3) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,
(IGFM 4) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function,
(IGFM 5) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,
(IGFM 6) $\mathcal{M}(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous,
(IGFM 7) $\mathcal{N}(x, y, z, t) > 0$,
(IGFM 8) $\mathcal{N}(x, y, z, t) = 0$ if and only if $x = y = z$,
(IGFM 9) $\mathcal{N}(x, y, z, t) = \mathcal{N}(p\{x, y, z\}, t)$, where p is a permutation function,
(IGFM 10) $\mathcal{N}(x, y, a, t) \diamond \mathcal{N}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,
(IGFM 11) $\mathcal{N}(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous.

Then, $(\mathcal{M}, \mathcal{N})$ is called an IGFM-Space on X.

The function $\mathcal{M}(x, y, z, t)$ and $\mathcal{N}(x, y, z, t)$ denote the degree of nearness and degree of non-nearness between x, y and z with respect to t, respectively.

Remark 1. Every fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is an IGFM-Space of the form $(X, \mathcal{M}, 1 - \mathcal{M}, *, \diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated, i.e., $x \diamond y = 1 - ((1 - x) * (1 - y))$ for any $x, y \in X$.

Example 1. Let X be an non-empty set and D is the D - metric space on X. Let $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ define $\mathcal{M}(x, y, z, t) = \frac{t}{t+D(x,y,z)}$ and $\mathcal{N}(x, y, z, t) = \frac{D(x,y,z)}{t+D(x,y,z)}$, for all $x, y, z \in X$. Then it is easy to see that $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is a IGFM- Space.

Example 2. Let X be an non-empty set and D is the D-metric space on X. Let $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ define $\mathcal{M}(x, y, z, t) = \mathcal{M}(x, y, t) * \mathcal{M}(y, z, t) * \mathcal{M}(z, x, t)$ and $\mathcal{N}(x, y, z, t) = \mathcal{N}(x, y, t) \diamond \mathcal{N}(y, z, t) \diamond \mathcal{N}(z, x, t)$ for all $x, y, z \in X$. Then it is easy to see that $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is a IGFM-Space.

Definition 2. A pair (A, S) of self-mappings of a IGFM-Space is said to be weak compatible or coincidentally commuting if A and S commute at their coincidence points, i.e., for $x \in X$ if $Ax = Sx$ then $ASx = SAx$.

Definition 3. Let A and B be maps from an IGFM-Space $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ into itself. The maps A and B are said to be compatible of type (β) if $\lim_{n\to\infty} \mathcal{M}(AAx_n, BBx_n, BBx_n, t) = 1$ and $\lim_{n\to\infty} \mathcal{N}(AAx_n, BBx_n, BBx_n, t) = 0$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Example 3. Let X be an IGFM-Space, where $X = [0, \infty]$ and $*$ and $\diamond$ are defined by $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define $\mathcal{M}(x, y, z, t) = \frac{t}{t+D(x,y,z)}$ and $\mathcal{N}(x, y, z, t) = \frac{D(x,y,z)}{t+D(x,y,z)}$ for all $x, y, z \in X$. Define self maps A and B on X as follows:

$$Ax = \frac{x}{4}; Bx = \frac{3x}{4}; x_n = \frac{1}{n},$$

then A and B be be compatible of type (β).

Definition 4. Let $I = [0,1]$, $*$ be a continuous t-norm and $\diamond$ be a continuous t-conorm, $\Psi = \{\phi, \psi\}$ be the set of all real continuous functions, $\phi, \psi : I^6 \to \mathbb{R}$ satisfying the following conditions;

(i) ϕ is decreasing and ψ is increasing in the fifth and sixth variables,
(ii) If, for some constant $k \in (0,1)$ we have

(a) $\phi\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) \geq 1$ or,
(b) $\phi\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right), 1\right) \geq 1$,
(c) $\psi\left(u(kt), v(t), v(t), u(t), 0, u\left(\frac{t}{2}\right) \diamond v\left(\frac{t}{2}\right)\right) \leq 1$ or,
(d) $\psi\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) \diamond v\left(\frac{t}{2}\right), 0\right) \leq 1$,

for any fixed $t > 0$ and any non-decreasing functions $u, v : (0,\infty) \to I$ with $0 \leq u(t), v(t) \leq 1$, then there exists $h \in (0,1)$ with $u(ht) \geq v(t) * u(t)$ and $u(ht) \leq v(t) \diamond u(t)$.

(iii) If, for some constant $k \in (0,1)$ we have $\phi(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$, for any fixed $t > 0$ and any non-decreasing function $u : (0,\infty) \to I$, then $u(kt) \geq u(t)$. Also, if, for some constant $k \in (0,1)$ we have $\psi(u(kt), u(t), 0, 0, u(t), u(t)) \leq 1$, for any fixed $t > 0$ and any non-increasing function $u : (0,\infty) \to I$, then $u(kt) \leq u(t)$.

Example 4. Let $\phi, \psi : [0,1]^6 \to R$ be defined by

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = \frac{t_1}{\min\{t_2, t_3, t_4, t_5, t_6\}},$$

and

$$\psi(t_1, t_2, t_3, t_4, t_5, t_6) = \frac{t_1}{\max\{t_2, t_3, t_4, t_5, t_6\}}.$$

Also $*$ and $\diamond$ be defined by $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$. For any fixed $t > 0$ and any non-decreasing function $u, v : (0,\infty) \to [0,1]$ with

$$0 < u(t), v(t) \leq 1, \phi\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right), 1\right) \geq 1,$$

and

$$\psi\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) \diamond v\left(\frac{t}{2}\right), 0\right) \leq 1,$$

we have,

$$\frac{u(kt)}{\min\left\{v(t), u(t), v(t), u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right), 1\right\}} \geq 1,$$

implies that

$$\frac{u(kt)}{u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)} \geq 1 * u(kt) \geq u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right).$$

Therefore, $u(pt) \geq u(t) * v(t)$, for $p = 2k \in (0,1)$ and

$$\frac{u(kt)}{\max\left\{v(t), u(t), v(t), u\left(\frac{t}{2}\right) \diamond v\left(\frac{t}{2}\right), 0\right\}} \leq 1,$$

implies that

$$u(kt) \leq u\left(\frac{t}{2}\right) \diamond v\left(\frac{t}{2}\right).$$

Therefore, $u(pt) \leq u(t) \diamond v(t)$, for $p = 2k \in (0,1)$. Also, from

$$\phi\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) \geq 1,$$

and

$$\psi\left(u(kt), v(t), v(t), u(t), 0, u\left(\frac{t}{2}\right) \diamond v\left(\frac{t}{2}\right)\right) \leq 1,$$

we have,

$$\frac{u(kt)}{\min\left\{v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right\}} \geq 1,$$

implies that

$$u(kt) \geq u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right).$$

Therefore, $u(pt) \geq u(t) * v(t)$, for $p = 2k \in (0,1)$ and

$$\frac{u(kt)}{\max\left\{v(t), v(t), u(t), 0, u\left(\frac{t}{2}\right) \diamond v\left(\frac{t}{2}\right)\right\}} \leq 1,$$

implies that

$$u(kt) \leq u\left(\frac{t}{2}\right) \diamond v\left(\frac{t}{2}\right).$$

Therefore, $u(pt) \leq u(t) \diamond v(t)$, for $p = 2k \in (0,1)$. Again, from

$$\phi(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1,$$

and

$$\psi(u(kt), u(t), 0, 0, u(t), u(t)) \leq 1,$$

we have,

$$\frac{u(kt)}{\min\{u(t), 1, 1, u(t), u(t)\}} \geq 1,$$

implies $u(kt) \geq u(t)$ and $\frac{u(kt)}{\max\{u(t),0,0,u(t),u(t)\}} \leq 1$, therefore $u(kt) \leq u(t)$, for $k \in (0,1)$. Thus, $\phi, \psi \in \Psi$.

Lemma 5. *In a IGFM-Space $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ limit of a sequence is unique.*

Proof. Let $\{x_n\}$ be a sequence in IGFM-Space X and suppose $x_n \to x$ and $x_n \to y$ for some $x, y \in X$. We shall show that $x = y$. We have $\mathcal{M}(x, x, y, t+s) \geq \mathcal{M}(x, x, x_n, t) * \mathcal{M}(x_n, y, y, s)$ and let $n \to \infty$. Then $\mathcal{M}(x, x, y, t+s) \geq \lim_{n\to\infty} \mathcal{M}(x, x, x_n, t) * \lim_{n\to\infty} \mathcal{M}(x_n, y, y, s) = 1 * 1 = 1$, thus $\mathcal{M}(x, x, y, t+s) = 1$. We have $\mathcal{N}(x, x, y, t+s) \leq \mathcal{N}(x, x, x_n, t) * \mathcal{N}(x_n, y, y, s)$ and let $n \to \infty$, then $\mathcal{N}(x, x, y, t+s) \leq \lim_{n\to\infty} \mathcal{N}(x, x, x_n, t) \diamond \lim_{n\to\infty} \mathcal{N}(x_n, y, y, s) = 0 \diamond 0 = 0$. Thus $\mathcal{N}(x, x, y, t+s) = 0$, for all $t, s > 0$. So, $x = y$. □

Lemma 6. *Let $(X, M, N, *, \diamond)$ be a IGFM-Space. Then*

(i) $\mathcal{M}(x, y, z, .)$ is a non-decreasing and $\mathcal{N}(x, y, z, .)$ is a non-increasing functions, for all $x, y, z \in X$.
(ii) If there exists $k \in (0,1)$ such that for all $x, y, z \in X$, $\mathcal{M}(x, y, z, kt) \geq \mathcal{M}(x, y, z, t)$, $\mathcal{N}(x, y, z, kt) \leq \mathcal{N}(x, y, z, t)$ for all $t > 0$, then $x = y = z$.
(iii) If there exists a number $k \in (0,1)$ such that $\mathcal{M}(x_{n+2}, x_{n+2}, x_{n+1}, kt) \geq \mathcal{M}(x_{n+1}, x_{n+1}, x_n, t)$, $\mathcal{N}(x_{n+2}, x_{n+2}, x_{n+1}, kt) \leq \mathcal{N}(x_{n+1}, x_{n+1}, x_n, t)$ for all $t > 0$ and $n \in \mathcal{N}$.

Then $\{x_n\}$ is a Cauchy sequence in X.

Proof. (i) By Definition 1(IGFM 5) for each $x, y, z, a \in X$ and $t, s > 0$, we have $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t+s)$. If $a = z$, then $\mathcal{M}(x, y, z, t) * \mathcal{M}(z, z, z, s) \leq \mathcal{M}(x, y, z, t+s)$, that is $\mathcal{M}(x, y, z, t) \leq \mathcal{M}(x, y, z, t+s)$.

Similarly by Definition 1(IGFM 10), we get $\mathcal{N}(x, y, z, t) \geq \mathcal{N}(x, y, z, t+s)$.

(ii) Assume that $\mathcal{M}(x, y, z, kt) \geq \mathcal{M}(x, y, z, t)$ and $\mathcal{N}(x, y, z, kt) \leq \mathcal{N}(x, y, z, t)$, for all $t > 0$. Since $kt < t$, by (i) we have $\mathcal{M}(x, y, z, kt) \leq \mathcal{M}(x, y, z, t)$ and $\mathcal{N}(x, y, z, kt) \geq \mathcal{N}(x, y, z, t)$. Thus $x = y = z$.

(iii) By induction with the condition $\mathcal{M}(x_{n+2}, x_{n+2}, x_{n+1}, kt) \geq \mathcal{M}(x_{n+1}, x_{n+1}, x_n, t)$, $\mathcal{N}(x_{n+2}, x_{n+2}, x_{n+1}, kt) \leq \mathcal{N}(x_{n+1}, x_{n+1}, x_n, t)$, for all $t > 0$ and $n \in \mathbb{N}$. So, $\mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t) \geq$

$\mathcal{M}\left(x_1, x_1, x_2, \frac{t}{k^n}\right), \mathcal{N}(x_{n+1}, x_{n+1}, x_{n+2}, t) \leq \mathcal{N}\left(x_1, x_1, x_2, \frac{t}{k^n}\right)$. Thus for any positive integer p and real number $t > 0$, we have

$$\begin{aligned}\mathcal{M}(x_n, x_n, x_{n+p}, t) &\geq \mathcal{M}\left(x_n, x_n, x_{n+1}, \frac{t}{p}\right) * \mathcal{M}\left(x_{n+1}, x_{n+1}, x_{n+2}, \frac{t}{p}\right) * \cdots * \mathcal{M}\left(x_{n+p-1}, x_{n+p-1}, x_{n+p}, \frac{t}{p}\right)\\ &\geq \mathcal{M}\left(x_1, x_1, x_2, \frac{t}{pk^{n-1}}\right) * \cdots * \mathcal{M}\left(x_1, x_1, x_2, \frac{t}{pk^{n+p-2}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(x_n, x_n, x_{n+p}, t) &\leq \mathcal{N}\left(x_n, x_n, x_{n+1}, \frac{t}{p}\right) \diamond \mathcal{N}\left(x_{n+1}, x_{n+1}, x_{n+2}, \frac{t}{p}\right) \diamond \cdots \diamond \mathcal{N}\left(x_{n+p-1} x_{n+p-1}, x_{n+p}, \frac{t}{p}\right)\\ &\leq \mathcal{N}\left(x_1, x_1, x_2, \frac{t}{pk^{n-1}}\right) \diamond \cdots \diamond \mathcal{N}\left(x_1, x_1, x_2, \frac{t}{pk^{n+p-2}}\right)\end{aligned}$$

Therefore, $\lim_{n\to\infty} \mathcal{M}(x_n, x_n, x_{n+p}, t) \geq 1 * \cdots * 1 \geq 1$, $\lim_{n\to\infty} \mathcal{N}(x_n, x_n, x_{n+p}, t) \leq 0 \diamond \cdots \diamond 0 \leq 0$, which implies that $\{x_n\}$ is a Cauchy sequence in X.

□

3. Main results

In this section, we prove a common fixed point theorems for compatible mappings of type (β) in IGFM-Space.

Theorem 7. *Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be a complete IGFM - Space and A, B, G, H, S, T, P, Q and R be mappings from X into itself such that the following conditions are satisfied:*

$$P(X) \subset ST(X), \quad Q(X) \subset AB(X) \quad \text{and} \quad R(X) \subset GH(X), \tag{1}$$

$$(P, AB) \text{ is compatible of type } (\beta) \text{ and } (Q, GH), (R, ST) \text{ are weak compatible,} \tag{2}$$

$$\text{There exists } k \in (0,1) \text{ such that for every } x, y, z \in X \text{ and } t > 0. \tag{3}$$

$$\phi\begin{pmatrix} \mathcal{M}^2(Px, Qy, Rz, kt), \mathcal{M}^2(ABx, STz, GHy, t), \mathcal{M}^2(Px, STz, GHy, t), \\ \mathcal{M}^2(Rz, Qy, STz, t), \mathcal{M}^2(GHy, Px, ABx, t), \mathcal{M}^2(ABxQy, Rz, t) \end{pmatrix} \geq 1,$$

$$\psi\begin{pmatrix} \mathcal{N}^2(Px, Qy, Rz, kt), \mathcal{N}^2(ABx, STz, GHy, t), \mathcal{N}^2(Px, STz, GHy, t), \\ \mathcal{N}^2(Rz, Qy, STz, t), \mathcal{N}^2(GHy, Px, ABx, t), \mathcal{N}^2(ABxQy, Rz, t) \end{pmatrix} \leq 1.$$

Then A, B, G, H, S, T, P, Q and R have a unique common fixed point in X.

Proof. Let $x_0 \in X$, then from (1), we have $x_1, x_2, x_3 \in X$ such that $Px_0 = STx_1, Qx_1 = ABx_2$ and $Rx_2 = GHx_3$. Inductively, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $n \in \mathbb{N}$

$$Px_{2n-2} = STx_{2n-1} = y_{2n-1}, Qx_{2n-1} = ABx_{2n} = y_{2n} \text{ and } Rx_{2n} = GHx_{2n+1} = y_{2n+1}.$$

Put $x = x_{2n}, y = x_{2n+1}$ and $z = x_{2n+1}$ in (2) then we have

$$\phi\begin{pmatrix} \mathcal{M}^2(Px_{2n}, Qx_{2n+1}, Rx_{2n+1}, kt), \mathcal{M}^2(ABx_{2n}, STx_{2n+1}, GHx_{2n+1}, t), \mathcal{M}^2(Px_{2n}, STx_{2n+1}, GHx_{2n+1}, t), \\ \mathcal{M}^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t), \mathcal{M}^2(GHx_{2n+1}, Px_{2n}, ABx_{2n}, t), \mathcal{M}^2(ABx_{2n}, Qx_{2n+1}, Rx_{2n+1}, t)) \end{pmatrix} > 1$$

$$\phi\begin{pmatrix} \mathcal{M}^2(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt), \mathcal{M}^2(y_{2n}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}^2(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), \\ \mathcal{M}^2(y_{2n+2}, y_{2n+2}, y_{2n+1}, t), \mathcal{M}^2(y_{2n+1}, y_{2n+1}, y_{2n}, t), \mathcal{M}^2(y_{2n}, y_{2n+2}, y_{2n+2}, t) \end{pmatrix} > 1$$

$$\phi\begin{pmatrix} \mathcal{M}^2(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt), \mathcal{M}^2(y_{2n}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}^2(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), \\ \mathcal{M}^2(y_{2n+2}, y_{2n+2}, y_{2n+1}, t), \mathcal{M}^2(y_{2n+1}, y_{2n+1}, y_{2n}, t), \\ \mathcal{M}^2(y_{2n}, y_{2n+1}, y_{2n+1}, \frac{t}{2}) * \mathcal{M}^2(y_{2n+1}, y_{2n+2}, y_{2n+2}, \frac{t}{2}) \end{pmatrix} > 1$$

$$\psi\begin{pmatrix} \mathcal{N}^2(Px_{2n}, Qx_{2n+1}, Rx_{2n+1}, kt), \mathcal{N}^2(ABx_{2n}, STx_{2n+1}, GHx_{2n+1}, t), \mathcal{N}^2(Px_{2n}, STx_{2n+1}, GHx_{2n+1}, t), \\ \mathcal{N}^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t), \mathcal{N}^2(GHx_{2n+1}, Px_{2n}, ABx_{2n}, t), \mathcal{N}^2(ABx_{2n}, Qx_{2n+1}, Rx_{2n+1}, t) \end{pmatrix} < 1,$$

$$\psi\begin{pmatrix} \mathcal{N}^2(y_{2n+1},y_{2n+2},y_{2n+2},kt),\mathcal{N}^2(y_{2n},y_{2n+1},y_{2n+1},t),\mathcal{N}^2(y_{2n+1},y_{2n+1},y_{2n+1},t), \\ \mathcal{N}^2(y_{2n+2},y_{2n+2},y_{2n+1},t),\mathcal{N}^2(y_{2n+1},y_{2n+1},y_{2n},t),\mathcal{N}^2(y_{2n},y_{2n+2},y_{2n+2},t) \end{pmatrix} < 1$$

$$\psi\begin{pmatrix} \mathcal{N}^2(y_{2n+1},y_{2n+2},y_{2n+2},kt),\mathcal{N}^2(y_{2n},y_{2n+1},y_{2n+1},t),\mathcal{N}^2(y_{2n+1},y_{2n+1},y_{2n+1},t), \\ \mathcal{N}^2(y_{2n+2},y_{2n+2},y_{2n+1},t),\mathcal{N}^2(y_{2n+1},y_{2n+1},y_{2n},t), \\ \mathcal{N}^2(y_{2n},y_{2n+1},y_{2n+1},\frac{t}{2})\diamond\mathcal{N}^2(y_{2n+1},y_{2n+2},y_{2n+2},\frac{t}{2}) \end{pmatrix} < 1.$$

From condition (1), we have

$$\mathcal{M}^2(y_{2n+1},y_{2n+2},y_{2n+2},kt) \geq \mathcal{M}^2(y_{2n},y_{2n+1},y_{2n+1},\frac{t}{2}) * \mathcal{M}^2(y_{2n+1},y_{2n+2},y_{2n+2},\frac{t}{2}).$$

Also, we have

$$\mathcal{M}^2(y_{2n+1},y_{2n+2},y_{2n+2},kt) \geq \mathcal{M}^2(y_{2n},y_{2n+1},y_{2n+1},\frac{t}{2}),$$

i.e.,

$$\mathcal{M}(y_{2n+1},y_{2n+2},y_{2n+2},kt) \geq \mathcal{M}(y_{2n},y_{2n+1},y_{2n+1},\frac{t}{2})$$

and

$$\mathcal{N}^2(y_{2n+1},y_{2n+2},y_{2n+2},kt) \leq \mathcal{N}^2(y_{2n},y_{2n+1},y_{2n+1},\frac{t}{2})\diamond\mathcal{N}^2(y_{2n+1},y_{2n+2},y_{2n+2},\frac{t}{2}).$$

So, we have

$$\mathcal{N}^2(y_{2n+1},y_{2n+2},y_{2n+2},kt) \leq \mathcal{N}^2(y_{2n},y_{2n+1},y_{2n+1},\frac{t}{2}),$$

i.e.,

$$\mathcal{N}(y_{2n+1},y_{2n+2},y_{2n+2},kt) \leq \mathcal{N}(y_{2n},y_{2n+1},y_{2n+1},\frac{t}{2}).$$

Similarly, we have

$$\mathcal{M}(y_{2n+2},y_{2n+3},y_{2n+3},kt) \geq \mathcal{M}(y_{2n+1},y_{2n+2},y_{2n+2},\frac{t}{2}),\mathcal{N}(y_{2n+2},y_{2n+3},y_{2n+3},kt) \leq \mathcal{N}(y_{2n+1},y_{2n+2},y_{2n+2},\frac{t}{2}).$$

Thus, we have

$$\mathcal{M}(y_{n+1},y_{n+2},y_{n+2},kt) \geq \mathcal{M}\left(y_n,y_{n+1},y_{n+1},\frac{t}{2}\right),$$
$$\mathcal{M}(y_{n+1},y_{n+2},y_{n+2},t) \geq \mathcal{M}\left(y_n,y_{n+1},y_{n+1},\frac{t}{2}^k\right),$$
$$\mathcal{M}(y_n,y_{n+1},y_{n+1},t) \geq \mathcal{M}\left(y_0,y_1,y_1,\frac{t}{2}^{nk}\right) \to 1.$$

and

$$\mathcal{N}(y_{n+1},y_{n+2},y_{n+2},kt) \leq \mathcal{N}\left(y_n,y_{n+1},y_{n+1},\frac{t}{2}\right),$$
$$\mathcal{N}(y_{n+1},y_{n+2},y_{n+2},t) \leq \mathcal{N}\left(y_n,y_{n+1},y_{n+1},\frac{t}{2}^k\right),$$
$$\mathcal{N}(y_n,y_{n+1},y_{n+1},t) \leq \mathcal{N}\left(y_0,y_1,y_1,\frac{t}{2}^{nk}\right) \to 0 \text{ as } n\to\infty, \text{for all } \ t>0, \text{for each } \ \epsilon>0.$$

We can choose $n_0 \in \mathcal{N}$ such that $\mathcal{M}(y_n,y_{n+1},y_{n+1},t) > 1-\epsilon, \mathcal{N}(y_n,y_{n+1},y_{n+1},t) < \epsilon$ for all $n > n_0$. For any $m,n \in \mathcal{N}$, we suppose that $m \geq n$, then we have

$$\mathcal{M}(y_n,y_m,y_m,t) \geq \mathcal{M}\left(y_n,y_{n+1},y_{n+1},\frac{t}{m-n}\right) * \mathcal{M}\left(y_{n+1},y_{n+2},y_{n+2},\frac{t}{m-n}\right) * \cdots * \mathcal{M}\left(y_{m-1},y_m,y_m,\frac{t}{m-n}\right),$$
$$\mathcal{M}(y_n,y_m,y_m,t) \geq (1-\epsilon) * (1-\epsilon) * \cdots * (1-\epsilon)(m-n)\text{times}\mathcal{M}(y_n,y_m,y_m,t) \geq (1-\epsilon),$$
$$\mathcal{N}(y_n,y_m,y_m,t) \leq \mathcal{N}\left(y_n,y_{n+1},y_{n+1},\frac{t}{m-n}\right) \diamond \mathcal{N}\left(y_{n+1},y_{n+2},y_{n+2},\frac{t}{m-n}\right) \diamond \cdots \diamond \mathcal{N}\left(y_{m-1},y_m,y_m,\frac{t}{m-n}\right),$$

$$\mathcal{N}(y_n, y_m, y_m, t) \leq (\epsilon) \diamond (\epsilon) \diamond \cdots \diamond (\epsilon)(m-n)\text{times}\mathcal{M}(y_n, y_m, y_m, t) \leq (\epsilon),$$

and hence $\{y_n\}$ is a Cauchy sequence in X.

Since $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is complete, $\{y_n\}$ converges to some point $w \in X$. Also, its subsequences converges to the same point $w \in X$. That is

$$\{Px_{2n+2}\} \to w, \qquad \{STx_{2n+1}\} \to w, \tag{4}$$
$$\{Qx_{2n+1}\} \to w, \qquad \{GHx_{2n}\} \to w, \tag{5}$$
$$\{Rx_{2n}\} \to w, \qquad \{ABx_{2n-1}\} \to w. \tag{6}$$

As (P, AB) is compatible pair of type (β), we have $\mathcal{M}(PPx_{2n}, (AB)(AB)x_{2n}, (AB)(AB)x_{2n}, t) = 1$ and $\mathcal{N}(PPx_{2n}, (AB)(AB)x_{2n}, (AB)(AB)x_{2n}, t) = 0$, for all $t > 0$. Equivalently, $\mathcal{M}(PPx_{2n}, ABw, ABw, t) = 1$ and $\mathcal{N}(PPx_{2n}, ABw, ABw, t) = 0$. Therefore, $PPx_{2n} \to ABw$.

Put $x = (AB)x_{2n}, y = x_{2n+1}$ and $z = x_{2n+1}$ in (3), we have

$$\phi\begin{pmatrix} \mathcal{M}^2(P(AB)x_{2n}, Qx_{2n+1}, Rx_{2n+1}, kt), \mathcal{M}^2(AB(AB)x_{2n}, STx_{2n+1}, GHx_{2n+1}, t), \\ \mathcal{M}^2(P(AB)x_{2n}, STx_{2n+1}, GHx_{2n+1}, t), \mathcal{M}^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t), \\ \mathcal{M}^2(GHx_{2n+1}, P(AB)x_{2n}, AB(AB)x_{2n}, t), \mathcal{M}^2(AB(AB)x_{2n}, Qx_{2n+1}, Rx_{2n+1}, t) \end{pmatrix} > 1$$
$$\psi\begin{pmatrix} \mathcal{N}^2(P(AB)x_{2n}, Qx_{2n+1}, Rx_{2n+1}, kt), \mathcal{N}^2(AB(AB)x_{2n}, STx_{2n+1}, GHx_{2n+1}, t), \\ \mathcal{N}^2(P(AB)x_{2n}, STx_{2n+1}, GHx_{2n+1}, t), \mathcal{N}^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t), \\ \mathcal{N}^2(GHx_{2n+1}, P(AB)x_{2n}, AB(AB)x_{2n}, t), \mathcal{N}^2(AB(AB)x_{2n}, Qx_{2n+1}, Rx_{2n+1}, t) \end{pmatrix} < 1.$$

Taking $n \to \infty$ and (1) we get

$$\mathcal{M}^2((AB)w, w, w, kt) \geq \mathcal{M}^2((AB)w, w, w, t), \mathcal{N}^2((AB)w, w, w, kt) \leq \mathcal{N}^2((AB)w, w, w, t).$$

That is

$$\mathcal{M}((AB)w, w, w, kt) \geq \mathcal{M}((AB)w, w, w, t), \mathcal{N}((AB)w, w, w, kt) \leq \mathcal{N}((AB)w, w, w, t).$$

Therefore we have, $ABw = w$.

Put $x = w, y = x_{2n+1}$ and $z = x_{2n+1}$ in (3) we have

$$\phi\begin{pmatrix} \mathcal{M}^2(Pw, Qx_{2n+1}, Rx_{2n+1}, kt), \mathcal{M}^2(ABw, STx_{2n+1}, GHx_{2n+1}, t), \mathcal{M}^2(Pw, STx_{2n+1}, GHx_{2n+1}, t), \\ \mathcal{M}^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t), \mathcal{M}^2(GHx_{2n+1}, Pw, ABw, t), \mathcal{M}^2(ABw, Qx_{2n+1}, Rx_{2n+1}, t) \end{pmatrix} \geq 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(Pw, Qx_{2n+1}, Rx_{2n+1}, kt), \mathcal{N}^2(ABw, STx_{2n+1}, GHx_{2n+1}, t), \mathcal{N}^2(Pw, STx_{2n+1}, GHx_{2n+1}, t), \\ \mathcal{N}^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t)\mathcal{N}^2(GHx_{2n+1}, Pw, ABw, t), \mathcal{N}^2(ABw, Qx_{2n+1}, Rx_{2n+1}, t) \end{pmatrix} \leq 1.$$

Taking $n \to \infty$ and (1) we get,

$$\mathcal{M}^2(Pw, w, w, kt) \geq \mathcal{M}^2(Pw, w, w, t), \mathcal{N}^2(Pw, w, w, kt) \leq \mathcal{N}^2(Pw, w, w, t),$$

that is

$$\mathcal{M}(Pw, w, w, kt) \geq \mathcal{M}(Pw, w, w, t)$$

and

$$\mathcal{N}(Pw, w, w, kt) \leq \mathcal{N}(Pw, w, w, t).$$

Therefore we have by using Lemma 6, we get $Pw = w$. So, we have $ABw = Pw = w$.

Putting $x = Bw, y = x_{2n+1}$ and $z = x_{2n+1}$ in (3) we have

$$\phi\begin{pmatrix} \mathcal{M}^2(PBw, Qx_{2n+1}, Rx_{2n+1}, kt), \mathcal{M}^2(ABBw, STx_{2n+1}, GHx_{2n+1}, t), \mathcal{M}^2(PBw, STx_{2n+1}, GHx_{2n+1}, t), \\ \mathcal{M}^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t), \mathcal{M}^2(GHx_{2n+1}, PBw, ABBw, t), \mathcal{M}^2(ABBw, Qx_{2n+1}, Rx_{2n+1}, t) \end{pmatrix} > 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(PBw, Qx_{2n+1}, Rx_{2n+1}, kt), \mathcal{N}^2(ABBw, STx_{2n+1}, GHx_{2n+1}, t), \mathcal{N}^2(PBw, STx_{2n+1}, GHx_{2n+1}, t), \\ \mathcal{N}^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t)\mathcal{N}^2(GHx_{2n+1}, PBw, ABBw, t), \mathcal{N}^2(ABBw, Qx_{2n+1}, Rx_{2n+1}, t) \end{pmatrix} < 1.$$

Taking $n \to \infty$ in (1) and using (4), we get $\mathcal{M}^2(Bw,w,w,kt) \geq \mathcal{M}^2(Bw,w,w,t)$ and $\mathcal{N}^2(Bw,w,w,kt) \leq \mathcal{N}^2(Bw,w,w,t)$. That is $\mathcal{M}(Bw,w,w,kt) \geq \mathcal{M}(Bw,w,w,t)$ and $\mathcal{N}(Bw,w,w,kt) \leq \mathcal{N}(Bw,w,w,t)$. Therefore by using Lemma 6, we have $Bw = w$, and also we have $ABw = w$ implies $Aw = w$. Therefore $Aw = Bw = Pw = w$. As $P(X) \subset ST(X)$, there exists $u \in X$ such that $w = Pw = STu$.

Putting $x = x_{2n}, y = x_{2n}$ and $z = u$ in (3) we have

$$\phi\begin{pmatrix} \mathcal{M}^2(Px_{2n},Qx_{2n},Ru,kt),\mathcal{M}^2(ABx_{2n},STu,GHx_{2n},t),\mathcal{M}^2(Px_{2n},STu,GHx_{2n},t), \\ \mathcal{M}^2(Ru,Qx_{2n},STu,t),\mathcal{M}^2(GHx_{2n},Px_{2n},ABx_{2n},t),\mathcal{M}^2(ABx_{2n},Qx_{2n},Ru,t) \end{pmatrix} > 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(Px_{2n},Qx_{2n},Ru,kt),\mathcal{N}^2(ABx_{2n},STu,GHx_{2n},t),\mathcal{N}^2(Px_{2n},STu,GHx_{2n},t), \\ \mathcal{N}^2(Ru,Qx_{2n},STu,t),\mathcal{N}^2(GHx_{2n},Px_{2n},ABx_{2n},t),\mathcal{N}^2(ABx_{2n},Qx_{2n},Ru,t) \end{pmatrix} < 1.$$

Taking $n \to \infty$ and using in (4) and (5) we get

$$\phi\begin{pmatrix} \mathcal{M}^2(w,w,Ru,kt),\mathcal{M}^2(w,STu,w,t),\mathcal{M}^2(w,STu,w,t), \\ \mathcal{M}^2(Ru,w,STu,t),\mathcal{M}^2(w,w,w,t),\mathcal{M}^2(w,w,Ru,t) \end{pmatrix} > 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(w,w,Ru,kt),\mathcal{N}^2(w,STu,w,t),\mathcal{N}^2(w,STu,w,t), \\ \mathcal{N}^2(Ru,w,STu,t),\mathcal{N}^2(w,w,w,t),\mathcal{N}^2(w,w,Ru,t) \end{pmatrix} < 1,$$

$\mathcal{M}^2(w,w,Ru,kt) \geq \mathcal{M}^2(w,w,Ru,t)$ and $\mathcal{N}^2(w,w,Ru,kt) \leq \mathcal{N}^2(w,w,Ru,t)$., that is $\mathcal{M}(w,w,Ru,kt) \geq \mathcal{M}(w,w,Ru,t)$ and $\mathcal{N}(w,w,Ru,kt) \leq \mathcal{N}(w,w,Ru,t)$, we have $Ru = w$. Hence $STu = w = Ru$. Hence (R,ST) is weak compatible, therefore, we have $RSTu = STRu$. Thus $Rw = STw$.

Putting $x = x_{2n}, y = x_{2n}$ and $z = w$ in (3) we get

$$\phi\begin{pmatrix} \mathcal{M}^2(Px_{2n},Qx_{2n},Rw,kt),\mathcal{M}^2(ABx_{2n},STw,GHx_{2n},t),\mathcal{M}^2(Px_{2n},STw,GHx_{2n},t), \\ \mathcal{M}^2(Rw,Qx_{2n},STw,t),\mathcal{M}^2(GHx_{2n},Px_{2n},ABx_{2n},t),\mathcal{M}^2(ABx_{2n},Qx_{2n},Rw,t) \end{pmatrix} > 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(Px_{2n},Qx_{2n},Rw,kt),\mathcal{N}^2(ABx_{2n},STw,GHx_{2n},t),\mathcal{N}^2(Px_{2n},STw,GHx_{2n},t), \\ \mathcal{N}^2(Rw,Qx_{2n},STw,t),\mathcal{N}^2(GHx_{2n},Px_{2n},ABx_{2n},t),\mathcal{N}^2(ABx_{2n},Qx_{2n},Rw,t) \end{pmatrix} < 1.$$

Taking $n \to \infty$ and using (5) we get

$$\phi\begin{pmatrix} \mathcal{M}^2(w,w,Rw,kt),\mathcal{M}^2(w,STw,w,t),\mathcal{M}^2(w,STw,w,t), \\ \mathcal{M}^2(Rw,w,w,t),\mathcal{M}^2(w,w,w,t),\mathcal{M}^2(w,w,Rw,t) \end{pmatrix} > 1,$$

and $\mathcal{M}^2(w,w,Ru,kt) \geq \mathcal{M}^2(w,w,Ru,t)$ and hence $\mathcal{M}(w,w,Ru,kt) \geq \mathcal{M}(w,w,Ru,t)$. Also,

$$\psi\begin{pmatrix} \mathcal{N}^2(w,w,Rw,kt),\mathcal{N}^2(w,STw,w,t),\mathcal{N}^2(w,STw,w,t), \\ \mathcal{N}^2(Rw,w,w,t),\mathcal{N}^2(w,w,w,t),\mathcal{N}^2(w,w,Rw,t) \end{pmatrix} < 1,$$

and $\mathcal{N}^2(w,w,Ru,kt) \leq \mathcal{N}^2(w,w,Ru,t)$ and hence $\mathcal{N}(w,w,Ru,kt) \leq \mathcal{N}(w,w,Ru,t)$, we get $Rw = w$.

Putting $x = x_{2n}, y = x_{2n}$ and $z = Tw$ in (3) we get

$$\phi\begin{pmatrix} \mathcal{M}^2(Px_{2n},Qx_{2n},RTw,kt),\mathcal{M}^2(ABx_{2n},STTw,GHx_{2n},t),\mathcal{M}^2(Px_{2n},STTw,GHx_{2n},t), \\ \mathcal{M}^2(RTw,Qx_{2n},STTw,t),\mathcal{M}^2(GHx_{2n},Px_{2n},ABx_{2n},t),\mathcal{M}^2(ABx_{2n}Qx_{2n},RTw,t) \end{pmatrix} > 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(Px_{2n},Qx_{2n},RTw,kt),\mathcal{N}^2(ABx_{2n},STTw,GHx_{2n},t),\mathcal{N}^2(Px_{2n},STTw,GHx_{2n},t), \\ \mathcal{N}^2(RTw,Qx_{2n},STTw,t)\mathcal{N}^2(GHx_{2n},Px_{2n},ABx_{2n},t),\mathcal{N}^2(ABx_{2n}Qx_{2n},RTw,t) \end{pmatrix} < 1.$$

As $RT = TR$ and $ST = TS$. We have $RTw = TRw = T$ and $ST(Tw) = T(STw) = TRw = Tw$. Taking $n \to \infty$ we get

$$\phi\left(\mathcal{M}^2(w,w,Tw,kt),\mathcal{M}^2(w,Tw,w,t),\mathcal{M}^2(w,Tw,w,t),\mathcal{M}^2(Tw,w,Tw,t),\mathcal{M}^2(w,w,w,t),\mathcal{M}^2(w,w,Tw,t)\right) > 1,$$

and

$$\mathcal{M}^2(w,w,Tw,kt) \geq \mathcal{M}^2(w,w,Tw,t).$$

Therefore $\mathcal{M}(w,w,Tw,kt) \geq \mathcal{M}(w,w,Tw,t)$. Taking $n \to \infty$ we get $\psi(\mathcal{N}^2(w,w,Tw,kt),\mathcal{N}^2(w,Tw,w,t),\mathcal{N}^2(w,Tw,w,t),\mathcal{N}^2(Tw,w,Tw,t),\mathcal{N}^2(w,w,w,t),\mathcal{N}^2(w,w,Tw,t)) < 1$ and $\mathcal{N}^2(w,w,Tw,kt) \leq \mathcal{N}^2(w,w,Tw,t)$. Therefore $\mathcal{N}(w,w,Tw,kt) \leq \mathcal{N}(w,w,Tw,t)$. Therefore by Lemma 6, we have $Tw = w$. Now, $STw = Tw = w$ implies $Sw = w$. Hence

$$Sw = Tw = Rw = w. \tag{7}$$

As $R(X) \subset GH(X)$, there exists $u \in X$ such that $w = Rw = GHu$.

Putting $x = x_{2n}, y = u$ and $z = x_{2n}$ in (3) we get

$$\phi\begin{pmatrix} \mathcal{M}^2(Px_{2n},Qu,Rx_{2n},kt),\mathcal{M}^2(ABx_{2n},STx_{2n},GHu,t),\mathcal{M}^2(Px_{2n},STx_{2n},GHu,t), \\ \mathcal{M}^2(Rx_{2n},Qu,STx_{2n},t),\mathcal{M}^2(GHu,Px_{2n},ABx_{2n},t),\mathcal{M}^2(ABx_{2n},Qu,Rx_{2n},t) \end{pmatrix} > 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(Px_{2n},Qu,Rx_{2n},kt),\mathcal{N}^2(ABx_{2n},STx_{2n},GHu,t),\mathcal{N}^2(Px_{2n},STx_{2n},GHu,t), \\ \mathcal{N}^2(Rx_{2n},Qu,STx_{2n},t),\mathcal{N}^2(GHu,Px_{2n},ABx_{2n},t),\mathcal{N}^2(ABx_{2n},Qu,Rx_{2n},t) \end{pmatrix} < 1.$$

Taking $n \to \infty$ and using in (4) and (5) we get,

$$\phi\begin{pmatrix} \mathcal{M}^2(w,Qu,w,kt),\mathcal{M}^2(w,w,GHu,t),\mathcal{M}^2(w,w,GHu,t), \\ \mathcal{M}^2(w,Qu,w,t),\mathcal{M}^2(GHu,w,w,t),\mathcal{M}^2(w,Qu,w,t) \end{pmatrix} > 1,$$

and $\mathcal{M}^2(w,Qu,w,kt) \geq \mathcal{M}^2(w,Qu,w,t)$, that is $\mathcal{M}(w,Qu,w,kt) \geq \mathcal{M}(w,Qu,w,t)$. Moreover,

$$\psi\begin{pmatrix} \mathcal{N}^2(w,Qu,w,kt),\mathcal{N}^2(w,w,GHu,t),\mathcal{N}^2(w,w,GHu,t), \\ \mathcal{N}^2(w,Qu,w,t),\mathcal{N}^2(GHu,w,w,t),\mathcal{N}^2(w,Qu,w,t) \end{pmatrix} < 1,$$

and $\mathcal{N}^2(w,Qu,w,kt) \leq \mathcal{N}^2(w,Qu,w,t)$, that is $\mathcal{N}(w,Qu,w,kt) \leq \mathcal{N}(w,Qu,w,t)$, we have $Qu = w$. Hence $GHu = w = Qu$, so that (Q,GH) is weak compatible, therefore, we have $QGHu = GHQu$. Thus $Qw = GHw$.

Putting $x = x_{2n}, y = w$ and $z = x_{2n}$ in (3) we get

$$\phi\begin{pmatrix} \mathcal{M}^2(Px_{2n},Qw,Rx_{2n},kt),\mathcal{M}^2(ABx_{2n},STx_{2n},GHw,t),\mathcal{M}^2(Px_{2n},STx_{2n},GHw,t), \\ \mathcal{M}^2(Rx_{2n},Qw,STx_{2n},t),\mathcal{M}^2(GHw,Px_{2n},ABx_{2n},t),\mathcal{M}^2(ABx_{2n},Qw,Rx_{2n},t) \end{pmatrix} > 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(Px_{2n},Qw,Rx_{2n},kt),\mathcal{N}^2(ABx_{2n},STx_{2n},GHw,t),\mathcal{N}^2(Px_{2n},STx_{2n},GHw,t), \\ \mathcal{N}^2(Rx_{2n},Qw,STx_{2n},t),\mathcal{N}^2(GHw,Px_{2n},ABx_{2n},t),\mathcal{N}^2(ABx_{2n},Qw,Rx_{2n},t) \end{pmatrix} < 1.$$

Taking $n \to \infty$ and using (5) we get

$$\phi\begin{pmatrix} \mathcal{M}^2(w,Qw,w,kt),\mathcal{M}^2(w,w,GHw,t),\mathcal{M}^2(w,w,GHw,t), \\ \mathcal{M}^2(w,Qw,w,t),\mathcal{M}^2(GHw,w,w,t),\mathcal{M}^2(w,Qw,w,t) \end{pmatrix} > 1,$$

and $\mathcal{M}^2(w,Qw,w,kt) \geq \mathcal{M}^2(w,Qw,w,t)$, and hence $\mathcal{M}(w,Qw,w,kt) \geq \mathcal{M}(w,Qw,w,t)$. Moreover

$$\psi\begin{pmatrix} \mathcal{N}^2(w,Qw,w,kt),\mathcal{N}^2(w,w,GHw,t),\mathcal{N}^2(w,w,GHw,t), \\ \mathcal{N}^2(w,Qw,w,t),\mathcal{N}^2(GHw,w,w,t),\mathcal{N}^2(w,Qw,w,t) \end{pmatrix} < 1,$$

and $\mathcal{N}^2(w,Qw,w,kt) \leq \mathcal{N}^2(w,Qw,w,t)$, and hence $\mathcal{N}(w,Qw,w,kt) \leq \mathcal{N}(w,Qw,w,t)$, so we get $Qw = w$.

Putting $x = x_{2n}, y = Hw$ and $z = x_{2n}$ in (3) we get

$$\phi\begin{pmatrix} \mathcal{M}^2(Px_{2n},QHw,Rx_{2n},kt),\mathcal{M}^2(ABx_{2n},STx_{2n},GHHw,t),\mathcal{M}^2(Px_{2n},STx_{2n},GHHw,t), \\ \mathcal{M}^2(Rx_{2n},QHw,STx_{2n},t),\mathcal{M}^2(GHHw,Px_{2n},ABx_{2n},t),\mathcal{M}^2(ABx_{2n}QHw,Rx_{2n},t) \end{pmatrix} > 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(Px_{2n},QHw,Rx_{2n},kt),\mathcal{N}^2(ABx_{2n},STx_{2n},GHHw,t),\mathcal{N}^2(Px_{2n},STx_{2n},GHHw,t), \\ \mathcal{N}^2(Rx_{2n},QHw,STx_{2n},t),\mathcal{N}^2(GHHw,Px_{2n},ABx_{2n},t),\mathcal{N}^2(ABx_{2n}QHw,Rx_{2n},t) \end{pmatrix} < 1.$$

As $QH = HQ$ and $GH = HG$, we have $QHw = HQw = H$ and $GH(Hw) = H(GHw) = HQw = Hw$. Taking $n \to \infty$ we get

$$\phi\begin{pmatrix} \mathcal{M}^2(w,Hw,w,kt), \mathcal{M}^2(w,w,Hw,t), \mathcal{M}^2(w,w,Hw,t), \\ \mathcal{M}^2(w,Hw,w,t), \mathcal{M}^2(Hw,w,w,t), \mathcal{M}^2(w,Hw,w,t) \end{pmatrix} > 1,$$

and $\mathcal{M}^2(w,Hw,w,kt) \geq \mathcal{M}^2(w,Hw,w,t)$, therefore $\mathcal{M}(w,Hw,w,kt) \geq \mathcal{M}(w,Hw,w,t)$. Moreover

$$\psi\begin{pmatrix} \mathcal{N}^2(w,Hw,w,kt), \mathcal{N}^2(w,w,Hw,t), \mathcal{N}^2(w,w,Hw,t), \\ \mathcal{N}^2(w,Hw,w,t), \mathcal{N}^2(Hw,w,w,t), \mathcal{N}^2(w,Hw,w,t) \end{pmatrix} < 1,$$

and $\mathcal{N}^2(w,Hw,w,kt) \leq \mathcal{N}^2(w,Hw,w,t)$, therefore $\mathcal{N}(w,Hw,w,kt) \leq \mathcal{N}(w,Hw,w,t)$. Now, by Lemma 6, we have $Hw = w$. Also, $GHw = Hw = w$ implies $Gw = w$. Hence

$$Gw = Hw = Qw = w. \tag{8}$$

Combining (7) and (8) we have $Aw = Bw = Rw = Pw = Sw = Tw = Qw = Gw = Hw = w$. Hence w is a common fixed point of A, B, G, H, S, T, P, Q and R.

Uniqueness

Let u be another common fixed point of A, B, G, H, S, T, P, Q and R. Then $Au = Bu = Ru = Pu = Su = Tu = Qu = Gu = Hu = u$. Putting $x = u, y = w$ and $z = w$ in (3) we get

$$\phi\begin{pmatrix} \mathcal{M}^2(Pu,Qw,Rw,kt), \mathcal{M}^2(ABu,STw,GHw,t), \mathcal{M}^2(Pu,STw,GHw,t), \\ \mathcal{M}^2(Rw,Qw,STw,t), \mathcal{M}^2(GHw,Pu,ABu,t), \mathcal{M}^2(ABu,Qw,Rw,t) \end{pmatrix} > 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(Pu,Qw,Rw,kt), \mathcal{N}^2(ABu,STw,GHw,t), \mathcal{N}^2(Pu,STw,GHw,t), \\ \mathcal{N}^2(Rw,Qw,STw,t), \mathcal{N}^2(GHw,Pu,ABu,t), \mathcal{N}^2(ABu,Qw,Rw,t) \end{pmatrix} < 1.$$

Taking limit both side then we get

$$\phi(\mathcal{M}^2(u,w,w,kt), \mathcal{M}^2(u,w,w,t), \mathcal{M}^2(u,w,w,t), \mathcal{M}^2(w,w,w,t), \mathcal{M}^2(w,u,u,t), \mathcal{M}^2(u,w,w,t)) > 1,$$

and $\mathcal{M}^2(u,w,w,kt) \geq \mathcal{M}^2(u,w,w,t)$, therefore $\mathcal{M}(u,w,w,kt) \geq \mathcal{M}(u,w,w,t)$. Moreover,

$$\psi(\mathcal{N}^2(u,w,w,kt), \mathcal{N}^2(u,w,w,t), \mathcal{N}^2(u,w,w,t), \mathcal{N}^2(w,w,w,t), \mathcal{N}^2(w,u,u,t), \mathcal{N}^2(u,w,w,t)) < 1,$$

and $\mathcal{N}^2(u,w,w,kt) \leq \mathcal{N}^2(u,w,w,t)$, therefore $\mathcal{N}(u,w,w,kt) \leq \mathcal{N}(u,w,w,t)$ we get $u = w$. That is w is a unique common fixed point of A, B, G, H, S, T, P, Q and R in X. □

Example 5. Let $X = [0,1], *, \diamond$ be a continuous t-norm and continuous t-conorm defined by $a * b = \min\{a,b\}, a \diamond b = \max\{a,b\}$, for all $a,b \in [0,1]$. For each $t \in (0,\infty)$ define $\mathcal{M}(x,y,z,t) = \frac{t}{t+D(x,y,z)}$ and $\mathcal{N}(x,y,z,t) = \frac{D(x,y,z)}{t+D(x,y,z)}$ for all $x,y,z \in X$. Then $(X,\mathcal{M},\mathcal{N},*,\diamond)$ is a complete IGFM-Space. Define $\phi,\psi : [0,1]^6 \to R$ be defined by $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = \frac{t_1}{\min\{t_2,t_3,t_4,t_5,t_6\}}$ and $\psi(t_1,t_2,t_3,t_4,t_5,t_6) = \frac{t_1}{\max\{t_2,t_3,t_4,t_5,t_6\}}$. Hence $\phi,\psi \in \Psi$. Let $Ax = Gx = Sx = 1; Bx = Hx = Tx = x, Px = Qx = Rx = \begin{cases} \frac{x+1}{2}, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$ be the self maps in X. Let $\{x_n\}$ be a sequence in X defined by $x_n = 1 - \frac{1}{n}$. Then $\lim_{n\to\infty} x_n = 1$. $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} P\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \frac{1}{2n} = 1$ and $\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} AB\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} A\left(1 - \frac{1}{n}\right) = 1$. Also, $\lim_{n\to\infty} \mathcal{M}(PPx_n, ABABx_n, ABABx_n, t) = \lim_{n\to\infty} \mathcal{M}(1 - \frac{1}{4n}, 1, 1, t) = 1$ and $\lim_{n\to\infty} \mathcal{N}(PPx_n, ABABx_n, ABABx_n, t) = \lim_{n\to\infty} \mathcal{N}(1 - \frac{1}{4n}, 0, 0, t) = 0, t > 0$. Therefore (P, AB) is compatible of type (β). $Q(1) = GH(1) = 1 \Rightarrow QGH(1) = GHQ(1) = 1$ and $R(1) = ST(1) = 1 \Rightarrow RST(1) = STR(1) = 1$. Thus, the pair (Q, GH) and (R, ST) are weak compatible. A,B,G,H,S,T,P,Q and R satisfies all the hypothesis of Theorem 7 and hence posses a unique common fixed point in X. i.e., $x = 1$.

If we take $B = T = H = I$ identity map on X in above theorem then we get the following corollary:

Corollary 8. *Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be a complete IGFM-Space and A, G, S, P, Q and R be mappings from X into itself such that the following conditions are satisfied:*
$P(X) \subset S(X), Q(X) \subset A(X)$ *and* $R(X) \subset G(X)$,
(P, A) *is compatible of type* (β) *and (Q, G) , (R, S) are weak compatible,*
there exists $k \in (0,1)$ *such that for every* $x, y, z \in X$ *and* $t > 0$,

$$\phi\begin{pmatrix} \mathcal{M}^2(Px, Qy, Rz, kt), \mathcal{M}^2(Ax, Sz, Gy, t), \mathcal{M}^2(Px, Sz, Gy, t), \\ \mathcal{M}^2(Rz, Qy, Sz, t), \mathcal{M}^2(Gy, Px, Ax, t), \mathcal{M}^2(Ax, Qy, Rz, t) \end{pmatrix} \geq 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(Px, Qy, Rz, kt), \mathcal{N}^2(Ax, Sz, Gy, t), \mathcal{N}^2(Px, Sz, Gy, t), \\ \mathcal{N}^2(Rz, Qy, Sz, t), \mathcal{N}^2(Gy, Px, Ax, t), \mathcal{N}^2(Ax, Qy, Rz, t) \end{pmatrix} \leq 1.$$

Then A, G, T, P, Q and R have a unique common fixed point in X.

If we take weakly compatible mapping in place of compatible mapping of type (β) then we get following result:

Corollary 9. *Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be a complete IGFM-Space and A, B, G, H, S, T, P, Q and R be mappings from X into itself such that the following conditions are satisfied:*
$P(X) \subset ST(X), Q(X) \subset AB(X)$ *and* $R(X) \subset GH(X)$,
$(P, AB), (Q, GH)$ *and* (R, ST) *are weak compatible,*
there exists $k \in (0,1)$ *such that for every* $x, y, z \in X$ *and* $t > 0$,

$$\phi\begin{pmatrix} \mathcal{M}^2(Px, Qy, Rz, kt), \mathcal{M}^2(ABx, STz, GHy, t), \mathcal{M}^2(Px, STz, GHy, t), \\ \mathcal{M}^2(Rz, Qy, STz, t), \mathcal{M}^2(GHy, Px, ABx, t), \mathcal{M}^2(ABx, Qy, Rz, t) \end{pmatrix} \geq 1,$$
$$\psi\begin{pmatrix} \mathcal{N}^2(Px, Qy, Rz, kt), \mathcal{N}^2(ABx, STz, GHy, t), \mathcal{N}^2(Px, STz, GHy, t), \\ \mathcal{N}^2(Rz, Qy, STz, t), \mathcal{N}^2(GHy, Px, ABx, t), \mathcal{N}^2(ABx, Qy, Rz, t) \end{pmatrix} \leq 1.$$

Then A, B, G, H, S, T, P, Q and R have a unique common fixed point in X.

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