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A note on extremal intersecting linear Ryser systems

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Abstract: A famous conjecture of Ryser states that any r-partite set system has transversal number at most $r-1$ times their matching number. This conjecture is only known to be true for $r \leq 3$ in general, for $r \leq 5$ if the set system is intersecting, and for $r \leq 9$ if the set system is intersecting and linear. In this note, we deal with Ryser's conjecture for intersecting r-partite linear systems: if τ is the transversal number for an intersecting r-partite linear system, then $\tau \leq r-1$. If this conjecture is true, this is known to be sharp for r for which there exists a projective plane of order $r-1$. There has also been considerable effort to find intersecting r-partite set systems whose transversal number is $r-1$. In this note, we prove that if $r \geq 2$ is an even integer, then $f_l(r) \geq 3r-5$, where $f_l(r)$ is the minimum number of lines of an intersecting r-partite linear system whose transversal number is $r-1$. Aharoni *et al.*, [R. Aharoni, J. Barát and I.M. Wanless, *Multipartite hypergraphs achieving equality in Ryser's conjecture*, Graphs Combin. **32**, 1–15 (2016)] gave an asymptotic lower bound: $f_l(r) \geq 3.052r + O(1)$ as $r \to \infty$. For some small values of r ($r \geq 2$ an even integer), our lower bound is better. Also, we prove that any r-partite linear system satisfies $\tau \leq r-1$ if $\nu_2 \leq r$ for all $r \geq 3$ odd integer and $\nu_2 \leq r-2$ for all $r \geq 4$ even integer, where ν_2 is the maximum cardinality of a subset of lines $R \subseteq \mathcal{L}$ such that any three elements chosen in R do not have a common point.

Keywords: Ryser's Conjecture; Linear systems; Transversal number; 2-packing number.

MSC: 05C65; 05C69.

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1. Introduction

A *set system* is a pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a finite family of subsets on a ground set X. A set system can be also thought of as a hypergraph, where the elements of X and $\mathcal{F}$ are called *vertices* and *hyperedges* respectively. The set system $(X, \mathcal{F})$ is called *r-uniform*, when all subsets of $\mathcal{F}$ are of size r. The set system $(X, \mathcal{F})$ is *r-partite* if the elements of X can be partitioned into r sets $X_1, \ldots, X_r$, called the *sides*, such that each element of $\mathcal{F}$ contains exactly one element of X_i, for every $i = 1, \ldots, r$. Thus, an r-partite set system is an r-uniform set system.

Let $(X, \mathcal{F})$ be a set system. A subset $T \subseteq X$ is a *transversal* of $(X, \mathcal{F})$ if $T \cap F \neq \emptyset$, for every $F \in \mathcal{F}$. The *transversal number* of $(X, \mathcal{F})$, $\tau = \tau(X, \mathcal{F})$, is the smallest possible cardinality of a transversal of $(X, \mathcal{F})$. The transversal number has been studied in the literature in many different contexts and names. For example, with the name of *piercing number* and *covering number*, see for instance [1–9].

Let $(X, \mathcal{F})$ be a set system. A subset $\mathcal{E} \subseteq \mathcal{F}$ is called a *matching* if $F \cap \hat{F} = \emptyset$, for every $F, \hat{F} \in \mathcal{E}$. The matching number of $(X, \mathcal{F})$, $\nu = \nu(X, \mathcal{F})$, is the cardinality of the largest matching of $(X, \mathcal{F})$. A set system is called *intersecting* if $\nu = 1$; that is, $F \cap \hat{F} \neq \emptyset$, for every $F, \hat{F} \in \mathcal{F}$.

It is not hard to see that any r-uniform set system $(X, \mathcal{F})$ satisfies the inequality $\tau \leq r\nu$. It is well-known that this bound is sharp, as shown by the family of all subsets of size r in a ground set of size $kr-1$, which has $\nu = k-1$ and $\tau = (k-1)r$. On the other hand, if $\nu = 1$, any projective plane of order $r-1$, Π_{r-1}, where $r-1$ is a prime power, satisfies $\tau = r$. However, for r-partite set systems, Ryser conjectured in the 1960's that the upper bound could be improved.

Ryser's Conjecture: Any r-partite set system satisfies $\tau \leq (r-1)\nu$, for every $r \geq 2$ an integer.

For the special case $r = 2$, Ryser's conjecture is equivalent to Kőnig's Theorem. Aharoni [10] proved the only other known general case of the conjecture when $r = 3$. However, Ryser's conjecture is also known to be true in some special cases. Tuza [11] verified Ryser's conjecture for $r \leq 5$ if the set system is intersecting. Furthermore, Francetić *et al.*, [12] verified Ryser's conjecture for $r \leq 9$ if the set system is linear, that is, a set system $(X, \mathcal{F})$ is a *linear system* if it satisfies $|E \cap F| \leq 1$, for every pair of distinct subsets $E, F \in \mathcal{F}$. In this note, a linear system will be written by $(P, \mathcal{L})$ instead of $(X, \mathcal{F})$; the elements of P and $\mathcal{L}$ are called *points* and *lines*, respectively. In the rest of this paper, only linear systems are considered. Most of the definitions can be generalized for set systems. Thus, we deal with Ryser's conjecture for intersecting r-partite linear systems, for every $r \geq 2$ an integer.

Intersecting linear Ryser's Conjecture: Every intersecting r-partite linear system satisfies $\tau \leq r-1$, for every $r \geq 2$ an integer.

In case the conjecture would be true, it is tight in the sense that for infinitely many r's there are constructions of intersecting r-partite linear systems with $\tau = r-1$. For example, if $r-1$ is a prime power, consider the finite projective plane of order $r-1$ as a linear system, Π_{r-1}. This linear system is r-uniform and intersecting. To make it r-partite, one just needs to delete one point from the projective plane. This truncated projective plane, Π'_{r-1}, gives an intersecting r-partite linear system with $\tau \geq r-1$, and $r(r-1)$ points and $(r-1)^2$ lines. However, the construction obtained from the projective plane is not the "optimal" extremal. Although the projective plane construction only contains $r(r-1)$ points (which is an optimal number of points), it has a lot of lines. Let $f(r)$ be the minimum integer so that there exists an intersecting r-partite set system $(X, \mathcal{F})$ with $\tau = r-1$ and $|\mathcal{F}| = f(r)$ lines. Analogously, let $f_l(r)$ be the minimum integer so that there exists an intersecting r-partite linear system $(P, \mathcal{L})$ with $\tau = r-1$ and $|\mathcal{L}| = f_l(r)$ lines. $f_l(r)$ probably does not exist for some values of r (if $r-1$ is a prime power, then Π'_{r-1} is known to exist, providing proof that $f_l(r)$ is well-defined). Hence, if $f_l(r)$ does exist, for some $r \geq 2$ integer, then $f(r) \leq f_l(r)$ (since any extremal linear system with $f_l(r)$ edges is in particular a set system).

It is not difficult to prove that $f_l(2) = 1$ and $f_l(3) = 3$, see [13]. Furthermore, Mansour *et al.*, [13] proved that $f_l(4) = 6$ and $f_l(5) = 9$. On the other hand, Aharoni *et al.*, [14] proved that $f_l(6) = 13$ and $f(7) = 17$ (even when the truncated projective plane does not exist, since it has been proved that finite projective planes of order six do not exist, see [15]); however Francetić *et al.*, [12] proved that $f_l(7)$ does not exist, that is, there is no intersecting 7-partite linear system such that $\tau = 6$. Abu-Khazneha *et al.*, [16] constructed a new infinite family of intersecting r-partite set systems extremal to Ryser's conjecture, which exist whenever a projective plane of order $r-2$ exists. That construction produces a large number of non-isomorphic extremal set systems. Finally, Aharoni *et al.*, [14] gave a lower bound on $f(r)$ when $r \to \infty$, showing that $f(r) \geq 3.052r + O(1)$, this lower bound is an improvement since Mansour *et al.*, [13] proved that $f(r) \geq (3 - \frac{1}{\sqrt{18}})r(1-o(1)) \approx 2.764(1-o(1))$, when $r \to \infty$.

In this note, we give a lower bound for $f_l(r)$ for small values of $r \geq 2$ an even integer.

Theorem 1. *If $r \geq 2$ is an even integer, then $3r - 5 \leq f_l(r)$.*

Our lower bound is better than that given in [14], for some small values of $r \geq 2$ an even integer. If $r \in \{2, 4, 6\}$, then $f_l(r) = 3r - 5$. Aharoni *et al.*, [14] proved that $18 \leq f(8)$ and $24 \leq f(10)$. Hence, Theorem 1 implies that $19 \leq f_l(8)$ and $25 \leq f_l(10)$.

2. Main Results

In this section, the main results of this paper are presented. Before this, Some definitions and results are necessary.

Let $(P, \mathcal{L})$ be a linear system and $p \in P$ be a point. The set $\mathcal{L}_p$ is the set of lines incident to p. In this context, the *degree* of p is $\deg(p) = |\mathcal{L}_p|$ and $\Delta = \Delta(P, \mathcal{L})$ is the maximum degree over all points of the linear system.

A subset R of lines of a linear system $(P, \mathcal{L})$ is a 2-*packing* of $(P, \mathcal{L})$ if any three elements chosen in R do not have a common point. The 2-packing number of $(P, \mathcal{L})$, $\nu = \nu_2(P, \mathcal{L})$, is the maximum cardinality of a 2-packing of $(P, \mathcal{L})$. There are some works that study this new parameter, see [17–25].

Theorem 2. *[20] Let $(P, \mathcal{L})$ be a linear system and $p \in P$ be a point such that $\Delta = \deg(p)$ and $\Delta' = \max\{\deg(x) : x \in P \setminus \{p\}\}$. If $|\mathcal{L}| \leq \Delta + \Delta' + \nu_2 - 3$, then $\tau \leq \nu_2 - 1$.*

Let $r \geq 3$ be an integer. If $(P, \mathcal{L})$ is an intersecting r-uniform linear system, then $\nu_2 \leq r+1$. However, if $\nu_2 = r+1$, for $r \geq 4$ an even integer, then $\tau = \lceil \nu_2/2 \rceil$, see [24]. Hence, we assume that $\nu_2 \leq r$ if $r \geq 4$ is an even integer.

Lemma 1. *[22] Let $(P, \mathcal{L})$ be an intersecting r-uniform linear system, with $r \geq 3$ an odd integer. If $\tau = r$, then $\nu_2 = r+1$.*

Lemma 2. *[24] Let $(P, \mathcal{L})$ be an intersecting r-uniform linear system, with $r \geq 4$ an even integer. If $\tau = r$, then $\nu_2 = r$.*

Lemma 3. *Let $(P, \mathcal{L})$ be an intersecting r-uniform linear system with $r \geq 4$ an even integer. If $\tau = r-1$, then $\nu_2 = r$.*

Proof. Let $(P, \mathcal{L})$ be an intersecting r-uniform linear system. Let $p \in P$ be a point such that $\Delta = \deg(p)$ and $\Delta' = \max\{\deg(x) : x \in P \setminus \{p\}\}$. By Theorem 2 if $|\mathcal{L}| \leq \Delta + \Delta' + \nu_2 - 3 \leq 3(r-1)$ (since $\Delta \leq r$ and $\nu_2 \leq r$), then $\tau \leq \nu_2 - 1$, which implies that $\nu_2 = r$, since $r-1 \leq \tau \leq \nu_2 - 1 \leq r-1$. □

By Lemmas 2 and 3, we have:

Corollary 1. *Let $(P, \mathcal{L})$ be an intersecting r-uniform linear system with $r \geq 4$ an even number. If $\tau \in \{r-1, r\}$, then $\nu_2 = r$.*

By Lemma 1 and Corollary 1, we have:

Theorem 3. *Let $r \geq 3$ be an integer. Then every intersecting r-partite linear system satisfies*

1. *$\tau \leq r-1$ If $\nu_2 \leq r$ and $r \geq 3$ an odd integer; and*
2. *$\tau \leq r-2$ if $\nu_2 \leq r-1$ and $r \geq 4$ an even integer.*

To prove intersecting linear Ryser's Conjecture it suffices to analyze the following two cases concerning the 2-packing number:

Conjecture 1. Let $r \geq 3$ be an integer. Then every intersecting r-partite linear system satisfies:

1. $\tau \leq r-1$ if $\nu_2 = r+1$, with $r \geq 3$ an odd number.
2. $\tau \leq r-1$ if $\nu_2 = r$, with $r \geq 4$ an even number.

Theorem 4. *Let $r \geq 4$ be an even integer, then $3r-5 \leq f_l(r)$.*

Proof. Assume that $\nu_2 \leq r-1$ (by Corollary 1). Let $p \in P$ be a point such that $\Delta = \deg(p)$ and $\Delta' = \max\{\deg(x) : x \in P \setminus \{p\}\}$. By Theorem 2 if $|\mathcal{L}| \leq \Delta + \Delta' + \nu_2 - 3 \leq 3r-6$ (since $\Delta \leq r-1$), then $\tau \leq \nu_2 - 1 \leq r-2$. Therefore, $3r-5 \leq f_l(r)$. □

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