Open Journal of
Discrete Applied Mathematics

PSR Press

Article

Non-isomorphic graphs with common degree sequences

Rikio Ichishima[1]* and Francesc Antoni Muntaner-Batle[2]

[1] Department of Sport and Physical Education, Faculty of Physical Education, Kokushikan University, 7-3-1 Nagayama, Tama-shi, Tokyo 206-8515, Japan.

[2] Graph Theory and Applications Research Group, School of Electrical Engineering and Computer Science, Faculty of Engineering and Built Environment, The University of Newcastle, NSW 2308 Australia.

* Correspondence: ichishim@kokushikan.ac.jp

Academic Editor: Muhammad Kamran Jamil
Received: 13 March 2023; Accepted: 18 April 2023; Published: 30 April 2023.

Abstract: For all positive even integers n, graphs of order n with degree sequence $S_n : 1, 2, \ldots, n/2, n/2, n/2 + 1, n/2 + 2, \ldots, n - 1$ naturally arose in the study of a labeling problem in [1].This fact motivated the authors of the aforementioned paper to study these sequences and as a result of this study they proved that there is a unique graph of order n realizing S_n for every even integer n. The main goal of this paper is to generalize this result.

Keywords: Vertex degree; Degree sequence; Isomorphism problems in graph theory; Graph operation.

MSC: 05C07; 05C60; 05C76.

1. Introduction

Unless stated otherwise, the graph-theoretical notation and terminology used here will follow Chartrand and Lesniak [2]. In particular, we assume that graphs considered in this paper are simple, that is, without loops or multiple edges. To indicate that a graph G has *vertex set* V and *edge set* E, we write $G = (V, E)$. To emphasize that V and E are the vertex set and edge set of a graph G, we will write V as $V(G)$ and E as $E(G)$.

The *removal of a vertex* v from a graph G results in that subgraph $G - v$ of G consisting of all vertices of G except v and all edges not incident with v. Thus, $G - v$ is the maximal subgraph of G not containing v. On the other hand, if v is not adjacent in G, *the addition of vertex* v results in the smallest supergraph $G + v$ of G containing the vertex v and all edges incident with v. The *union* $G \cong G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

The *degree* of a vertex v in a graph G denoted by $\deg_G v$ is the number of edges incident with v. A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers is called a *degree sequence* of a graph G of order n if the vertices of G can be labeled $v_1, v_2, \ldots, v_n$ so that $\deg v_i = d_i$ for $1 \leq i \leq n$. Throughout this paper, we write the degree sequence of a graph as an increasing sequence. A finite sequence s of nonnegative integers is *graphical* if there exists some graph that realizes s, that is, s is a degree sequence of some graph.

A necessary and sufficient condition for a sequence to be graphical was found by Havel [3] and later rediscovered by Hakimi [4]. This result actually provides an efficient algorithm for determining whether a given finite sequence of nonnegative integers is graphical. Another well-known characterization for graphical sequences was provided by Erdös and Gallai [5]. All these references provide excellent sources for the interested reader.

The concepts of graph isomorphism and isomorphic graphs are also crucial for the development of this paper, and although they are very basic in graph theory, we introduce them as a matter of completeness. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. They are *isomorphic* (written $G_1 \cong G_2$) if there exists a bijective function $\phi : V_1 \to V_2$ such that $xy \in E_1$ if and only if $\phi(x)\phi(y) \in E_2$. In this case, the function ϕ is called an *isomorphism* from G_1 to G_2.

The following two lemmas regarding isomorphism of graphs are very elementary and fundamental, but nevertheless, necessary for the proof of our main result of this paper. Hence, we state and prove them next.

Lemma 1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs of order n for which there exist unique vertices $v_1 \in V_1$ and $v_2 \in V_2$ such that*

$$\deg_{G_1} v_1 = \deg_{G_2} v_2 = n - 1.$$

Then $G_1 \cong G_2$ if and only if $G_1 - v_1 \cong G_2 - v_2$.

Proof. First, assume that $G_1 \cong G_2$. Then there exists an isomorphism $\phi : V_1 \to V_2$. Since v_i $(i = 1, 2)$ are the only vertices of V_i with degree $n - 1$ and each isomorphism preserves degrees, it follows that $\phi(v_1) = v_2$. Thus, if we consider $G_1 - v_1$ and $G_2 - v_2$, it follows that the function $\phi' : V_1 \backslash \{v_1\} \to V_2 \backslash \{v_2\}$ defined by $\phi'(a) = \phi(a)$ for all $a \in V_1 \backslash \{v_1\}$ is well defined and bijective. Furthermore, $ab \in E_1 \backslash \{v_1 x \mid x \in V_1 \backslash \{v_1\}\}$ if and only if $\phi'(a)\phi'(b) \in E_2 \backslash \{v_2 x \mid x \in V_2 \backslash \{v_2\}\}$. This implies that $\phi' : V_1 \backslash \{v_1\} \to V_2 \backslash \{v_2\}$ is an isomorphism and hence $G_1 \cong G_2$.

Next, assume that $H_1 = (V_1', E_1')$ and $H_2 = (V_2', E_2')$ are two isomorphic graphs with an isomorphism $\phi : V_1' \to V_2'$. Also, let v_1 and v_2 be two new vertices and consider two graphs $H_1 + v_1$ and $H_2 + v_2$. We show that $H_1 + v_1 \cong H_2 + v_2$. To do this, consider the function $\phi' : V(H_1 + v_1) \to V(H_2 + v_2)$ defined by

$$\phi'(v) = \begin{cases} \phi(v) & \text{if } v \in V_1' \\ v_2 & \text{if } v = v_1. \end{cases}$$

We will show that ϕ' is an isomorphism from $H_1 + v_1$ to $H_2 + v_2$. Since ϕ is an isomorphism from H_1 to H_2, it follows that $ab \in E(H_1 + v_1)$ and $\{a, b\} \cap \{v_1\} = \varnothing$ if and only if $\phi'(a)\phi'(b) \in E(H_2 + v_2)$. On the other hand, if $av_1 \in E(H_1 + v_1)$ for all $a \in V_1'$ and $bv_2 \in E(H_2 + v_2)$ for all $b \in V_2'$, then $\phi'(a)\phi'(v_1) = \phi(a)v_2 \in E(H_2 + v_2)$. This implies that ϕ' is an isomorphism from $H_1 + v_1$ to $H_2 + v_2$ so that $H_1 + v_1 \cong H_2 + v_2$. □

Lemma 2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. If v_1 and v_2 are two new vertices, then $G_1 \cong G_2$ if and only if $G_1 \cup v_1 \cong G_2 \cup v_2$.*

Proof. First, assume that $G_1 \cong G_2$. Then there exists an isomorphism $\phi : V_1 \to V_2$. Now, consider the function $\phi' : V_1 \cup \{v_1\} \to V_2 \cup \{v_1\}$ defined by

$$\phi'(v) = \begin{cases} \phi(v) & \text{if } v \in V_1 \\ v_2 & \text{if } v = v_1. \end{cases}$$

Since no edge of the form av_1 exists in $G_1 \cup v_1$ and no edge of the form bv_2 exists in $G_2 \cup v_2$, it follows that ϕ' is an isomorphism from $G_1 \cup v_1$ to $G_2 \cup v_2$ and hence $G_1 \cup v_1 \cong G_2 \cup v_2$.

Next, assume that $G_1 \cup v_1 \cong G_2 \cup v_2$. Then there exists an isomorphism $\phi : V_1 \to V_2$. Since the image under ϕ of any isolated vertex is an isolated vertex, we may assume, without loss of generality, that $\phi(v_1) = v_2$. This implies that the function $\phi' : V_1 \to V_2$ defined by $\phi(v) = v$ for all $v \in V_1$ is clearly well defined, bijective and an isomorphism from G_1 to G_2. Therefore, $G_1 \cong G_2$. □

2. Main results

With the information provided in the introduction, we are ready to present our main results.

Let $S_0 : 0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_{n-1} \leqq a_n$ be a graphical sequence. If we assume that there exist exactly k $(k \geq 1)$ graphs that realize S_0, then we have the following result.

Theorem 1. *The sequences*

$S_0^{(1)} : 1, a_1 + 1, a_2 + 1, \ldots, a_n + 1, n + 1;$

$S_0^{(2)} : 1, 2, a_1 + 2, a_2 + 2, \ldots, a_n + 2, n + 2, n + 3;$

$S_0^{(3)} : 1, 2, 3, a_1 + 3, a_2 + 3, \ldots, a_n + 3, n + 3, n + 4, n + 5;$

$$\vdots$$

$S_0^{(i)} : 1, 2, 3, \ldots, i, a_1 + i, a_2 + i, \ldots, a_n + i, n + i, n + i + 1, \ldots, n + 2i - 1$

$$\vdots$$

are all graphical. Furthermore, there exist exactly k, $k \geq 1$, connected non-isomorphic graphs that realize each one of the sequences $S_0^{(1)}, S_0^{(2)}, \ldots, S_0^{(i)}, \ldots$.

Proof. We start by showing that each sequence $S_0^{(i)}$ is graphical for any positive integer i. To do this, we only need to take a graph that realizes S_0, introduce two new vertices and join one of these two new vertices with all remaining vertices. Hence, $S_0^{(1)}$ is graphical. To obtain a graph that realizes $S_0^{(2)}$, we just need to take a graph that realizes $S_0^{(1)}$ and once again introduce two new vertices joining one of these new two vertices with all remaining vertices. If we continue this process inductively, we obtain a graph that realizes $S_0^{(i)}$ for any positive integer i.

Now, observe that since each graph realizing $S_0^{(i)}$ ($i \geq 1$) has a vertex which is adjacent to all the other vertices, it follows that all these graphs are connected. Thus, it remains to show that each one of these sequences realizes exactly k ($k \geq 1$) graphs. To see this, let $S_0 = S_0^{(0)}$ and proceed by induction on the super subscript i of $S_0^{(i)}$ for $i \geq 0$. First, observe that $S_0^{(0)}$ has the property that there exist exactly k ($k \geq 1$) non-isomorphic graphs that realize $S_0^{(0)}$ by assumption.

Next, let $i = l$ ($l \geq 0$) and assume that there exist exactly k ($k \geq 1$) non-isomorphic graphs realizing $S_0^{(l)}$. Consider the sequence

$$S_0^{(l+1)} : 1, 2, \ldots, l, l+1, a_1 + l + 1, a_2 + l + 1, \ldots, a_n + l + 1, n + l + 1, n + l + 2, \ldots, n + 2l + 1.$$

and let $G_0^{(l+1)}$ be any graph that realizes $S_0^{(l+1)}$. It is now clear that the vertex of degree $n + 2l + 1$ is adjacent to all other vertices of $V\left(G_0^{(l+1)}\right)$. It is also true that if we eliminate this vertex, then we obtain a new graph with degree sequence $0, S_0^{(l)}$. By inductive hypothesis, there exist exactly k ($k \geq 1$) non-isomorphic graphs with degree sequence $S_0^{(l)}$. Then Lemma 2 yields that there are exactly k ($k \geq 1$) non-isomorphic graphs with degree sequence $0, S_0^{(l)}$, and Lemma 1 implies that there are exactly k ($k \geq 1$) non-isomorphic graphs realizing $S_0^{(l+1)}$. Therefore, the result follows. □

To conclude this section, notice that it is clear that the only graph that realizes the sequence $s : 1, 1$ is the complete graph K_2 of order 2. From this observation together with Theorem 1, the next result found in [1] follows as an immediate corollary.

Corollary 1. *For all positive integers n, there exists a unique graph of order n that realizes the sequence S_n : $1, 2, \ldots, n/2.n/2, n/2 + 1, n/2 + 2, \ldots, n - 1$.*

In summary, what we have proved in this paper is that if a degree sequence is realized by exactly k ($k \geq 1$) non-isomorphic graphs of order n, then there exist infinitely many sequences that realize exactly k ($k \geq 1$) non-isomorphic graphs. Furthermore, all these graphs have the additional property that they are all connected.

Acknowledgments: The authors would like to dedicate this paper to Susana Clara López Masip who passed away on December 26, 2022 after a life dedicated to graph theory. The authors are also gratefully indebted to Yukio Takahashi for his technical assistance.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] Ichishima, R., Muntaner-Batle, F. A., & Oshima, A. (2022). Minimum degree conditions for the strength and bandwidth of graphs, *Discrete Applied Mathematics, 340*, 191–198.

[2] Chartrand, G., & Lesniak, L . (1996). *Graphs & Digraphs* 3th ed. CRC Press, Boca Raton.

[3] Havel, V. (1955). A remarkon the existence of finite graphs (Czech), *Časopis Pěst. Math*, 80, 477–480.

[4] Hakimi, S. (1962). On therealizability of a set of integers as degrees of the vertices of a graph, *SIAM Journal on Applied Mathematics, 10*, 496–506.

[5] Erdös, P., & Gallai, T. (1960). Graphs withprescribed degrees of vertices (Hungarian), *Matematikai Lapok, 11*, 264–274.