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Arc coloring of odd graphs for hamiltonicity

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Abstract: Coloring the arcs of biregular graphs was introduced with possible applications to industrial chemistry, molecular biology, cellular neuroscience, etc. Here, we deal with arc coloring in some non-bipartite graphs. In fact, for $1 < k \in \mathbb{Z}$, we find that the odd graph O_k has an arc factorization with colors $0, 1, \ldots, k$ such that the sum of colors of the two arcs of each edge equals k. This is applied to analyzing the influence of such arc factorizations in recently constructed uniform 2-factors in O_k and in Hamilton cycles in O_k as well as in its double covering graph known as the middle-levels graph M_k.

Keywords: Arc coloring; Hamilton cycle; odd graphs; k-germs; Dyck words

MSC: 05C15, 05C38, 05C75, 68R15

1. Introduction

Let $0 < k \in \mathbb{Z}$, let $n = 2k + 1$ and let O_k be the *k-odd graph* [1], that we consider as the graph whose vertices are the k-subsets of the cyclic group $\mathbb{Z}_n$ over the set $[0, 2k] = \{0, 1, \ldots, 2k\}$ having an edge, denoted uv, between each two vertices u, v if and only if $u \cap v = \varnothing$.

Coloring the arcs of biregular graphs was considered in [2], with potential applications to the design of experiments for industrial chemistry, molecular biology, cellular neuroscience and solving 3-dimensional puzzles like the one known as Great Circle Challenge. It would be also valuable to find likewise applications of arc coloring to graphs other than bipartite graphs, like the odd graphs, for example, and any other similar graphs in that all vertices have departing arcs with all weights (colors) from 0 to k such that the sum of oppositely oriented arcs is constantly k. In this work, coloring the arcs of O_k occupies the place of missing 1-factorizations, since the Petersen graph O_3 is 4-edge-colorable and if k is a power of 2 then O_k is $k + 1$-edge-colorable [1]. Resulting arc-factorizations in Section 4 are seen in Section 5 to influence recent uniform 2-factors and Hamilton cycles of O_k [3,4].

In fact, we recur in Section 4 to an *edge-supplementary 1-arc factorization* $\mathbb{A}_k$ of O_k, meaning that the two oppositely oriented arcs (1-arcs, in [5, p. 59]) of each edge of O_k are assigned *colors* $a, b \in \{0, \ldots, k\} = [0, k]$ by means of $\mathbb{A}_k$ such that $a + b = k$ (so a, b are said to be *k-supplementary* or *supplementary in k*), in such a way that the arcs departing from each vertex are in one-to-one correspondence with $[0, k]$.

To define the claimed edge-supplementary 1-arc factorization $\mathbb{A}_k$ of O_k, we consider in Section 4 a partition of $V(O_k)$ into $\mathbb{Z}_n$*-classes*, namely the cyclic equivalence classes mod n.

To get these $\mathbb{Z}_n$-classes, we take each vertex u of O_k expressed as the characteristic vector of the subset $u \subset \mathbb{Z}_n$ it represents. Each such vector is a binary string, or *bitstring*, namely a sequence of digits 0 and 1 said to be 0-*bits* and 1-*bits*, respectively.

The number of bits (resp., 1-bits) of a bitstring u is said to be its *length* (resp., its *weight*). Each $u \in V(O_k)$ can be seen as a bitstring of length n said to be an *n-bitstring*.

Example 1. In O_1, the subsets $\{i\}$ of $\mathbb{Z}_3$, $(i = 0, 1, 2)$ are denoted $100, 010, 001$, respectively.

We also consider the vertices of O_k as corresponding polynomials mod $x^n + 1$ in the ring $\mathbb{Z}[x]$, [6,7], namely in Example 1: x^0, x^1 and x^2 mod $x^3 + 1$.

The $\mathbb{Z}_n$-classes of O_k are obtained by successive multiplication of such polynomials by x mod $x^n + 1$. The resulting equivalence relation defines a quotient graph of O_k whose vertices are those $\mathbb{Z}_n$-classes. In Example 1, O_1 has just one such equivalence class, and O_2 has two.

Theorem 2 below asserts that there is a bijection between the $\mathbb{Z}_n$-classes of O_k and the *Dyck words* of length $2k$, defined in Subsection 3.2 via Example 4 (to Subsection 3.1), namely with the roles of 0- and 1-bits exchanged with respect to the Dyck words of [4]. This allows to determine $\mathbb{A}_k$ in Section 4 and an arc-coloring analysis through $\mathbb{A}_k$ (not covered in [4]) of:

- **(i)** the uniform 2-factors of O_k [3] (as in Theorem 3, via Subsections 4.1-5.2) and the Hamilton cycles [4] of O_k, for $k > 2$ (as in Theorem 4, via Subsections 5.1-6.1);
- **(ii)** the double covering graph M_k of O_k, namely the *middle-levels graph* of the *Boolean lattice* B_n induced by the levels L_k and L_{k+1} of B_n, formed by the n-bitstrings of weight k and $k+1$ (with Hamilton cycles lifted from those in item (i), see Corollary 6);
- **(iii)** the explicit *modular* 1-factorization of the graphs M_k [8], with factor colors in $[1, k+1]$ obtained from the color set $[0, k]$ in Section 4 by uniformly adding 1.

The modular 1-factorization of M_k cited in item (iii) differ from the *lexical* 1-factorization of M_k [13]. For example, there are at least two different approaches to Hamilton cycles in M_k, namely: via the modular 1-factorization [3] for O_k in [4], these represented below in Corollary 6, as well as via the lexical 1-factorization for M_k (never O_k) in [9,10].

2. Restricted growth strings and k-germs

To unify presentation of the odd graphs O_k, let us consider the sequence $S_{(\infty)}$ [14, A239903] of *restricted-growth strings* (*RGS*) [11, p. 325] and the k-th Catalan number $C_k = \frac{(2k)!}{k!(k+1)!}$ [14, A000108]. The first C_k terms of $S_{(\infty)}$ form a set $S_{(u)}$ [12, p. 222] equivalent to the set $S_{(i)}$ of Dick paths from $(0, 0)$ to $(2k, 0)$ [12, p. 221]. Both $S_{(u)}$ and $S_{(i)}$ are items in [12, ex. 6.19].

The sequence $S_{(\infty)}$ is expressible as $S_{(\infty)} = (\beta(0), \beta(1), \beta(2), \ldots, \beta(17), \ldots) =$

$$(0, 1, 10, 11, 12, 100, 101, 110, 111, 112, 120, 121, 122, 123, 1000, 1001, 1010, 1011, \ldots),$$

with the lengths of any two contiguous terms $\beta(m-1)$ and $\beta(m)$ $(1 \leq m \in \mathbb{Z})$ constant unless $m = C_k$, for some integer $k > 1$, in which case $\beta(m-1) = \beta(C_k - 1) = 12\cdots k$ has length k and $\beta(m) = \beta(C_k) = 10^k = 10\cdots 0$ has length $k+1$.

To manipulate O_k and M_k $(k > 1)$ in relation to their Hamilton cycles [3,9,10], we *dress* the RGS's $\beta = \beta(m)$ with length$(\beta) \leq k$ as strings of fixed length $k-1$ that we call *k-germs* [6, p. 138] [7, p. 8] in order to show (via the *nested castling* of Theorem 1 and Subsection 3.1) that k-germs form the domain of a bijection f onto the Dyck words of length $2k$. Concretely, we make the *k-germ* of any such RGS $\beta = \beta(m)$ to be the $(k-1)$-string $\alpha = \alpha(m) = a_{k-1}a_{k-2}\cdots a_2a_1$ obtained from β by prefixing $k-$length(β) zeros to it. This makes a *k-germ* to be a $(k-1)$-string $\alpha = a_{k-1}a_{k-2}\cdots a_2a_1$ such that:

- **(1)** the leftmost position of α, namely position $k-1$, has entry $a_{k-1} \in \{0, 1\}$;
- **(2)** given $1 < i < k$, the entry a_{i-1} at position $i-1$ satisfies $0 \leq a_{i-1} \leq a_i + 1$.

Note that every k-germ $a_{k-1}a_{k-2}\cdots a_2a_1$ yields a $(k+1)$-germ $0a_{k-1}a_{k-2}\cdots a_2a_1$.

To *undress* a k-germ $\alpha = \alpha(m) = a_{k-1}a_{k-2}\cdots a_1 \neq 00\cdots 0$ means that a *non-null RGS* is obtained by stripping α of its null entries to the left of its leftmost entry equal to 1, in which case we denote such a non-null RGS

also by $\alpha = \alpha(m)$. To complement this notion, we say that the *null RGS* $\alpha = \alpha(0) = 0$ corresponds to all null k-germs $\alpha = \alpha(0)$, for $0 < k \in \mathbb{Z}$.

We consider also the *empty RGS*, denoted $\alpha = \phi$, that yields for $k = 1$ the only *empty* k-germ $\alpha = 0^{k-1} = 0^{1-1} = \phi$, using the same notation ϕ both for the empty RGS and the empty 1-germ and extending this way the general notation $\alpha = 0^{k-1}$ $(k > 1)$ to every $k > 0$.

There are exactly C_k k-germs $\alpha = \alpha(m) < 10^k$, $\forall k > 0$. Given two k-germs $\alpha = a_{k-1}\cdots a_2 a_1$ and $\beta = b_{k-1}\cdots b_2 b_1$, $(\alpha \neq \beta)$, α is said to be less than β, written $\alpha < \beta$, if

(i) either $0 = a_{k-1} < b_{k-1} = 1$
(ii) or $\exists i \in [2, k-1]$ such that $a_{k-i} < b_{k-i}$ with $a_{k-j} = b_{k-j}$, $\forall j \in [1, i-1]$.

The resulting order on k-germs $\alpha(m)$ corresponds bijectively with the natural order of the integers $m \in [0, C_k]$, via the assignment $m \to \alpha(m)$.

3. Ordered trees of k-germs and Dyck words

We recall from [6, Theorem 3.1] or [7, Theorem 1] that the k-germs are the nodes of an ordered tree $\mathcal{T}_k$ rooted at 0^{k-1} and such that each k-germ $\alpha = a_{k-1}\cdots a_2 a_1 \neq 0^{k-1}$ with rightmost nonzero entry a_i $(1 \leq i = i(\alpha) < k)$ has parent $\beta(\alpha) = b_{k-1}\cdots b_2 b_1 < \alpha$ in $\mathcal{T}_k$ with $b_i = a_i - 1$ and $a_j = b_j$, for every $j \neq i$ in $[1, k-1]$.

Example 2. By representing $\mathcal{T}_k$ with each node β having its children α enclosed between parentheses following β and separating siblings with commas, we can write:

$$\mathcal{T}_4 = 000(001, 010(011(012)), 100(101, 110(111(121)), 120(121(122(123))))).$$

Theorem 1. ***(i-nested castling)*** *To each k-germ $\alpha = a_{k-1}\cdots a_1$ corresponds an n-string $F(\alpha)$ whose entries are the numbers $0, 1, \ldots, k$, once each, and k "=" signs. Moreover,*

$$F(0^{k-1}) = \text{“}012\cdots(k-2)(k-1)k = \cdots = \text{”}, \ (e.g,\ F(0) = \text{“}01 = \text{”},\ F(00) = \text{“}012 == \text{”}).$$

Furthermore, if $\alpha \neq 0^{k-1}$, let

1. *W^i and Z^i be the leftmost and rightmost, respectively, substrings of length $i = i(\alpha)$ in $F(\beta)$, where β is the parent of α in $\mathcal{T}_k$;*
2. *$c > 0$ be the leftmost entry of $F(\beta) \setminus (W^i \cup Z^i)$, and*
3. *$F(\beta) \setminus (W^i \cup Z^i)$ be the concatenation $X|Y$, where Y starts at the entry $c + 1$ of $F(\beta)$.*

Then $F(\alpha) = W^i|Y|X|Z^i$ is the i-nested castling of $F(\beta) = W^i|X|Y|Z^i$. In addition, if an entry $b' \in [0, k]$ of $F(\alpha)$ is followed immediately to its right by an entry $b \in [0, k]$, then $k \neq b' < b$. Also, W^i is an ascending number i-substring, Z^i is formed by i signs"=", and "k = " is a substring of $F(\alpha)$, but " = k" is not.

Proof. It was proved in [6, Theorem 3.2], as well as in [7, Theorem 2], where asterisks, "$*$", were used instead of the present "=" signs. Examples 3 and 4 yield ideas on the proof. □

Example 3. Figure 1 shows each tree $\mathcal{T}_k$ $(k = 1, 2, 3, 4)$, with its root 0^{k-1} represented in a box containing the order $ord(0^{k-1}) = 0$, the root 0^{k-1} and $F(0^{k-1})$. Each other node α of $\mathcal{T}_k$ is represented by a box of two levels: the top level contains the order $ord(\beta(\alpha))$, the parent $\beta(\alpha)$ and $F(\beta(\alpha))$; the lower level contains the order $ord(\alpha)$, α and $F(\alpha)$. In these presentations of $\beta(\alpha)$ and α, the entries b_i and a_i are colored red and the remaining entries black. In all boxes, $F(\beta(\alpha))$ = "$W^i|X|Y|Z^i$" and $F(\alpha)$ = "$W^i|Y|X|Z^i$" have X and Y colored blue and red, respectively, while W^i and Z^i are left black. In addition, the edge leading from $\beta(\alpha)$ to α is labeled with its subindex i.

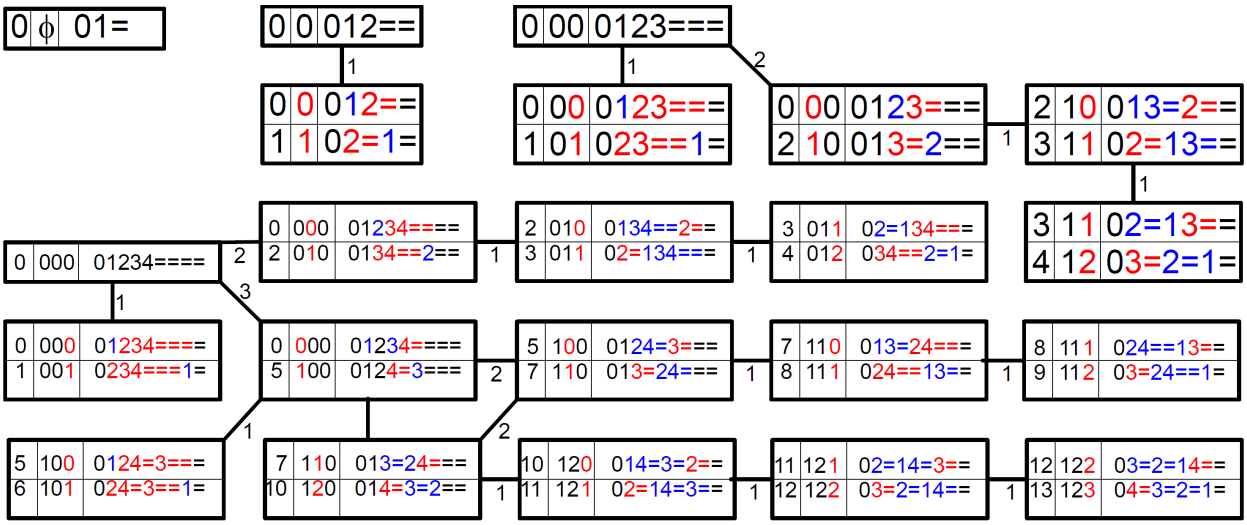


Figure 1. Exemplifying Theorem 1 for the ordered trees $\mathcal{T}_k$ ($k = 1,2,3,4$)

3.1. Bitstring forms out of nested castling

For each k-germ α ($k > 1$), let us define the bitstring form $f(\alpha)$ of $F(\alpha)$ by replacing each number entry of $F(\alpha)$ by a 0-bit and each "=" sign by a 1-bit. (0-bits and 1-bits here correspond respectively to the 1-bits and 0-bits of [4]). Such $f(\alpha)$ is an n-bitstring of weight k whose support $supp(f(\alpha))$ is in $V(O_k)$. So, we consider both $F(\alpha)$ and the characteristic vector $f(\alpha)$ of $supp(f(\alpha))$ to represent the vertex $supp(f(\alpha))$ of O_k.

Example 4. We can recover $F(\alpha)$ from $f(\alpha)$, exemplified for $k = 1,2,3$ in Figure 2. In it, for each one of the $1+2+5 = 8$ cases in the figure, a piecewise-linear curve $PLC(\alpha)$ is constructed iteratively that starts at the shown origin O in the Cartesian plane Π by replacing successively the 0-bits and 1-bits of $f(\alpha)$ by *up-steps* and *down-steps*, namely diagonal segments $(x,y)(x+1,y+1)$ and $(x,y)(x+1,y-1)$, respectively. To each down-step of $PLC(\alpha)$, we assign the "="- sign. We assign the integers in the interval $[0,k]$ in decreasing order (from k to 0) to the up-steps of $PLC(\alpha)$, from the top unit layer of $PLC(\alpha)$ in Π to the bottom one and from left to right at each pertaining unit layer between contiguous lines $y, y+1 \in \mathbb{Z}$. Then, by reading and successively writing the number entries and "=" signs assigned to the steps of $PLC(\alpha)$, the n-tuple $F(\alpha)$ is obtained. Figure 2 is provided, underneath each instance, with the corresponding k-germ α followed by $F(\alpha)$ and its (underlined) order of presentation via Theorem 1. We assume that all elements of $V(O_k)$ are represented by means of such piecewise-linear curves, for each fixed integer $k > 0$.

Theorem 1 is exemplified in Figure 2 too, where *i-nested castling* is occurring via *layer polygons* (either isosceles trapezoids or triangles) with their interiors pairwise disjoint, as follows. For $k = 2$: between the blue layer polygon (delimited by the up-step "1" on the left and the down-step "=" on the right) and the yellow layer polygon (delimited by the up-step "2" on the left and the down-step "=" on the right). For $k = 3$: between the blue, green and yellow layer polygons (delimited on the left by the up-steps "1", "2" and "3", respectively, and corresponding down-steps "=" on their right).

Specifically in Fig 2: For $k = 2$, the 1-nested castling from the root 2-RGS (0) to the 2-RGS (1) is depicted as a permutation of the contiguously labelled up-steps of the (possibly shortened) layer polygons. For $k = 3$, the 1- and 2-nested castlings from the root RGS (00) to the RGS's (01) and (10), respectively, permute the order of the contiguously labelled up-steps of the (possibly shortened) layer polygons as indicated in the figure. Similarly for the 1-nested castlings from (10) to (11) and from (11) to (12).

3.2. Dyck paths

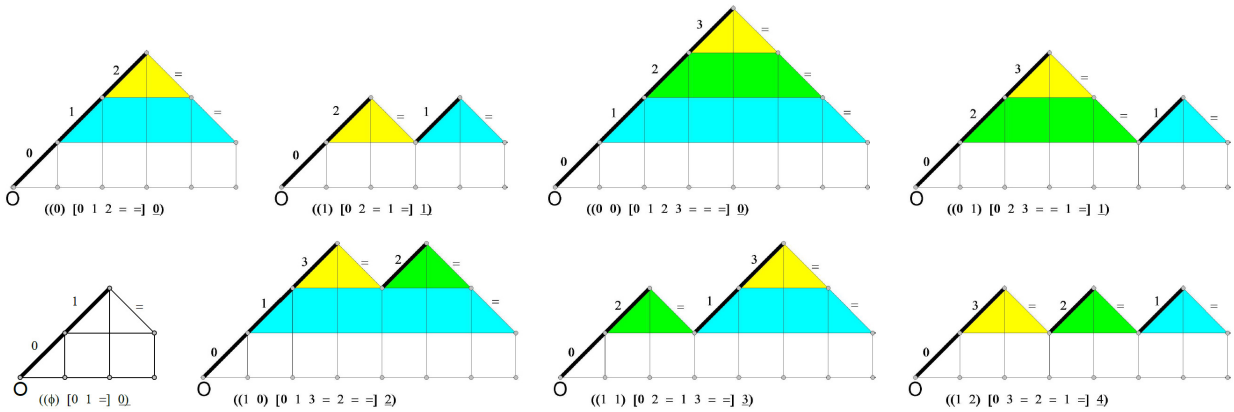


Figure 2. Recovering $F(\alpha)$ from $f(\alpha)$: $PLC(\alpha)$ for triples $((\alpha)\ [F(\alpha)], \underline{ord(\alpha)})$, $k = 1, 2, 3$

Let $0 < k \in \mathbb{Z}$ and let α be a k-germ. The curve $PLC(\alpha)$ (Example 4 and Figure 2) yields a *Dyck path* $DP(\alpha)$ via the removal of its first up-step $(0,0)(1,1)$ and a change of coordinates from $(1,1)$ to $(0,0)$. Such Dyck path $DP(\alpha)$ represents a corresponding *Dyck word* $DW(\alpha)$ = "0···1" of length $2k$, a particular case for $\ell = k$ of a *Dick word of length* 2ℓ ($0 < \ell \in \mathbb{Z}$), defined as a 2ℓ-bitstring of weight ℓ such that in every prefix the number of 0-bits is at least the number of 1-bits (differing from the Dyck words of [4], in which, on the contrary, the number of 1-bits is at least equal to the number of 0-bits). The concept of *empty Dyck word* ϵ also makes sense here and is used for example in Section 5, display (1). The Dyck paths $DP(\alpha)$ corresponding to the curves $PLC(\alpha)$ in Figure 2 are represented in the lower-left quarter of Figure 4, with notation specified in Examples 6 and 8, and preserving the colors of Figure 2. In Subsection 5.2, the down-steps on the right of the layer polygons of Fig 2 will have their labels "=" changed to "$\underline{j}$", if "j" is the label of the associated up-step. This takes $F(\alpha)$ into an n-string $\underline{F}(\alpha)$, whose substrings $[j, \underline{j}]$ project in $f(\alpha)$ as Dyck subwords.

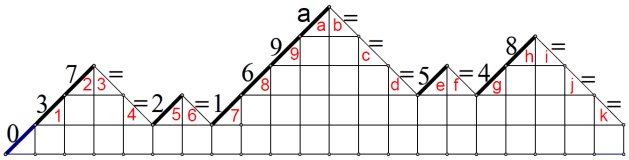
Theorem 2. *There exists a bijection λ from the $\mathbb{Z}_n$-classes of $V(O_k)$ onto the Dyck words of length $2k$. In fact, each $\mathbb{Z}_n$-class Γ of $V(O_k)$ has a Dyck word $f(\alpha)$ of length $2k$ as sole representative. The other n-tuples in Γ are obtained by translations $f(\alpha).j$ mod n of $f(\alpha)$, where $j \in [0.2k]$ is the position of the null entry in $f(\alpha).j$. Also, $f(\alpha)$ may be interpreted as its corresponding $F(\alpha)$ and the other n-tuples $f(\alpha).j$ above may be interpreted as the corresponding translations $F(\alpha).j$ mod n.*

Proof. The n-tuples $F(\alpha)$ were obtained via the i-nested castlings of Theorem 1 associated to the indices $i = i(\alpha)$ of the oriented edges $\beta\alpha$ of the tree $\mathcal{T}_k$ of Section 3 from the parent β of each non-root k-germ α to α. Note that there are just C_k Dick words of length $2k$ (Subsection 3.2) corresponding bijectively to the n-tuples $F(\alpha)$, and to their binary versions $f(\alpha)$ (Subsection 3.1). Also, there are exactly C_k $\mathbb{Z}_n$-classes Γ in $V(O_k)$. Then, each $\mathbb{Z}_n$-class Γ of O_k contains a sole $f(\alpha)$, which correspond to a sole $F(\alpha)$ by the approach in Example 4. As a result, the correspondence from the $\mathbb{Z}_n$-classes Γ onto such Dyck words is a bijection λ with $\lambda(\Gamma) = f(\alpha)$, as claimed, and we may write $\Gamma = \Gamma_\alpha = \lambda^{-1}(f(\alpha))$.

With respect to the last two sentences in the statement, note that $\mathbb{Z}_n$ acts on O_k yielding, for each k-germ α, the orbit of $f(\alpha)$, including the translations of $f(\alpha)$, namely:

$$f(\alpha) = f(\alpha).0 = f_0 f_1 \cdots f_{2k}, \quad f(\alpha).1 = f_1 f_2 \cdots f_{2k} f_0, \quad f(\alpha).2 = f_2 f_3 \cdots f_{2k} f_0 f_1, \quad \ldots,$$

037==2=169a===5=48===	012223142	1	3715
02=169a===5=48==37===	012233341	1	3714
0169a===5=48==37==2==	012233340	2	3713
015=48==37==269a=====	012233330	3	3711
0148==37==269a===5===	012233320	4	3708
0137==269a===5=48====	012233310	5	3704
01269a===5=48==37===	012233300	5	3699
0125=48==37=69a=====	012233200	9	3694
01248==37=69a===5===	012233100	14	3680
01237=69a===5=48====	012233000	14	3666
012369a===5=48=7====	012232000	28	3652
01235=48=7=69a======	012231000	48	3624
012348=7=69a===5=====	012230000	42	3576
012347=69a===58=====	012220000	90	3534
0123469a===58=7=====	012210000	165	3444
0123458=7=69a=======	012200000	132	3279
0123457=69a==8=======	012100000	297	3147
01234569a==8=7=======	012000000	429	2850
01234568=79a========	011000000	1001	2431
012345679a==8========	010000000	1430	1430
0123456789a==========	000000000	0	0



(0(0,1)1)(0,1)(0(0(0(0,1)1)1)(0,1)(0(0,1)1)1)
(1(2,3)4)(5,6)(7(8(9(a,b)c)d)(e, f)(g(h, i) j) k)
(4(3,2)1)(6,5)(k(j(i(h,g) f)e)(d, c)(b(a,9)8)7)
(4(2,3)1)(6,5)(k(e(f(g,h) i) j)(c, d)(8(9,a)b)7)
(4(2,3)1)(6,5)(k(e(i(h,g) f) j)(c, d)(8(a,9)b)7)
(4(2,3)1)(6,5)(k(e(i(g,h) f) j)(c, d)(8(a,9)b)7)
(4(2,3)1)(6,5)(k(g(i(h, j) e)f)(8, c)(a(b,9)d)7)
(0(0,1)1)(0,1)(0(0(0(0,1)1)1)(0,1)(0(0,1)1)1)
(3(7,7)3)(2,2)(1(6(9(a,a)9,6)(5,5)(4(8,8)4)1)

1								
1	1							
1	2	2						
1	3	5	5					
1	4	9	14	14				
1	5	14	28	42	42			
1	6	20	48	90	132	132		
1	7	27	75	165	297	429	429	
1	8	35	110	275	572	1001	1430	1430

00								
01	11							
02	12	22						
03	13	23	33					
04	14	24	34	44				
05	15	25	35	45	55			
06	16	26	36	46	56	66		
07	17	27	37	47	57	67	77	
08	18	28	38	48	58	68	78	88

Figure 3. Illustration for Examples 5 and 7 and Subsection 4.1

where $f_j = f_j(\alpha) \in \{0,1\}$ ($j \in [0,2k]$), with general term given by

$$f(\alpha).j = f_j f_{j+1} \cdots f_{2k} f_0 f_1 \cdots f_{j-1},$$

and ending up with $f(\alpha).2k = f_{2k} f_0 f_1 \cdots f_{2k-1}$.

Similar treatment holds by taking $F(\alpha) = F(\alpha).0 = F_0 F_1 \cdots F_{2k}$, where $F_j =$ " = " if f_j is a 1-bit and $F_j \in [0,k]$ if f_j is a 0-bit (the value of F_j provided by the said approach in Example 4), with general translation term given by

$$F(\alpha).j = F_j F_{j+1} \cdots F_{2k} F_0 F_1 \cdots F_{j-1}.$$

This covers all the vertices of $\mathbb{Z}_n$-classes of $V(O_k)$, seen either from the $f(\alpha)$ point of view or from the $F(\alpha)$ point of view. □

Example 5. In this and subsequent examples, we express integers in their hexadecimal form (e.g., $a = 10, b = 11$, etc.). To clarify concepts, let us determine the n-germ α_1 ($n = 21$) corresponding to the bitstring $f(\alpha_1) = 00110100001110100111$. We proceed by determining $PLC(\alpha_1)$ (as indicated in Example 4), drawn in the upper-right of Figure 3, where the black hexadecimal number entries and "=" signs form the n-string $F(\alpha_1)$, while the red symbols are the first twenty positive hexadecimal numbers, (that appear in that order in the expression $h_0(\alpha) = h_0(\alpha_1)$ of Subsection 4.1, item 2). To associate the k-germ α_1 to the n-string $F(\alpha_1)$, we build a list $\mathbb{L}(\alpha_1)$ shown on the left of Figure 3. The first lines of $\mathbb{L}(\alpha_1)$ contain data concerning the path $P(\alpha_1)$ from α_1 to the root $0^{20} = \alpha_{21}$ in $\mathcal{T}_{21}$, namely: $F(\alpha_i)$, α_i, $ord(\alpha_i) - ord(\alpha_{i+1})$ and $ord(\alpha_i)$, for $i = 1, 2, \ldots, 20$. The first sublist $\mathbb{L}'(\alpha_1)$ in $\mathbb{L}(\alpha_1)$, composed successively by $F(\alpha_1), \ldots, F(\alpha_{21})$, shows each of the 21-strings $F(\alpha_j)$, ($j = 1, \ldots, 20$), as a concatenation "$W^{i_j}|X|Y|Z^{i_j}$", where i_j is the first index in $F(\alpha_j) = c_0 c_1 \cdots c_{20}$ such that $c_j > j$ with blue X, red Y, and black for both W^{i_j} and Z^{i_j}, showing in the following line the 21-string $F(\alpha_{j+1}) =$ "$W^{i_j}|Y|X|Z^{i_j}$", just under $F(\alpha_j)$. To the right of $\mathbb{L}'(\alpha_1)$ and starting at the red $\alpha_{21} = 0^{k-1}$ in line 21, we went up and built a sublist $\mathbb{L}''(\alpha_1)$ by reconstructing each $\alpha_j = a_{k-1} \cdots a_1$, setting in red the terminal

substring $a_{i_j}\cdots a_1$ and in black the initial substring $a_{k-1}\cdots a_{i_j+1}$. To the right of $\mathbb{L}''(\alpha_1)$, we constructed an accompanying blue sublist $\mathbb{L}'''(\alpha_1)$ formed by Catalan numbers taken as increments that determine the corresponding orders of the vertices in $P(\alpha_1)$. These orders, appearing in the final sublist $\mathbb{L}''''(\alpha_1)$, are obtained as the partial sums of Catalan numbers. This takes to $ord(\alpha_1) = 3715 < 4862 = |V(\mathcal{T}_{21})|$. The blue sublist $\mathbb{L}'''(\alpha_1)$ arises from the red entries in the first nine lines of Catalan's triangle in the lower part of Figure 3 with entries τ_i^j as in [6, pp. 139–140] represented as pairs ij to the right of the said nine lines, ($i \in [0,8]$, $j \in [1,8]$).

4. Edge-supplementary arc factorizations

Each arc (u,v) of O_k ($u,v \in V(O_k)$) is represented by translations $F(\alpha_u).j_u$ and $F(\alpha_v).j_v$ mod n of the n-strings $F(\alpha_u)$ and $F(\alpha_v)$. Looking u and v upon as $u = F(\alpha_u).j_u$ and $v = F(\alpha_v).j_v$ and comparing, we see that apart from a specific number entry $i \in [0,2k]$ in both u and v, all other number entries of one of them correspond to "=" sign entries of the other one, and vice versa. Moreover, the entries u_i of u and v_i of v satisfy $u_i + v_i = k$, so they are said to be *k-supplementary*. Then, the *edge-supplementary 1-arc factorization* $\mathbb{A}_k$ of O_k claimed in Section 1 is given by the values of those entries u_i and v_i taken as colors of the arcs (u,v) and (v,u), respectively, for all pairs of adjacent vertices u and v of O_k.

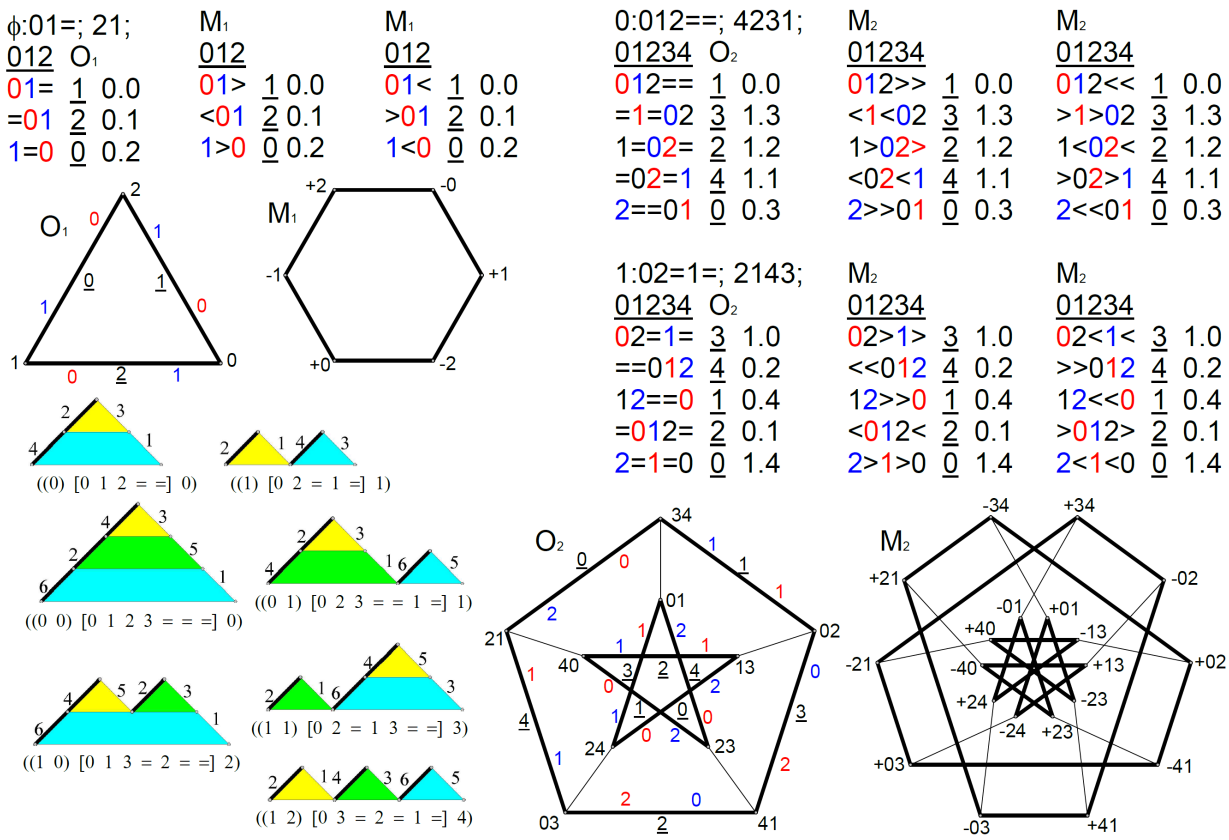


Figure 4. Illustration for Section 4 and Examples 6 and 8

Example 6. Edge-supplementarity is illustrated for $k = 1,2$ in Figure 4. In it, the n-tuples $F(\alpha)$ are shown as the initial lines of corresponding vertical lists $L(\alpha)$ in which arcs (u,v) of O_k appear as ordered pairs $(F(\alpha_u).j_u, F(\alpha_v).j_v)$ disposed on contiguous lines, except for all arcs from bottom lines, taken as n-tuples $F(\alpha_u).j_u$, to corresponding top lines, taken as associated n-tuples $F(\alpha_v).0$, thus closing the lists $L(\alpha)$ into

oriented cycles $C(\alpha)$. In each such pair $(F(\alpha_u).j_u, F(\alpha_v).j_v)$, the i-entries u_i and v_i are colored respectively blue in $F(\alpha_u).j_u$ and red in $F(\alpha_v).j_v$ (the other entries in black) with the exception of the bottom and top n-tuples in each list: these are also adjacent, with the entry $u_i = u_0$ holding blue value k on the bottom n-tuple u and the entry $v_i = v_0$ holding red value 0 on the top n-tuple $v = F(\alpha_v) = F(\alpha_v)_0$. The position i of the blue entry u_i in each line of the lists is cited underlined ("$\underline{i}$") to the right of its n-tuple u; the vertex $u \in O_k$ represented in such a line is still cited to the right of its "$\underline{i}$" as "$ord(\alpha).j_u$", where $ord(\alpha)$ refers to the $\mathbb{Z}_n$-class Γ_α (so denoted in the proof of Theorem 2) of u in O_k. Such vertical lists are used in Section 5 (Figure 5, 6, 7) in order to yield Hamilton cycles of O_k, for $k > 2$, as in [3,4]; ($k = 2$ is excluded; indeed, O_2 is the hypohamiltonian Petersen graph).

4.1. String reversals in properly nested parentheses

Assignment of a $2k$-permutation $\pi(\alpha)$ to each k-germ α.
Consider the Dyck path $DP(\alpha)$ obtained from $PLC(\alpha)$ by the removal of its first up-step and subsequent change of coordinates from $(1,1)$ to $(0,0)$.

1. Let f' (resp., F') be the $2k$-string obtained from f (resp. F) by removing its first entry. Set parentheses or commas between each two entries of f', so that the four substrings

   "01" , "10" , "00" and "11" are transformed into the substrings
   "0,1" , "1)(0" , "0(0" and "1)1" , respectively, resulting in a string f''.

   Add a terminal parenthesis to f'', so that the last "1" in f'' is transformed into "1)". Denote by g the string resulting from such addition of a closing parenthesis to f''.
2. By proceeding from left to right, replace the bits of g by the successive integers from 1 to $|g|$, keeping all pre-inserted parentheses and commas in g unchanged in position. This yields a version h_0 of g.
3. Note h_0 is a concatenation $(w_1)|(w_2)|\cdots|(w_t)$ of expressions (w_i), $(i \in [1,t])$, for some $t \geq 1$, each (w_i) with terminal ")" being the closing ")" nearest to its opening "(". Let w_i' be the number string obtained from w_i by removal of parentheses and commas. For $i = 1,\ldots,t$, perform a recursive step $\mathcal{R}$ consisting in transforming w_i' into its reverse substring w_i'' and then resetting w_i'' in place of w_i' in (w_i), with the parentheses and commas of (w_i) kept in place. Denote the resulting expression by $\mathcal{R}(w_i)$. This yields a string $h_1 = \mathcal{R}(w_1)|\mathcal{R}(w_2)|\cdots|\mathcal{R}(w_t)$.
4. For $i \in [1,t]$, let $\mathcal{R}(w_i) = (a^1_{i,1}\eta^1_{i,1}b^1_{i,1})|(a^1_{i,2}\eta^1_{i,2}b^1_{i,2})|\cdots|(a^1_{i,t_i}\eta^1_{i,t_i}b^1_{i,t_i})$. Apply item 3 to each $(w_{i,j}) = \eta_{i,j} \neq$"," with terminal ")" being the closing ")" nearest to its opening "(", for $j \in [1,t_i]$. Replace the resulting strings $\mathcal{R}(w_{i,j})$ in place of the corresponding $(w_{i,j})$ in $\mathcal{R}(w_i)$, yielding a modified version $\mathcal{R}^2(w_i)$ of $\mathcal{R}(w_i)$. Let $h_2 = \mathcal{R}^2(w_1)|\mathcal{R}^2(w_2)|\cdots|\mathcal{R}^2(w_t)$.
5. Each $\mathcal{R}(w_{i,j})$ is a concatenation of terms of the form $a^2_I\eta^2_I b^2_I$ with $I = \{i, j_1, j_2\}$, where $j_1 = j$. In each such concatenation, the strings $\eta^2_I \neq$"," are of the form (w_I) and must be treated as $(w_{i,j})$ is in item 4 (or (w_i) in item 3), producing a modified string $\mathcal{R}(w_I)$ that forms part of the subsequent string h_3. Eventually ahead, to pass from $h_{\ell-1}$ to h_ℓ ($\ell > 3$), each $\mathcal{R}(w_I)$ in $h_{\ell-1}$ with $I = \{i, j_1, \ldots, j_{\ell-2}\}$ would be a concatenation of terms of the form $a^{\ell-1}_{I'}|\eta^{\ell-1}_{I'}|b^{\ell-1}_{I'}$ with $I' = \{i, j_1, \ldots, j_{\ell-1}\}$. In each such concatenation, those $\eta^{\ell-1}_{I'} \neq$"," would be of the form $(w_{I'})$, to be treated again as in items 3–4.
6. A sequence $(h_0, \ldots, h_{s+1})$ is eventually obtained for some $s \geq 0$ when all innermost expressions $(w_I) = (a, a \pm 1)$ with $a, a \pm 1 \in [1, 2k]$ are already processed. Disregarding parentheses and commas in h_{s+1} yields a $2k$-string g' and an assignment $i \to p(i)$, for $i \in [1, 2k]$, by corresponding the places $i \in [1, 2k]$ of g' to the values in the places of g'. Define $\pi = p^{-1}$, the inverse $2k$-permutation of $p = (p(1)p(2)\cdots p(2k))$ [4].

Example 7. (Continuation of Example 5) The middle right of Figure 3 (just under the upper-right representation of the curve $PLC(\alpha_1)$) contains a list, call it ℓ, whose first line represents $g(\alpha_1)$

(Subsection 4.1, item 1), with α_1 as in Example 5, and whose second line represents $h_0(\alpha_1)$ (Subsection 4.1, item 2), in an hexadecimal-notation continuation. In this representation of $h_0(\alpha_1)$, the red substrings w_1 = "$1(2,3)4$", w_2 = "$5,6$" and w_3 = "$7(8(9(a,b)c)d)(e,f)(g,h,i)j)k$" are to be reversed according to the first instances of $\mathcal{R}(w)$ in Subsection 4.1. This yields the third line, representing $h_1(\alpha_1)$ in the list ℓ. In $h_1(\alpha_1)$, the red substrings are to be reversed according to the next instances of $\mathcal{R}(w)$, and so on. In the end, the sixth line of ℓ, represents $h_4(\alpha_1)$ =

$$p(\alpha_1) = (4,3,2,1,6,5,20,14,18,16,17,15,19,12,13,8,10,9,11,7).$$

The inverse of this is $\pi(\alpha_1)$ =

$$p^{-1}(\alpha_1) = (4,3,2,1,6,5,20,16,18,17,19,14,15,8,12,10,11,9,13,7),$$

represented as a blue string under the mentioned sixth line $h_4(\alpha_1) = p(\alpha_1)$ and in a similar format with inserted parentheses and commas.

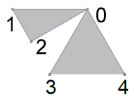
Example 8. (Continuation of Example 6) Figure 4 contains one oriented 3-cycle for O_1 and two oriented 5-cycles for O_2. Their lists $L(\alpha)$ are headed by two lines: a first line reading "$ord(\alpha)$:$F(\alpha)$;$\pi(\alpha)$", with "$F(\alpha)$" as the first line of the cycle and "$\pi(\alpha)$" (as in Subsection 4.1), formed by the different entries at which a blue-to-red k-supplementation takes place in the cycle; the second line contains the (underlined) positions 0 to $2k$ of the vertices (as n-tuples) in the cycle, followed by "O_k". The arcs of O_k receive colors in the set $[0,k]$ so that the edge between each two adjacent vertices in those cycles has its two composing arcs bearing *k-supplementary* colors b (for blue) and r (for red), meaning that $b,r \in [0,k]$ are such that $b+r=k$. To the immediate right of each of these three cycles, for lists $L(\epsilon), L(0), L(1)$ of respective lengths 3, 5, 5, are also represented vertical lists $L^M(\epsilon), L^M(0), L^M(1)$, (occupying two contiguous columns each) closing into corresponding cycles $C^M(\epsilon), C^M(0), C^M(1)$ of respective double lengths 6, 10, 10, obtained by replacing the "=" signs by the ">" signs and "<" signs uniformly on alternate lines. These cycles can be interpreted as cycles in the middle levels graphs M_1, M_2, obtained by reading the subsequent lines in the concatenation of two subsequent columns as follows: from left to right if they bear ">" signs, and from right to left if they bear "<" signs. In addition, the graphs O_1, O_2, M_1, M_2 are represented in Figure 4 in thick trace for the edges containing the arcs of the oriented cycles $C(\alpha)$; recalling $\mathbb{A}_k$ from Section 1, each vertex (resp., edge) of O_1, O_2 is denoted by the support of its corresponding bitstring $f(\alpha)$ (resp., denoted centrally by its underlined color in $\mathcal{E}_k$ and marginally by its blue-red arc-color pair in $\mathbb{A}_k$). In M_1, M_2, a plus or minus sign precedes each such support indicating respectively a vertex in level L_k or in level L_{k+1} of B_n; if in L_{k+1}, as the complement $\overline{f(\alpha,<)}$ of the right-to-left reading $f(\alpha,<)$ of the bitstring $f(\alpha) = f(\alpha,>)$; if in L_k, as $f(\alpha) = f(\alpha,>)$ itself. The resulting readings of n-tuples of M_1, M_2 inherit the mentioned arc colors for O_1, O_2, corresponding to the *modular matchings* of [8], only that the colors in [8] are in $[1,k+1]$ with supplementary sum $k+1$ while our colors are in $[0,k]$ with supplementary sum k.

5. Uniform 2-factors and Hamilton cycles

Let $k > 1$. A vertical list $L(\alpha)$ as in Examples 6 and 8, illustrated in Fig 4, can be formed for each k-germ α. In fact, there are C_k such lists $L(\alpha) = (L_0(\alpha), L_1(\alpha), \ldots, L_{2k}(\alpha))^t$, where t stands for transpose, each $L(\alpha)$ representing in O_k an oriented n-path $P(\alpha)$ whose end-vertices $L_0(\alpha) = F(\alpha)$ and $L_{2k}(\alpha)$, are adjacent in O_k, thus completing an oriented cycle $C(\alpha)$ in O_k by the addition of the arc $(L_{2k}(\alpha), L_0(\alpha))$.

Those paths $P(\alpha)$ arose in [3, Theorem 4] and [4, Lemma 4], in the latter case leading to Hamilton cycles in O_k and M_k. Construction of these $L(\alpha)$ is controlled by the $2k$-permutation $\pi(\alpha)$ assigned to α via the procedure contained in items 1–6 of Subsection 4.1, as will be established in Theorem 3.

```
0 : 000111
1 : 001101
2 : 001011

=2==013 5 2.4 0
1=03=2= 3 4.2 0
=03=2=1 6 4.1 2
3==1=02 0 1.5 2
=02=13= 4 3.1 1
2=13==0 2 3.6 1

0
0123=== 1 0.0 0
=2==013 5 2.4 0
1=03=2= 3 4.2 0
=1=03=2 4 4.3 0
2=1=03= 2 4.4 0
=023==1 6 0.1 0
3===012 0 0.4 0

1
023==1= 5 1.0 1
===0123 6 0.3 1
123===0 1 0.6 1
=1=023= 3 1.3 1
2=13==0 2 3.6 1
=02=13= 4 3.1 1
3==02=1 0 3.3 1

2
013=2== 1 2.0 2
=2=1=03 3 4.5 2
1=023== 2 1.2 2
=13==02 5 3.5 2
2==013= 4 2.3 2
=03=2=1 6 4.1 2
3==1=02 0 1.5 2

=2==013 5 2.4 0
0123=== 1 0.0 0
3===012 0 0.4 0
=023==1 6 0.1 0
2=1=03= 2 4.4 0
=1=03=2 4 4.3 0
1=03=2= 3 4.2 0
=03=2=1 6 4.1 2
2==013= 4 2.3 2
=13==02 5 3.5 2
1=023== 2 1.2 2
=2=1=03 3 4.5 2
013=2== 1 2.0 2
3==1=02 0 1.5 2
=02=13= 4 3.1 1
3==02=1 0 3.3 1
023==1= 5 1.0 1
===0123 6 0.3 1
123===0 1 0.6 1
=1=023= 3 1.3 1
2=13==0 2 3.6 1

0 : 000111
3 : 001101
4 : 010101

=023==1 6 0.1 0
3===012 0 0.4 0
=013=2= 2 2.1 4
23==1=0 1 1.6 4
==02=13 5 3.2 3
13==02= 4 3.4 3

0
0123=== 1 0.0 0
=2==013 5 2.4 0
1=03=2= 3 4.2 0
=1=03=2 4 4.3 0
2=1=03= 2 4.4 0
=023==1 6 0.1 0
3===012 0 0.4 0

3
02=13== 3 3.0 3
==02=13 5 3.2 3
13==02= 4 3.4 3
==013=2 6 2.2 3
23===01 1 0.5 3
=0123== 2 0.1 3
3=2==01 0 2.5 3

4
03=2=1= 5 4.0 4
==1=023 6 1.4 4
13=2==0 3 2.6 4
==0123= 4 0.2 4
23==1=0 1 1.6 4
==02=13 5 3.2 3
02=13== 3 3.0 3
3=2==01 0 2.5 3
=0123== 2 0.1 3
23===01 1 0.5 3
==013=2 6 2.2 3
13==02= 4 3.4 3

=023==1 6 0.1 0
2=1=03= 2 4.4 0
=1=03=2 4 4.3 0
1=03=2= 3 4.2 0
=2==013 5 2.4 0
0123=== 1 0.0 0
3===012 0 0.4 0
=013=2= 2 2.1 4
3=2=1=0 0 0.6 4
03=2=1= 5 4.0 4
==1=023 6 1.4 4
13=2==0 3 2.6 4
==0123= 4 0.2 4
23==1=0 1 1.6 4
==02=13 5 3.2 3
02=13== 3 3.0 3
3=2==01 0 2.5 3
=0123== 2 0.1 3
23===01 1 0.5 3
==013=2 6 2.2 3
13==02= 4 3.4 3
```




```
=2==013 5 2.4 0
0123=== 1 0.0 0
3===012 0 0.4 0
=013=2= 2 2.1 4
3=2=1=0 0 0.6 4
03=2=1= 5 4.0 4
==1=023 6 1.4 4

13=2==0 3 2.6 4
==0123= 4 0.2 4
23==1=0 1 1.6 4
==02=13 5 3.2 3
02=13== 3 3.0 3
3=2==01 0 2.5 3
=0123== 2 0.1 3

23===01 1 0.5 3
==013=2 6 2.2 3
13==02= 4 3.4 3
=023==1 6 0.1 0
2=1=03= 2 4.4 0
=1=03=2 4 4.3 0
1=03=2= 3 4.2 0

=03=2=1 6 4.1 2
2==013= 4 2.3 2
=13==02 5 3.5 2
1=023== 2 1.2 2
=2=1=03 3 4.5 2
013=2== 1 2.0 2
3==1=02 0 1.5 2

=02=13= 4 3.1 1
3==02=1 0 3.3 1
023==1= 5 1.0 1
===0123 6 0.3 1
123===0 1 0.6 1
=1=023= 3 1.3 1
2=13==0 2 3.6 1
```

Figure 5. Illustration for Section 5 and Example 10

5.1. Flippable tuples and flipping cycles

Figure 5 for $k = 3$ and Figure 7 for $k = 4$ (Example 11) contain the lists $L(\alpha)$ assembled in triples and/or quadruples. For $k = 3$, Figure 5 shows two such triples, that we call $\tau_0 = (L(\alpha_0), L(\alpha_1), L(\alpha_2))$ on the upper-left of the figure and $\tau_1 = (L(\alpha_0), L(\alpha_3), L(\alpha_4))$ on the upper-right, where each list $L(\alpha_i)$ is distinguished on its upper-left corner with the subindex i of its k-germ α_i. Two concepts from [4] are used here:

1. **Flippable tuples:** In each such $L(\alpha_i)$, there is at least one pair of contiguous red lines, apart from, or including, its first red line, $F(\alpha_i)$, except for their initial black entries and the unique vertical pair of number k-supplementary blue entries (Section 4). These unique colored-line pairs $FT(\alpha_i, j)$, where $j \in [0, 2k]$ are the respective positions counted from the right at which the k-supplementary pairs determining adjacency in O_k occurs, are said to be *flippable tuples* [4].
2. **Flipping κ-cycles:** For $j = 0, 1$, the three pairs $FT(\alpha_i, j)$ with $L(\alpha_i) \in \tau_j$ are combined into a 6-cycle $F_6C(\tau_j)$ in O_3, which for $\kappa = 2k = 6$, is an example of a *flipping κ-cycle* [4], that we denote $F_\kappa C(\tau_j)$. Such flipping 6-cycle $F_6C(\tau_j)$ is shown in the middle left of the respective upper-left part and upper-right part, respectively, in Figure 5, sided each on its right and below by its three participating lists $L(\alpha_i)$. Above such flipping 6-cycle $F_6C(\tau_j)$ ($j = 0, 1$), a triple of Dyck words of length 6 headed each by the subindex $i \in \{0, 1, 2\}$ or $i \in \{0, 3, 4\}$ of the corresponding $L(\alpha_i)$ is shown in red except for one blue entry at the position of the blue k-supplementary number entries in the two contiguous red-blue substrings of the corresponding flippable tuples $FT(\alpha_i, j)$.

For any $k > 1$, red-blue flippable tuples as those six in this subsection (see Fig 5) were shown to exist in [4, pp. 1261–1265]. These six flippable tuples were shown to form part of a bitstring family [4, display (3.3)] (see displays (1) and (3) in Subsection 6.1 below). They were used in the construction of Hamilton cycles in [4], reconsidered below.

012344321	1	0.0 0	023443211	7	1.0 1	013443221	1	2.0 2	022134431	3	3.0 3
433210124	7	5.5 0	432101234	8	0.4 1	433102214	5	b.4 2	421331024	7	8.6 3
110344322	3	4.2 0	123443210	1	0.8 1	110332442	3	9.2 2	144103322	5	c.7 3
322110443	5	d.5 0	322101344	5	2.4 1	322144103	4	c.7 2	310221443	6	c.2 3
221104433	4	d.4 0	221443310	3	b.8 1	221103443	2	4.4 2	244211033	4	9.6 3
211044332	6	d.3 0	211033244	4	9.3 1	213443102	7	3.7 2	210134432	8	2.2 3
332210144	2	a.5 0	332214410	2	c.8 1	332101244	6	5.4 2	344321012	1	0.6 3
102344321	8	1.1 0	102442133	6	8.1 1	103443221	8	4.1 2	101234432	2	0.1 3
443210123	0	0.5 0	442101332	0	7.4 1	443102213	0	3.4 2	443321012	0	5.6 3
034432211	7	4.0 4	012443321	1	5.0 5	024433211	7	6.0 6	013324421	1	7.0 7
431022134	8	3.3 4	433211024	7	6.6 5	432110234	8	1.5 6	433221014	3	a.6 7
134432210	5	2.8 4	110443322	5	d.2 5	124433210	1	5.8 6	110234432	2	1.2 7
321012344	6	0.3 4	322101443	6	a.4 5	322110344	3	4.5 6	324421013	7	7.6 7
234432110	1	1.8 4	221441033	3	c.6 5	221344310	2	3.8 6	221014433	5	a.3 7
211024433	3	6.3 4	211034432	4	4.3 5	213310244	5	8.5 6	214410332	6	c.4 7
332442110	2	9.8 4	332211044	2	d.6 5	331024421	4	8.4 6	331022144	4	b.3 7
102214433	4	b.1 4	102443321	8	6.1 5	103322144	6	c.1 6	103324421	8	c.1 7
441032221	0	c.3 4	443211023	0	1.5 5	442110332	0	9.5 6	443221013	0	2.6 7
024421331	5	8.0 8	033244211	7	9.0 9	014433221	1	a.0 a	022144331	3	b.0 b
421013324	7	7.3 8	432210134	8	2.5 9	433221104	3	d.7 a	410332214	5	c.2 b
134431022	6	3.6 8	133244210	3	7.8 9	120244331	2	6.2 a	133102442	4	8.4 b
321012443	8	5.3 8	310221344	5	3.2 9	324421103	5	9.7 a	310244213	7	8.5 b
234432101	1	0.7 8	244213310	4	8.8 9	221013443	4	2.3 a	244210133	6	7.5 b
211023443	3	1.3 8	210133244	6	7.2 9	214433102	7	b.7 a	210144332	8	a.2 b
332442101	2	7.7 8	344310221	1	3.5 9	332110244	6	6.5 a	344321102	1	1.7 b
102213443	4	3.1 8	101344322	2	2.1 9	104433221	8	d.1 a	101244332	2	5.1 b
442133102	0	8.7 8	443310221	0	b.5 9	443221103	0	4.7 a	443321102	0	6.7 b
033221441	5	c.0 c	044332211	7	d.0 d						
421103324	7	9.4 c	432211034	8	4.6 d						
144331022	6	b.6 c	144332210	5	a.8 d						
321102443	8	6.4 c	321102344	6	1.4 d						
244332101	3	5.7 c	244332110	3	6.8 d						
210123443	4	0.2 c	210124433	4	5.2 d						
344322101	1	2.7 c	344322110	1	4.8 d						
101332442	2	7.1 c	101443322	2	a.1 d						
443322101	0	a.7 c	443322110	0	d.8 d						

Figure 6. Illustration for Subsection 5.2 and Example 11

Example 9. For $k = 3$, Figure 5 contains, on the right of each of the two cases of τ_j in Subsection 5.1, the symmetric difference of the corresponding flipping 6-cycle $F_6C(\tau_j)$ and the union of the three 7-cycles $C(\alpha_i)$, for each $i \in (0,1,2)$ or $i \in (0,3,4)$, yielding a 21-cycle in each case. The two 21-cycles are then recombined into a Hamilton cycle of O_3, shown on the lower part of Figure 5 as a list sectioned from left to right into five sublists. To the left of these five sublists, there is a drawing of an hypergraph as defined in Subsection 6.1 below.

5.2. Modified n-tuples

Each n-tuple $F(\alpha)$ gives place to a modified n-tuple $\underline{F}(\alpha)$ formed by the number entries $j \in [0,k]$ of $F(\alpha)$ set in the same positions they have in $F(\alpha)$ together with k underlined number entries $\underline{j}$ in place of the "=" signs, where $j \in [1,k]$ (or $\underline{j} \in \{\underline{1},\ldots,\underline{k}\}$), in a fashion determined by the fact that a nonempty Dyck word is expressible uniquely as a string $0u1v = 0^v_u u 1^v_u v$ (modified from $1u0v$ [4, p. 1260]), where u and v are (possibly empty) Dyck words. Each number entry $j \in [0,k]$ in $F(\alpha)$ corresponds to the starting entry 0^v_u of a Dyck word $0^v_u u 1^v_u v$ in $f(\alpha)$, with its 1^v_u represented in $F(\alpha)$ by an "=" sign. Its $\underline{F}(\alpha)$ has each number entry j ($\neq \underline{j}$) in its same position as in $F(\alpha)$, with a corresponding entry 0^v_u of a Dyck word $0^v_u u 1^v_u v$ in $f(\alpha)$.

Moreover, $\underline{F}(\alpha)$ has each "=" sign of $F(\alpha)$ replaced by a corresponding underlined integer $\underline{j}$ in the position of an accompanying 1^v_u. As an example, the right side of Figure 3 contains, under the list ℓ of Example 7 and the blue string containing $\pi(\alpha_1) = p^{-1}(\alpha_1)$, a red line repeating the first line $g(\alpha_1)$ of ℓ, and a subsequent red line with the 0-bits and 1-bits of $g(\alpha_1)$ replaced by the respective number entries j and $\underline{j}$ of $\underline{F}(\alpha_1)$.

For $k = 4$, Figure 6 contains vertical lists $\underline{L}(\alpha) = (\underline{L}_0(\alpha), \underline{L}_1(\alpha), \ldots, \underline{L}_{2k}(\alpha))^T$ similar to the lists $L(\alpha)$ but corresponding instead to the n-strings $\underline{F}(\alpha) = \underline{L}_0(\alpha)$, $\underline{L}_1(\alpha)$, ..., $\underline{L}_{2k}(\alpha)$, where α runs over the total of fourteen 4-germs and the only non-black entries are those corresponding to the 4-supplementary vertical blue-red pairs realizing the adjacency of each pair of contiguous lines, including the pair formed by the initial blue "4" in the last line $\underline{L}_8(\alpha)$ and the initial red 0 in the first line $\underline{L}_0(\alpha)$ in each list. All the first columns of the fourteen lists form the same column vector, with transpose row vector $(0,\underline{4},1,\underline{3},2,\underline{2},3,\underline{1},4)$.

The sole representative $f(\alpha)$ of a $\mathbb{Z}_n$-class of $V(O_k)$, as in Theorem 2, may not only be interpreted as the n-tuple $F(\alpha)$ but also as the corresponding $\underline{F}(\alpha)$, so the other n-tuples of that class may be interpreted as its translations mod n. The lines of each $L(\alpha)$ and the lines of its associated $\underline{L}(\alpha)$ are translations mod n of respective n-tuples $F(\alpha_\iota)$ and $\underline{F}(\alpha_\iota)$ that depend on the orders $\iota \in [0,2k]$ of such lines. These facts are used in the statement of Theorem 3, where the subindex j is $j = 2k - \iota$ in relation to the subindex ι.

6. Iterative generation of modified n-tuples

Theorem 3. *For each k-germ α:*

(i) *$L(\alpha)$ is generated by transforming iteratively for $j = 2k, 2k-1, \ldots, 2, 1$ and with initial n-tuple $L_0(\alpha) = F(\alpha)$ the n-tuple $L_{2k-j}(\alpha)$ into the uniquely feasible next n-tuple, $L_{2k-j+1}(\alpha)$, via k-supplementation of its $\pi(j)$-th entry and exchange of its k remaining number entries by its k "=" sign entries;*

(ii) *the first column of $\underline{L}(\alpha)$ has transpose row vector*

$$(0,\underline{k},1,\underline{k-1},2,k-2,\cdots,\underline{3},k-2,\underline{2},k-1,\underline{1},k),$$

obtained by alternating the entries of the vectors

$$(0,1,2,\ldots,k-1,k) \text{ and } (\underline{k},\underline{k-1},\ldots,\underline{2},\underline{1});$$

moreover, $k\underline{k}$ and $\underline{1}0$ are substrings mod *n of each $\underline{L}_j(\alpha)$.*

The resulting lists $L(\alpha)$ and $\underline{L}(\alpha)$, yield a uniform 2-factor of O_k formed by C_k n-cycles.

Proof. Item (i) is an adaptation of [4, Lemma 5] to the k-germ setting of Subsections 3.1-3.2 and 4.1-5.2 as well as the following argument.

The Dyck path of length $2k$ defined in Subsection 3.2 corresponds to the Dyck paths with $2k$ steps and 0 flaws of [3], presented in each list $L(\alpha)$ as $L_0(\alpha)$.

In the same way, $L_2(\alpha), L_4(\alpha), \ldots, L_{2k}(\alpha)$ correspond respectively to the Dyck paths with $2k$ steps and $1, 2, \ldots, k$ flaws of [3], obtained in our cases again as in Subsection 4.1 by the removal of its first up-step and change of coordinates from $(1,1)$ to $(0,0)$.

In fact, passing from each $L_{2i}(\alpha)$ to $L_{2i+1}(\alpha)$ corresponds to applying the function g defined in the second paragraph of [3, Subsection 1.1].

Passing from $L_{2i+1}(\alpha)$ to $L_{2k+2}(\alpha)$ corresponds to applying the function h composing the mapping $f = h \circ g$ of Theorem 2 [3].

For item (ii), note that the n-tuples $\underline{L}_j(\alpha)$ having a common initial entry in $[0,k] \cup [\underline{1}, \underline{k}]$ are at the same height j in all vertical lists $\underline{L}(\alpha)$ so that the entries of the first column $(a_0, \underline{b}_0, a_1, \underline{b}_1, \ldots, a_k, \underline{b}_k, a_{k+1})^T$ of each such $\underline{L}(\alpha)$ satisfy both $a_i + b_i = k$ and $b_i + a_{i+1} = k+1$, for $i \in [0,k]$.

Thus, the alternating first-entry column in each vertical list characterizes and controls the formation of the claimed uniform 2-factor. □

6.1. Dyck-word collections

Consider the following Dyck-word collections (triples, quadruple, etc.):

$$\begin{array}{lllllllll}
S_1(w) & = & \{\xi^1_{1(w)} = 0w00\underline{1}1, & \xi^2_{1(w)} = 0w\underline{0}1101, & \xi^3_{1(w)} = 0w0101\underline{1} & & & \}, \\
S_2 & = & \{\xi^1_2 = 0\underline{0}110011, & \xi^2_2 = 00100111, & \xi^3_2 = 00010\underline{1}11 & & & \}, \\
S_3 & = & \{\xi^1_3 = 000111, & \xi^2_3 = 010011, & \xi^3_3 = \underline{0}10101 & & & \}, \\
S_4 & = & \{\xi^1_4 = 000111, & \xi^2_4 = 00101\underline{1}, & \xi^3_4 = 01\underline{0}011, & \xi^4_4 = \underline{0}10101 & & \},
\end{array} \tag{1}$$

(based on [4, display (4.2)]) where w is any (possibly empty) Dyck word. Consider also the sets $\underline{S}_1(w)$, $\underline{S}_2$, $\underline{S}_3$, $\underline{S}_4$ obtained respectively from $S_1(w)$, S_2, S_3, S_4 by having their component Dyck paths $\underline{\xi}^j_{1(w)}$, $\underline{\xi}^j_2$, $\underline{\xi}^j_3$, $\underline{\xi}^j_4$ defined as the complements of the reversed strings of the corresponding Dyck paths $\xi^j_{1(w)}$, ξ^j_2, ξ^j_3, ξ^j_4. Note that each Dyck word in the subsets of display (1) has an underlined entry. By denoting

$$\xi^j_{i(w)} = x_s x_{s-1} \cdots x_2 x_1 x_0 \text{ and } \xi^j_i = x_s x_{s-1} \cdots x_2 x_1 x_0, \text{ for } i = 2,3,4, \tag{2}$$

where $j = 1,2,3$ for $i = 1,2,3$ and $j = 1,2,3,4$ for $j = 4$ and adequate s in each case, the underlined positions in (1) are the targets of the following correspondence Φ:

$$\begin{array}{llllllll}
\Phi(\xi^1_{1(w)}) = 1, & \Phi(\xi^2_{1(w)}) = 4, & \Phi(\xi^3_{1(w)}) = 0, & \\
\Phi(\xi^1_2) = 6, & \Phi(\xi^2_2) = 0, & \Phi(\xi^3_2) = 2, & \\
\Phi(\xi^1_3) = 0, & \Phi(\xi^1_3) = 1, & \Phi(\xi^3_3) = 5, & \\
\Phi(\xi^1_4) = 0, & \Phi(\xi^2_4) = 1, & \Phi(\xi^3_4) = 3, & \Phi(\xi^4_4) = 5.
\end{array} \tag{3}$$

The correspondence Φ is extended over the Dyck words $\underline{\xi}^j_{1(w)}$, $\underline{\xi}^j_2$, $\underline{\xi}^j_3$, $\underline{\xi}^j_4$ with their barred positions taken reversed with respect to the corresponding barred positions in $\xi^j_{1(w)}$, ξ^j_2, ξ^j_3, ξ^j_4, respectively.

Recall the ordered tree $\mathcal{T}_k$ from Theorem 1. Adapting from [4], we define an hypergraph H_k with $V(H_k) = V(\mathcal{T}_k)$ and having as hyperedges the subsets $\{\alpha^j; j \in \{1,2,3\}\} \subset V(H_k)$ and $\{\alpha^j; j \in \{1,2,3,4\}\} \subset V(H_k)$ whose member k-germs α^j have associated bitstrings $f(\alpha^j)$, for $j = 1,2,3$ or $j = 1,2,3,4$, containing respective Dyck words in $\{\xi^j_{1(w)}, \xi^j_2, \xi^j_3, \xi^j_4, \underline{\xi}^j_{1(w)}, \underline{\xi}^j_2, \underline{\xi}^j_3, \underline{\xi}^j_4\}$ in the same 6 or 8 fixed positions x_i (for

specific indices $i \in \{0,1,\ldots,s\}$ in (2)) and forming respective subsets $\{\xi^j_{1(w)}(w); j = 1,2,3\}$, $\{\underline{\xi}^j_{1(w)}; j = 1,2,3\}$, $\{\xi^j_4; j = 1,2,3,4\}$, $\{\underline{\xi}^j_4; j = 1,2,3,4\}$, $\{\xi^j_i; j = 1,2,3\}$ and $\{\underline{\xi}^j_i; j = 1,2,3\}$, for both $i = 2$ and 3.

Example 10. Two hyperedges h_0, h_1 of H_3 are shown heading on the upper-left and upper-right sides of Figure 5, respectively, with strings $\xi^j_{1(\epsilon)}$ or ξ^j_i ($j = 3,4$) having their constituent entries in red except for one barred entry, in blue. For h_0 (resp., h_1), $f(\alpha_0)$, $f(\alpha_1)$, $f(\alpha_2)$, (resp., $f(\alpha_0)$, $f(\alpha_3)$, $f(\alpha_4)$), represented by the respective subindices 0, 1, 2 (resp., 0, 3, 4), are shown stacked in the upper-left (-right) of the figure, those subindices indicating (each via a colon) respectively the Dyck words $\xi^1_{1(\epsilon)}$, $\underline{\xi}^2_3$, $\xi^3_{1(\epsilon)}$ (resp., $\xi^3_{1(\epsilon)}$, ξ^2_3, ξ^3_4), with their entries in red except for the entries in positions $\Phi(\xi^1_{1(\epsilon)}) = 1$, $\Phi(\underline{\xi}^2_3) = 4$, $\Phi(\xi^3_{1(\epsilon)}) = 0$ (resp., $\Phi(\xi^3_{1(\epsilon)}) = 0$, $\Phi(\xi^2_3) = 1$, $\Phi(\xi^3_4) = 5$), which are blue. Then H_3 contains the connected subhypergraph H'_3 depicted on the lower-left of the figure. This is used to construct the shown Hamilton cycle. The hyperedges of H'_3 are denoted by the triples of subindices i of their composing 4-germs α_i. So, the hyperedges of H'_3 are taken to be $h_0 = (0,1,2)$ and $h_1 = (0,3,4)$. This type of notation is used in Example 11, as well.

Example 11. For $k = 4$, let us represent each k-germ α_i by its respective order $i = ord(\alpha_i)$. In a likewise manner to that of Example 10. Figure 7 shows on its lower-left corner a depiction of a subhypergraph H'_4 of H_4 with the hyperedges

$$h_0 = (0,2,a),\ h_1 = (8,7,5),\ h_2 = (7,6,a),\ h_3 = (1,4,6),\ h_4 = (1,9,d),\ h_5 = (3,b,c,d).$$

The respective triples of Dyck words $\xi^j_{1(w)}$ or $\underline{\xi}^j_{1(w)}$ or ξ^j_i or $\underline{\xi}^j_i$ ($j = 2,3,4$) may be expressed as follows by replacing the Greek letters ξ by the values of the correspondence Φ:

$$(\underline{5}^1_3, \underline{4}^2_3, \underline{0}^3_3),\ (6^1_2, 0^2_2, 2^3_2),\ (1^1_{1(01)}, 4^2_{1(01)}, 0^3_{1(01)}),\ (1^1_{1(\epsilon)}, 4^2_{1(\epsilon)}, 0^3_{1(\epsilon)}),\ (0^1_3, 1^2_3, 5^3_3),\ (0^1_4, 1^2_4, 3^3_4, 5^4_4),$$

where we can also write $(\underline{5}^1_3, \underline{4}^2_3, \underline{0}^3_3) = \underline{(0^1_3, 1^2_3, 5^3_3)}$. From top to bottom in Figure 7, excluding the said depiction of H'_4, the vertical lists corresponding to the composing 4-germs of those six hyperedges are presented side by side, in a fashion similar to that of Figure 5, except that the first line in each such vertical list has its corresponding substring ξ (a member of one of the sets presented in display (3)) in red but for its blue entry $\Phi(\xi)$ to stress their roles in the respective $L(\alpha)$ and $FT(\alpha, j)$. The flippable tuples $FT(\alpha, j)$ allow to compose five flipping 6-cycles and one flipping 8-cycle, presented to the right of each triple or quadruple of vertical lists, allowing to integrate, by symmetric differences, a Hamilton cycle comprising all the vertices in the cycles provided by the vertical lists. Below those 6- or 8-cycles, the corresponding red-blue substrings ξ^j_i appear separated by a hyphen in each case from the associated (multicolored) first lines.

We represent H_k as a simple graph $\psi(H_k)$ with $V(\psi(H_k)) = V(H_k)$ by replacing each hyperedge e of H_k by the clique $K(e) = K(V(e))$ so that $\psi(H_k[e]) = K(e)$, being such replacements the only source of cliques of $\psi(H_k)$. A *tree* T of H_k is a subhypergraph of H_k such that: **(a)** $\psi(T)$ is a connected union of cliques $K(V(e))$; **(b)** for each cycle C of $\psi(H_k)$, there exist a unique clique $K(V(e))$ such that C is a subgraph of $K(e)$. A *spanning tree* T of H_K is a tree of H_k with $V(T) = V(H_k)$. Clearly, the subhypergraphs H'_k of H_k depicted in Figure 5 and 7 for $k = 3$ and 4 are corresponding spanning trees.

A subset G of hyperedges of H_k is said to be *conflict-free* [4] if: **(a)** any two hyperedges of G have at most one vertex in common; **(b)** for any two hyperedges g, g' of G with a vertex in common, the corresponding images by Φ (as in display (3)) in g and g' are distinct. A proof of the following final result is included, as our viewpoint and notation differs from that of [4].

01234====	1	0.0 0	0134==2==	1	2.0 2	014=3=2==	1	a.0 a	3=2==014=	2	a.5 0
=3===0124	7	5.5 0	=3==02=14	5	b.4 2	=3=2=1=04	3	d.7 a	=0234===1	8	1.1 0
1=034==2=	3	4.2 0	1=03=24==	3	9.2 2	1=024=3==	2	6.2 a	3==1=024=	6	6.5 a
=2=1=04=3	5	d.5 0	=2=14==03	4	c.7 2	=24==1=03	5	9.7 a	=14=3==02	7	b.7 a
2=1=04=3=	4	d.4 0	2=1=034==	2	4.4 2	2==0134==	4	2.3 a	1=03=24==	3	9.2 2
=1=04=3=2	6	d.3 0	=134===02	7	3.7 2	=14=3==02	7	b.7 a	=2=14==03	4	c.7 2
3=2==014=	2	a.5 0	3===0124=	6	5.4 2	3==1=024=	6	6.5 a	000111-1234====		0
=0234===1	8	1.1 0	=034==2=1	8	4.1 2	=04=3=2=1	8	d.1 a	001101-134==2==		2
4====0123	0	0.5 0	4===02=13	0	3.4 2	4==2=1=03	0	4.7 a	010101-14=3=2==		a

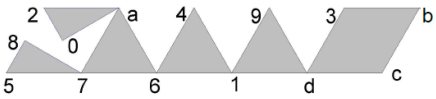
024==13==	5	8.0 8	013=24===	1	7.0 7	0124=3===	1	5.0 5	3=24===01	2	7.7 8
===013=24	7	7.3 8	=3=2==014	3	a.6 7	=3==1=024	7	6.6 5	=02=134==	4	3.1 8
134===02=	6	3.6 8	1=0234===	2	1.2 7	1=04=3=2=	5	d.2 5	4==0==013	0	2.6 7
===0124=3	8	5.3 8	=24===013	7	7.6 7	=2==014=3	6	a.4 5	=03=24==1	8	c.1 7
234====01	1	0.7 8	2==014=3=	5	a.3 7	2=14==03=	3	c.6 5	2=14==03=	3	c.6 5
=1=0234==	3	1.3 8	=14==03=2	6	c.4 7	=1=034==2	4	4.3 5	=2==014=3	6	a.4 5
3=24===01	2	7.7 8	3==02=14=	4	b.3 7	3=2=1=04=	2	d.6 5	00110011-24==13==		8
=02=134==	4	3.1 8	=03=24==1	8	c.1 7	=024=3==1	8	6.1 5	00100111-13=24===		7
4==13==02	0	8.7 8	4==2==013	0	2.6 7	4===1=023	0	1.5 5	00010111-124=3===		5

013=24===	1	7.0 7	024=3==1=	7	6.0 6	014=3=2==	1	a.0 a	=24===013	7	7.6 7
=3=2==014	3	a.6 7	===1=0234	8	1.5 6	=3=2=1=04	3	d.7 a	2==014=3=	5	a.3 7
1=0234===	2	1.2 7	124=3===0	1	5.8 6	1=024=3==	2	6.2 a	=04=3=2=1	8	d.1 a
=24===013	7	7.6 7	=2=1=034=	3	4.5 6	=24==1=03	5	9.7 a	4==2=1=03	0	4.7 a
2==014=3=	5	a.3 7	2=134===0	2	3.8 6	2==0134==	4	2.3 a	=03=2=14=	6	c.1 6
=14==03=2	6	c.4 7	=13==024=	5	8.5 6	=14=3==02	7	b.7 a	3==024==1	4	8.4 6
3==02=14=	4	b.3 7	3==024==1	4	8.4 6	3==1=024=	6	6.5 a	00100111-13=24===		7
=03=24==1	8	c.1 7	=03=2=14=	6	c.1 6	=04=3=2=1	8	d.1 a	00101101-24=3==1=		6
4==2==013	0	2.6 7	4==1=03=2	0	9.5 6	4==2=1=03	0	4.7 a	00101011-14=3=2==		a

0234===1=	7	1.0 1	034==2=1=	7	4.0 4	024=3==1=	7	6.0 6	=2==0134=	5	2.4 1
====01234	8	0.4 1	===02=134	8	3.3 4	===1=0234	8	1.5 6	2=14=3==0	3	b.8 1
1234====0	1	0.8 1	134==2==0	5	2.8 4	124=3===0	1	5.8 6	=03=2=14=	6	c.1 6
=2==0134=	5	2.4 1	===01234=	6	0.3 4	=2=1=034=	3	4.5 6	4==1=03=2	0	9.5 6
2=14=3==0	3	b.8 1	234===1=0	1	1.8 4	2=134===0	2	3.8 6	=02=14=3=	4	b.1 4
=1=03=24=	4	9.3 1	=1=024=3=	3	6.3 4	=13==024=	5	8.5 6	3=24==1=0	2	9.8 4
3=2=14==0	2	c.8 1	3=24==1=0	2	9.8 4	3==024==1	4	8.4 6	000111-234===1=		1
=024==13=	6	8.1 1	=02=14=3=	4	b.1 4	=03=2=14=	6	c.1 6	001101-34==2=1=		4
4===013=2	0	7.4 1	4==03=2=1	0	c.3 4	4==1=03=2	0	9.5 6	001011-24=3==1=		6

0234===1=	7	1.0 1	03=24==1=	7	9.0 9	04=3=2=1=	7	d.0 d	=024==13=	6	8.1 1
====01234	8	0.4 1	==2==0134	8	2.5 9	==2=1=034	8	4.6 d	4===013=2	0	7.4 1
1234====0	1	0.8 1	13=24===0	3	7.8 9	14=3=2==0	5	a.8 d	=014=3=2=	2	a.1 d
=2==0134=	5	2.4 1	==02=134=	5	3.2 9	==1=0234=	6	1.4 d	34==2=1=0	1	4.8 d
2=14=3==0	3	b.8 1	24==13==0	4	8.8 9	24=3==1=0	3	6.8 d	==02=134=	5	3.2 9
=1=03=24=	4	9.3 1	==013=24=	6	7.2 9	==0124=3=	4	5.2 d	24==13==0	4	8.8 9
3=2=14==0	2	c.8 1	34===02=1	1	3.5 9	34==2=1=0	1	4.8 d	000111-234===1=		1
=024==13=	6	8.1 1	=0134==2=	2	2.1 9	=014=3=2=	2	a.1 d	010011-3=24==1=		9
4===013=2	0	7.4 1	4=3==02=1	0	b.5 9	4=3=2=1=0	0	d.8 d	010101-4=3=2=1=		d

02=134===	3	3.0 3	02=14=3==	3	b.0 b	03=2=14==	5	c.0 c	03=2=14==	5	c.0 c
==13==024	7	8.6 3	==03=2=14	5	c.2 b	==1=03=24	7	9.4 c	==1=03=24	7	9.4 c
14==03=2=	5	c.7 3	13==024==	4	8.4 b	14=3==02=	6	b.6 c	24=3==1=0	3	6.8 d
==02=14=3	6	c.2 3	==024==13	7	8.5 b	==1=024=3	8	6.4 c	==0124=3=	4	5.2 d
24==1=03=	4	9.6 3	24==013=	6	7.5 b	24=3===01	3	5.7 c	34===012	1	0.6 3
==0134==2	8	2.2 3	==014=3=2	8	a.2 b	==01234==	4	0.2 c	==0134==2	8	2.2 3
34===012	1	0.6 3	34===1=02	1	1.7 b	34==2==01	1	2.7 c	24==013=	6	7.5 b
=01234===	2	0.1 3	=0124=3==	2	5.1 b	=013=24==	2	7.1 c	==024==13	7	8.5 b
4=3===012	0	5.6 3	4=3==1=02	0	6.7 b	4=3=2==01	0	a.7 c	000111-2=134===		3
									001011-2=14=3==		b
									010011-3=2=14==		c
									010101-4=3=2=1=		d



04=3=2=1=	7	d.0 d
==2=1=034	8	4.6 d
14=3=2==0	5	a.8 d
==1=0234=	6	1.4 d
24=3==1=0	3	6.8 d
==0124=3=	4	5.2 d
34==2=1=0	1	4.8 d
=014=3=2=	2	a.1 d
4=3=2=1=0	0	d.8 d

Figure 7. Illustration for Section 5 and Example 11

Theorem 4. *([4]) A conflict-free spanning tree of H_k yields a Hamilton cycle of O_k, for every $k \geq 3$. Moreover, distinct conflict-free spanning trees of H_k yield distinct Hamilton cycles of H_k, for every $k \geq 6$.*

Proof. Let D_k be the set of all Dyck words of length $2k$ and, recalling display (1), let

$E_2 = \{0101\}$	$E_3 = S_4$	$E_k = 01D_{k-1}, \forall k > 3$
$F_2 = \{0011\}$	$F_3 = D_3 \setminus E_3 = \{001101\}$	$F_k = D_k \setminus 01D_{k-1}, \forall k > 3$

(4)

In particular, $0101(01)^{k-2} \in E_k$ and $0011(01)^{k-2} \in F_k$. Now, let

$\mathcal{E}_2 = \varnothing$	$\mathcal{E}_3 = \{S_4\}$	$\mathcal{T}_3 = \{S_1(\epsilon), S_3\}$	$\mathcal{E}_k = 01\mathcal{T}_{k-1}, \forall k > 3$
$\mathcal{F}_2 = \varnothing$	$\mathcal{F}_3 = \varnothing$	$\mathcal{F}_4 = \{S_1(01), S_2, 0\underline{S_3}1, S_1(\epsilon)01\}$	

(5)

Let us set $\mathcal{F}_k$ as a function of $\mathcal{E}_2, \ldots, \mathcal{E}_{k-1}, \mathcal{F}_2, \ldots, \mathcal{F}_{k-1}, \mathcal{T}_{k-2}$, as follows: For $1 < j \leq k$, let $F_k^j = \cup_{i=2}^{j}\{0\underline{u}1v; u \in D_{i-1}, v \in D_{k-1}\}$. Since $F_k = F_k^k$, then the following implies the existence of a spanning tree of $H_k[F_k]$.

Lemma 5. *For every $1 < j \leq k$, there exists a spanning tree $\mathcal{F}_k^j$ of $H_k[F_k^j]$.*

Proof. Lemma 7 [4] asserts that if τ is a flippable tuple and u, v are Dyck words, then: **(i)** $u\tau v$ is a flippable tuple if $|u|$ is even; **(ii)** $u\underline{\tau}v$ is a flippable tuple if $|u|$ is odd. Lemma 8 [4] insures that the collections in (1) are flippable tuples. Using those two lemmas of [4], we define Ψ as the set of all the flippable tuples $u\tau v$ and $u\underline{\tau}v$ arising from (1). Moreover, we define $\Psi_2 = \varnothing$ and $\Psi_k = \Psi \cap D_k$, for $k > 2$.

Since $F_k^2 = 0011D_{k-2}$, we let $\mathcal{F}_k^2 = 0011\mathcal{T}_{k-2}$. Assuming $2 < j \leq k$, since $D_{j-2} = E_{j-1} \cup F_{j-1}$ is a disjoint union, then we have the following partition:

$$F_k^j = F_k^{j-1} \cup_{v \in D_{k-j}} (0\underline{D}_{j-1}1v) = F_k^{j-1} \cup_{v \in D_{k-j}} ((0\underline{E}_{j-1}1v) \cup (0\underline{F}_{j-1}1v)). \tag{6}$$

For every $v \in D_{k-j}$, the elements of $\tau(v) = S_1((01)^{j-3})v \in \Psi_k$ are:

$0(01)^{j-3}001\underline{1}1v \in 0\underline{F}_{j-1}1v$	$0(01)^{j-3}0101\underline{1}v \in 0\underline{E}_{j-1}1v$	$0(01)^{j-3}\underline{0}1101v \in F_k^{j-1}$

(7)

Now, we let

$$\mathcal{F}_k^j = \mathcal{F}_k^{j-1} \cup (\cup_{v \in D_{k-j}}(\{\tau(v)\} \cup (0\underline{\mathcal{E}}_{j-1}1v) \cup (0\underline{\mathcal{F}}_{j-1}1v))), \tag{8}$$

which defines a spanning tree of $H_k[F_k^j]$. □

Now, the elements of $\tau = S_3(01)^{k-3} \in \Psi_k$ are:

$000111\underline{1}(01)^{k-3} \in F_k, (k > 3)$	$0100\underline{1}1(01)^{k-3} \in 01E_{k-1}$	$\underline{0}10101(01)^{k-3} \in 01F_{k-1}$

(9)

The sets F_k, $01E_{k-1}$ and $01F_{k-1}$ form a partition of D_k. We take the spanning trees of the subhypergraphs induced by these three sets and connect them into a single spanning tree of H_k by means of the triple τ, that is:

$$H_k' = \mathcal{F}_k \cup \{\tau\} \cup 01\mathcal{E}_{k-1} \cup 01\mathcal{F}_{k-1}. \tag{10}$$

□

Example 12. Example 10 uses $\mathcal{T}_3$ in display (5), with $S_1(\epsilon) = 012$ and $S_3 = 034$ yielding the hypergraph $\mathcal{T}_3$ depicted in the lower left of Figure 5. Example 11 uses H'_k in display (10) for $k = 4$, $\mathcal{F}_4$ and $\mathcal{E}_3$ in display (5) and τ in display (9), with $S_1(01) = 67a$, $S_2 = 875$, $0\underline{S}_3 1 = 02a$, $S_1(\epsilon) = 146$, being these four triples the elements in $\mathcal{F}_4$; $01S_4 = 3bcd$, this one as the only element of $01\mathcal{E}_3$, (while $\mathcal{F}_3 = \varnothing$); and $\tau = 02a$, yielding the hypergraph H'_4 depicted at the lower left corner of Figure 7.

Corollary 6. *To each Hamilton cycle in O_k produced by Theorem 4 corresponds a Hamilton cycle in M_k.*

Proof. For each vertical list $L(\alpha)$ provided by Theorem 3, let $L^M(\alpha)$ be a vertical list as exemplified in Example 8 and Figure 4, which is obtained from $L(\alpha)$ by replacing its "=" signs by: **(a)** ">" signs (meaning left-to-right string-reading) for the strings $L_{2j}(\alpha)$ ($j \in [0,k]$) of $L(\alpha)$ and **(b)** "<" signs (meaning right-to-left string-reading) for the strings $L_{2j+1}(\alpha)$ ($j \in [0,k-1]$) of $L(\alpha)$. Then, Theorem 4 can be adapted to producing Hamilton cycles in the M_k by repeating the argument in its proof in replacing the lists $L(\alpha)$ by lists $L^M(\alpha)$, since they have locally similar behavior, being the cycles provided by the lists $L^M(\alpha)$ twice as long as the corresponding lists $L(\alpha)$, so the said local behavior happens twice around opposite (rather short) subpaths. Combining Dyck-word triples and quadruples as in display (1) into adequate pullback liftings (of the covering graph map $M_k \to O_k$ associated to item (ii), Section 1) in the lists $L^M(\alpha)$ of those parts of the lists $L(\alpha)$ in which the necessary symmetric differences take place to produce the Hamilton cycles in O_k will produce corresponding Hamilton cycles in M_k. □

Historical Note. The k-edge ordered trees appearing in [12, p. 221, item (e)] as "plane trees with" $k+1$ vertices and in [9] as "ordered rooted trees", represent Dyck paths of length $2k$ (see Subsection 3.2). These trees are equivalent to k-strings $0b_{k-1}\cdots b_1$ called *k-RGS*'s in [6] and tailored from the RGS's of Section 2 via items (r) and (u) in [12, p. 222] in a different way from that of the k-germs of Section 2. An equivalence of these k-germs and those k-RGS's was presented in [6] via their distinct relation to the k-edge ordered trees, whose purpose in [9,10] was using their plane rotations toward Hamilton cycles in M_k, not related to the odd-graph approach to Hamilton cycles of [4] to which we applied our ideas in Section 5.

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