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# Chromatically unique 6-bridge graph $\theta(r, r, s, s, t, u)$

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**Abstract:** Let  $A$  and  $B$  be two graph and  $P(A, z)$  and  $P(B, z)$  are their chromatic polynomial, respectively. The two graphs  $A$  and  $B$  are said to be *chromatic equivalent* denoted by  $A \sim B$  if  $P(A, z) = P(B, z)$ . A graph  $A$  is said to be *chromatically unique*(or simply  $\chi$ - unique) if for any graph  $B$  such that  $A \sim B$ , we have  $A \cong B$ , that is  $A$  is isomorphic to  $B$ . In this paper, the chromatic uniqueness of a new family of 6-bridge graph  $\theta(r, r, s, s, t, u)$  where  $2 \leq r \leq s \leq t \leq u$  is investigated.

**Keywords:** Chromatic polynomial, Chromatically unique, multi-bridge graph.

**MSC:** 05C15.

## 1. Introduction

Let  $A$  be a finite undirected graph with a vertex set  $V(A)$  and an edge set  $E(A)$ . A function  $f : V(A) \rightarrow \{1, \dots, k\}$  is called a proper coloring if for any two adjacent vertices  $x$  and  $y$  (i.e.,  $xy \in E(A)$ ), it holds that  $f(x) \neq f(y)$ . The chromatic polynomial of the graph  $A$ , denoted by  $P(A, z)$ , is defined as the number of all proper colorings of  $A$ . Consider two graphs  $A$  and  $B$  with their respective chromatic polynomials  $P(A, z)$  and  $P(B, z)$ . These graphs are said to be *chromatically equivalent*, denoted by  $A \sim B$ , if  $P(A, z) = P(B, z)$ . A graph  $A$  is termed a *chromatically unique* graph if no other graph shares the same chromatic polynomial as  $A$ .

For each integer  $k \geq 2$ , let  $\theta_k$  denote the multi-graph with two vertices and  $k$  edges. Any subdivision of  $\theta_k$  is referred to as a multi-bridge graph or a  $k$ -bridge graph, denoted by  $\theta(y_1, y_2, y_3, \dots, y_k)$ , where  $y_1, y_2, \dots, y_k \in \mathbb{N}$  and  $y_1 \leq y_2 \leq \dots \leq y_k$ . The graph  $\theta(y_1, y_2, y_3, \dots, y_k)$  is obtained by replacing the edges of  $\theta_k$  with paths of lengths  $y_1, y_2, y_3, \dots, y_k$ , respectively. Consequently, the graph  $\theta(y_1, y_2, y_3, \dots, y_k)$  possesses  $y_1 + y_2 + \dots + y_k - k + 2$  vertices and  $y_1 + y_2 + \dots + y_k$  edges.

## 2. Chromaticity Of $k$ -bridge graphs

The chromaticity of  $k$ -bridge graphs has been extensively studied by numerous researchers. A 2-bridge graph, which is essentially a cycle graph, is known to be  $\chi$ -unique. The theta graph, a type of 3-bridge graph, is denoted by  $\theta(1, y_1, y_2)$ . Chao and Whitehead [1] established that every theta graph is  $\chi$ -unique. Extending their work, Loerinc [2] demonstrated that all 3-bridge graphs are  $\chi$ -unique. The chromaticity of 4-bridge graphs was successfully addressed by Chen et al. [3] and Xu et al. [4]. Research on the chromaticity of 5-bridge graphs has been conducted by several scholars, as cited in [5–8].

**Theorem 1.** (Xu et al. [3]) For  $k \geq 2$ , the graph  $\theta_k(h)$  is  $\chi$  unique.

**Theorem 2.** (Dong et al. [9]) If  $2 \leq y_1 \leq y_2 \leq \dots \leq y_k < y_1 + y_2$  where  $k \geq 3$ , then the graph  $\theta(y_1, y_2, \dots, y_k)$  is  $\chi$ -unique.

For any graph  $A$  and real number  $z$ , write

$$Q(A, z) = (-1)^{1+|E(A)|} (1-z)^{|V(A)|+|E(A)|+1} P(A, 1-z).$$

**Theorem 3.** (Dong et al. [9]) For any  $k, y_1, y_2, \dots, y_k \in N$ ,

$$Q(\theta(y_1, y_2, \dots, y_k), z) = z \prod_{i=1}^k (z^{y_i} - 1) - \prod_{i=1}^k (z^{y_i} - z) \quad (1)$$

**Theorem 4.** (Dong et al. [9]) For any graph  $A$  and  $B$ ,

1. If  $B \sim A$ , then  $Q(B, z) = Q(A, z)$ .
2. If  $Q(B, z) = Q(A, z)$  and  $v(B) = v(A)$ , then  $B \sim A$ .

**Theorem 5.** (Dong et al. [9]) Suppose that  $\theta(y_1, y_2, \dots, y_k) \sim \theta(x_1, x_2, \dots, x_k)$ , where  $k \geq 3$ ,  $2 \leq y_1 \leq y_2 \leq \dots \leq y_k$  and  $2 \leq x_1 \leq x_2 \leq \dots \leq x_k$ , then  $y_i = x_i$  for all  $i = 1, 2, 3, \dots, k$ .

**Theorem 6.** (Dong et al. [9]) Let  $B \sim \theta(y_1, y_2, \dots, y_k)$ , where  $k \geq 3$  and  $y_i \geq 2$  for all  $i$ , then one of them is true:

1.  $B \cong \theta(y_1, y_2, \dots, y_k)$
2.  $B \in g_e(\theta(x_1, x_2, \dots, x_k) C_{x_{i+1}}, \dots, C_{x_{k+1}})$ , where  $3 \leq t \leq k-1$  and  $x_i \geq 2$  for all  $i = 1, 2, 3, \dots, k$ .

**Theorem 7.** (Dong et al. [9]) Let  $k, t, x_1, x_2, \dots, x_k \in N$  where  $3 \leq t \leq k-1$  and  $x_i \geq 2$  for all  $i = 1, 2, 3, \dots, k$ . If  $B \in g_e(\theta(x_1, x_2, \dots, x_t), C_{x_{t+1}}, \dots, C_{x_{k+1}})$ , then

$$Q(B, z) = z \prod_{i=1}^k (z^{x_i} - 1) - \prod_{i=1}^t (z^{x_i} - z) \prod_{i=t+1}^k (z^{x_i} - 1). \quad (2)$$

**Theorem 8.** (Koh & Teo [10]) If  $A \sim B$ , then

1.  $v(A) = v(B)$ ,
2.  $e(A) = e(B)$ ,
3.  $g(A) = g(B)$ ,
4.  $A$  and  $B$  have the same number of shortest cycle.

where  $v(A)$ ,  $v(B)$ ,  $e(A)$ ,  $e(B)$ ,  $g(A)$  and  $g(B)$  denote the number of vertices, the number of edges and the girth of  $A$  and  $B$ , respectively.

The chromaticity on several families of 6-bridge graph has been done by several authors which are given below.

**Lemma 9.** [11] A 6-bridge graph  $\theta(y_1, y_2, \dots, y_6)$  is  $\chi$  unique if the positive integer  $y_1, y_2, \dots, y_6$  assume exactly two distinct values.

**Lemma 10.** [12] The graph 6-bridge  $\theta(3, 3, 3, s, s, t)$ , where  $r \leq s \leq t$ , is  $\chi$ -unique.

**Lemma 11.** [13] The graph 6-bridge  $\theta(r, r, r, s, s, t)$ , where  $r \leq s \leq t$ , is  $\chi$ -unique.

**Lemma 12.** [14] The graph 6-bridge  $\theta(3, 3, 3, s, t, u)$ , where  $3 \leq s \leq t$ , is  $\chi$ -unique.

**Lemma 13.** [15] The graph 6-bridge  $\theta(r, r, s, s, t, t)$ , where  $r \leq s \leq t$ , is  $\chi$ -unique.

**Lemma 14.** [16] The graph 6-bridge  $\theta(r, r, s, s, s, t)$ , where  $r \leq s \leq t$ , is  $\chi$ -unique.

**Lemma 15.** [18] The graph 6-bridge  $\theta(r, r, r, s, t, u)$ , where  $r \leq s \leq t \leq u$ , is  $\chi$ -unique.

In this paper, we have extended this study to a new family of 6-bridge graph  $\theta(r, r, s, s, t, u)$  where  $2 \leq r \leq s \leq t \leq u$  and showed that this family of 6-bridge graph is chromatically unique.

### 3. Discussion and Main Results

In this section we present our main result on the chromaticity of 6-bridge graph.

**Theorem 16.** The 6-bridge graph  $\theta(r, r, s, s, t, u)$  where  $r \leq s \leq t \leq u$  is chromatically unique.

**Proof.** Let  $A$  be the 6-bridge graph of the form  $\theta(r, r, s, s, t, u)$  and  $2 \leq r \leq s \leq t \leq u$ . By Theorem 2,  $A$  is  $\chi$  unique if  $u < 2r$ . Suppose  $r \geq 2$  and  $B \sim A$ . We shall solve  $Q(A) = Q(B)$  to get all the solutions. The lowest remaining power (l.r.p) means the lowest remaining power of  $z$  in the expression after simplification and highest remaining power (h.r.p) mean the maximum power of  $z$  in the expression after simplification. By Theorem 8,  $g(A) = g(B) = 2r$  and  $B$  has the same number of shortest cycles as  $A$ . Thus, we have

$$2r + 2s + t + u = x_1 + x_2 + x_3 + x_4 + x_5 + x_6. \quad (3)$$

By Theorem 6 and 7, there are three cases to consider, that are

$B \in g_e(\theta(x_1, x_2, x_3), C_{x_4+1}, C_{x_5+1}, C_{x_6+1})$ , where  $2 \leq x_1 \leq x_2 \leq x_3$  and  $2 \leq x_4, x_5, x_6$ , or

$B \in g_e(\theta(x_1, x_2, x_3, x_4), C_{x_5+1}, C_{x_6+1})$ , where  $2 \leq x_1 \leq x_2 \leq x_3 \leq x_4$  and  $2 \leq x_5, x_6$ , or

$B \in g_e(\theta(x_1, x_2, x_3, x_4, x_5), C_{x_6+1})$ , where  $2 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$  and  $2 \leq x_6$ .

**Case A**  $B \in g_e(\theta(x_1, x_2, x_3), C_{x_4+1}, C_{x_5+1}, C_{x_6+1})$ , where  $2 \leq x_1 \leq x_2 \leq x_3$  and  $2 \leq x_4, x_5, x_6$ .

As  $A \cong \theta(r, r, r, s, t, u)$  and  $B \in g_e(\theta(x_1, x_2, x_3), C_{x_4+1}, C_{x_5+1}, C_{x_6+1})$ , then by Theorem 7, we have

$$Q(A) = z(z^r - 1)^2(z^s - 1)^2(z^t - 1)(z^u - 1) - (z^r - z)^2(z^s - z)^2(z^t - z)(z^u - z).$$

$$Q(B) = z(z^{x_1} - 1)(z^{x_2} - 1)(z^{x_3} - 1)(z^{x_4} - 1)(z^{x_5} - 1)(z^{x_6} - 1) - (z^{x_1} - z)(z^{x_2} - z)(z^{x_3} - z)(z^{x_4} - z)(z^{x_5} - z)(z^{x_6} - z).$$

Let  $Q_1(A)$  is a new polynomial obtained by comparing  $Q(A) = Q(B)$ .

$$\begin{aligned} Q_1(A) = & z^{2r+2s+1} + z^{2r+t+u+1} + 2z^{2r+s+u+1} + 2z^{2r+s+t+1} + 2z^{2r+u+t+1} + z^{2r+1} + z^{2s+1} + z^{t+u+1} + 2z^{s+t+1} + 2z^{s+u+1} + \\ & 2z^{r+s+1} + 2z^{r+u+1} + 2z^{r+2s+t+1} + 2z^{r+2s+u+1} + 4z^{r+s+t+u+1} + 2z^{s+t+u+3} + 2z^{r+t+u+3} + 2z^{r+2s+t+u+1} + z^{2r+t+3} + z^{2s+t+3}z^{t+5} + \\ & 4z^{r+s+t+3} + z^{2r+u+3} + z^{2s+u+3}z^{u+5} + 4z^{r+s+u+3} + z^{r+s+3} + z^{r+2s+3} + 2z^{s+5} + 2z^{r+5} + 4z^{r+s+1} - (z^{2r+t+1} + z^{2r+s+1} + z^{2r+u+1} + \\ & z^{2s+t+1} + z^{2s+u+1} + z^{t+1} + 2z^{s+1} + 2z^{s+t+u+1} + 2z^{r+1} + z^{u+1} + 2z^{r+2s+t+u+1} + 2z^{r+t+u+1} + 4z^{r+s+t+1} + 4z^{r+s+u+1} + 2z^{r+2s+1} + \\ & z^{2r+2s+2} + z^{2r+t+u+2} + z^{2r+4} + 2z^{2r+s+t+2} + 2z^{2r+s+u+2} + 2z^{2r+t+u+2} + z^{2s+4} + z^{t+u+4} + 2z^{s+t+4} + 2z^{s+u+4} + 2z^{r+t+4} + \\ & 2z^{r+u+4} + 2z^{r+2s+t+2} + 2z^{r+2s+u+2} + 4z^{r+s+t+u+2} + 4z^{r+s+4} + z^6). \end{aligned}$$

$$\begin{aligned} Q_1(B) = & z^{x_1+x_2+x_3+x_4+x_5} + z^{x_1+x_2+x_3+x_4+x_6} + z^{x_1+x_2+x_3+x_4+1} + z^{x_1+x_2+x_3+x_5+x_6} + z^{x_1+x_2+x_3+x_5+1} + z^{x_1+x_2+x_3+x_6+1} + \\ & z^{x_1+x_2+x_3} + z^{x_1+x_4+x_5+x_6+1} + z^{x_1+x_4+x_5+2} + z^{x_1+x_4+x_6+2} + z^{x_1+x_5+x_6+2} + z^{x_1+x_4+1} + z^{x_1+x_5+1} + z^{x_1+x_6+1} + z^{x_1+2} + \\ & z^{x_2+x_4+x_5+x_6+1} + z^{x_2+x_4+x_5+2} + z^{x_2+x_4+x_6+2} + z^{x_2+x_4+1} + z^{x_2+x_5+x_6+2} + z^{x_2+x_5+x_5+1} + z^{x_2+x_6+1} + z^{x_2+2} + z^{x_3+x_4+x_5+x_6+1} + \\ & z^{x_3+x_4+x_5+2} + z^{x_3+x_4+x_6+2} + z^{x_3+x_4+1} + z^{x_3+x_5+x_6+2} + z^{x_3+x_5+1} + z^{x_3+x_6+1} + z^{x_3+2} + z^{x_4+x_5+x_6+3} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + \\ & z^{x_4+3} + z^{x_5+x_6+1} + z^{x_5+3} + z^{x_6+3} - (z^{x_1+x_2+x_3+x_4+x_5+1} + z^{x_1+x_2+x_3+x_4+x_6+1} + z^{x_1+x_2+x_3+x_4} + z^{x_1+x_2+x_3+x_5+x_6+1} + \\ & z^{x_1+x_2+x_3+x_5} + z^{x_1+x_2+x_3+x_6} + z^{x_1+x_2+x_3+1} + z^{x_1+x_4+x_5+x_6+2} + z^{x_1+x_4+x_5+1} + z^{x_1+x_4+x_6+1} + z^{x_1+x_5+x_6+1} + z^{x_1+x_4+2} + \\ & z^{x_1+x_5+2} + z^{x_1+x_6+2} + z^{x_1+1} + z^{x_2+x_4+x_5+x_6+2} + z^{x_2+x_4+x_5+1} + z^{x_2+x_4+x_6+1} + z^{x_2+x_4+2} + z^{x_2+x_5+x_6+2} + z^{x_2+x_5+2} + z^{x_2+x_6+2} + \\ & z^{x_2+1} + z^{x_3+x_4+x_5+x_6+2} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+x_6+1} + z^{x_3+x_5+x_6+1} + z^{x_3+x_5+x_5+2} + z^{x_3+x_6+2} + z^{x_3+1} + z^{x_4+x_5+x_6+1} + \\ & z^{x_4+x_5+3} + z^{x_4+x_6+3} + z^{x_4+1} + z^{x_5+x_6+3} + z^{x_5+1} + z^{x_6+1} + z^3). \end{aligned}$$

Compare the l.r.p in  $Q_1(A)$  and the l.r.p in  $Q_1(B)$ . Thus,  $r = 2$ . Therefore,  $g(A) = g(B) = 2r = 4$ . Since  $A$  has one cycle of length four, therefore  $B$  has one cycle of length 4. Without loss of generality, we have four cases to consider.

1.  $x_4 = x_5 = x_6 = 3$  or

2.  $x_4 = x_5 = 3, x_6 \neq 3$  or

3.  $x_4 = 3, x_5 \neq 3, x_6 \neq 3$  or

4.  $x_4 \neq 3, x_5 \neq 3, x_6 \neq 3$ .

**Case 1:**  $x_4 = x_5 = x_6 = 3$ .

Therefore,  $B$  has at least three cycles of length 4. While  $A$  has one cycle of the same length, by Theorem 8 a contradiction.

**Case 2:**  $x_4 = x_5 = 3, x_6 \neq 3$

Therefore,  $B$  has at least two cycles of length 4. While  $A$  has one cycle of the same length, by Theorem 8 this is a contradiction.

**Case 3:**  $x_4 = 3, x_5 \neq 3, x_6 \neq 3$

$$2s + t + u + 1 = x_1 + x_2 + x_3 + x_5 + x_6. \quad (4)$$

$$\begin{aligned} Q_2(A) = & 3z^{2s+5} + 3z^{t+u+5} + 6z^{s+u+5} + 6z^{s+t+5} + z^{t+u+1} + z^{2s+1} + 2z^{s+t+1} + 2z^{s+u+1} + 3z^{2s+u+3} + 3z^{2s+t+3} + 2z^{t+3} + \\ & 2z^{u+3} + 4z^{s+3} + 6z^{s+t+u+3} + z^{t+7} + z^{u+7} + 2z^7 + z^5 - (z^{2s+t+1} + z^{2s+u+1} + z^{t+1} + 2z^{s+1} + 2z^{s+t+u+1} + 2z^{t+u+3} + 4z^{s+t+3} + \end{aligned}$$

$$4z^{s+u+3} + z^{u+1} + 2z^{2s+3} + z^{2s+6} + z^{t+u+6} + 2z^{s+t+6} + 2z^{s+u+6} + z^{2s+4} + z^{t+u+4} + 2z^{s+t+4} + 2z^{s+u+4} + 2z^{2s+t+4} + 2z^{2s+u+4} + 2z^{t+6} + 2z^{u+6} + 4z^{s+6} + 4z^{s+t+u+3} + z^8 + z^6).$$

$$Q_2(B) = z^{x_1+x_2+x_3+x_5+3} + z^{x_1+x_2+x_3+x_6+3} + z^{x_1+x_2+x_3+4} + z^{x_1+x_2+x_3+x_5+x_6} + z^{x_1+x_2+x_3+x_5+1} + z^{x_1+x_2+x_3+x_6+1} + z^{x_1+x_2+x_3+x_6+5} + z^{x_1+x_5+x_6+4} + z^{x_1+x_5+x_5+5} + z^{x_1+x_6+5} + z^{x_1+x_5+x_6+2} + z^{x_1+4} + z^{x_1+x_5+1} + z^{x_1+x_6+1} + z^{x_1+2} + z^{x_2+x_5+x_6+4} + z^{x_2+x_5+5} + z^{x_2+x_6+5} + z^{x_2+4} + z^{x_2+x_5+x_6+2} + z^{x_2+x_5+1} + z^{x_2+x_6+1} + z^{x_2+2} + z^{x_3+x_5+x_6+4} + z^{x_3+x_5+5} + z^{x_3+x_6+5} + z^{x_3+4} + z^{x_3+x_5+x_6+2} + z^{x_3+x_5+1} + z^{x_3+x_6+1} + z^{x_3+2} + z^{x_5+x_6+5} + z^{x_5+4} + z^{x_6+4} + z^6 + z^{x_5+x_6+1} + z^{x_5+3} + z^{x_6+3} - (z^{x_1+x_2+x_3+x_5+4} + z^{x_1+x_2+x_3+x_6+4} + z^{x_1+x_2+x_3+x_5+3} + z^{x_1+x_2+x_3+x_6+1} + z^{x_1+x_2+x_3+x_5} + z^{x_1+x_2+x_3+x_6} + z^{x_1+x_2+x_3+1} + z^{x_1+x_5+x_6+5} + z^{x_1+x_5+4} + z^{x_1+x_6+4} + z^{x_1+x_5+x_6+1} + z^{x_1+5} + z^{x_1+x_5+2} + z^{x_1+x_6+2} + z^{x_1+1} + z^{x_2+x_5+x_6+5} + z^{x_2+x_5+4} + z^{x_2+x_6+4} + z^{x_2+5} + z^{x_2+x_5+x_6+2} + z^{x_2+x_5+2} + z^{x_2+x_6+2} + z^{x_2+1} + z^{x_3+x_5+x_6+5} + z^{x_3+x_5+4} + z^{x_3+x_6+4} + z^{x_3+5} + z^{x_3+x_5+x_6+1} + z^{x_3+x_5+x_6+2} + z^{x_3+x_6+2} + z^{x_3+1} + z^{x_5+x_6+4} + z^{x_5+6} + z^4 + z^{x_5+x_6+3} + z^{x_5+1} + z^{x_6+1} + z^3).$$

Compare the l.r.p in  $Q_2(A)$  and the l.r.p in  $Q_2(B)$ . We have  $x_1 = 2$  or  $x_2 = 2$  or  $x_3 = 2$

**Case 3.1:**  $x_1 = 2$ . Then  $2 \leq x_2 \leq x_3$ .

$$Q_3(B) = z^{x_2+x_3+x_5+5} + z^{x_2+x_3+x_6+5} + z^{x_2+x_3+6} + z^{x_2+x_3+x_5+3} + z^{x_2+x_3+x_6+3} + z^{x_2+x_3+2} + z^{x_5+x_6+6} + z^{x_5+7} + z^{x_6+7} + z^6 + z^{x_5+x_6+4} + z^{x_5+3} + z^{x_6+3} + z^{x_2+x_5+x_6+4} + z^{x_2+x_5+5} + z^{x_2+x_6+5} + z^{x_2+4} + z^{x_2+x_5+x_6+2} + z^{x_2+x_5+1} + z^{x_2+x_6+1} + z^{x_3+x_5+x_6+4} + z^{x_3+x_5+5} + z^{x_3+x_6+5} + z^{x_3+4} + z^{x_3+x_5+x_6+2} + z^{x_3+x_5+1} + z^{x_3+x_6+1} + z^{x_3+2} + z^{x_5+x_6+6} + z^{x_5+4} + z^{x_6+4} + z^6 + z^{x_5+x_6+1} + z^{x_5+3} + z^{x_6+3} - (z^{x_2+x_3+x_5+6} + z^{x_2+x_3+x_6+6} + z^{x_2+x_3+x_5+2} + z^{x_2+x_3+x_6+2} + z^{x_2+x_3+3} + z^{x_5+x_6+7} + z^{x_5+6} + z^{x_6+6} + z^7 + z^{x_5+x_6+3} + z^{x_5+4} + z^{x_6+4} + z^{x_2+x_5+x_6+5} + z^{x_2+x_5+4} + z^{x_2+x_6+4} + z^{x_2+5} + z^{x_2+x_5+x_6+1} + z^{x_2+x_5+2} + z^{x_2+x_6+2} + z^{x_2+1} + z^{x_3+x_5+x_6+5} + z^{x_3+x_5+4} + z^{x_3+x_6+4} + z^{x_3+5} + z^{x_3+x_5+x_6+1} + z^{x_3+x_5+x_6+2} + z^{x_3+x_6+2} + z^{x_3+1} + z^{x_5+x_6+4} + z^{x_5+6} + z^{x_5+x_6+3} + z^{x_5+1} + z^{x_6+1}).$$

Consider the l.r.p in  $Q_2(A)$  and the l.r.p in  $Q_3(B)$ , we have  $s = 4$  or  $t = 4$  or  $u = 4$ .

**Case 3.1.1:**

$$s = 4.$$

Since the coefficient of  $-z^{s+1}$  in  $Q_2(A)$  is 2, then there shall have one  $-x^5$  in  $Q_3(B)$ . Hence, we have to consider for  $x_2 = 4$  or  $x_3 = 4$  or  $x_5 = 4$  or  $x_6 = 4$ .

**Case 3.1.1.1:**  $x_2 = 4$ .

$$Q_3(A) = 5z^{t+9} + 2z^{t+5} + 3z^{t+11} + 2z^{t+3} + 5z^{u+9} + 2z^{u+5} + 3z^{u+11} + 2z^{u+3} + z^{t+u+5} + 3z^{13} + 6z^7 + z^9 - (z^{t+1} + 3z^{t+7} + 2z^{t+10} + 2z^{t+8} + 2z^{t+12} + 2z^{t+6} + z^{u+1} + 3z^{u+7} + 2z^{u+10} + 2z^{u+8} + 2z^{u+12} + 2z^{u+6} + 2z^{t+u+3} + z^{t+u+6} + z^{t+u+4} + 2z^{t+u+7} + 4z^{10} + 2z^{14} + 2z^{11} + z^{12} + z^8 + z^6).$$

$$Q_4(B) = z^{x_3+x_5+9} + z^{x_3+x_6+9} + z^{x_3+10} + z^{x_3+x_5+7} + z^{x_3+x_6+7} + z^{x_3+6} + z^{x_5+x_6+6} + z^{x_5+7} + z^{x_6+7} + 3z^6 + z^{x_5+x_6+4} + z^{x_5+3} + z^{x_6+3} + z^{x_5+x_6+8} + z^{x_5+9} + z^{x_2+x_6+9} + z^8 + z^{x_5+x_6+6} + z^{x_5+5} + z^{x_6+5} + z^{x_3+x_5+x_6+4} + z^{x_3+x_5+5} + z^{x_3+x_6+5} + z^{x_3+x_5+x_6+2} + z^{x_3+x_5+1} + z^{x_3+x_6+1} + z^{x_3+2} + z^{x_5+x_6+6} + z^{x_5+4} + z^{x_6+4} + z^{x_5+x_6+1} + z^{x_5+3} + z^{x_6+3} - (z^{x_3+x_5+10} + z^{x_3+x_6+10} + z^{x_3+9} + z^{x_3+x_5+6} + z^{x_3+x_6+6} + z^{x_3+7} + z^{x_5+x_6+7} + z^{x_5+6} + z^{x_6+6} + z^7 + z^{x_5+x_6+3} + z^{x_5+4} + z^{x_6+4} + z^{x_5+x_6+9} + z^{x_5+8} + z^{x_6+8} + z^9 + z^{x_5+x_6+5} + z^{x_5+6} + z^{x_6+6} + z^{x_3+x_5+x_6+5} + z^{x_3+x_5+4} + z^{x_3+x_6+4} + z^{x_3+5} + z^{x_3+x_5+x_6+1} + z^{x_3+x_5+x_6+2} + z^{x_3+x_6+2} + z^{x_3+1} + z^{x_5+x_6+4} + z^{x_5+6} + z^{x_5+x_6+3} + z^{x_5+1} + z^{x_6+1}).$$

Consider the l.r.p in  $Q_3(A)$  and the l.r.p in  $Q_4(B)$ , we have  $x_3 = x_5 = x_6 = 5$ .

$$Q_5(B) = 2z^{19} + 2z^{17} + z^{18} + 3z^{15} + z^{10} + 5z^8 + z^6 - (3z^{20} + z^{14} + z^{12} + 2z^{11} + 4z^{13})$$

$Q_3(A) \neq Q_5(B)$ , a contradiction.

**Case 3.1.1.2:**  $x_3 = 4$ .

$$Q_6(B) = z^{x_2+x_5+9} + z^{x_2+x_6+9} + z^{x_2+10} + z^{x_2+x_5+7} + z^{x_2+x_6+7} + z^{x_2+6} + z^{x_5+x_6+6} + z^{x_5+7} + z^{x_6+7} + 3z^6 + z^{x_5+x_6+4} + z^{x_5+3} + z^{x_6+3} + z^{x_2+x_5+x_6+4} + z^{x_2+x_5+5} + z^{x_2+x_6+5} + z^{x_2+4} + z^{x_2+x_5+x_6+2} + z^{x_2+x_5+1} + z^{x_2+x_6+1} + z^{x_2+2} + z^{x_5+x_6+8} + z^{x_5+9} + z^{x_6+9} + z^9 + z^{x_5+x_6+6} + z^{x_5+5} + z^{x_6+5} + z^{x_5+x_6+6} + z^{x_5+4} + z^{x_6+4} + z^{x_5+x_6+1} + z^{x_5+3} + z^{x_6+3} - (z^{x_2+x_5+10} + z^{x_2+x_6+10} + z^{x_2+9} + z^{x_2+x_5+6} + z^{x_2+x_6+6} + z^{x_2+7} + z^{x_5+x_6+7} + z^{x_5+6} + z^{x_6+6} + z^7 + z^{x_5+x_6+3} + z^{x_5+4} + z^{x_6+4} + z^{x_2+x_5+x_6+5} + z^{x_2+x_5+4} + z^{x_2+x_6+4} + z^{x_2+5} + z^{x_2+x_5+x_6+1} + z^{x_2+x_5+2} + z^{x_2+x_6+2} + z^{x_2+1} + z^{x_3+x_5+x_6+5} + z^{x_3+x_5+4} + z^{x_3+x_6+4} + z^{x_3+5} + z^{x_3+x_5+x_6+1} + z^{x_3+x_5+x_6+2} + z^{x_3+x_6+2} + z^{x_3+1} + z^{x_5+x_6+4} + z^{x_5+6} + z^{x_5+x_6+3} + z^{x_5+1} + z^{x_6+1}).$$

Consider the l.r.p in  $Q_3(A)$  is  $-z^6$  and the l.r.p in  $Q_6(B)$  is  $3z^6$ , since  $2 \leq x_2 \leq 4$  and  $x_5, x_6 \geq 4$

$Q_3(A) \neq Q_6(B)$ , a contradiction.

**Case 3.1.1.3:**  $x_5 = 4$ .

$$Q_7(B) = z^{x_2+x_3+9} + z^{x_2+x_6+9} + z^{x_2+x_3+6} + z^{x_2+x_3+8} + z^{x_2+x_3+x_6+3} + z^{x_2+x_3+2} + z^{x_6+10} + z^{11} + z^{x_6+7} + 2z^6 + z^{x_6+10} + z^7 + z^{x_6+3} + z^{x_2+x_6+8} + z^{x_2+9} + z^{x_2+x_6+5} + z^{x_2+4} + z^{x_2+x_6+6} + z^{x_2+5} + z^{x_2+x_6+1} + z^{x_2+2} + z^{x_3+x_6+8} + z^{x_3+9} + z^{x_3+x_6+5} + z^{x_3+4} + z^{x_3+x_6+6} + z^{x_3+5} + z^{x_3+x_6+1} + z^{x_3+2} + z^{x_6+10} + z^{x_6+4} + z^{x_6+5} + z^7 + z^{x_6+3} - (z^{x_2+x_3+10} + z^{x_2+x_3+x_6+6} + z^{x_2+x_3+5} + z^{x_2+x_3+6} + z^{x_2+x_3+x_6+2} + z^{x_2+x_3+x_6+3} + z^{x_6+11} + z^{10} + z^{x_6+6} + z^7 + z^{x_6+7} + z^{x_6+4} + z^{x_2+x_6+9} + z^{x_2+8} + z^{x_2+x_6+4} + z^{x_2+5} + z^{x_2+x_6+5} + z^{x_2+6} + z^{x_2+x_6+2} + z^{x_2+1} + z^{x_3+x_6+9} + z^{x_3+8} + z^{x_3+x_6+4} + z^{x_3+5} + z^{x_3+x_6+5} + z^{x_3+6} + z^{x_3+x_6+2} + z^{x_3+1} + z^{x_6+8} + z^{10} + z^{x_6+6} + z^{x_6+7} + z^{x_6+1}).$$

Consider the l.r.p in  $Q_3(A)$  and the l.r.p in  $Q_7(B)$ , we have  $x_2 = x_3 = x_6 = 5$ .

$$Q_4(A) = 5z^{t+9} + 2z^{t+5} + 3z^{t+11} + 2z^{t+3} + 5z^{u+9} + 2z^{u+5} + 3z^{u+11} + 2z^{u+3} + z^{t+u+5} + 3z^{13} + 6z^7 + z^9 - (z^{t+1} + 3z^{t+7} + 2z^{t+10} + 2z^{t+8} + 2z^{t+12} + 2z^{t+6} + z^{u+1} + 3z^{u+7} + 2z^{u+10} + 2z^{u+8} + 2z^{u+12} + 2z^{u+6} + 2z^{t+u+3} + z^{t+u+6} + z^{t+u+4} + 2z^{t+u+7} + 4z^{10} + 2z^{14} + 2z^{11} + z^{12} + z^8).$$

$$Q_8(B) = 3z^{18} + z^{15} + z^{17} + z^{16} + 3z^7 + 2z^9 + 2z^8 - (z^{21} + z^{11} + 4z^{12} + 3z^{13} + z^{19}).$$

Compare the l.r.p in  $Q_4(A)$  and the l.r.p in  $Q_8(B)$ , we have

$Q_4(A) \neq Q_8(B)$ , a contradiction.

**Case 3.1.1.4 :**  $x_6 = 4$ .

Similar to Case 3.1.1.3, we obtain a contradiction.

**Case 3.1.2 :**  $t = 4$ .

Therefore  $s = 2$  or  $s = 3$  or  $s = 4$

If  $s = 2$ ,  $A \cong \theta(2, 2, 2, 2, 4, u)$  implies  $A$  is  $\chi$ -unique by Lemma 15.

If  $s = 4$ ,  $A \cong \theta(2, 2, 4, 4, 4, u)$  implies  $A$  is  $\chi$ -unique by Lemma 14.

If  $s = 3$ ,

$$Q_5(A) = 6z^{u+9} + 3z^{u+8} + z^{u+11} + z^{u+5} + 2z^{u+4} + 2z^{u+3} + 3z^{13} + 6z^{12} + 4z^{11} + 5z^7 + 4z^6 - (2z^{u+7} + 4z^{u+6} + z^{u+1} + z^{u+10} + 2z^{u+7} + 2z^{u+6} + z^{11} + 6z^9 + 7z^{10} + 2z^4 + z^{12} + 2z^{14} + 2z^{11} + z^6).$$

$$Q_9(B) = z^{x_2+x_3+x_5+5} + z^{x_2+x_3+x_6+5} + z^{x_2+x_3+x_6+6} + z^{x_2+x_3+x_5+3} + z^{x_2+x_3+x_6+3} + z^{x_2+x_3+x_2} + z^{x_5+x_6+6} + z^{x_5+7} + z^{x_6+7} + z^6 + z^{x_5+x_6+4} + z^{x_5+3} + z^{x_6+3} + z^{x_2+x_5+x_6+4} + z^{x_2+x_5+5} + z^{x_2+x_6+5} + z^{x_2+4} + z^{x_2+x_5+x_6+2} + z^{x_2+x_5+1} + z^{x_2+x_6+1} + z^{x_2+2} + z^{x_3+x_5+x_6+4} + z^{x_3+x_5+5} + z^{x_3+x_6+5} + z^{x_3+4} + z^{x_3+x_5+x_6+2} + z^{x_3+x_5+1} + z^{x_3+x_6+1} + z^{x_3+2} + z^{x_5+x_6+6} + z^{x_5+4} + z^{x_6+4} + z^6 + z^{x_5+x_6+1} + z^{x_5+3} + z^{x_6+3} - (z^{x_2+x_3+x_5+6} + z^{x_2+x_3+x_6+6} + z^{x_2+x_3+5} + z^{x_2+x_3+x_5+2} + z^{x_2+x_3+x_6+2} + z^{x_2+x_3+3} + z^{x_5+x_6+7} + z^{x_5+6} + z^{x_6+6} + z^7 + z^{x_5+x_6+3} + z^{x_5+4} + z^{x_6+4} + z^{x_2+x_5+x_6+5} + z^{x_2+x_5+4} + z^{x_2+x_6+4} + z^{x_2+5} + z^{x_2+x_5+x_6+1} + z^{x_2+x_5+2} + z^{x_2+x_6+2} + z^{x_2+1} + z^{x_3+x_5+x_6+5} + z^{x_3+x_5+4} + z^{x_3+x_6+4} + z^{x_3+5} + z^{x_3+x_5+x_6+1} + z^{x_3+x_5+2} + z^{x_3+x_6+2} + z^{x_3+1} + z^{x_5+x_6+4} + z^{x_5+6} + z^{x_6+6} + z^{x_5+x_6+3} + z^{x_5+1} + z^{x_6+1}).$$

Compare the l.r.p in  $Q_5(A)$  and the l.r.p in  $Q_9(B)$ , we have  $x_2 = x_3 = 3$ .

$$Q_6(A) = 6z^{u+9} + 3z^{u+8} + z^{u+11} + z^{u+5} + 2z^{u+4} + 2z^{u+3} + 3z^{13} + 6z^{12} + 4z^{11} + 5z^7 + z^6 - (2z^{u+7} + 4z^{u+6} + z^{u+1} + z^{u+10} + 2z^{u+7} + 2z^{u+6} + z^{11} + 6z^9 + 7z^{10} + z^{12} + 2z^{14} + 2z^{11}).$$

$$Q_{10}(B) = z^{x_5+x_6+6} + z^{x_5+x_6+4} + 2z^{x_5+x_6+7} + 2z^{x_5+x_6+5} + z^{x_5+x_6+6} + z^{x_5+x_6+1} + z^{x_5+11} + z^{x_5+9} + 2z^{x_5+3} + 2z^{x_5+8} + 2z^{x_5+4} + z^{x_6+11} + z^{x_6+9} + 2z^{x_6+3} + 2z^{x_6+8} + 2z^{x_6+4} + z^{12} + 2z^7 + 2z^5 - (2z^{x_5+x_6+8} + 2z^{x_5+x_6+3} + 2z^{x_5+x_6+7} + 3z^{x_5+x_6+4} + z^{x_5+12} + 2z^{x_5+6} + 2z^{x_5+5} + z^{x_5+8} + z^{x_5+7} + z^{x_5+1} + z^{x_6+12} + z^{x_6+8} + 2z^{x_6+5} + 2z^{x_6+7} + z^{x_6+6} + z^{x_6+1} + z^{11} + z^9 + z^7 + z^8).$$

Compare the l.r.p in  $Q_6(A)$  and the l.r.p in  $Q_{10}(B)$ , we have  $u = 5$ ,  $x_5 = x_6 = 4$ .

$$Q_7(A) = z^{16} + 4z^{14} + 4z^{13} + z^{12} + 2z^9 + 3z^8 - (5z^{11} + z^{15} + 6z^{10} + 6z^9 + 2z^{14})$$

$$Q_{11}(B) = 4z^{15} + 2z^{14} + 4z^{13} + 6z^{12} - (4z^{16} + 2z^{15} + 6z^{11} + 5z^{12} + 3z^{10} + z^8)$$

Compare the l.r.p in  $Q_7(A)$  and the l.r.p in  $Q_{11}(B)$ , we have

$Q_7(A) \neq Q_{11}(B)$ , a contradiction.

**Case 3.1.3 :**  $u = 4$ .

Therefore, we have  $2 \leq s \leq t \leq 4$ .

If  $s = 2$ , then  $t = 2$  or  $t = 3$  or  $t = 4$

$s = 2, t = 2, A \cong \theta(2, 2, 2, 2, 2, 4)$ ,  $A$  is  $\chi$ -unique, by Lemma 9.

$s = 2, t = 3, A \cong \theta(2, 2, 2, 2, 3, 4)$ ,  $A$  is  $\chi$ -unique by Lemma 15.

$s = 2, t = 4, A \cong \theta(2, 2, 2, 2, 4, 4)$ ,  $A$  is  $\chi$ -unique, by Lemma 9.

If  $s = 3$ ,

$s = 3, t = 3, A \cong \theta(2, 2, 3, 3, 3, 4)$ ,  $A$  is  $\chi$ -unique, by Lemma 14.

$s = 3, t = 4, A \cong \theta(2, 2, 3, 3, 4, 4)$ ,  $A$  is  $\chi$ -unique, by Lemma 13.

If  $s = 4$ ,

$s = 4, t = 4, A \cong \theta(2, 2, 4, 4, 4, 4)$ ,  $A$  is  $\chi$ -unique by Lemma 9.

**Case 3.2 :**  $x_2 = 2$ .

Then  $x_1 = 2$ . Hence  $B$  has at least two cycles of length 4, a contradiction.

**Case 3.3 :**  $x_3 = 2$ . Then  $x_1 = x_2 = 2$ . Hence  $B$  has at least four cycles of length 4, a contradiction.

**Case 4 :**  $x_4 \neq 3, x_5 \neq 3, x_6 \neq 3$

We know that  $x_4, x_5, x_6 > 3$ . Given that  $B$  shall has one cycle of length 4, then  $x_1 + x_2 = 4$ ,  $x_1 = x_2 = 2$ .

Then  $s = 2$  or  $t = 2$  or  $u = 2$ .

If  $s = 2, A \cong \theta(2, 2, 2, 2, t, u)$ ,  $A$  is  $\chi$ -unique by Lemma 15.

If  $t = 2, A \cong \theta(2, 2, 2, 2, 2, u)$ ,  $A$  is  $\chi$ -unique, by Lemma 9.

If  $u = 2$ ,  $A \cong \theta(2, 2, 2, 2, 2)$ ,  $A$  is  $\chi$ -unique, by Theorem 1.

**Case B**:  $B \in g_e(\theta(x_1, x_2, x_3, x_4), C_{x_5+1}, C_{x_6+1})$ , where  $2 \leq x_1 \leq x_2 \leq x_3 \leq x_4$  and  $2 \leq x_5, x_6$ .

As  $A \cong \theta(r, r, s, t, u)$  and  $B \in g_e(\theta(x_1, x_2, x_3, x_4), C_{x_5+1}, C_{x_6+1})$ , then by Theorem 7, we have

$$Q_8(A) = z(z^r - 1)^2(z^s - 1)^2(z^t - 1)(z^u - 1) - (z^r - z)^2(z^s - z)^2(z^t - z)(z^u - z).$$

$$Q_{12}(B) = z(z^{x_1} - 1)(z^{x_2} - 1)(z^{x_3} - 1)(z^{x_4} - 1)(z^{x_5} - 1)(z^{x_6} - 1) - (z^{x_1} - z)(z^{x_2} - z)(z^{x_3} - z)(z^{x_4} - z)(z^{x_5} - 1)(z^{x_6} - 1).$$

$Q_8(A) = Q_{12}(B)$ , yields

$$\begin{aligned} Q_9(A) &= z^{2r+2s+1} + z^{2r+t+u+1} + 2z^{2r+s+u+1} + 2z^{2r+s+t+1} + 2z^{2r+u+t+1} + z^{2r+1} + z^{2s+1} + z^{t+u+1} + 2z^{s+t+1} + 2z^{s+u+1} + \\ &2z^{r+s+1} + 2z^{r+u+1} + 2z^{r+2s+t+1} + 2z^{r+2s+u+1} + 4z^{r+s+t+u+1} + 2z^{s+t+u+3} + 2z^{r+t+u+3} + 2z^{r+2s+t+u+1} + z^{2r+t+3} + z^{2s+t+3}z^{t+5} + \\ &4z^{r+s+t+3} + z^{2r+u+3} + z^{2s+u+3}z^{u+5} + 4z^{r+s+u+3} + z^{r+s+3} + z^{r+2s+3} + 2z^{s+5} + 2z^{r+5} + 4z^{r+s+1} - (z^{2r+t+1} + z^{2r+s+1} + z^{2r+u+1} + \\ &z^{2s+t+1} + z^{2s+u+1} + z^{t+1} + 2z^{s+1} + 2z^{s+t+u+1} + 2z^{r+1} + z^{u+1} + 2z^{r+2s+t+u+1} + 2z^{r+t+u+1} + 4z^{r+s+t+1} + 4z^{r+s+u+1} + 2z^{r+2s+1} + \\ &z^{2r+2s+2} + z^{2r+t+u+2} + z^{2r+4} + 2z^{2r+s+t+2} + 2z^{2r+s+u+2} + 2z^{2r+t+u+2} + z^{2s+4} + z^{t+u+4} + 2z^{s+t+4} + 2z^{s+u+4} + 2z^{r+t+4} + \\ &2z^{r+u+4} + 2z^{r+2s+t+2} + 2z^{r+2s+u+2} + 4z^{r+s+t+u+2} + 4z^{r+s+4} + z^6). \end{aligned}$$

$$\begin{aligned} Q_{13}(B) &= z^{x_1+x_2+x_3+x_4+x_5} + z^{x_1+x_2+x_3+x_4+x_6} + z^{x_1+x_2+x_3+x_4+1} + z^{x_1+x_2+x_5+x_6+1} + z^{x_1+x_2+x_5+2} + z^{x_1+x_2+x_6+2} + z^{x_1+x_2+1} + \\ &z^{x_1+x_3+x_5+x_6+1} + z^{x_1+x_3+x_5+2} + z^{x_1+x_3+x_6+2} + z^{x_1+x_3+1} + z^{x_1+x_4+x_5+x_6+1} + z^{x_1+x_4+x_5+2} + z^{x_1+x_4+x_6+2} + z^{x_1+x_4+1} + \\ &z^{x_1+x_5+x_6+3} + z^{x_1+x_5+1} + z^{x_1+x_6+1} + z^{x_1+3} + z^{x_2+x_3+x_5+x_6+1} + z^{x_2+x_3+x_5+2} + z^{x_2+x_3+x_6+2} + z^{x_2+x_3+1} + z^{x_2+x_4+x_5+x_6+1} + \\ &z^{x_2+x_4+x_5+2} + z^{x_2+x_4+x_6+2} + z^{x_2+x_4+1} + z^{x_2+x_5+x_6+3} + z^{x_2+x_5+1} + z^{x_2+x_6+1} + z^{x_2+3} + z^{x_3+x_4+x_5+x_6+1} + z^{x_3+x_4+x_5+2} + \\ &z^{x_3+x_4+x_6+2} + z^{x_3+x_4+1} + z^{x_3+x_5+x_6+2} + z^{x_3+x_5+1} + z^{x_3+x_6+1} + z^{x_3+3} + z^{x_4+x_5+x_6+3} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + z^{x_4+3} + \\ &z^{x_5+x_6+1} + z^{x_5+4} + z^{x_6+4} - (z^{x_1+x_2+x_3+x_4+x_5+1} + z^{x_1+x_2+x_3+x_4+x_6+1} + z^{x_1+x_2+x_3+x_4} + z^{x_1+x_2+x_3+x_5+x_6+2} + z^{x_1+x_2+x_5+1} + \\ &z^{x_1+x_2+x_6+1} + z^{x_1+x_2+2} + z^{x_1+x_3+x_5+x_6+2} + z^{x_1+x_3+x_5+1} + z^{x_1+x_3+x_6+1} + z^{x_1+x_3+2} + z^{x_1+x_4+x_5+x_6+2} + z^{x_1+x_4+x_5+1} + \\ &z^{x_1+x_4+x_6+1} + z^{x_1+x_5+x_6+1} + z^{x_1+x_4+2} + z^{x_1+x_5+3} + z^{x_1+x_6+3} + z^{x_1+1} + z^{x_2+x_3+x_5+x_6+2} + z^{x_2+x_3+x_5+1} + z^{x_2+x_3+x_6+1} + \\ &z^{x_2+x_3+2} + z^{x_2+x_4+x_5+x_6+2} + z^{x_2+x_4+x_5+1} + z^{x_2+x_4+x_6+1} + z^{x_2+x_4+2} + z^{x_2+x_5+x_6+1} + z^{x_2+x_5+3} + z^{x_2+x_6+3} + z^{x_2+1} + \\ &z^{x_3+x_4+x_5+x_6+2} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+x_6+1} + z^{x_3+x_4+2} + z^{x_3+x_5+x_6+1} + z^{x_3+x_5+3} + z^{x_3+x_6+3} + z^{x_3+1} + z^{x_4+x_5+x_6+1} + \\ &z^{x_4+x_5+3} + z^{x_4+x_6+3} + z^{x_4+1} + z^{x_5+x_6+4} + z^{x_5+1} + z^{x_6+1} + z^4). \end{aligned}$$

Since  $2 \leq r \leq s \leq t \leq u$ . Therefore, by comparing the l.r.p in  $Q_9(A)$  and the l.r.p in  $Q_{13}(B)$ , we have  $r = 2$  or  $r = 3$ .

**Case 1** :  $r = 2$

Then  $g(A) = g(B) = 2r = 4$ . Since  $A$  has one cycle of length four, therefore  $B$  has one cycle of length 4. Without loss of generality, we have three cases to consider ,

1.  $x_5 = x_6 = 3$  or

2.  $x_5 = 3, x_6 \neq 3$  or

3.  $x_5 \neq 3, x_6 \neq 3$

**Case 1.1** :  $x_5 = x_6 = 3$ .

Since  $B$  has at least two cycles of length 4, a contradiction.

**Case 1.2** :  $x_5 = 3, x_6 \neq 3$

We know that  $x_6 > 3$ . Substituting into  $Q_9(A)$  and  $Q_{13}(B)$ . We obtain that there is  $-2z^3$  in  $Q_9(A)$ . Hence there are six cases to be considered, that are  $x_1 = x_2 = 2$  or  $x_1 = x_3 = 2$  or  $x_1 = x_4 = 2$  or  $x_2 = x_3 = 2$  or  $x_3 = x_4 = 2$  or  $x_2 = x_4 = 2$ .

For  $x_1 = x_2 = 2$ ,  $B$  has at least two cycles of length 4, a contradiction.  $B$  has at least three cycles of length 4 for all other cases. Thus, a contradiction.

**Case 1.3** :  $2 \leq x_5 \leq x_6$

We know that  $x_5, x_6 > 3$ . Hence  $x_1 + x_2 = 4$ , implying  $x_1 = x_2 = 2$ .

Considering the l.r.p in  $Q_9(A)$  and the l.r.p in  $Q_{13}(B)$ , we have  $s = 3$  or  $t = 3$  or  $u = 3$ .

**Case 1.3.1** :  $s = 3$ .

Note that the term  $-z^{s+1}$  in  $Q_9(A)$  has coefficient 2, then  $x_3 = 3$  or  $x_4 = 3$ .

**Case 1.3.1.1** :  $x_3 = 3$

$$\begin{aligned} Q_{14}(B) &= z^{x_4+x_5+7} + z^{x_4+x_6+7} + z^{x_4+8} + z^{x_5+x_6+5} + z^{x_5+6} + z^{x_6+6} + 2z^5 + 3z^{x_5+x_6+6} + 2z^{x_5+7} + 2z^{x_6+7} + 3z^{x_4+x_5+x_6+3} + \\ &2z^{x_4+x_5+4} + 2z^{x_4+x_6+4} + 3z^{x_4+3} + 2z^{x_5+x_6+5} + 2z^{x_5+3} + 2z^{x_6+3} + z^{x_4+x_5+x_6+4} + z^{x_4+x_5+5} + z^{x_4+x_6+5} + z^{x_4+4} + 2z^{x_5+4} + \\ &2z^{x_6+4} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + z^{x_5+x_6+1} - (z^{x_4+x_5+8} + z^{x_4+x_6+8} + z^{x_4+7} + z^{x_5+x_6+6} + 3z^{x_5+5} + 3z^{x_6+5} + z^6 + z^{x_5+x_6+7} + \\ &3z^{x_5+6} + 3z^{x_6+6} + 2z^7 + 2z^{x_4+x_5+x_6+4} + 3z^{x_4+x_5+3} + 3z^{x_4+x_6+3} + 2z^{x_4+4} + 2z^{x_5+x_6+3} + z^{x_5+x_6+7} + z^{x_4+x_5+x_6+5} + z^{x_4+x_5+4} + \\ &z^{x_4+x_6+4} + z^{x_4+5} + 2z^{x_5+x_6+4} + z^{x_4+x_5+x_6+1} + z^{x_4+1} + z^{x_5+1} + z^{x_6+1}). \end{aligned}$$

Considering the l.r.p in  $Q_9(A)$  and the l.r.p in  $Q_{14}(B)$ , we have  $x_4 = x_5 = 4$ , or  $x_4 = x_6 = 4$ , or  $x_5 = x_6 = 4$  .

**Case 1.3.1.1.1** :  $x_4 = x_5 = 4$ .

$$Q_{15}(B) = z^{15} + 3z^{x_6+11} + 2z^{12} + 3z^{x_6+9} + 2z^{x_6+10} + 2z^{x_6+3} + z^{x_6+12} + z^{13} + 2z^8 + 2z^{x_6+4} - (z^{16} + 3z^{x_6+6} + 3z^{11} + 3z^9 + \\ z^{x_6+5} + z^6 + 2z^{10} + 2z^{x_6+6} + 3z^{x_6+7} + z^{13} + z^{x_6+8} + z^{x_6+1}).$$

Compare the l.r.p in  $Q_9(A)$  and the l.r.p in  $Q_{15}(B)$ , we have  $t = 5$  or  $u = 5$ .

If  $u = 5$   $3 \leq t \leq 5$

If  $t = 3$ ,  $A \cong \theta(2, 2, 3, 3, 3, 5)$ ,  $A$  is  $\chi$ -unique by Lemma 14.

$$\text{If } t = 4, Q_{10}(A) = 4z^{14} + 6z^{12} + 7z^{13} + z^{10} + z^{15} + 3z^{11} - (3z^{13} + 5z^{12} + 7z^{10} + 8z^{11} + 2z^{14} + 2z^{15}).$$

$$Q_{16}(B) = z^{15} + 3z^{x_6+11} + 2z^{12} + 3z^{x_6+9} + 2z^{x_6+10} + 2z^{x_6+3} + z^{x_6+12} + z^{13} + 2z^8 + 2z^{x_6+4} - (z^{16} + 3z^{x_6+6} + 3z^{11} + 3z^9 + z^{x_6+5} + z^6 + 2z^{10} + 2z^{x_6+6} + 3z^{x_6+7} + z^{13} + z^{x_6+8} + z^{x_6+1}).$$

Compare the l.r.p in  $Q_{10}(A)$  and the l.r.p in  $Q_{16}(B)$ , we have  $x_6 = 4$ .

$$Q_{11}(A) = 4z^{14} + 6z^{12} + 7z^{13} + z^{10} + z^{15} + 3z^{11} - (3z^{13} + 5z^{12} + 7z^{10} + 8z^{11} + 2z^{14} + 2z^{15}).$$

$$Q_{17}(B) = 5z^{15} + 5z^{13} + 2z^{14} - (3z^{16} + 6z^{11} + z^{15} + 4z^{10} + z^{13}). Q_{11}(A) \neq Q_{17}(B), \text{ a contradiction.}$$

If  $t = 5$ ,  $A \cong \theta(2, 2, 3, 3, 5, 5)$ ,  $A$  is  $\chi$ -unique by Lemma 13.

**Case 1.3.1.1.2 :**  $x_4 = x_6 = 4$ .

Similarly to Case 1.3.1.1.1, we obtain a contradiction.

**Case 1.3.1.1.3 :**  $x_5 = x_6 = 4$ .

Similarly to Case 1.3.1.1.1, we obtain a contradiction.

**Case 1.3.1.2 :**  $x_4 = 3$ .

Similarly to Case 1.3.1.1.1, we obtain a contradiction.

**Case 1.3.2 :**  $t = 3$ .

Since  $2 \leq s \leq 3$ . We know that  $s = 2$  or  $s = 3$ .

If  $s = 2$ ,  $A \cong \theta(2, 2, 2, 2, 3, u)$ ,  $A$  is  $\chi$ -unique by Lemma 15.

If  $s = 3$ ,  $A \cong \theta(2, 2, 3, 3, 3, u)$ ,  $A$  is  $\chi$ -unique by Lemma 14.

**Case 1.3.3 :**  $u = 3$ .  $2 \leq s \leq t \leq 3$ . We know that  $s = t = 2$  or  $s = 2, t = 3$  or  $s = t = 3$ .

If  $s = t = 2$ ,  $A \cong \theta(2, 2, 2, 2, 2, u)$ ,  $A$  is  $\chi$ -unique by Lemma 8.

If  $s = 2, t = 3$ ,  $A \cong \theta(2, 2, 2, 2, 3, u)$ ,  $A$  is  $\chi$ -unique by Lemma 15.

If  $s = t = 3$ ,  $A \cong \theta(2, 2, 3, 3, 3, u)$ ,  $A$  is  $\chi$ -unique by Lemma 14.

**Case 2 :**  $r = 3$ .

Then  $g(A) = g(B) = 2r = 6$ . Since  $A$  has one cycle of length 6, therefore  $B$  has one cycle of length 6. Without loss of generality, we have three cases to consider,

1.  $x_5 = x_6 = 5$  or

2.  $x_5 = 5, x_6 \neq 5$  or

3.  $x_5 \neq 5, x_6 \neq 5$

**Case 2.1 :**  $x_5 = x_6 = 5$ .

$B$  has at least two cycles of length 6, a contradiction.

**Case 2.2 :**

$x_5 = 5, x_6 \neq 5$ .

compare the l.r.p in  $Q_{10}(A)$  and  $Q_{21}(B)$ , we have a contradiction.

**Case 2.3 :**  $x_5 \neq 5, x_6 \neq 5$ .

We know that  $x_5, x_6 \geq 5$ . Hence  $x_1 + x_2 = 6$ .

$x_1 = x_2 = 3$ .

$$Q_{11}(A) = z^{2s+7} + z^{t+u+7} + z^7 + 2z^{s+u+7} + 2z^{s+t+7} + z^{t+u+1} + z^{2s+1} + 2z^{s+t+1} + 2z^{s+u+1} + 2z^{2s+t+4} + 2z^{2s+u+4} + 2z^{t+4} + 2z^{u+4} + 4z^{s+t+u+4} + 2z^{s+t+u+3} + 2z^{s+u+6} + z^{t+9} + z^{2s+t+3} + z^{t+5} + 4z^{s+t+6} + z^{u+9} + z^{2s+u+3} + z^{u+5} + 4z^{s+u+6} + 2z^{s+6} + 2z^{s+5} + 2z^8 + 2z^{2s+6} + 4z^{s+4} - (3z^{t+7} + 3z^{u+7} + 5z^{s+7} + z^{2s+t+1} + z^{2s+u+1} + z^{t+1} + 2z^{s+t+u+1} + 4z^{s+t+4} + 4z^{s+u+4} + z^{u+1} + 2z^{2s+4} + 2z^{s+1} + z^{t+u+4} + 2z^{2s+8} + z^{t+u+8} + z^{10} + 2z^{2s+t+u+2} + 2z^{s+t+8} + 2z^{s+u+8} + z^{2s+4} + z^{t+u+4} + 2z^{s+t+4} + 2z^{s+u+4} + 2z^{2s+t+5} + 2z^{2s+u+5} + 4z^{s+t+u+4} + z^6).$$

$$Q_{22}(B) = z^{x_3+x_4+x_5+6} + z^{x_3+x_4+x_6+6} + z^{x_3+x_4+7} + z^{x_5+x_6+7} + z^{x_5+8} + z^{x_6+8} + z^7 + z^{x_3+x_5+x_6+4} + z^{x_3+x_5+5} + z^{x_3+x_6+5} + z^{x_3+4} + z^{x_4+x_5+x_6+4} + z^{x_4+x_5+5} + z^{x_4+x_6+5} + z^{x_4+4} + z^{x_5+x_6+6} + z^{x_5+4} + z^{x_6+4} + z^6 + z^{x_3+x_5+x_6+4} + z^{x_3+x_5+5} + z^{x_3+x_6+5} + z^{x_3+4} + z^{x_4+x_5+x_6+4} + z^{x_4+x_5+5} + z^{x_4+x_6+5} + z^{x_4+4} + z^{x_5+x_6+6} + z^{x_5+4} + z^{x_6+4} + z^6 + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+x_6+2} + z^{x_3+x_5+x_6+2} + z^{x_3+x_5+x_6+1} + z^{x_3+x_6+1} + z^{x_3+x_5+x_6+3} + z^{x_4+x_5+x_6+3} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + z^{x_4+3} + z^{x_5+x_6+1} + z^{x_5+4} + z^{x_6+4} - (z^{x_3+x_4+x_5+7} + z^{x_3+x_4+x_6+7} + z^{x_3+x_4+x_6+8} + z^{x_3+x_5+x_6+8} + z^{x_5+7} + z^{x_6+7} + z^8 + z^{x_3+x_5+x_6+5} + z^{x_3+x_5+4} + z^{x_3+x_6+4} + z^{x_3+5} + z^{x_4+x_5+x_6+5} + z^{x_4+x_5+4} + z^{x_4+x_6+4} + z^{x_5+x_6+4} + z^{x_4+5} + z^{x_5+6} + z^{x_6+6} + z^{x_3+x_5+x_6+5} + z^{x_3+x_5+4} + z^{x_3+x_6+4} + z^{x_3+5} + z^{x_4+x_5+x_6+5} + z^{x_4+x_5+4} + z^{x_4+x_6+4} + z^{x_4+5} + z^{x_5+6} + z^{x_5+6} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+x_6+1} + z^{x_3+x_4+x_6+1} + z^{x_3+x_5+x_6+1} + z^{x_3+x_5+x_6+3} + z^{x_3+x_6+3} + z^{x_3+1} + z^{x_4+x_5+x_6+1} + z^{x_4+x_5+3} + z^{x_4+x_6+3} + z^{x_4+1} + z^{x_5+x_6+4} + z^{x_5+1} + z^{x_6+1} + z^4).$$

since  $2 \leq r \leq s \leq t \leq u$ , Compare the l.r.p in  $Q_{11}(A)$  and the l.r.p in  $Q_{22}(B)$  we have  $s = 3$  or  $t = 3$  or  $u = 3$ .

**Case 2.3.1 :**  $s = 3$ .

Then  $A \cong \theta(3, 3, 3, 3, t, u)$ ,  $A$  is  $\chi$ -unique by Lemma 15.

**Case 2.3.2 :**  $t = 3$ .

Since  $3 \leq s \leq 3 \leq u$ . We know that  $s = 3$ .

Then  $A \cong \theta(3, 3, 3, 3, 3, u)$ ,  $A$  is  $\chi$ -unique by Lemma 9.

**Case 2.3.3 :**  $u = 3$ .

Since  $3 \leq s \leq t \leq 3$ . We know that  $s = t = 3$ .

Then  $A \cong \theta(3, 3, 3, 3, 3, 3)$ ,  $A$  is  $\chi$ -unique by Theorem 1.

**Case C :**  $B \in g_e(\theta(x_1, x_2, x_3, x_4, x_5), C_{x_6+1})$ , where  $2 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$  and  $2 \leq x_6$ .

As  $A \cong \theta(r, r, s, s, t, u)$  and  $B \in g_e(\theta(x_1, x_2, x_3, x_4, x_5), C_{x_6+1})$ , then by Theorem 7, we have

$$Q_{12}(A) = z(z^r - 1)^2(z^s - 1)^2(z^t - 1)(z^u - 1) - (z^r - z)^3(z^s - z)(z^t - z)(z^u - z).$$

$$Q_{23}(B) = z(z^{x_1} - 1)(z^{x_2} - 1)(z^{x_3} - 1)(z^{x_4} - 1)(z^{x_5} - 1)(z^{x_6} - 1) - (z^{x_1} - z)(z^{x_2} - z)(z^{x_3} - z)(z^{x_4} - z)(z^{x_5} - z)(z^{x_6} - z).$$

$Q_{12}(A) = Q_{23}(B)$ , yields

$$\begin{aligned} Q_{13}(A) &= z^{2r+2s+1} + z^{2r+t+u+1} + 2z^{2r+s+u+1} + 2z^{2r+s+t+1} + z^{2r+1} + z^{2s+1} + z^{t+u+1} + 2z^{s+t+1} + 2z^{s+u+1} + \\ &2z^{r+s+1} + 2z^{r+u+1} + 2z^{r+2s+t+1} + 2z^{r+2s+u+1} + 4z^{r+s+t+u+1} + 2z^{s+t+u+3} + 2z^{r+t+u+3} + 2z^{r+2s+t+u+1} + z^{2r+t+3} + z^{2s+t+3}z^{t+5} + \\ &4z^{r+s+t+3} + z^{2r+u+3} + z^{2s+u+3} + z^{u+5} + 4z^{r+s+u+3} + z^{r+s+3} + z^{r+2s+3} + 2z^{s+5} + 2z^{r+5} + 4z^{r+s+1} - (z^{2r+t+1} + z^{2r+s+1} + \\ &z^{2r+u+1} + z^{2s+t+1} + z^{2s+u+1} + z^{t+1} + 2z^{s+1} + 2z^{s+t+u+1} + 2z^{r+1} + z^{u+1} + 2z^{r+2s+t+u+1} + 2z^{r+t+u+1} + 4z^{r+s+u+1} + \\ &2z^{r+2s+1} + z^{2r+2s+2} + z^{2r+t+u+2} + z^{2r+4} + 2z^{2r+s+t+2} + 2z^{2r+s+u+2} + 2z^{2s+t+u+2} + z^{2s+4} + z^{t+u+4} + 2z^{s+t+4} + 2z^{s+u+4} + \\ &2z^{r+t+4} + 2z^{r+u+4} + 2z^{r+2s+t+2} + 2z^{r+2s+u+2} + 4z^{r+s+t+u+2} + 4z^{r+s+4} + z^6). \end{aligned}$$

$$\begin{aligned} Q_{24}(B) &= z^{x_1+x_2+x_3+x_4+x_5} + z^{x_1+x_2+x_3+x_6+1} + z^{x_1+x_2+x_3+2} + z^{x_1+x_2+x_4+x_6+1} + z^{x_1+x_2+x_5+x_6+1} + z^{x_1+x_2+x_5+2} + \\ &z^{x_1+x_2+x_6+3} + z^{x_1+x_2+1} + z^{x_1+x_3+x_4+x_6+1} + z^{x_1+x_3+x_4+2} + z^{x_1+x_3+x_5+x_6+1} + z^{x_1+x_3+x_5+2} + z^{x_1+x_3+x_6+3} + z^{x_1+x_3+1} + \\ &z^{x_1+x_4+x_5+x_6+1} + z^{x_1+x_4+x_5+2} + z^{x_1+x_4+x_6+3} + z^{x_1+x_4+1} + z^{x_1+x_5+x_6+3} + z^{x_1+x_5+1} + z^{x_1+x_6+1} + z^{x_1+4} + z^{x_2+x_3+x_4+x_6+1} + \\ &z^{x_2+x_3+x_4+2} + z^{x_2+x_3+x_5+x_6+1} + z^{x_2+x_3+x_5+2} + z^{x_2+x_3+x_6+3} + z^{x_2+x_3+1} + z^{x_2+x_4+x_5+x_6+1} + z^{x_2+x_4+x_5+2} + z^{x_2+x_4+x_6+3} + \\ &z^{x_2+x_4+1} + z^{x_2+x_5+x_6+3} + z^{x_2+x_5+1} + z^{x_2+x_6+1} + z^{x_2+4} + z^{x_3+x_4+x_5+x_6+1} + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+x_6+3} + z^{x_3+x_4+1} + \\ &z^{x_3+x_5+x_6+3} + z^{x_3+x_5+1} + z^{x_3+x_6+1} + z^{x_3+4} + z^{x_4+x_5+x_6+3} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + z^{x_4+4} + z^{x_5+x_6+1} + z^{x_5+4} + z^{x_6+5} - \\ &(z^{x_1+x_2+x_3+x_4+x_5+1} + z^{x_1+x_2+x_3+x_6+2} + z^{x_1+x_2+x_3+1} + z^{x_1+x_2+x_4+x_6+2} + z^{x_1+x_2+x_4+1} + z^{x_1+x_2+x_5+x_6+2} + z^{x_1+x_2+x_5+1} + \\ &z^{x_1+x_2+x_6+1} + z^{x_1+x_2+3} + z^{x_1+x_3+x_4+x_6+1} + z^{x_1+x_3+x_4+1} + z^{x_1+x_3+x_5+x_6+2} + z^{x_1+x_3+x_5+1} + z^{x_1+x_3+x_6+1} + z^{x_1+x_3+3} + \\ &z^{x_1+x_4+x_5+x_6+2} + z^{x_1+x_4+x_5+1} + z^{x_1+x_4+x_6+1} + z^{x_1+x_5+x_6+1} + z^{x_1+x_4+3} + z^{x_1+x_5+3} + z^{x_1+x_6+4} + z^{x_1+1} + z^{x_2+x_3+x_4+x_6+2} + \\ &z^{x_2+x_3+x_4+1} + z^{x_2+x_3+x_5+x_6+2} + z^{x_2+x_3+x_5+1} + z^{x_2+x_3+x_6+1} + z^{x_2+x_3+3} + z^{x_2+x_4+x_5+x_6+2} + z^{x_2+x_4+x_5+1} + z^{x_2+x_4+x_5+x_6+2} + \\ &z^{x_2+x_4+x_5+1} + z^{x_2+x_4+x_6+1} + z^{x_2+x_4+3} + z^{x_2+x_5+x_6+1} + z^{x_2+x_5+3} + z^{x_2+x_6+4} + z^{x_2+1} + z^{x_3+x_4+x_5+x_6+2} + z^{x_3+x_4+x_5+1} + \\ &z^{x_3+x_4+x_6+1} + z^{x_3+x_4+3} + z^{x_3+x_5+x_6+1} + z^{x_3+x_5+3} + z^{x_3+x_6+4} + z^{x_3+1} + z^{x_4+x_5+x_6+1} + z^{x_4+x_5+3} + z^{x_4+x_6+4} + z^{x_4+1} + \\ &z^{x_5+x_6+4} + z^{x_5+1} + z^{x_6+1} + z^5). \end{aligned}$$

Consider the l.r.p in  $Q_{13}(A)$  that is  $r+1$  and the l.r.p in  $Q_{24}(B)$  that is 5.

Since  $r = 4$  and  $r \geq 2$ . We have three cases to consider

1)  $r = 2$  or

2)  $r = 3$  or

3)  $r = 4$ .

**Case 1 :**  $r = 2$ .

Then  $g(A) = g(B) = 2r = 4$ . Since  $A$  has one cycle of length four, therefore  $B$  has one cycle of length 4. Without loss of generality, we have two cases to consider,

1.  $x_6 = 3$  or

2.  $x_6 \neq 3$ .

**Case 1.1 :**  $x_6 = 3$ .

Substituting  $r = 2$  in  $Q_{13}(A)$  and  $x_6 = 3$  in  $Q_{24}(B)$ . We obtain that there is  $-z^3$  in  $Q_{13}(A)$ . Hence there are cases to be considered,  $x_1 = x_2 = 2$  or  $x_1 = x_3 = 2$  or  $x_1 = x_4 = 2$  or  $x_1 = x_5 = 2$  or  $x_2 = x_3 = 2$  or  $x_2 = x_4 = 2$  or  $x_2 = x_5 = 2$  or  $x_3 = x_4 = 2$  or  $x_3 = x_5 = 2$  or  $x_4 = x_5 = 2$ . We know that  $B$  has at least three cycles of length 4 for all cases. Thus, a contradiction.

**Case 1.2 :**  $x_6 \neq 3$

$x_6 \geq 4$ .

Given that  $B$  has one cycle of length 4, then  $x_1 + x_2 = 4$ , implying  $x_1 = x_2 = 2$ . Consider the l.r.p in  $Q_{13}(A)$  and the l.r.p in  $Q_{24}(B)$ . We have  $s = 4$  or  $t = 4$  or  $u = 4$ .

**Case 1.2.1 :**  $s = 4$ .

$$t + u + 8 = x_3 + x_4 + x_5 + x_6. \quad (5)$$

$$Q_{14}(A) = 3z^{13} + 3z^{t+u+5} + 5z^{u+9} + 5z^{t+9} + z^{t+u+1} + 3z^9 + 2z^{t+5} + 2z^{u+5} + 3z^{u+11} + 3z^{t+11} + 2z^{t+3} + 2z^{u+3} + 6z^{t+u+7} + 6z^7 - (z^{t+1} + z^{u+1} + z^5 + 2z^{t+u+5} + 2z^{t+u+3} + 3z^{t+7} + 3z^{u+7} + 2z^{11} + z^{14} + z^{t+u+6} + 2z^{t+10} + 2z^{u+10} + z^{12} + z^{t+u+4} + 2z^{t+8} + 2z^{u+8} + 2z^{t+12} + 2z^{u+12} + 2z^{t+6} + 2z^{u+6} + 4z^{10} + z^8 + z^6).$$

$$Q_{25}(B) = z^{x_3+x_4+x_5+4} + z^{x_3+x_6+5} + z^{x_3+6} + z^{x_4+x_6+5} + z^{x_4+6} + 3z^{x_5+x_6+5} + z^{x_5+6} + z^{x_6+7} + 3z^{x_3+x_4+x_6+3} + 2z^{x_3+x_5+4} + 2z^{x_3+x_6+5} + 2z^{x_4+3} + 2z^{x_5+3} + 2z^{x_3+3} + 3z^{x_3+x_5+x_6+3} + 2z^{x_3+x_5+4} + 2z^6 + 2z^{x_3+x_6+5} + 2z^{x_6+3} + 3z^{x_4+x_5+x_6+3} + 2z^{x_4+x_5+4} + 2z^{x_4+x_6+5} + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+1} + z^{x_3+x_5+1} + z^{x_3+x_6+1} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + z^{x_5+x_6+1} + z^{x_4+4} + z^{x_5+4} + z^{x_6+5} - (z^{x_3+x_4+x_5+5} + z^{x_3+x_6+6} + z^{x_3+5} + z^{x_4+x_6+6} + z^{x_4+5} + z^{x_5+x_6+6} + 3z^{x_5+5} + z^{x_6+5} + z^7 + z^{x_3+x_4+x_6+3} + 3z^{x_3+x_4+3} + 2z^{x_3+x_5+x_6+4} + 3z^{x_3+x_5+3} + 2z^{x_3+x_6+3} + 2z^{x_3+5} + 2z^{x_4+x_5+x_6+4} + 3z^{x_4+x_5+3} + 2z^{x_4+x_6+3} + 2z^{x_4+5} + 2z^{x_5+x_6+3} + 2z^{x_6+6} + z^{x_3+x_4+x_6+4} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+x_6+1} + z^{x_3+x_5+x_6+1} + z^{x_3+x_6+4} + z^{x_3+1} + z^{x_4+x_5+x_6+1} + z^{x_4+x_6+4} + z^{x_4+1} + z^{x_5+x_6+4} + z^{x_5+1} + z^{x_6+1}).$$

Compare the l.r.p in  $Q_{25}(A)$  and the l.r.p in  $Q_{25}(B)$  we have  $x_3 = 4$  or  $x_4 = 4$  or  $x_5 = 4$  or  $x_6 = 4$ .

**Case 1.2.1.1 :**  $x_3 = 4$ .

$$Q_{26}(B) = z^{x_4+x_5+8} + z^{x_6+9} + z^{10} + z^{x_4+x_6+5} + z^{x_4+6} + 3z^{x_5+x_6+5} + z^{x_5+6} + z^{x_6+7} + 3z^{x_4+x_6+7} + 2z^{x_5+8} + 2z^{x_6+9} + 2z^{x_4+3} + 2z^{x_5+3} + 2z^7 + 3z^{x_5+x_6+7} + 2z^{x_5+8} + 2z^6 + 2z^{x_6+9} + 2z^{x_6+3} + 3z^{x_4+x_5+x_6+3} + 2z^{x_4+x_5+4} + 2z^{x_4+x_6+5} + z^{x_4+x_5+6} + z^{x_4+5} + z^{x_5+5} + z^{x_6+5} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + z^{x_5+x_6+1} + z^{x_4+4} + z^{x_5+4} + z^{x_6+5} - (z^{x_4+x_5+9} + z^{x_6+10} + z^9 + z^{x_4+x_6+6} + z^{x_4+5} + z^{x_5+x_6+6} + 3z^{x_5+5} + z^{x_6+5} + z^7 + z^{x_4+x_6+7} + 3z^{x_4+7} + 2z^{x_5+x_6+8} + 3z^{x_5+7} + 2z^{x_6+7} + 2z^9 + 2z^{x_4+x_5+x_6+4} + 3z^{x_4+x_5+3} + 2z^{x_4+x_6+3} + 2z^{x_4+5} + 2z^{x_5+x_6+3} + 2z^{x_6+6} + z^{x_4+x_6+8} + z^{x_4+x_5+5} + z^{x_4+x_6+5} + z^{x_5+x_6+5} + z^{x_6+8} + z^{x_4+x_5+x_6+1} + z^{x_4+x_6+4} + z^{x_4+1} + z^{x_5+x_6+4} + z^{x_5+1} + z^{x_6+1}).$$

Compare the l.r.p in  $Q_{13}(A)$  and the l.r.p in  $Q_{26}(B)$  we have  $x_4 = x_5 = x_6 = 5$ .

$$t + u = 11. \quad (6)$$

Since  $2 \leq 4 \leq t \leq u$

If  $t = 4$  then  $u = 7$ ,  $A \cong \theta(2, 2, 4, 4, 7)$ ,  $A$  is  $\chi$  unique by Lemma 10.

If  $t = 5$  then  $u = 6$ .

We obtain  $Q_{13}(A) \neq Q_{26}(B)$ , a contradiction.

**Case 1.2.1.2 :**  $x_4 = 4$ .

$$t + u + 4 = x_3 + x_5 + x_6. \quad (7)$$

Similarly to Case 1.2.1.1, we obtain a contradiction.

**Case 1.2.1.3 :**  $x_5 = 4$ .

$$t + u + 4 = x_3 + x_4 + x_6. \quad (8)$$

Similarly to Case 1.2.1.1, we obtain a contradiction.

**Case 1.2.1.4 :**  $x_6 = 4$ .

$$t + u + 4 = x_3 + x_5 + x_4. \quad (9)$$

Similarly to Case 1.2.1.1, we obtain a contradiction.

**Case 1.2.2 :**  $t = 4$ .

We know that either  $s = 2$  or  $s = 3$  or  $s = 4$ .

If  $s = 2$ ,  $A \cong \theta(2, 2, 2, 2, 4, u)$ ,  $A$  is  $\chi$ -unique by by Lemma 15.

If  $s = 3$ .

$$Q_{15}(A) = 3z^{13} + 6z^{u+8} + 6z^{u+9} + 6z^{12} + z^{u+11} + 5z^7 + 2z^{u+5} + 6z^8 + 2z^{u+4} + 4z^{11} + 2z^{u+3} + 6z^{u+10} + 2z^{u+13} + z^{u+7} + 3z^6 - (z^{u+5} + z^{u+1} + z^8 + z^{11} + 3z^{u+7} + 3z^{u+8} + 2z^{u+13} + 5z^{10} + z^{12} + 6z^{u+6} + 6z^9 + z^{u+10} + z^8 + 2z^{u+9} + z^{u+12} + 4z^{u+11} + 2z^{13} + z^8 + 2z^4).$$

$$Q_{27}(B) = z^{x_3+x_4+x_5+4} + z^{x_3+x_6+5} + z^{x_3+6} + z^{x_4+x_6+5} + z^{x_4+6} + 3z^{x_5+x_6+5} + z^{x_5+6} + z^{x_6+7} + 3z^{x_3+x_4+x_6+3} + 2z^{x_3+x_5+4} + 2z^{x_3+x_6+5} + 2z^{x_4+3} + 2z^{x_5+3} + 2z^{x_3+3} + 3z^{x_3+x_5+x_6+3} + 2z^{x_3+x_5+4} + 2z^6 + 2z^{x_3+x_6+5} + 2z^{x_6+3} + 3z^{x_4+x_5+x_6+3} + 2z^{x_4+x_5+4} + 2z^{x_4+x_6+5} + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+1} + z^{x_3+x_5+1} + z^{x_3+x_6+1} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + z^{x_5+x_6+1} + z^{x_4+4} + z^{x_5+4} + z^{x_6+5} -$$

$$(z^{x_3+x_4+x_5+5} + z^{x_3+x_6+6} + z^{x_3+5} + z^{x_4+x_6+6} + z^{x_4+5} + z^{x_5+x_6+6} + 3z^{x_5+5} + z^{x_6+5} + z^7 + z^{x_3+x_4+x_6+3} + 3z^{x_3+x_4+3} + 2z^{x_3+x_5+x_6+4} + 3z^{x_3+x_5+3} + 2z^{x_3+x_6+3} + 2z^{x_3+5} + 2z^{x_4+x_5+x_6+4} + 3z^{x_4+x_5+3} + 2z^{x_4+x_6+3} + 2z^{x_4+5} + 2z^{x_5+x_6+3} + 2z^{x_6+6} + z^{x_3+x_4+x_6+4} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+x_6+1} + z^{x_3+x_5+x_6+1} + z^{x_3+x_6+4} + z^{x_3+1} + z^{x_4+x_5+x_6+1} + z^{x_4+x_6+4} + z^{x_4+1} + z^{x_5+x_6+4} + z^{x_5+1} + z^{x_6+1}).$$

Compare the l.r.p in  $Q_{15}(A)$  and the l.r.p in  $Q_{27}(B)$  we have  $x_3 = x_4 = 3$  or  $x_3 = x_5 = 3$  or  $x_4 = x_5 = 3$ .

Without loss of generality, suppose that  $x_3 = x_4 = 3$ .

$$Q_{28}(B) = z^{x_5+10} + 6z^{x_6+8} + 2z^9 + 3z^{x_5+x_6+5} + z^{x_5+6} + z^{x_6+7} + 3z^{x_6+9} + 2z^{10} + 6z^{x_5+x_6+6} + 4z^{x_5+7} + z^{x_5+8} + 6z^6 + 2z^{x_5+3} + 2z^{x_6+3} + z^7 + 2z^{x_5+4} + 2z^{x_6+4} + z^{x_5+x_6+1} + z^{x_5+4} + z^{x_6+4} + z^7 - (z^{x_5+11} + z^{x_6+9} + 6z^8 + z^{x_6+10} + z^8 + z^{x_5+x_6+6} + 3z^{x_5+5} + z^{x_6+5} + z^7 + z^{x_6+9} + 3z^9 + 2z^{x_5+x_6+7} + 3z^{x_6+6} + 2z^{x_5+6} + 2z^{x_5+x_6+7} + 3z^{x_5+6} + 2z^{x_5+x_6+3} + 2z^{x_6+6} + z^{x_6+10} + z^{x_5+7} + z^{x_6+7} + z^{x_5+1} + 3z^{x_5+x_6+4} + 2z^{x_6+7} + z^{x_6+1}).$$

Compare the l.r.p in  $Q_{15}(A)$  and  $Q_{28}(B)$ , we have  $Q_{15}(A) \neq Q_{27}(B)$ , a contradiction.

If  $s = 4$

$$Q_{16}(A) = z^{13} + z^{u+9} + 2z^{u+9} + 2z^{13} + 2z^{u+9} + z^5 + z^9 + z^{u+5} + 2z^9 + 2z^{u+5} + 2z^7 + 2z^{u+3} + 2z^{15} + 2z^{u+11} + 4z^{u+11} + 2z^{u+11} + 2z^{u+9} + 2z^{u+15} + z^{11} + z^{15}z^9 + 4z^{13} + z^{u+7} + z^{u+11} + z^{u+5} + 4z^{u+9} + z^9 + z^{13} + 2z^9 + 2z^7 + 4z^7 - (z^9 + z^9 + z^{u+5} + z^{13} + z^{u+9} + z^5 + 2z^5 + 2z^{u+9} + 2z^3 + z^{u+1} + 2z^{u+15} + 2z^{u+7} + 4z^{11} + 4z^{u+7} + 2z^{11} + z^{14} + z^{u+10} + z^8 + 2z^{14} + 2z^{u+10} + 2z^{u+14} + z^{12} + z^{u+8} + 2z^{12} + 2z^{u+8} + 2z^{10} + 2z^{u+6} + 2z^{16} + 2z^{u+12} + 4z^{u+12} + 4z^{10} + z^6).$$

$$Q_{29}(B) = z^{x_3+x_4+x_5+4} + z^{x_3+x_6+5} + z^{x_3+6} + z^{x_4+x_6+5} + z^{x_4+6} + 3z^{x_5+x_6+5} + z^{x_5+6} + z^{x_6+7} + 3z^{x_3+x_4+x_6+3} + 2z^{x_3+x_5+4} + 2z^{x_3+x_6+5} + 2z^{x_4+3} + 2z^{x_5+3} + 2z^{x_3+3} + 3z^{x_3+x_5+x_6+3} + 2z^{x_3+x_5+4} + 2z^6 + 2z^{x_3+x_6+5} + 2z^{x_6+3} + 3z^{x_4+x_5+x_6+3} + 2z^{x_4+x_5+4} + 2z^{x_4+x_6+5} + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+1} + z^{x_3+x_5+1} + z^{x_3+x_6+1} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + z^{x_5+x_6+1} + z^{x_4+4} + z^{x_5+4} + z^{x_6+5} - (z^{x_3+x_4+x_5+5} + z^{x_3+x_6+6} + z^{x_3+5} + z^{x_4+x_6+6} + z^{x_4+5} + z^{x_5+x_6+6} + 3z^{x_5+5} + z^{x_6+5} + z^7 + z^{x_3+x_4+x_6+3} + 3z^{x_3+x_4+3} + 2z^{x_3+x_5+x_6+4} + 3z^{x_3+x_5+3} + 2z^{x_3+x_6+3} + 2z^{x_3+5} + 2z^{x_4+x_5+x_6+4} + 3z^{x_4+x_5+3} + 2z^{x_4+x_6+3} + 2z^{x_4+5} + 2z^{x_5+x_6+3} + 2z^{x_6+6} + z^{x_3+x_4+x_6+4} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+x_6+1} + z^{x_3+x_5+x_6+1} + z^{x_3+x_6+4} + z^{x_3+1} + z^{x_4+x_5+x_6+1} + z^{x_4+x_6+4} + z^{x_4+1} + z^{x_5+x_6+4} + z^{x_5+1} + z^{x_6+1}).$$

Compare the l.r.p in  $Q_{16}(A)$  and the l.r.p in  $Q_{29}(B)$  we have  $x_3 = x_4 = 2$  or  $x_3 = x_5 = 2$  or  $x_4 = x_5 = 2$ .

Without loss of generality, suppose that  $x_3 = x_4 = 3$ .

$$Q_{30}(B) = z^{x_5+8} + z^{x_6+7} + z^8 + z^{x_6+7} + z^8 + 3z^{x_5+x_6+5} + z^{x_5+6} + z^{x_6+7} + 3z^{x_6+7} + 2z^{x_5+6} + 2z^{x_6+7} + 2z^5 + 2z^{x_5+3} + 2z^5 + 3z^{x_5+x_6+5} + 2z^{x_5+6} + 2z^6 + 2z^{x_6+7} + 2z^{x_6+3} + 3z^{x_5+x_6+5} + 2z^{x_5+6} + 2z^{x_6+7} + z^{x_5+6} + z^5 + z^{x_5+3} + z^{x_6+3} + z^{x_5+1} + z^6 + z^{x_5+4} + z^{x_6+5} - (z^{x_5+9} + z^{x_6+8} + z^7 + z^{x_6+8} + z^7 + z^{x_5+x_6+6} + 3z^{x_5+5} + z^{x_6+5} + z^7 + z^{x_6+7} + 3z^7 + 2z^{x_5+x_6+6} + 3z^{x_5+5} + 2z^{x_6+5} + 2z^7 + 2z^{x_5+x_6+6} + 3z^{x_5+5} + 2z^{x_6+5} + 2z^7 + 2z^{x_5+x_6+3} + 2z^{x_5+6} + 2z^{x_6+8} + z^{x_5+5} + z^{x_6+5} + z^{x_5+x_6+3} + z^{x_6+6} + z^3 + z^{x_5+x_6+3} + z^{x_6+6} + z^3 + z^{x_5+x_6+4} + z^{x_5+1} + z^{x_6+1}).$$

Compare the l.r.p in  $Q_{16}(A)$  and  $Q_{23}(B)$ , we have  $Q_{16}(A) \neq Q_{29}(B)$ , a contradiction.

### Case 1.2.3 : $u = 4$ .

Since  $2 \leq s \leq t \leq 4$ .

If  $s = 2$ , then

$t = 2$ ,  $A \cong \theta(2, 2, 2, 2, 2, 4)$ ,  $A$  is  $\chi$ -unique by Lemma 9.

$t = 3$ ,  $A \cong \theta(2, 2, 2, 2, 3, 4)$ ,  $A$  is  $\chi$ -unique by Lemma 15.

$t = 4$ ,  $A \cong \theta(2, 2, 2, 2, 4, 4)$ ,  $A$  is  $\chi$ -unique by Lemma 9.

If  $s = 3$ , then

$t = 3$ ,  $A \cong \theta(2, 2, 3, 3, 3, 4)$ ,  $A$  is  $\chi$ -unique by Lemma 14.

$t = 4$ ,  $A \cong \theta(2, 2, 3, 3, 4, 4)$ ,  $A$  is  $\chi$ -unique by Lemma 13.

If  $s = 4$ , then

$t = 4$ ,  $A \cong \theta(2, 2, 4, 4, 4, 4)$ ,  $A$  is  $\chi$ -unique by Lemma 9.

### Case 2 : $r = 3$ .

Then  $g(A) = g(B) = 2r = 6$ . Since  $A$  has a cycle of length 6, therefore  $B$  has one cycle of length 6. Without loss of generality, we have two cases to consider ,

1.  $x_6 = 5$  or

2.  $x_6 \neq 5$ .

### Case 2.1 : $x_6 = 5$ .

Substituting  $r = 3$  in  $Q_{23}(A)$ , we obtain that there is  $-2z^4$  in  $Q_{23}(A)$ . There are cases to be considered  $x_1 = x_2 = 3$  or  $x_1 = x_3 = 3$  or  $x_1 = x_4 = 3$  or  $x_1 = x_5 = 3$  or  $x_2 = x_3 = 3$  or  $x_2 = x_4 = 3$  or  $x_2 = x_5 = 3$  or  $x_3 = x_4 = 3$  or  $x_3 = x_5 = 3$  or  $x_4 = x_5 = 3$ .

We know that  $B$  has at least three cycles of length 6 for all cases. Thus a contradiction.

**Case 2.2:**  $x_6 \neq 5$ .

Since the girth of  $B$  is 6, then  $x_6 \geq 6$ . Given that  $B$  has one cycle of length 6, then  $x_1 + x_2 = 6$  implying  $x_1 = x_2 = 3$ .

$$Q_{17}(A) = z^{2s+7} + z^{t+u+7} + 2z^{s+u+7} + 2z^{t+u+1} + z^{2s+1} + 2z^{s+t+1} + z^7 + z^{t+u+1} + 2z^{s+u+1} + 2z^{2s+u+4} + 2z^{2s+t+4} + 2z^{t+4} + 2z^{u+4} + 4z^{s+t+u+4} + 2z^{s+t+u+3} + 2z^{t+u+6} + z^{t+9} + z^{2s+t+3} + z^{t+5} + 4z^{s+t+6} + z^{u+9} + z^{2s+u+3} + z^{u+5} + 4z^{s+u+6} + 2z^{s+6} + 2z^{s+5} + 2z^8 + 2z^{2s+6} + 4z^{s+4} - (3z^{t+7} + 3z^{u+7} + 5z^{s+7} + z^{2s+t+1} + z^{2s+u+1} + z^{t+1} + 2z^{s+t+u+1} + 2z^{s+1} + 4z^{t+u+4} + 4z^{s+t+4} + z^{2s+t+u+2} + 2z^{2s+4} + z^{t+u+4} + z^{s+u+4} + z^{s+t+4} + z^6 + 2z^{2s+t+5} + 2z^{2s+u+5} + 4z^{s+t+u+5} + 4z^{s+u+4} + z^{u+1} + 2z^{2s+4} + 2z^{2s+8} + 2z^{t+u+8} + z^{10} + 2z^{s+u+8} + 2z^{s+t+8}).$$

$$Q_{31}(B) = z^{x_3+x_4+x_5+6} + z^{x_3+x_6+7} + z^{x_3+8} + z^{x_4+x_6+7} + z^{x_4+8} + z^{x_5+x_6+7} + z^{x_5+8} + z^{x_6+8} + 3z^7 + 2z^{x_3+x_4+x_6+4} + 2z^{x_3+x_4+5} + 2z^{x_3+x_5+x_6+4} + 2z^{x_3+x_5+5} + 2z^{x_3+x_6+6} + 3z^{x_3+4} + 2z^{x_4+x_5+x_6+4} + 2z^{x_4+x_5+5} + 2z^{x_4+x_6+6} + 3z^{x_4+4} + 2z^{x_5+x_6+6} + 3z^{x_5+4} + 2z^{x_6+4} + z^{x_3+x_4+x_5+x_6+1} + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+x_6+3} + z^{x_3+x_4+1} + z^{x_3+x_5+1} + z^{x_3+x_5+x_6+3} + z^{x_3+x_6+1} + z^{x_4+x_5+x_6+3} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + z^{x_6+5} + z^{x_5+x_6+1} - (z^{x_3+x_4+x_5+7} + z^{x_3+x_6+8} + z^{x_3+7} + z^{x_4+x_6+8} + z^{x_4+7} + z^{x_5+x_6+8} + z^{x_5+7} + z^{x_6+7} + z^9 + z^{x_3+x_4+x_6+4} + 2z^{x_3+x_4+x_6+5} + 2z^{x_3+x_5+4} + 3z^{x_3+x_6+4} + 2z^{x_3+6} + 2z^{x_4+x_5+x_6+5} + 2z^{x_4+x_5+4} + 3z^{x_4+x_6+4} + 2z^{x_4+x_6+6} + 3z^{x_5+x_6+4} + 2z^{x_5+6} + 2z^{x_6+7} + 2z^{x_3+x_4+x_6+5} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+x_6+1} + z^{x_3+x_4+3} + z^{x_3+x_5+x_6+1} + z^{x_3+x_5+3} + z^{x_3+1} + z^{x_4+x_5+x_6+1} + z^{x_4+x_5+3} + z^{x_4+1} + z^{x_5+1} + z^{x_6+1} + z^5).$$

Compare the l.r.p in  $Q_{17}(A)$  and the l.r.p in  $Q_{31}(B)$ , we have  $s = 4$  or  $t = 4$  or  $u = 4$ .

**Case 2.2.1:**  $s = 4$ .

$$Q_{18}(A) = z^{13} + 3z^{t+11} + 3z^{u+11} + 3z^{t+u+1} + 3z^9 + 3z^{t+5} + 3z^{u+5} + z^7 + 2z^{t+4} + 2z^{u+4} + 3z^{t+u+8} + 2z^{t+u+7} + 2z^{t+u+6} + z^{t+u+7} + 4z^{t+10} + 4z^{u+10} + z^{10} + 6z^8 + 2z^{14} - (3z^{t+7} + 3z^{u+7} + 5z^{11} + z^{t+1} + z^{u+1} + 2z^{t+u+5} + 3z^{t+u+4} + 6z^{t+8} + 6z^{u+8} + 2z^{16} + z^{t+u+10} + z^{12} + 2z^{t+13} + 2z^{u+13} + 4z^{t+u+9} + z^6 + 2z^5).$$

Compare the l.r.p in  $Q_{18}(A)$  and the l.r.p in  $Q_{31}(B)$ , we have  $-2z^5$  in  $Q_{18}(A)$  and  $-z^5$  in  $Q_{31}(B)$ , so  $x_3 = 4$  or  $x_4 = 4$  or  $x_5 = 4$ .

**Case 2.2.1.1:**  $x_3 = 4$ .

$$Q_{32}(B) = z^{x_4+x_5+10} + z^{x_6+11} + z^{12} + z^{x_4+x_6+7} + z^{x_4+8} + z^{x_5+x_6+7} + z^{x_5+8} + z^{x_6+8} + 3z^7 + 2z^{x_4+x_6+8} + 2z^{x_4+9} + 2z^{x_5+x_6+8} + 2z^{x_5+9} + 2z^{x_6+10} + 3z^8 + 2z^{x_4+x_5+x_6+4} + 2z^{x_4+x_6+5} + 2z^{x_4+x_5+6} + 3z^{x_4+4} + 2z^{x_5+x_6+6} + 3z^{x_5+4} + 2z^{x_6+4} + z^{x_4+x_5+x_6+5} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + z^{x_5+x_6+1} + z^{x_6+5} - (z^{x_4+x_5+11} + z^{x_6+12} + z^{11} + z^{x_4+x_6+8} + z^{x_4+7} + z^{x_5+x_6+8} + z^{x_5+7} + z^{x_6+7} + z^9 + z^{x_4+x_6+8} + 2z^{x_4+8} + 2z^{x_5+x_6+9} + 2z^{x_5+8} + 3z^{x_6+8} + 2z^{10} + 2z^{x_4+x_5+4} + 3z^{x_4+x_6+4} + 2z^{x_4+6} + 3z^{x_5+x_6+4} + 2z^{x_5+6} + 2z^{x_6+7} + 2z^{x_4+x_6+9} + 2z^{x_4+x_5+5} + z^{x_4+x_6+5} + z^{x_4+7} + z^{x_5+x_6+5} + z^{x_5+7} + z^{x_4+x_5+3} + z^{x_4+1} + z^{x_5+1} + z^{x_6+1}).$$

Compare the l.r.p in  $Q_{18}(A)$  and the l.r.p in  $Q_{32}(B)$ , we have  $x_4 = 5$  or  $x_5 = 5$ .

**Case 2.2.1.1.1:**  $x_4 = 5$ .

$$Q_{33}(B) = z^{x_5+15} + 3z^{x_6+11} + z^{x_6+12} + z^{x_5+x_6+7} + z^{x_5+8} + z^{x_6+8} + 2z^7 + 2z^{x_5+x_6+8} + z^{x_5+10} + z^{x_6+10} + 2z^{x_5+x_6+6} + 3z^{x_5+4} + 2z^{x_6+4} + z^{x_5+x_6+10} + z^{x_6+6} + z^{x_5+x_6+1} + z^{x_6+5} - (z^{x_5+16} + z^{x_6+12} + z^{11} + 2z^{x_5+7} + 3z^{x_6+7} + z^9 + 2z^{13} + 3z^{x_5+8} + 3z^{x_6+8} + 2z^{10} + 3z^{x_6+9} + z^{x_6+1} + 3z^{x_5+x_6+4} + z^{x_5+6} + 2z^{x_6+14} + z^{x_5+x_6+5} + z^{x_5+1}).$$

Compare the l.r.p in  $Q_{18}(A)$  and the l.r.p in  $Q_{33}(B)$ , we have  $x_5 = x_6 = 6$ .

$$Q_{19}(A) = 3z^{t+11} + 3z^{u+11} + 3z^{t+u+1} + 3z^{t+5} + 3z^{u+5} + z^7 + 2z^{t+4} + 2z^{u+4} + 3z^{t+u+8} + 2z^{t+u+7} + 2z^{t+u+6} + 4z^{t+10} + 4z^{u+10} + z^{10} + 6z^8 - (3z^{t+7} + 3z^{u+7} + 5z^{11} + z^{t+1} + z^{u+1} + z^{t+u+5} + 3z^{t+u+4} + 6z^{t+8} + 6z^{u+8} + 2z^{16} + z^{t+u+10} + 2z^{t+13} + 2z^{u+13} + 2z^{t+u+9}).$$

$$Q_{34}(B) = z^{21} + z^{19} + 2z^{18} + 2z^{17} + z^{16} + z^{12} + 3z^{10} - (3z^{15} + 4z^{14} + 6z^{13} + 2z^{11} + z^9).$$

Compare the l.r.p in  $Q_{19}(A)$  and the l.r.p in  $Q_{34}(B)$ , we obtain  $Q_{18}(A) \neq Q_{32}(B)$ , a contradiction.

**Case 2.2.1.1.2:**  $x_5 = 5$ .

Similar to Case 2.2.1.1.1, we obtain a contradiction.

**Case 2.2.1.2.1:**  $x_4 = 4$ .

Similar to Case 2.2.1.1.1, we obtain a contradiction.

**Case 2.2.1.2.2:**  $x_5 = 4$ .

Similar to Case 2.2.1.1, we obtain a contradiction.

**Case 2.2.2:**  $t = 4$ .

Since  $3 \leq s \leq 4 \leq u$ . We know that  $s = 3$  or  $s = 4$ .

If  $s = 3$ ,  $A \cong \theta(3, 3, 3, 4, u)$ ,  $A$  is  $\chi$ -unique by Lemma 12.

If  $s = 4$ ,  $A \cong \theta(3, 3, 4, 4, u)$ ,  $A$  is  $\chi$ -unique by Lemma 12.

**Case 2.2.3:**  $u = 4$ .

Since  $3 \leq s \leq t \leq 4$ . We know that  $s = t = 3$  or  $s = 3, t = 4$  or  $s = t = 4$ .

If  $s = t = 3$ ,  $A \cong \theta(3, 3, 3, 3, 4)$ ,  $A$  is  $\chi$ -unique by Lemma 9.

If  $s = 3, t = 4$ ,  $A \cong \theta(3, 3, 3, 3, 4, 4)$ ,  $A$  is  $\chi$ -unique by Lemma 9.

If  $s = t = 4$ ,  $A \cong \theta(3, 3, 4, 4, 4, 4)$ ,  $A$  is  $\chi$ -unique by Lemma 9.

**Case 3 :  $r = 4$ .**

Then  $g(A) = g(B) = 2r = 8$ . Since  $A$  has a cycle of length 8, therefore  $B$  has one cycle of length 8. Without loss of generality, we have two cases to consider ,

1.  $x_6 = 7$  or

2.  $x_6 \neq 7$ .

**Case 3.1 :  $x_6 = 7$ .**

Since  $B$  has one cycle of length 8.

$$Q_{20}(A) = z^{2s+9} + z^{t+u+9} + 3z^9 + 2z^{s+u+9} + 2z^{s+t+9} + z^{t+u+1} + z^{2s+1} + 2z^{s+t+1} + 2z^{s+u+1} + 2z^{2s+t+5} + 2z^{2s+u+5} + 3z^{t+5} + 3z^{u+5} + 4z^{s+t+u+5} + z^{2s+t+u+1} + 2z^{s+t+u+3} + 2z^{t+u+7} + z^{t+11} + z^{2s+t+3} + 4z^{s+t+7} + z^{u+11} + z^{2s+u+3} + 4z^{s+u+7} + 2z^{s+7} + 6z^{s+5} + 2z^{2s+7} - (z^{t+9} + z^{u+9} + 2z^{s+9} + z^{2s+t+1} + z^{2s+u+1} + z^{t+1} + 2z^{s+t+u+1} + 2z^{s+1} + z^6 + 2z^{t+u+5} + z^5 + 4z^{s+t+5} + 4z^{s+u+5} + z^{u+1} + 2z^{2s+5} + z^{2s+10} + z^{t+u+10} + z^{12} + 2z^{s+t+10} + 2z^{s+u+10} + z^{2s+4} + z^{t+u+4} + 2z^{s+u+4} + 2z^{s+t+4} + 2z^{2s+t+6} + 2z^{2s+u+6} + 2z^{t+8} + z^{2s+t+u+2} + 2z^{u+8} + 4z^{s+t+u+6} + 4z^{s+8}).$$

$$Q_{34}(B) = z^{x_1+x_2+x_3+8} + z^{x_1+x_2+x_3+2} + z^{x_1+x_2+x_4+8} + z^{x_1+x_2+x_5+2} + z^{x_1+x_2+x_10} + z^{x_1+x_2+1} + z^{x_1+x_3+x_4+8} + z^{x_1+x_3+x_4+2} + z^{x_1+x_3+x_5+8} + z^{x_1+x_3+x_5+2} + z^{x_1+x_3+10} + z^{x_1+x_3+1} + z^{x_1+x_4+x_5+8} + z^{x_1+x_4+x_5+2} + z^{x_1+x_4+10} + z^{x_1+x_4+1} + z^{x_1+x_5+10} + z^{x_1+x_5+1} + z^{x_1+8} + z^{x_1+4} + z^{x_2+x_3+x_4+8} + z^{x_2+x_3+x_4+2} + z^{x_2+x_3+x_5+8} + z^{x_2+x_3+x_5+2} + z^{x_2+x_3+10} + z^{x_2+x_3+1} + z^{x_2+x_4+x_5+8} + z^{x_2+x_4+x_5+2} + z^{x_2+x_4+10} + z^{x_2+x_4+1} + z^{x_2+x_5+10} + z^{x_2+x_5+1} + z^{x_2+x_5+8} + z^{x_2+x_4+8} + z^{x_2+x_4+4} + z^{x_3+x_4+x_5+8} + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+10} + z^{x_3+x_4+1} + z^{x_3+x_5+10} + z^{x_3+x_5+1} + z^{x_3+x_3+8} + z^{x_3+x_3+4} + z^{x_4+x_5+10} + z^{x_4+x_5+1} + z^{x_4+x_4+8} + z^{x_4+x_4+4} + z^{x_5+x_5+8} + z^{x_5+x_5+4} + z^{x_1+x_2+x_3+9} + z^{x_1+x_2+x_3+1} + z^{x_1+x_2+x_4+9} + z^{x_1+x_2+x_4+1} + z^{x_1+x_2+x_5+9} + z^{x_1+x_2+x_5+1} + z^{x_1+x_2+8} + z^{x_1+x_2+3} + z^{x_1+x_3+x_4+8} + z^{x_1+x_3+x_4+1} + z^{x_1+x_3+x_5+9} + z^{x_1+x_3+x_5+1} + z^{x_1+x_3+8} + z^{x_1+x_3+3} + z^{x_1+x_4+x_5+9} + z^{x_1+x_4+x_5+1} + z^{x_1+x_3+x_5+1} + z^{x_1+x_4+8} + z^{x_1+x_5+8} + z^{x_1+x_4+3} + z^{x_1+x_5+3} + z^{x_1+11} + z^{x_1+1} + z^{x_2+x_3+x_4+9} + z^{x_2+x_3+x_4+1} + z^{x_2+x_3+x_5+9} + z^{x_2+x_3+x_5+1} + z^{x_2+x_3+8} + z^{x_2+x_3+3} + z^{x_2+x_4+x_5+9} + z^{x_2+x_4+x_5+1} + z^{x_2+x_4+x_5+1} + z^{x_2+x_4+8} + z^{x_2+x_4+3} + z^{x_2+x_5+8} + z^{x_2+x_5+3} + z^{x_2+x_6+4} + z^{x_2+1} + z^{x_3+x_4+x_5+9} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+8} + z^{x_3+x_4+3} + z^{x_3+x_5+8} + z^{x_3+x_5+3} + z^{x_3+11} + z^{x_3+1} + z^{x_4+x_5+8} + z^{x_4+x_5+3} + z^{x_4+11} + z^{x_4+1} + z^{x_5+11} + z^{x_5+1} + z^8).$$

Compare the l.r.p in  $Q_{20}(A)$  and the l.r.p in  $Q_{34}(B)$ , we have  $x_1 = 4$ .

$$Q_{35}(B) = z^{x_2+x_3+12} + z^{x_2+x_3+6} + z^{x_2+x_4+12} + z^{x_2+x_4+6} + z^{x_2+x_5+12} + z^{x_2+x_5+6} + z^{x_2+14} + z^{x_2+5} + z^{x_3+x_4+12} + z^{x_3+x_4+6} + z^{x_3+x_5+12} + z^{x_3+x_5+6} + z^{x_3+14} + z^{x_3+5} + z^{x_4+x_5+12} + z^{x_4+x_5+6} + z^{x_4+x_4+14} + z^{x_4+5} + z^{x_5+14} + z^{x_5+4} + z^{x_12} + z^{x_2+x_3+x_4+8} + z^{x_2+x_3+x_4+2} + z^{x_2+x_3+x_5+8} + z^{x_2+x_3+x_5+2} + z^{x_2+x_3+10} + z^{x_2+x_3+1} + z^{x_2+x_4+x_5+8} + z^{x_2+x_4+x_5+2} + z^{x_2+x_4+10} + z^{x_2+x_4+1} + z^{x_2+x_5+10} + z^{x_2+x_5+1} + z^{x_2+x_5+1} + z^{x_2+x_5+8} + z^{x_2+x_2+8} + z^{x_2+x_4+4} + z^{x_3+x_4+x_5+8} + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+10} + z^{x_3+x_4+1} + z^{x_3+x_5+10} + z^{x_3+x_5+1} + z^{x_3+x_5+1} + z^{x_3+x_3+8} + z^{x_3+x_4+x_5+10} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+x_5+9} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+x_5+8} + z^{x_3+x_4+x_5+4} + z^{x_4+x_5+10} + z^{x_4+x_5+1} + z^{x_4+x_4+8} + z^{x_4+x_4+4} + z^{x_5+8} + z^{x_5+4} + z^{x_12} - (z^{x_2+x_3+13} + z^{x_2+x_3+5} + z^{x_2+x_4+13} + z^{x_2+x_4+5} + z^{x_2+x_5+13} + z^{x_2+x_5+5} + z^{x_2+x_2+12} + z^{x_2+x_2+7} + z^{x_3+x_4+12} + z^{x_3+x_4+5} + z^{x_3+x_5+13} + z^{x_3+x_5+5} + z^{x_3+x_3+12} + z^{x_3+x_3+7} + z^{x_4+x_5+13} + z^{x_4+x_5+5} + z^{x_4+x_5+1} + z^{x_4+x_5+12} + z^{x_5+12} + z^{x_5+7} + z^{x_5+15} + z^{x_5+1} + z^{x_5+1} + z^{x_2+x_3+x_4+9} + z^{x_2+x_3+x_4+1} + z^{x_2+x_3+x_5+9} + z^{x_2+x_3+x_5+1} + z^{x_2+x_3+x_3+8} + z^{x_2+x_3+x_3+3} + z^{x_2+x_3+x_5+9} + z^{x_2+x_3+x_5+1} + z^{x_2+x_4+x_5+9} + z^{x_2+x_4+x_5+1} + z^{x_2+x_4+x_5+1} + z^{x_2+x_4+x_5+8} + z^{x_2+x_4+x_5+3} + z^{x_2+x_5+8} + z^{x_2+x_5+3} + z^{x_2+x_6+4} + z^{x_2+1} + z^{x_3+x_4+x_5+9} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+8} + z^{x_3+x_4+3} + z^{x_3+x_5+8} + z^{x_3+x_5+3} + z^{x_3+11} + z^{x_3+1} + z^{x_4+x_5+8} + z^{x_4+x_5+3} + z^{x_4+11} + z^{x_4+1} + z^{x_5+11} + z^{x_5+1}).$$

Compare the l.r.p in  $Q_{20}(A)$  and the l.r.p in  $Q_{35}(B)$ , we have  $x_2 = 5$ .

$$Q_{36}(B) = z^{x_3+17} + z^{x_3+11} + z^{x_4+17} + z^{x_4+11} + z^{x_5+17} + z^{x_5+11} + z^{19} + z^{10} + z^{x_3+x_4+12} + z^{x_3+x_4+6} + z^{x_3+x_5+12} + z^{x_3+x_5+6} + z^{x_3+14} + z^{x_3+5} + z^{x_4+x_5+12} + z^{x_4+x_5+6} + z^{x_4+x_4+14} + z^{x_4+5} + z^{x_5+14} + z^{x_5+4} + z^{x_12} + z^{x_3+x_4+13} + z^{x_3+x_4+7} + z^{x_3+x_5+13} + z^{x_3+x_5+7} + z^{x_3+15} + z^{x_3+6} + z^{x_4+x_5+13} + z^{x_4+x_5+7} + z^{x_4+15} + z^{x_4+6} + z^{x_5+15} + z^{x_5+6} + z^{13} + z^{x_3+x_4+x_5+8} + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+10} + z^{x_3+x_4+1} + z^{x_3+x_5+10} + z^{x_3+x_5+1} + z^{x_3+x_5+8} + z^{x_3+x_4+10} + z^{x_3+x_4+1} + z^{x_3+x_5+8} + z^{x_3+x_5+4} + z^{x_4+x_5+10} + z^{x_4+x_5+1} + z^{x_4+x_4+8} + z^{x_4+x_4+4} + z^{x_5+8} + z^{x_5+4} + z^{x_12} - (z^{x_3+18} + z^{x_3+10} + z^{x_4+18} + z^{x_4+10} + z^{x_5+18} + z^{x_5+10} + z^{17} + z^{12} + z^{x_3+x_4+12} + z^{x_3+x_4+5} + z^{x_3+x_5+13} + z^{x_3+x_5+5} + z^{x_3+12} + z^{x_3+7} + z^{x_4+x_5+13} + z^{x_4+x_5+5} + z^{x_4+12} + z^{x_5+12} + z^{x_4+7} + z^{x_5+7} + z^{15} + z^{x_2+x_3+x_4+9} + z^{x_2+x_3+x_4+1} + z^{x_2+x_3+x_5+9} + z^{x_2+x_3+x_5+1} + z^{x_2+x_3+x_3+8} + z^{x_2+x_3+x_3+3} + z^{x_2+x_3+x_5+9} + z^{x_2+x_3+x_5+1} + z^{x_2+x_4+x_5+9} + z^{x_2+x_4+x_5+1} + z^{x_2+x_4+x_5+1} + z^{x_2+x_4+x_5+8} + z^{x_2+x_4+x_5+3} + z^{x_2+x_5+8} + z^{x_2+x_5+3} + z^{x_2+x_6+4} + z^{x_2+1} + z^{x_3+x_4+x_5+9} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+8} + z^{x_3+x_4+3} + z^{x_3+x_5+8} + z^{x_3+x_5+3} + z^{x_3+11} + z^{x_3+1} + z^{x_4+x_5+8} + z^{x_4+x_5+3} + z^{x_4+11} + z^{x_4+1} + z^{x_5+11} + z^{x_5+1}).$$

Compare the l.r.p in  $Q_{20}(A)$  and the l.r.p in  $Q_{36}(B)$ , we have  $s = 5$  or  $s = 6$  or  $s = 7$  or  $s = 8$  or  $s \geq 9$ .

**Case 3.1.1 :  $s = 4$ .**

Since  $5 \leq x_3 \leq x_4 \leq x_5$ . We have  $Q_{19}(A) \neq Q_{34}(B)$ .

**Case 3.1.2 :  $s = 5$ .**

$$Q_{21}(A) = z^{19} + z^{t+u+9} + 2z^{t+14} + 2z^{u+14} + z^{t+u+1} + z^{11} + 2z^{t+6} + 2z^{u+6} + z^{u+13} + 3z^{t+5} + 3z^{u+5} + 3z^{t+u+10} + 2z^{t+u+8} + 4z^{u+12} + 2z^{t+u+7} + z^{t+13} + 4z^{t+12} + 2z^{12} + 6z^{10} + 2z^{17} + 2z^9 - (3z^{t+9} + 3z^{u+9} + 3z^{14} + z^{t+1} + 2z^{t+u+6} + 2z^6 + 2z^{t+u+5} + 4z^{t+10} + 4z^{u+10} + z^{u+1} + 2z^{15} + z^{20} + z^{t+u+4} + 2z^{t+16} + 2z^{u+16} + 2z^{t+8} + 2z^{u+8} + 4z^{t+u+11} + 4z^{13}).$$

$$Q_{37}(B) = z^{x_3+17} + z^{x_3+11} + z^{x_4+17} + z^{x_4+11} + z^{x_5+17} + z^{x_5+11} + z^{19} + z^{10} + z^{x_3+x_4+12} + z^{x_3+x_4+6} + z^{x_3+x_5+12} + z^{x_3+x_5+6} + z^{x_3+14} + z^{x_3+5} + z^{x_4+x_5+12} + z^{x_4+x_5+6} + z^{x_4+14} + z^{x_5+4} + z^{12} + z^{x_3+x_4+13} + z^{x_3+x_4+7} + z^{x_3+x_5+13} + z^{x_3+x_5+7} + z^{x_3+8} + z^{x_4+x_5+14} + z^{x_4+x_5+6} + z^{x_4+x_5+14} + z^{x_4+x_5+6} + z^{x_4+13} + z^{x_4+8} + z^{x_5+13} + z^{x_5+8} + z^{x_6+9} + z^{x_3+x_4+x_5+9} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+8} + z^{x_3+x_4+3} + z^{x_3+x_5+8} + z^{x_3+x_5+3} + z^{x_3+11} + z^{x_3+1} + z^{x_4+x_5+8} + z^{x_4+x_5+3} + z^{x_4+11} + z^{x_4+1} + z^{x_5+11} + z^{x_5+1})$$

$$z^{x_3+15} + z^{x_3+6} + z^{x_4+x_5+13} + z^{x_4+x_5+7} + z^{x_4+15} + z^{x_4+6} + z^{x_5+15} + z^{x_5+6} + z^{13} + z^{x_3+x_4+x_5+8} + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+10} + z^{x_3+x_4+1} + z^{x_3+x_5+10} + z^{x_3+x_5+1} + z^{x_3+8} + z^{x_3+4} + z^{x_4+x_5+10} + z^{x_4+x_5+1} + z^{x_4+8} + z^{x_4+4} + z^{x_5+8} + z^{x_5+4} + 2z^{12} - (z^{x_3+18} + z^{x_3+10} + z^{x_4+18} + z^{x_4+10} + z^{x_5+18} + z^{x_5+10} + z^{17} + z^{12} + z^{x_3+x_4+12} + z^{x_3+x_4+5} + z^{x_3+x_5+13} + z^{x_3+x_5+5} + z^{x_3+12} + z^{x_3+7} + z^{x_4+x_5+13} + z^{x_4+x_5+5} + z^{x_4+12} + z^{x_5+12} + z^{x_4+7} + z^{x_5+7} + z^{15} + z^{x_2+x_3+x_4+9} + z^{x_3+x_4+6} + z^{x_3+x_5+14} + z^{x_3+x_5+6} + z^{x_3+13} + z^{x_3+8} + z^{x_4+x_5+14} + z^{x_4+x_5+6} + z^{x_4+x_5+14} + z^{x_4+x_5+6} + z^{x_4+13} + z^{x_4+8} + z^{x_5+13} + z^{x_5+8} + z^{x_6+9} + z^{x_3+x_4+x_5+9} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+8} + z^{x_3+x_4+3} + z^{x_3+x_5+8} + z^{x_3+x_5+3} + z^{x_3+11} + z^{x_3+1} + z^{x_4+x_5+8} + z^{x_4+x_5+3} + z^{x_4+11} + z^{x_4+1} + z^{x_5+11} + z^{x_5+1}).$$

Compare the l.r.p in  $Q_{21}(A)$  and the l.r.p in  $Q_{37}(B)$ , we have  $x_3 = x_4 = 5$ .

$$Q_{38}(B) = 2z^{x_5+17} + z^{x_5+14} + 2z^{x_5+12} + 3z^{x_5+6} + z^{x_5+22} + z^{x_5+15} + z^{x_5+4} + 2z^{22} + 2z^{19} + 3z^{20} + 2z^{11} - (3z^{x_5+10} + z^{x_5+13} + z^{x_5+11} + 2z^{x_5+8} + z^{x_5+7} + z^{x_5+1} + z^{24} + z^{23} + 3z^{18} + 2z^{17} + z^{16} + 2z^{15} + z^{12}).$$

Compare the l.r.p in  $Q_{21}(A)$  and the l.r.p in  $Q_{38}(B)$ , we have  $t = 5$  or  $t = 6$  or  $t = 7$  or  $t = 8$  or  $t \geq 9$ .

**Case 3.1.2.1 :**  $t = 5$ .

$A \cong \theta(4, 4, 5, 5, u)$ ,  $A$  is  $\chi$ -unique by Lemma 14.

**Case 3.1.2.2 :**  $t = 6, x_5 = 6$ .

$$Q_{22}(A) = 4z^{u+14} + 2z^{u+6} + z^{u+13} + 3z^{u+5} + 2z^{u+12} + z^{u+7} + z^{u+16} + 2z^{18} + z^{12} + z^{11} + 2z^{17} + 2z^{10} - (3z^{u+9} + 4z^{u+10} + z^{u+1} + 2z^{u+8} + 3z^{13} + z^{15} + 2z^{22} + 5z^{14} + z^{u+17}).$$

$$Q_{39}(B) = z^{28} + z^{21} + 2z^{22} + 3z^{20} - (3z^{17} + z^{24} + 3z^{18} + z^{12}).$$

We have  $Q_{22}(A) \neq Q_{39}(B)$ , a contradiction.

**Case 3.1.2.3 :**  $t = 7, x_5 = 7$ .

$$Q_{23}(A) = 4z^{u+14} + 2z^{u+6} + z^{u+13} + 3z^{u+5} + 2z^{u+12} + 2z^{u+15} + 3z^{u+17} + z^{21} + 4z^{12} + 3z^{10} - (3z^{u+9} + 4z^{u+10} + z^{u+1} + z^{u+8} + 4z^{13} + z^{16} + z^{23} + 2z^{u+16} + 2z^{u+18} + 2z^{u+13} + 2z^{14}).$$

$$Q_{40}(B) = z^{24} + z^{13} + 2z^{29} + 3z^{22} + 3z^{11} + z^{20} - (3z^{17} + z^{25} + 4z^{18} + z^{12}).$$

We have  $Q_{23}(A) \neq Q_{40}(B)$ , a contradiction.

**Case 3.1.2.4 :**  $t = 8, x_5 = 8$ .

$$Q_{24}(A) = 3z^{u+18} + 2z^{u+6} + z^{u+13} + 3z^{u+5} + 4z^{u+12} + z^{21} + 2z^{12} + 3z^{10} - (2z^{u+8} + z^{u+19} + z^{24} + z^{14} + z^{13} + 2z^{u+9} + z^{u+1} + 4z^{u+10}).$$

$$Q_{41}(B) = 2z^{25} + 3z^{14} + z^{30} + z^{22} + z^{19} + 2z^{20} + 2z^{11} - (z^{21} + z^{16} + 3z^{15} + 2z^{18} + z^{17} + z^{12}).$$

We have  $Q_{24}(A) \neq Q_{41}(B)$ , a contradiction.

**Case 3.1.2.5 :**  $t \geq 9, x_5 \geq 9$

$$Q_{25}(A) = 2z^{u+14} + 2z^{u+6} + z^{u+13} + 3z^{u+5} + 4z^{u+12} + z^{u+10} + 3z^{u+19} + 2z^{u+17} + 2z^{23} + 2z^{15} + 3z^{10} + 2z^{12} + 4z^{21} - (z^{u+20} + 4z^{13} + 3z^{u+9} + 4z^{u+10} + z^{u+1} + 2z^{u+16} + 2z^{u+8} + 2z^{25} + z^{19} + z^{20} + 2z^{u+15}).$$

$$Q_{42}(B) = 2z^{26} + 2z^{21} + z^{15} + z^{31} + z^{13} + 2z^{11} + 2z^{19} + 2z^{20} - (4z^{17} + 2z^{16} + z^{12}).$$

We have  $Q_{25}(A) \neq Q_{42}(B)$ , a contradiction.

**Case 3.1.3 :**  $s = 6$ .

$$Q_{26}(A) = z^{21} + 3z^{t+u+9} + 3z^{u+15} + 3z^{t+15} + z^{t+u+1} + 3z^{13} + 3z^{u+13} + 2z^{u+7} + 2z^{t+7} + 2z^{t+17} + 2z^{u+17} + 3z^{t+5} + 3z^{u+5} + 4z^{t+u+11} + 3z^{t+13} + 6z^{11} + 2z^{19} + 2z^9 - (z^{t+9} + z^{u+9} + 2z^{15} + z^{t+1} + 2z^7 + 2z^{t+u+5} + 3z^{t+11} + 3z^{u+11} + z^{u+1} + 2z^{17} + z^{22} + z^{t+u+10} + 2z^{t+16} + 2z^{u+16} + z^{16} + z^{t+u+4} + 2z^{t+10} + 2z^{u+10} + 2z^{t+18} + 2z^{u+18} + 2z^{t+8} + 2z^{u+8} + 4z^{t+u+12} + 4z^{14}).$$

$$Q_{43}(B) = z^{x_3+17} + z^{x_3+11} + z^{x_4+17} + z^{x_4+11} + z^{x_5+17} + z^{x_5+11} + z^{19} + z^{10} + z^{x_3+x_4+12} + z^{x_3+x_4+6} + z^{x_3+x_5+12} + z^{x_3+x_5+6} + z^{x_3+14} + z^{x_3+5} + z^{x_4+x_5+12} + z^{x_4+x_5+6} + z^{x_4+14} + z^{x_4+5} + z^{x_5+14} + z^{x_5+4} + z^{12} + z^{x_3+x_4+13} + z^{x_3+x_4+7} + z^{x_3+x_5+13} + z^{x_3+x_5+7} + z^{x_3+15} + z^{x_3+6} + z^{x_4+x_5+13} + z^{x_4+x_5+7} + z^{x_4+15} + z^{x_4+6} + z^{x_5+15} + z^{x_5+6} + z^{13} + z^{x_3+x_4+x_5+8} + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+10} + z^{x_3+x_4+1} + z^{x_3+x_5+10} + z^{x_3+x_5+1} + z^{x_3+8} + z^{x_3+4} + z^{x_4+x_5+10} + z^{x_4+x_5+1} + z^{x_4+8} + z^{x_4+4} + z^{x_5+8} + z^{x_5+4} + 2z^{12} - (z^{x_3+18} + z^{x_3+10} + z^{x_4+18} + z^{x_5+18} + z^{x_5+10} + z^{17} + z^{12} + z^{x_3+x_4+12} + z^{x_3+x_4+5} + z^{x_3+x_5+13} + z^{x_3+x_5+5} + z^{x_3+12} + z^{x_3+7} + z^{x_4+x_5+13} + z^{x_4+x_5+5} + z^{x_4+12} + z^{x_5+12} + z^{x_4+7} + z^{x_5+7} + z^{15} + z^{x_2+x_3+x_4+9} + z^{x_3+x_4+6} + z^{x_3+x_5+14} + z^{x_3+x_5+6} + z^{x_3+13} + z^{x_3+8} + z^{x_4+x_5+14} + z^{x_4+x_5+6} + z^{x_4+x_5+14} + z^{x_4+x_5+6} + z^{x_4+13} + z^{x_4+8} + z^{x_5+13} + z^{x_5+8} + z^{x_6+9} + z^{x_3+x_4+x_5+9} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+8} + z^{x_3+x_4+3} + z^{x_3+x_5+8} + z^{x_3+x_5+3} + z^{x_3+11} + z^{x_3+1} + z^{x_4+x_5+8} + z^{x_4+x_5+3} + z^{x_4+11} + z^{x_4+1} + z^{x_5+11} + z^{x_5+1}).$$

Compare the l.r.p in  $Q_{26}(A)$  and the l.r.p in  $Q_{43}(B)$ , we have  $x_3 = x_4 = 6$ .

$$Q_{44}(B) = 2z^{x_5+16} + z^{x_5+7} + z^{x_5+24} + z^{x_5+20} + z^{x_5+6} + z^{x_5+4} + z^{x_5+15} + 2z^{23} + z^{20} + z^{21} + 3z^{10} + 3z^{12} + z^{22} - (2z^{x_5+11} + 2z^{x_5+9} + z^{x_5+14} + z^{x_5+21} + z^{x_5+10} + 2z^{24} + z^{26} + 2z^{19} + 2z^{18} + 2z^{16} + z^{x_5+1}).$$

Compare the l.r.p in  $Q_{26}(A)$  and the l.r.p in  $Q_{44}(B)$ , we have  $t = 6$  or  $t = 7$  or  $t = 8$  or  $t \geq 9$ .

**Case 3.1.3.1 :**  $t = 6$ .

$A \cong \theta(4, 4, 6, 6, 6, u)$ ,  $A$  is  $\chi$ -unique by Lemma 14.

**Case 3.1.3.2 :**  $t = 7, x_5 = 7$ .

$$Q_{27}(A) = z^{22} + 3z^{u+15} + 3z^{u+13} + 2z^{14} + 2z^{u+7} + 2z^{24} + z^{u+8} + z^{u+17} + 3z^{11} + 3z^{u+5} + 2z^{u+18} + 2z^{20} + 2z^9 + 2z^{13} - (4z^{14} + 2z^{15} + 2z^{u+19} + z^{u+9} + 2z^{u+12} + 3z^{u+11} + z^{u+1} + 2z^{u+10} + 2z^{23} + 2z^{25} + z^{17}).$$

$$Q_{45}(B) = 4z^{23} + z^{14} + z^{31} + z^{27} + 3z^{10} + z^{25} - (3z^{16} + z^{28} + 2z^{24} + z^{26} + 2z^{19} + z^{18}).$$

We have  $Q_{27}(A) \neq Q_{45}(B)$ , a contradiction.

**Case 3.1.3.3 :**  $t = 8, x_5 = 8$ .

$$Q_{28}(A) = z^{25} + 3z^{u+15} + z^{u+13} + z^{15} + 2z^{u+7} + 2z^{21} + 4z^{11} + 4z^{u+17} + 6z^{13} + 3z^{u+5} + 4z^{u+19} + 2z^9 - (3z^{u+18} + 2z^{u+20} + 2z^{u+10} + 2z^{u+8} + 3z^{u+11} + z^{u+1} + 4z^{14} + 2z^{24} + 2z^{u+16} + z^{17} + z^{26} + z^{22}).$$

$$Q_{46}(B) = z^{23} + z^{28} + z^{14} + 4z^{12} + 3z^{10} + z^{20} - (z^{29} + z^{19} + z^{18} + 2z^{18}).$$

We have  $Q_{28}(A) \neq Q_{46}(B)$ , a contradiction.

**Case 3.1.3.4 :**  $t \geq 9, x_5 \geq 9$ .

$$Q_{29}(A) = 3z^{24} + 3z^{u+15} + 3z^{u+13} + 2z^{u+7} + 2z^{u+17} + 3z^{u+5} + 2z^{16} + z^{u+10} + 2z^{26} + 4z^{u+20} + z^{22} + 2z^9 + 3z^{13} + 4z^{11} - (z^{u+9} + 3z^{u+11} + z^{u+1} + z^{u+19} + 2z^{u+16} + 2z^{u+10} + 2z^{u+8} + 2z^{u+21} + 2z^{u+14} + z^{14} + z^{20} + 2z^{25} + 2z^{27} + 2z^{17}).$$

$$Q_{47}(B) = 3z^{25} + z^{33} + z^{29} + z^{15} + z^{13} + 3z^{12} + z^{23} + z^{20} + z^{21} + 3z^{10} - (5z^{18} + z^{30} + z^{26} + z^{24} + z^{19} + 2z^{16}).$$

We have  $Q_{29}(A) \neq Q_{47}(B)$ , a contradiction.

**Case 3.1.4 :**  $s = 7$ .

$$Q_{30}(A) = z^{23} + 3z^{t+u+9} + 3z^{u+16} + 3z^{t+16} + z^{t+u+1} + z^{15} + 4z^{u+14} + 2z^{t+19} + 2z^{u+19} + 3z^{t+5} + 3z^{u+5} + 4z^{t+u+12} + z^{t+u+10} + 2z^{t+u+7} + 4z^{t+14} + 6z^{12} + 2z^{14} + 2z^9 + 2z^{21} - (z^{t+9} + z^{u+9} + 2z^{16} + z^{t+1} + 2z^8 + 2z^{t+u+8} + z^{u+15} + z^{t+15} + z^{t+u+5} + 4z^{t+12} + 4z^{u+12} + z^{u+1} + 2z^{19} + z^{24} + z^{t+17} + z^{u+17} + z^{18} + z^{t+u+4} + 2z^{t+20} + z^{t+11} + z^{u+11} + 2z^{u+20} + 4z^{t+u+13} + 4z^{15}).$$

Compare the l.r.p in  $Q_{30}(A)$  and the l.r.p in  $Q_{43}(B)$ , we have  $t = 7$  or  $t = 8$  or  $t \geq 9$ .

**Case 3.1.4.1 :**  $t = 7$ .

$A \cong \theta(4, 4, 7, 7, u)$ ,  $A$  is  $\chi$ -unique by Lemma 14.

**Case 3.1.4.2 :**

$$Q_{31}(A) = z^{23} + 3z^{u+16} + 3z^{t+16} + z^{t+u+1} + z^{15} + 4z^{u+14} + 2z^{t+19} + 2z^{u+19} + 3z^{t+5} + 3z^{u+5} + 4z^{t+u+12} + z^{t+u+10} + z^{t+u+7} + 4z^{t+14} + 6z^{12} + 2z^{14} + 2z^9 - (z^{t+9} + z^{u+9} + z^{16} + z^{t+1} + 2z^8 + 2z^{t+u+8} + z^{u+15} + z^{t+15} + z^{t+u+5} + 4z^{t+12} + 4z^{u+12} + z^{u+1} + z^{24} + z^{t+17} + z^{u+17} + z^{18} + z^{t+20} + z^{t+11} + z^{u+11} + 2z^{u+20} + 2z^{t+u+13} + 3z^{15}).$$

$$Q_{48}(B) = 2z^{x_5+17} + 2z^{x_5+19} + 3z^{x_5+14} + z^{x_5+5} + z^{x_5+6} + z^{x_5+22} + z^{x_5+26} + 2z^{x_5+8} + z^{x_5+4} + 3z^{24} + z^{27} + z^{20} + z^{22} + 3z^{13} + 2z^{11} + z^{10} - (z^{x+18} + 3z^{x_5+15} + z^{x_5+23} + z^{x_5+10} + z^{x_5+7} + z^{x_5+12} + 2z^{25} + z^{28} + 3z^{20} + 4z^{17} + 2z^{14} + z^{x_5+1}).$$

Compare the l.r.p in  $Q_{31}(A)$  is  $2z^9$ . We have  $t = u = 8$ .

$$Q_{32}(A) = 7z^{22} + 6z^{13} + 4z^{27} + z^{16} + z^{23} + z^{14} + 6z^{12} - (4z^{20} + 2z^{25} + 2z^{19} + 2z^{29} + z^{21} + z^{17} + 4z^{20} + 3z^{15} + 3z^{18}).$$

Compare the l.r.p in  $Q_{31}(B)$  is  $z^{10}$ . We have  $x_5 = 9$ .

$$Q_{33}(A) = 7z^{22} + 6z^{13} + 4z^{27} + z^{16} + z^{23} + z^{14} + 6z^{12} - (4z^{20} + 2z^{25} + 2z^{19} + 2z^{29} + z^{21} + z^{17} + 4z^{20} + 3z^{15} + 3z^{18}).$$

$$Q_{49}(B) = 2z^{26} + z^{28} + 3z^{23} + z^{31} + z^{35} + 4z^{13} + z^{22} + 2z^{11} - (3z^{24} + z^{32} + z^{19} + z^{16} + z^{21} + 2z^{25} + 2z^{20} + 2z^{14}).$$

We have  $Q_{33}(A) \neq Q_{49}(B)$ , a contradiction.

**Case 3.1.4.3 :**  $t \geq 9, x_5 \geq 9$ .

$$Q_{34}(A) = 3z^{u+16} + 3z^{u+14} + 3z^{u+5} + 3z^{u+19} + z^{u+10} + 4z^{u+21} + 2z^{25} + 2z^{23} + 4z^{14} + 2z^{28} + 6z^{12} + 2z^9 - (z^{u+9} + z^{u+15} + 4z^{u+12} + z^{u+1} + z^{u+17} + z^{u+11} + 2z^{u+20} + 2z^{u+22} + 2z^{u+17} + 4z^{18} + 2z^{24} + z^{10} + 4z^{21} + z^{26} + z^{20} + 2z^{29} + 3z^{15}).$$

$$Q_{50}(B) = 2z^{26} + z^{15} + z^{31} + z^{35} + z^{27} + 4z^{13} + z^{22} + z^{27} + z^{22} - (z^{32} + z^{19} + z^{16} + z^{21} + 2z^{25} + 2z^{20} + 2z^{17} + 2z^{14}).$$

We have  $Q_{34}(A) \neq Q_{50}(B)$ , a contradiction.

**Case 3.1.5 :**  $s = 8$ .

$$Q_{35}(A) = z^{25} + z^{t+u+9} + z^{t+17} + z^{u+17} + z^{t+u+1} + 4z^{u+15} + z^{t+9} + z^{u+9} + 2z^{t+21} + 2z^{u+21} + 3z^{t+5} + 3z^{u+5} + z^{u+11} + 4z^{t+u+13} + 2z^{t+u+11} + 2z^{t+u+7} + z^{t+11} + z^{t+19} + 4z^{t+15} + z^{u+19} + 2z^{15} + 6z^{13} + 2z^{23} - (z^{17} + z^{t+17} + z^{u+17} + z^{t+1} + 2z^{t+u+9} + 2z^{t+u+5} + 4z^{t+13} + 4z^{u+13} + z^{u+1} + 2z^{21} + z^{26} + z^{t+u+10} + 2z^{t+8} + 2z^{u+8} + z^{20} + z^{t+u+4} + 2z^{t+12} + 2z^{u+12} + 2z^{t+22} + 2z^{u+22} + 2z^{t+8} + 2z^{u+8} + 4z^{t+u+14} + 4z^{16}).$$

We have  $t = 8$  or  $t \geq 9$ .

**Case 3.1.5.1 :**  $t = 8$ .

$A \cong \theta(4, 4, 8, 8, u)$ ,  $A$  is  $\chi$ -unique by Lemma 14.

**Case 3.1.5.2 :**  $t \geq 9$ .

$$Q_{36}(A) = z^{u+18} + z^{u+17} + z^{u+10} + 4z^{u+15} + z^{u+9} + 2z^{u+21} + 3z^{u+5} + z^{u+11} + 4z^{u+22} + 2z^{u+20} + 2z^{u+16} + z^{25} + z^{18} + 2z^{30} + 3z^{14} + z^{20} + z^{28} + 4z^{24} + 2z^{15} + 6z^{13} + 2z^{23} - (4z^{u+18} + 2z^{u+14} + 4z^{u+13} + z^{u+1} + 2z^{u+12} + 2z^{u+22} + 2z^{u+8} + z^{u+19} + 4z^{u+23} + 3z^{17} + z^{u+13} + 4z^{22} + 2z^{27} + 2z^{31} + 4z^{16} + z^{10}).$$

Consider the l.r.p in  $Q_{35}(A)$  is  $-z^{10}$ . We have  $x_3 = 9$ .

$$Q_{51}(B) = z^{26} + z^{x_4+15} + z^{x_4+16} + z^{x_5+21} + z^{x_5+15} + z^{23} + z^{14} + z^{x_4+x_5+6} + z^{x_4+5} + z^{x_5+5} + z^{x_4+22} + z^{24} + z^{x_5+16} + z^{x_4+6} + z^{x_5+6} + z^{x_4+x_5+17} + z^{x_4+x_5+21} + z^{x_4+19} + z^{x_5+19} + 2z^{13} + z^{x_4+4} + z^{x_5+4} + z^{19} + z^{12} + z^{10} - (z^{x_5+18} + z^{x_4+18} + z^{19} + z^{x_4+12} + z^{x_4+7} + z^{x_5+12} + z^{x_5+7} + z^{x_4+23} + z^{x_5+23} + z^{x_4+13} + z^{x_5+13} + z^{x_4+x_5+18} + z^{x_4+11} + z^{x_5+17} + z^{x_5+11} + z^{x_4+x_5+3} + z^{x_4+1} + z^{x_5+1} + z^{15} + z^{17}).$$

Consider the l.r.p in  $Q_{36}(B)$  is  $z^{10}$ . We have  $x_4 = 9$ .

$$Q_{37}(A) = z^{u+18} + z^{u+17} + z^{u+10} + 3z^{u+15} + z^{u+9} + 2z^{u+21} + 3z^{u+5} + z^{u+11} + 4z^{u+22} + z^{u+20} + 2z^{u+16} + z^{25} + z^{18} + 2z^{30} + z^{14} + z^{20} + 2z^{24} + 3z^{13} + z^{23} - (4z^{u+18} + z^{u+14} + 4z^{u+13} + z^{u+1} + 2z^{u+12} + z^{u+22} + 2z^{u+8} + z^{u+19} + 4z^{u+23} + 2z^{17} + 2z^{22} + 2z^{21} + 2z^{31} + 2z^{16}).$$

$$Q_{52}(B) = z^{x_5+16} + z^{x_5+15} + z^{x_5+26} + z^{x_5+30} + z^{x_5+19} + z^{x_5+21} + z^{x_5+6} + z^{x_5+5} + z^{x_5+4} + z^{26} + z^{12} - (z^{x_5+27} + z^{x_5+23} + z^{x_5+18} + z^{x_5+17} + z^{x_5+13} + 2z^{x_5+12} + z^{x_5+11} + z^{x_5+7} + z^{x_5+1} + z^{32} + z^{20}).$$

Consider the l.r.p in  $Q_{37}(B)$  is  $z^{12}$ . We have  $x_5 = 11$   $u \geq 9$ .

$$Q_{53}(B) = 2z^{26} + z^{16} + z^{17} + z^{37} + z^{41} + z^{30} + z^{15} + z^{31} - (2z^{23} + z^{29} + z^{34} + z^{18} + z^{38} + z^{28} + z^{22} + z^{20} + z^{24}).$$

We have  $Q_{37}(A) \neq Q_{53}(B)$ , a contradiction.

**Case 3.1.6 :  $s \geq 9$ .**

$$Q_{38}(A) = z^{27} + z^{t+u+9} + 2z^{u+18} + 2z^{t+18} + z^{t+u+1} + z^{19} + 4z^{u+16} + 2z^{t+10} + 2z^{u+10} + 2z^{t+23} + 2z^{u+23} + 3z^{t+5} + 3z^{u+5} + z^{u+11} + 4z^{t+u+14} + 2z^{t+u+12} + 2z^{t+u+7} + z^{t+11} + z^{t+21} + 4z^{t+16} + z^{u+21} + 2z^{16} + 6z^{14} + 2z^{25} - (z^{t+9} + z^{u+9} + 2z^{18} + z^{t+19} + z^{u+19} + z^{t+1} + 2z^{t+u+10} + 2z^{10} + 2z^{t+u+5} + 4z^{t+14} + 4z^{u+14} + z^{u+1} + 2z^{23} + z^{28} + z^{t+u+10} + 2z^{t+19} + 2z^{u+19} + z^{22} + z^{t+u+4} + 2z^{t+13} + 2z^{u+13} + 2z^{t+24} + 2z^{u+24} + 2z^{t+8} + 2z^{u+8} + 4z^{t+u+15} + 4z^{17}).$$

Consider the l.r.p in  $Q_{38}(A)$  is  $2z^9$ . We have  $x_3 = x_4 = 5$ .

$$Q_{54}(B) = 2z^{22} + 3z^{x_5+17} + z^{x_5+11} + z^{x_5+18} + z^{17} + 3z^{19} + 3z^{10} + z^{x_5+14} + z^{x_5+5} + z^{23} + 2z^{x_5+12} + 3z^{20} + 3z^{11} + 3z^{x_5+15} + 3z^{x_5+6} + z^{x_5+22} + z^{x_5+4} + z^{13} + z^{12} - (2z^{23} + 3z^{15} + 3z^{x_5+10} + 2z^{17} + 2z^{12} + z^{x_5+12} + z^{x_5+7} + z^{24} + 2z^{x_5+19} + z^{x_5+11} + 2z^{18} + z^{13} + 2z^{x_5+13} + 2z^6 + z^{x_5+1} + z^{16} + z^{15} + z^{17}).$$

Since  $9 \leq t \leq u$ , the the l.r.p in  $Q_{54}(B)$  is  $-2z^6$  but not in  $Q_{38}(A)$ .

We have  $Q_{38}(A) \neq Q_{54}(B)$ , a contradiction.

If  $x_3 = x_4 = 9$ .

We have  $Q_{38}(A) \neq Q_{54}(B)$ , a contradiction.

**Case 3.2 :  $x_6 \neq 7$ .**

We know that  $x_6 > 7$ . Hence  $x_1 + x_2 = 8$  implies that  $x_1 = x_2 = 4$ .

$$Q_{39}(A) = z^{2s+9} + z^{t+u+9} + 3z^9 + 2z^{s+u+9} + 2z^{s+t+9} + z^{t+u+1} + z^{2s+1} + 2z^{s+u+1} + 2z^{2s+t+5} + 2z^{2s+u+5} + 3z^{t+5} + 3z^{u+5} + 4z^{s+t+u+5} + z^{2s+t+u+1} + 2z^{s+t+u+3} + 2z^{t+u+7} + z^{t+11} + z^{2s+t+3} + 4z^{s+t+7} + z^{u+11} + z^{2s+u+3} + 4z^{s+u+7} + 2z^{s+7} + 6z^{s+5} + 2z^{2s+7} - (z^{t+9} + z^{u+9} + 2z^{s+9} + z^{2s+t+1} + z^{2s+u+1} + z^{t+1} + 2z^{s+t+u+1} + 2z^{s+1} + z^6 + 2z^{t+u+5} + 4z^{s+t+5} + 4z^{s+u+5} + z^{u+1} + 2z^{2s+5} + z^{2s+10} + z^{t+u+10} + z^{12} + 2z^{s+t+10} + 2z^{s+u+10} + z^{2s+4} + z^{t+u+4} + 2z^{s+u+4} + 2z^{s+t+4} + 2z^{2s+u+6} + 2z^{t+8} + 2z^{u+8} + 4z^{s+t+u+6} + 4z^{s+8}).$$

$$Q_{55}(B) = z^{x_3+x_4+x_5+8} + z^{x_3+x_6+9} + z^{x_3+10} + z^{x_4+x_6+9} + z^{x_4+10} + z^{x_5+x_6+9} + z^{x_5+10} + z^{x_6+11} + z^9 + 2z^{x_3+x_4+x_6+5} + 2z^{x_3+x_4+6} + 2z^{x_3+x_5+x_6+5} + 2z^{x_3+x_5+6} + 2z^{x_3+x_6+7} + 2z^{x_3+5} + 2z^{x_4+x_5+x_6+5} + 2z^{x_4+x_5+6} + 2z^{x_4+x_6+7} + 2z^{x_4+5} + 2z^{x_5+x_6+7} + 2z^{x_5+5} + 2z^{x_6+5} + 2z^8 + z^{x_3+x_4+x_5+x_6+1} + z^{x_3+x_4+x_5+2} + z^{x_3+x_4+x_6+3} + z^{x_3+x_4+x_4+1} + z^{x_3+x_5+x_6+3} + z^{x_3+x_5+1} + z^{x_3+x_6+1} + z^{x_3+4} + z^{x_4+x_5+x_6+3} + z^{x_4+x_5+1} + z^{x_4+x_6+1} + z^{x_4+4} + z^{x_5+x_6+1} + z^{x_5+4} + z^{x_6+5} - (z^{x_3+x_4+x_5+} + z^{x_3+x_6+10} + z^{x_3+9} + z^{x_4+x_6+10} + z^{x_4+9} + z^{x_5+x_6+10} + z^{x_5+9} + z^{x_6+9} + z^{11} + z^{x_3+x_4+x_6+5} + z^{x_3+x_4+x_6+6} + 2z^{x_3+x_4+5} + 2z^{x_3+x_5+x_6+6} + 2z^{x_3+x_5+5} + 2z^{x_3+x_6+5} + 2z^{x_3+7} + 2z^{x_4+x_5+x_6+6} + 2z^{x_4+x_5+5} + 2z^{x_4+x_6+5} + 2z^{x_5+x_6+5} + 2z^{x_4+7} + 2z^{x_5+7} + z^{x_6+7} + z^{x_6+8} + z^{x_3+x_4+x_5+1} + z^{x_3+x_4+x_6+1} + z^{x_3+x_4+3} + z^{x_3+x_5+x_6+1} + z^{x_3+x_5+3} + z^{x_3+x_6+4} + z^{x_3+1} + z^{x_4+x_5+x_6+1} + z^{x_4+x_5+3} + z^{x_4+x_6+4} + z^{x_4+1} + z^{x_5+x_6+4} + z^{x_5+1} + z^{x_6+1} + z^5).$$

Compare the l.r.p in  $Q_{39}(A)$  and the l.r.p in the  $Q_{55}(B)$ . We have  $s = 4$  or  $t = 4$  or  $u = 4$ .

**Case 3.2.1 :  $s = 4$**

$A \cong \theta(4, 4, 4, 4, t, u)$ ,  $A$  is  $\chi$ -unique by Lemma 15.

**Case 3.2.2 :  $t = 4$**

We know that  $s = 4$ .

$A \cong \theta(4, 4, 4, 4, 4, u)$ ,  $A$  is  $\chi$ -unique by Lemma 9.

**Case 3.2.3 :  $u = 4$**

We know that  $s = t = 4$ .

$A \cong \theta(4, 4, 4, 4, 4, 4)$ ,  $A$  is  $\chi$ -unique by Theorem 1.

□

#### 4. Conclusion

Many researchers have studied the chromaticity of  $k$ -bridge graphs. A 2-bridge graph is essentially a  $\chi$ -unique cycle graph. The theta graph, represented as  $\theta(1, y_1, y_2)$ , serves as an example of a 3-bridge graph. Chao and Whitehead [1] demonstrated that each such theta graph is  $\chi$ -unique. Extending this result, Loerinc

[2] showed that all 3-bridge graphs are  $\chi$ -unique. The chromaticity of 4-bridge graphs was addressed by Chen et al. [3] and Xu et al. [4]. The chromaticity of 5-bridge graphs has been the subject of investigation in several studies [5–8]. The chromaticity of the 6-bridge graph has been explored by various researchers [11–19]. In this paper, we extend the existing research on the chromaticity of 6-bridge graphs by investigating the chromaticity of a specific 6-bridge graph, denoted as  $\theta(r, r, s, t, u)$ , where  $2 \leq r \leq s \leq t \leq u$ .

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