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The bounds for topological invariants of a weighted graph using traces

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Abstract: In this paper, we obtain the bounds for the Laplacian eigenvalues of a weighted graph using traces. Then, we find the bounds for the Kirchhoff and Laplacian Estrada indices of a weighted graph. Finally, we define the Laplacian energy of a weighted graph and get the upper bound for this energy.

Keywords: Laplacian energy; Kirchhoff index; Laplacian Estrada index; weighted graph.

MSC: 05C50; 05C22.

1. Introduction

Let G be a weighted graph with n vertices and m edges. The Laplacian matrix L of a weighted graph G is the $n \times n$ matrix defined as follows:

$$l_{ij} = \begin{cases} w_i, & \text{if } i = j \\ -w_{ij}, & \text{if } i \sim j \\ 0, & \text{otherwise.} \end{cases}$$

Here, the weight $w_i = \sum\limits_{i \sim j} w_{ij}$ is the sum of the weights of edges incident on vertex i. There are some studies on indices of graphs using Laplacian eigenvalues. For any connected graph G with n vertices, the Kirchhoff index is

$$Kf(G) = n \sum_{k=1}^{n-1} \frac{1}{\lambda_k},$$

where λ_k are the Laplacian eigenvalues of G [1].

Let G be a graph with n vertices without loops and multiple edges. Then, the Laplacian Estrada index of G is

$$LEE(G) = \sum_{i=1}^{n} e^{\lambda_i},$$

where λ_i are the Laplacian eigenvalues of G [2].

The energy $E(G)$ of a graph G defined as the sum of the absolute values of its eigenvalues. In 2006, Gutman and Zhou defined the Laplacian energy of a graph as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where $n \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ [3].

In the following theorems, the bounds for the eigenvalues of a matrix is given by using the trace of the matrix [4].

Theorem 1. *Let A be an $n \times n$ complex matrix. A^* denotes the conjugate transpose of A. Let $B = AA^*$ with eigenvalues $\lambda_n(B) \leq \cdots \leq \lambda_1(B)$. Then,*

$$m - s\sqrt{n-1} \le \lambda_n(B) \le m - \frac{s}{\sqrt{n-1}}$$

and

$$m + \frac{s}{\sqrt{n-1}} \le \lambda_1(B) \le m + s\sqrt{n-1},$$

where $m = \dfrac{trB}{n}$ *and* $s^2 = \dfrac{trB^2}{n} - m^2$.

Theorem 2. *Let A, m and s^2 be defined as in Theorem 1, then*

$$m - s\sqrt{\frac{k-1}{n-k+1}} \le \lambda_k(B) \le m + s\sqrt{\frac{n-1}{k}}. \tag{1}$$

By using the above theorems, the bounds for the Laplacian eigenvalues for a connected simple graph were established [5].

In the next chapter, we will obtain the bounds for the Laplacian eigenvalues of a weighted graph. Then, we will find the bounds for the Kirchhoff and Laplacian Estrada indices of a weighted graph. Finally, we will define the Laplacian energy of a weighted graph and get the upper bound for this energy.

2. Discussion and Main Results

Theorem 3. *Let G be a weighted graph with n vertices and L be the Laplacian matrix of G with the eigenvalues $\lambda_n(L) \le \cdots \le \lambda_1(L)$. Then,*

$$m - s\sqrt{n-1} \le \lambda_n(L) \le m - \frac{s}{\sqrt{n-1}}$$

and

$$m + \frac{s}{\sqrt{n-1}} \le \lambda_1(L) \le m + s\sqrt{n-1},$$

where

$$m = \frac{1}{n}\sum_{i=1}^{n} w_i$$

and

$$s^2 = \frac{n-1}{n^2}\sum_{i=1}^{n} w_i^2 + \frac{2}{n}\sum_{i\sim j} w_{ij}^2 - \frac{2}{n^2}\sum_{i<j}^{n} w_i w_j.$$

Proof. By Theorem 1, we can write $m = \dfrac{trL}{n}$. So, by the definition of weighted Laplacian matrix, we have

$$m = \frac{trL}{n} = \frac{1}{n}\sum_{i=1}^{n} w_i.$$

Moreover, evaluating the matrix L^2, when $i = j$ we find the $(i,j)\_th$ elements as

$$L_{ij} = w_i^2 + \sum_{i\sim j} w_{ij}^2. \tag{2}$$

Again, by Theorem 1, we can write $s^2 = \dfrac{trL^2}{n} - m^2$. So, by using (2), we have

$$s^2 = \frac{1}{n}\left(\sum_{i=1}^{n} w_i^2 + 2\sum_{i\sim j} w_{ij}^2\right) - \frac{1}{n^2}\left(\sum_{i=1}^{n} w_i\right)^2 = \frac{n-1}{n^2}\sum_{i=1}^{n} w_i^2 + \frac{2}{n}\sum_{i\sim j} w_{ij}^2 - \frac{2}{n^2}\sum_{i<j}^{n} w_i w_j.$$

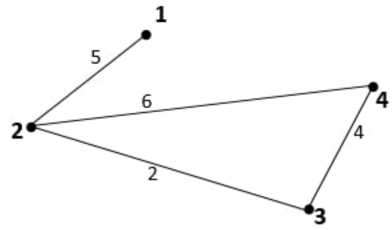

Figure 1. Weighted graph G

□

Example 1. Let we have the weighted graph G shown in the following figure. weighted Laplacian matrix of G is

$$L = \begin{bmatrix} 5 & -5 & 0 & 0 \\ -5 & 13 & -2 & -6 \\ 0 & -2 & 6 & -4 \\ 0 & -6 & -4 & 10 \end{bmatrix}$$

and the eigenvalues are $\lambda_1 = 18.89$, $\lambda_2 = 10.79$, $\lambda_3 = 4.31$ and $\lambda_4 = 0$.

By using Theorem 3, we get $m = 8.5$, $s^2 = 50.75$ and so $s = 7.12$.Also, by the inequality (1) we get some bounds for the eigenvalues λ_2 and λ_3 as

$$4.38 \leq \lambda_2 = 10.79 \leq 15.62,$$

$$1.38 \leq \lambda_3 = 4.31 \leq 12.61.$$

By Theorem 3, we get a bound for the largest eigenvalue λ_1 as

$$12.61 \leq \lambda_1 = 18.89 \leq 20.83.$$

Now, we will give an upper and lower bound for the Kirchhoff index of a weighted graph using the trace of the weighted Laplacian matrix.

Theorem 4. *Let G be a weighted graph with n vertices, then*

$$\frac{n(n-1)}{m + s\sqrt{n-1}} \leq Kf(G) \leq \frac{n(n-1)}{m - s\sqrt{\frac{n-2}{2}}}, \tag{3}$$

where m and s^2 are defined in Theorem 3.

Proof. We know that the largest eigenvalue of Laplacian matrix of G has the following bound

$$m + \frac{s}{\sqrt{n-1}} \leq \lambda_1(L) \leq m + s\sqrt{n-1}.$$

Assume that all eigenvalues of L are equal to the largest eigenvalue $\lambda_1(L)$, then,

$$\begin{aligned} Kf(G) \quad &\geq n(n-1)\frac{1}{\lambda_1(L)} \\ &\geq \frac{n(n-1)}{m + s\sqrt{n-1}}. \end{aligned}$$

Conversely, let all eigenvalues of L be equal to the smallest eigenvalue $\lambda_{n-1}(L)$ different from zero, then by the inequality (1) we can write

$$m - s\sqrt{\frac{n-2}{2}} \leq \lambda_{n-1}(L) \leq m + s\sqrt{\frac{1}{n-1}}, \tag{4}$$

and so,

$$Kf(G) \quad \leq n(n-1)\frac{1}{\lambda_{n-1}(L)} \leq \frac{n(n-1)}{m-s\sqrt{\frac{n-2}{2}}}.$$

□

Example 2. Let we have the graph defined in Example 1 with $m = 8.5$ and $s = 7.12$.

The Kirchhoff index of G is evaluated as

$$Kf(G) = 4\left(\frac{1}{18.89} + \frac{1}{10.79} + \frac{1}{4.31}\right) = 1.50$$

By using the inequality (3) we get the following bound for Kirchhoff index of G

$$0.57 \leq Kf(G) = 1.50 \leq 8.69.$$

Now, we will give an upper and lower bound for the Laplacian Estrada index of a weighted graph using the trace of the weighted Laplacian matrix.

Theorem 5. *Let G be a weighted graph with n vertices, then*

$$1 + (n-1)e^{m-s\sqrt{\frac{n-2}{2}}} \leq LEE(G) \leq 1 + (n-1)e^{m+s\sqrt{n-1}}, \tag{5}$$

where m and s^2 are defined in Theorem 3.

Proof. We know that the largest eigenvalue of Laplacian matrix of G has the following bound

$$m + \frac{s}{\sqrt{n-1}} \leq \lambda_1(L) \leq m + s\sqrt{n-1}.$$

Assume that all eigenvalues of L are equal to the largest eigenvalue $\lambda_1(L)$, then,

$$\begin{aligned} LEE(G) \quad &\leq 1 + (n-1)e^{\lambda_1(L)} \\ &\leq 1 + (n-1)e^{m+s\sqrt{n-1}}. \end{aligned}$$

Conversely, let all eigenvalues of L be equal to the smallest eigenvalue $\lambda_{n-1}(L)$ different from zero, then by using the bounds (4) for $\lambda_{n-1}(L)$, we get

$$\begin{aligned} LEE(G) \quad &\geq 1 + (n-1)e^{\lambda_{n-1}(L)} \\ &\geq 1 + (n-1)e^{m-s\sqrt{\frac{n-2}{2}}}. \end{aligned}$$

□

Example 3. Let we have the graph defined in Example 1 with $m = 8.5$ and $s = 7.12$.

The Laplacian Estrada index of G is evaluated as

$$LEE(G) = 1 + e^{18.89} + e^{10.79} + e^{4.31} = 1599391195.21$$

By using the inequality (5) we get the following bound for Laplacian Estrada index of G

$$33.41 \leq LEE(G) = 1599391195.21 \leq 3296780717.01.$$

Definition 6. Laplacian energy of a weighted graph G is defined as

$$LE^{w}(G) = \sum_{i=1}^{n-1}\left|\mu_i - \frac{\sum_{i=1}^{n} w_i}{n}\right|,$$

where μ_i $(i = 1, ...n)$ are Laplacian eigenvalues and w_i $(i = 1, ...n)$ are the weights of the edges of the weighted graph G.

Theorem 7. *The upper bound for the Laplacian energy of a weighted graph G is*

$$LE^w(G) \leq (n-1)\left[m + s\sqrt{n-1} - w_{\min}\right],$$

where $m = \dfrac{trL(G)}{n}$ *and* $s^2 = \dfrac{trL(G)^2}{n} - m^2$.

Proof. By the above definition, Laplacian energy of a weighted graph G is

$$LE^w(G) = \sum_{i=1}^{n-1} \left| \mu_i - \frac{\sum_{i=1}^{n} w_i}{n} \right|.$$

If we take the maximum Laplacian eigenvalue μ_1 and the minimum edge weight $w_{\min}$ instead of all eigenvalues and all edges, respectively, we get an upper bound for Laplacian energy. Then,

$$LE^w(G) \quad \leq \sum_{i=1}^{n-1} |\mu_1 - w_{\min}| \leq (n-1)\left[\mu_1 - w_{\min}\right] \leq (n-1)\left[m + s\sqrt{n-1} - w_{\min}\right].$$

□

Example 4. Let we have the graph defined in Example 1 with $m = 8.5$ and $s = 7.12$.

The Laplacian energy of G is evaluated as

$$LE^w(G) = 16.865.$$

Using the above theorem, we get the upper bound for the Laplacian energy of G as

$$LE^w(G) \leq 47.493.$$

3. Conclusion

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