## Article

# Calculating Degree-Based Topological Indices and M-Polynomials for Various Interconnection Networks 

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#### Abstract

There are three different kinds of topological indices: spectrum-based, degree-based, and distance-based. We presented the $K$-swapped network for $t$-regular graphs in this study. We also computed various degree-based topological indices of the $K$-swapped network for $t$-regular graphs, eye, and $n$-dimensional twisted cube network. The metrics used to analyze the abstract structural characteristics of networks are called topological indices. We also calculate each of the aforementioned networks M-polynomials. A graph can be used to depict an interconnection network's structure. The processing nodes in the network are represented by vertices, while the links connecting the processor nodes are represented by edges. We can quickly determine the diameter and degree between the nodes based on the graph's topology. A key component of graph theory are graph invariants, which identify the structural characteristics of networks and graphs. Furthermore described by graph invariants are computer, social, and internet networks.


Keywords: Interconnection networks; topological indices; M-polynomials.
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## 1. Introduction

Interconnection networks are very important in computer networking and useful for the transformation between the computer and processor. In last few years, many researchers designed the new interconnection networks. In parallel processsing system, interconnection networks plays an important role to increase the performance of computers. In graph theory, interconnection network can be expressed by a graph. In this expression, processor represented by vertex and connection between the units represented by an edge. From the topology of a network, we can easily determine certain properties. The number of links (edges) connected to a node known as the degree of that node (vertex). If the degree is same for all nodes in a network, then network is called regular network [1]. The degree of a vertex is represented by $\left\{d_{v} \mid v \in V(\aleph)\right\}$ of a graph $\kappa$. The sum of the neighborhood degree of a vertex $v \in \aleph$ is the sum of degrees of all adjacent vertices of a vertex $v$ and it is denoted by $S_{v}$.

In graph theory and the characterization of the structural characteristics of graphs and networks, graph invariants are crucial. The chemical structure of molecules, the internet, chemical reactions, computers, and social networks are all described in this. Topological indices are these indices that we employ to investigate the abstract structural features of networks. There are three types of topological indices: degree-based [2], distance-based $[3,4]$, and spectrum-based topological indices [5,6].

The first degree-based topological index was put forward in 1975 by Milan Randić, [7]. It is define as

$$
\begin{equation*}
R_{\alpha}(\aleph)=\sum_{u v \in E(\aleph)}\left(d_{u} d_{v}\right)^{\alpha}, \quad \alpha=-1,-\frac{1}{2}, \frac{1}{2}, 1 . \tag{1}
\end{equation*}
$$

Later, this index was generalized for any real number $\alpha$, and known as the generalized Randić index, [8] and its mathematical expression is

$$
\begin{equation*}
R R_{\alpha}(\aleph)=\sum_{u v \in E(\aleph)} \frac{1}{\left(d_{u} d_{v}\right)^{\alpha}} \tag{2}
\end{equation*}
$$

Gutman and Trinajstic, introduced the first Zagreb index in [9], mathematically can be written as

$$
\begin{equation*}
M_{1}(\aleph)=\sum_{v \in E(\aleph)} d_{v}^{2}=\sum_{u v \in E(\aleph)}\left(d_{u}+d_{v}\right) \tag{3}
\end{equation*}
$$

The general sum-connectivity index [10], defined as

$$
\begin{equation*}
S C I(\aleph)=\sum_{u v \in E(\aleph)} \frac{1}{\sqrt{d_{u}+d_{v}}} \tag{4}
\end{equation*}
$$

Vukicevic et al. [11], introduced the geometric arithmetic index and expressed as

$$
\begin{equation*}
G A(\aleph)=\sum_{u v \in E(\aleph)} \frac{2 \sqrt{d_{u} d_{v}}}{d_{u}+d_{v}} \tag{5}
\end{equation*}
$$

The $5^{\text {th }}$ geometric arithmetic index was introduced by Graovac et al. [12], its formulation is

$$
\begin{equation*}
G A_{5}(\aleph)=\sum_{u v \in E(\aleph)} \frac{2 \sqrt{S_{u} S_{v}}}{S_{u}+S_{v}} \tag{6}
\end{equation*}
$$

There are many other types of degree based topological indices like, general Randić index, general sum connectivity index, second Zagreb index, third Zagreb index, first multiple Zagreb index, second multiple Zagreb index, hyper Zagreb index, augmented Zagreb index, harmonic index, reduced second Zagreb index, reduced reciprocal Randić index, modified Zagreb index, symmetric division index and inverse sum index, briefly defined in [13].

The M-polynomial was introduced by Deutsch and Klavzar [14] in 2015. It functions similarly to the Hosoya polynomial [15] in determining the closed form of numerous degree-based topological indices. For more detail on the recent study of topological indices and their applications see [16-24]. This motivates us to study M-polynomial of some graph operations and some interconnection networks. Recently, the study of M-polynomial are reported in [25-33].

The M-polynomial can be defines as

$$
\begin{equation*}
M(\kappa ; x, y)=\sum_{i \leq j} m_{i, j}(\kappa) x^{i} y^{j} \tag{7}
\end{equation*}
$$

where $m_{i, j}(\aleph)$ is the number of edges of graph $\aleph$ such that $i \leq j$. where

$$
\begin{array}{lll}
D_{x}=x \frac{\partial f}{\partial x}, & S_{x}=\int_{0}^{x} \frac{f(z, y)}{z} d z, & Q_{a}=x^{a} f(x, y), \\
D_{y}=x \frac{\partial f}{\partial y}, & S_{y}=\int_{0}^{y} \frac{f(x, z)}{z} d z & J=f(x, x) \tag{8}
\end{array}
$$

## 2. K-Swapped networks

Wenjun et. al [34] introduced the new family of communication architectures called "biswapped networks". Given any $n$-node basis network $\kappa$, the associated biswapped network $B s w(\aleph)$ is built of $2 n$ copies

Table 1. M-Polynomials

| Topological index | Derivation from M( $\Gamma ; \mathbf{x}, \mathbf{y})$ |
| :--- | ---: |
| First Zagreb | $\left(D_{x}+D_{y}\right)(M(\kappa ; x, y))$ |
| Second Zagreb | $\left(D_{x} D_{y}\right)(M(\kappa ; x, y))$ |
| Second modified Zagreb | $\left(S_{x} S_{y}\right)(M(\kappa ; x, y))$ |
| General Randić | $\left(D_{x}^{\alpha} D_{y}^{\alpha}\right)(M(\kappa ; x, y))$ |
| General inverse Randić | $\left(S_{x}^{\alpha} S_{y}^{\alpha}\right)(M(\kappa ; x, y))$ |
| Symmetric division index | $\left(D_{x} S_{y}+D_{y} S_{x}\right)(M(\kappa ; x, y))$ |
| Harmonic index | $2 S_{x} J(M(\kappa ; x, y))$ |
| Inverse sum index | $S_{x} J D_{x} D_{y}(M(\kappa ; x, y))$ |
| Augmented Zagreb | $S_{x}^{3} Q_{-2} J D_{x}^{3} D_{y}^{3}(M(\kappa ; x, y))$ |



Figure 1. 3-swapped network of $C_{3}$
of $\kappa$. We generalized this biswapped network to K-swapped network for $t$ regular graphs. For a base graph $\aleph$, the K-swapped interconnection networks $\left(K s w\left(\aleph_{t}\right)\right)$ are the graphs with vertex set and edge set defined as

$$
\begin{gather*}
V\left(K s w\left(\aleph_{t}\right)\right)=\{\langle i, p, g\rangle,\langle j, p, g>| p, g \in V(\aleph), \\
i=0,1,2, \ldots, k \& j=1,2, \ldots, k \text { with } i \neq j\} .  \tag{9}\\
E\left(K_{s w}\left(\aleph_{t}\right)\right)=\left\{\left(<i, p, g_{1}>,<i, p, g_{2}\right\rangle\right),\left(<j, p, g_{1}>,<j, p, g_{1}>\right), \\
\left.\mid\left(g_{1}, g_{2}\right) \in E(\aleph), p \in V(\aleph), i=0,1,2, \ldots, k \& j=1,2, \ldots, k \text { with } i=j\right\} . \tag{10}
\end{gather*}
$$

The size and degree of $K$-swapped networks are $\frac{(t+K-1) K n^{2}}{2}, t+K-1$ respectively, where $t$ is the regularity of base graphs.

In this section, we study some topological properties of K -swapped networks of $t$-regular graphs.

### 2.1. Results for K-swapped networks $\left(\operatorname{Ksw}\left(\aleph_{t}\right)\right)$

Theorem 1. Let $\Gamma$ be the K-swapped network of t-regular graphs, then Randić, general and reduced reciprocal Randić indices are

1: $\quad R_{\alpha}(\Gamma)=\frac{K n^{2}}{2} d^{2 \alpha+1}$.
2: $\quad R R_{\alpha}(\Gamma)=\frac{K n^{2}}{2 d^{2 \alpha-1}}$.
3: $\quad R R R(\Gamma)=\frac{K n^{2}}{2} d(d-1)$.
Proof. K-swapped network has degree, $d=t+K-1$ and size $\frac{d K n^{2}}{2}$, putting degree, $d$ and number of edges in "Equation (1) and (2)",

1: $\quad R_{\alpha}(\Gamma)=\frac{d K n^{2}}{2}(d \times d)^{\alpha} .=\frac{d\left(K n^{2}\right)}{2} d^{2 \alpha}=\frac{K n^{2}}{2} d^{2 \alpha+1}$.
2: $\quad R R_{\alpha}(\Gamma)=\frac{d K n^{2}}{2(d \times d)^{\alpha}}=\frac{d K n^{2}}{2 d^{2 \alpha}}=\frac{K n^{2}}{2 d^{2 \alpha-1}}$.
The reduced reciprocal Randić index defined and used to get the result for $\operatorname{Ksw}\left(\aleph_{t}\right)$ below,
3: $\quad R R R(\Gamma)=\sum_{u v \in E(\Gamma)} \sqrt{\left(d_{u}-1\right)\left(d_{v}-1\right)}=\frac{d K n^{2}(d-1)}{2}=\frac{K n^{2}}{2} d(d-1)$.

Theorem 2. Let $\Gamma$ be the K-swapped network of t-regular graphs, then family of Zagreb indices are
1: $\quad M_{1}(\Gamma)=K n^{2} d^{2}$.
2: $\quad M_{2}(\Gamma)=\frac{K n^{2} d^{3}}{2}$.
3: $\quad M_{3}(\Gamma)=0$.
4: $\quad P M_{1}(\Gamma)=(2 d)^{\frac{K n^{2} d}{2}}$.
5: $\quad P M_{2}(\Gamma)=d^{K n^{2} d}$.
6: $\quad A Z I(\Gamma)=\frac{K n^{2} d^{7}}{2\left(d^{2}-2\right)^{3}}$.
7: $\quad R M_{2}(\Gamma)=\frac{K n^{2}}{2} d(d-1)^{2}$.
8: ${ }^{m} M_{2}(\Gamma)=\frac{K n^{2}}{2 d}$.
9: $\quad H M(\Gamma)=2 K n^{2} d^{3}$.

Proof. As we already computed the degree, $d=t+K-1$ and size $\frac{d K n^{2}}{2}$ of $K$-swapped network, using these parameters in the formulation of different Zagreb indices, after some simplification get the results as defined below,

1: $\quad M_{1}(\Gamma)=\sum_{u v \in E(\Gamma)}\left(d_{u}+d_{v}\right)=\frac{d K n^{2}}{2}(d+d)=\frac{d K n^{2}}{2} 2(d)=K n^{2} d^{2}$.
2: $\quad M_{2}(\Gamma)=\sum_{u v \in E(\Gamma)}\left(d_{u} d_{v}\right)=\frac{d K n^{2}}{2}(d \times d)=\frac{K n^{2} d^{3}}{2}$.
3: $\quad M_{3}(\Gamma)=\sum_{u v \in E(\Gamma)}\left(d_{u}-d_{v}\right)=\frac{d K n^{2}}{2}(d-d)=0$.
4: $\quad P M_{1}(\Gamma)=\prod_{u v \in E(\Gamma)}\left(d_{u}+d_{v}\right)=\prod_{u v \in E(\Gamma)}(d+d)=(2 d)^{\frac{d K n^{2}}{2}}$.
5: $\quad P M_{2}(\Gamma)=\prod_{u v \in E(\Gamma)}\left(d_{u} d_{v}\right)=d^{2} \times d^{2} \times \cdots \times\left(\frac{d K n^{2}}{2}\right)=d^{K n^{2} d}$.

6: $\quad \operatorname{AZI}(\Gamma)=\sum_{u v \in E(\Gamma)}\left(\frac{d_{u} d_{v}}{d_{u} d_{v}-2}\right)^{3}=\frac{d K n^{2}}{2}\left(\frac{d^{2}}{d^{2}-2}\right)^{3}=\frac{K n^{2} d^{7}}{2\left(d^{2}-2\right)^{3}}$.
7: $\quad R M_{2}(\Gamma)=\sum_{u v \in E(\Gamma)}\left(d_{u}-1\right)\left(d_{v}-1\right)=\frac{d K n^{2}}{2}(d-1)(d-1)=\frac{d K n^{2}}{2}(d-1)^{2}$.
8: ${ }^{m} M_{2}(\Gamma)=\sum_{u v \in E(\Gamma)} \frac{1}{d_{u} d_{v}}=\frac{d K n^{2}}{2 d^{2}}=\frac{K n^{2}}{2 d}$.
9: $\quad H M(\Gamma)=\sum_{u v \in E(\Gamma)}\left(d_{u}+d_{v}\right)^{2}=\frac{d K n^{2}}{2}(d+d)^{2}=\frac{d K n^{2}}{2}(2 d)^{2}=2 K n^{2} d^{3}$.

Theorem 3. Let $\Gamma$ be the K-swapped network of $t$-regular graphs, then sum connectivity and general sum connectivity indices are

1: $\quad \operatorname{SCI}(\Gamma)=\frac{K n^{2} d}{2 \sqrt{2 d}}$.
2: $\quad \chi_{\alpha}(\Gamma)=2^{\alpha-1} K n^{2} d^{\alpha+1}$.
Proof. Consider, we have a K-swapped network of $t$-regular graph, then by using degree and size of derived network, making some calculations their indices are defined below:

1: $\quad \operatorname{SCI}(\Gamma)=\sum_{u v \in E(\Gamma)} \frac{1}{\sqrt{d_{u}+d_{v}}}=\frac{d K n^{2}}{2} \frac{1}{\sqrt{2 d}}=\frac{K n^{2} d}{2 \sqrt{2 d}}$.
2: $\quad \chi_{\alpha}(\Gamma)=\sum_{u v \in E(\Gamma)}\left(d_{u}+d_{v}\right)^{\alpha}=2^{\alpha-1} K n^{2} d^{\alpha+1}$.

Theorem 4. Let $\Gamma$ be the K-swapped network of t-regular graphs, then the geometric arithmetic, $5^{\text {th }}$ geometric arithmetic, symmetric division and inverse sum indices are

1: $\quad G A(\Gamma)=\frac{d K n^{2}}{2}$.
2: $\quad G A_{5}(\Gamma)=\frac{d K n^{2}}{2}$.
3: $\quad \operatorname{SDI}(\Gamma)=d K n^{2}$.
4: $\quad I S(\Gamma)=\frac{K n^{2} d^{2}}{4}$.
Proof. Consider, we have a K-swapped network of $t$-regular graphs, then by using degree and size parameters in "Equation (5) and (6)",

1: $\quad G A(\Gamma)=\frac{K n^{2}}{2}(d) \frac{2 \sqrt{(d)^{2}}}{2 d}=\frac{d K n^{2}}{2}$.
2: $\quad G A_{5}(\Gamma)=\frac{K n^{2}}{2}(d) \frac{2 \sqrt{(d)^{2}}}{2 d}=\frac{d K n^{2}}{2}$.

3: $\operatorname{SDI}(\Gamma)=\sum_{u v \in E(\Gamma)} \frac{d_{u}^{2}+d_{v}^{2}}{d_{u} d_{v}}=\frac{d K n^{2}}{2} \times \frac{(d)^{2}+(d)^{2}}{(d)^{2}}=K n^{2} d$.
4: $\quad I S(\Gamma)=\sum_{u v \in E(\Gamma)} \frac{d_{u} d_{v}}{d_{u}+d_{v}}=\frac{d K n^{2}}{2} \frac{d^{2}}{2 d}=\frac{K n^{2} d^{2}}{4}$.

Following theorem define the M-polynomials of different K-swapped networks derived from $t$-regular, different graphs.

Theorem 5. : Let $\Gamma$ be the K-swapped network of t-regular graphs, then M-polynomial is:

$$
M(\Gamma ; x, y)=\frac{d K n^{2}}{2} x^{d} y^{d} .
$$

Proof. As we know that

$$
M(\Gamma ; x, y)=\sum_{i \leq j} m_{i j}(\Gamma) x^{i} y^{j}
$$

where, $m_{i j}(\Gamma)=|E(\Gamma)|$ and $i, j$ are the degree of adjacent vertices $u$, $v$ respectively.
$M(\Gamma ; x, y)=\frac{K n^{2}}{2}(t+K-1) x^{t+K-1} y^{t+K-1}$ and already computed parameter $d=t+K-1$ then we have $M(\Gamma ; x, y)=\frac{d K n^{2}}{2} x^{d} y^{d}$.
This completed the proof of M-polynomials for different K-swapped networks derived from t-regular graphs.

Following theorem is about the subdivided M-polynomials for different topological indices like, family of Zagreb indices and some from Randić index.

Theorem 6. : Let $\Gamma$ be the K-swapped network of t-regular graphs, then different M-polynomials Table:1 for different indices are:

1: $\quad M_{1}(\Gamma ; x, y)=d^{2} K n^{2} x^{d} y^{d}$.
2: $\quad M_{2}(\Gamma ; x, y)=\frac{d^{4} K^{2} n^{4}}{4} x^{2 d} y^{2 d}$.
3: ${ }^{m} M(\Gamma ; x, y)=\frac{K^{2} n^{4}}{4} x^{2 d} y^{2 d}$.
4: $\quad R_{\alpha}(\Gamma ; x, y)=\left(\frac{d^{4} K^{2} n^{4}}{4} x^{2 d} y^{2 d}\right)^{\alpha}$.
$5: \quad R R_{\alpha}(\Gamma ; x, y)=\left(\frac{K^{2} n^{4}}{4} x^{2 d} y^{2 d}\right)^{\alpha}$.
6: $\quad \operatorname{SDI}(\Gamma ; x, y)=\frac{d^{2} K^{2} n^{4}}{2} x^{2 d} y^{2 d}$.
7: $\quad H(\Gamma ; x, y)=\frac{d K^{2} n^{4}}{2} x^{3 d} y^{2 d}$.
8: $\quad \operatorname{IS}(\Gamma ; x, y)=\frac{d^{5} K^{4} n^{8}}{16} x^{5 d} y^{3 d}$.
9: $A Z I(\Gamma ; x, y)=\frac{d^{14} K^{11} n^{22}}{2^{11}} x^{12 d-2} y^{10 d}$.
Proof. Consider, we have K-swapped networks of $t$-regular graphs, using "Equation (8)", in which computed different differential formulas with same $d=K+t-1$ like,

$$
\begin{aligned}
D_{x} & =D_{y}=\frac{d^{2} K n^{2}}{2}(x y)^{d}, \quad J=\frac{d K n^{2} x^{2 d}}{2} . \\
S_{x} & =S_{y}=\frac{K n^{2}(x y)^{d}}{2}, \quad Q_{-2}=\frac{d K n^{2}}{2} x^{d-2} y^{d} .
\end{aligned}
$$

then we have the following M-polynomials of Table: 1

1: $\quad M_{1}(\Gamma ; x, y)=\left(\frac{d^{2} K n^{2}}{2}(x y)^{d}+\frac{d^{2} K n^{2}}{2}(x y)^{d}\right)=d^{2} K n^{2} x^{d} y^{d}$.
2: $\quad M_{2}(\Gamma ; x, y)=\frac{d^{2} K n^{2}}{2}(x y)^{d} \times \frac{d^{2} K n^{2}}{2}(x y)^{d}=\frac{d^{4} K^{2} n^{4}}{4} x^{2 d} y^{2 d}$.


Figure 2. 5-dimensional twisted cube

3: ${ }^{m} M(\Gamma ; x, y) .=\frac{K n^{2}(x y)^{d}}{2} \frac{K n^{2}(x y)^{d}}{2}=\frac{K^{2} n^{4}}{4} x^{2 d} y^{2 d}$.
4: $\quad R_{\alpha}(\Gamma ; x, y)=\left(\frac{d^{2} K n^{2}}{2}(x y)^{d}\right)^{\alpha}\left(\frac{d^{2} K n^{2}}{2}(x y)^{d}\right)^{\alpha}=\left(\frac{x^{2 d} y^{2 d} d^{4} K^{2} n^{4}}{4}\right)^{\alpha}$.
5: $\quad R R_{\alpha}(\Gamma ; x, y)=\left(\frac{d K n^{2}(x y)^{d}}{2}\right)^{\alpha}\left(\frac{d K n^{2}(x y)^{d}}{2}\right)^{\alpha}=\left(\frac{d K n^{2}(x y)^{d}}{2}\right)^{2 \alpha}=\left(\frac{K^{2} n^{4}}{4} x^{2 d} y^{2 d}\right)^{\alpha}$.
6: $\quad \operatorname{SDI}(\Gamma ; x, y)=\left(\frac{d^{2} K n^{2}(x y)^{d}}{2} \frac{K n^{2}(x y)^{d}}{2}+\frac{K n^{2}(x y)^{d}}{2} \frac{d^{2} K n^{2}(x y)^{d}}{2}\right)$.

$$
=\frac{d^{2} K^{2} n^{4}(x y)^{2 d}}{4}+\frac{d^{2} K^{2} n^{4}(x y)^{2 d}}{4}=\frac{d^{2} K^{2} n^{4}}{2} x^{2 d} y^{2 d}
$$

7: $\quad H(\Gamma ; x, y)=2 \frac{K n^{2}(x y)^{d}}{2} \frac{d K n^{2} x^{2 d}}{2}=\frac{d K^{2} n^{4}}{2} x^{3 d} y^{2 d}$.
8: $\quad I S(\Gamma ; x, y)=\frac{d K n^{2}(x y)^{d}}{2} \times \frac{d K n^{2} x^{2 d}}{2} \times \frac{d^{2} K n^{2}}{2}(x y)^{d} \times \frac{d^{2} K n^{2}}{2}(x y)^{d}=\frac{d^{5} K^{4} n^{8}}{16} x^{5 d} y^{3 d}$.
9: $\quad A Z I(\Gamma ; x, y)=\left(\frac{d K n^{2}(x y)^{d}}{2}\right)^{3}\left(\frac{d^{2} K^{2} n^{4} x^{3 d-2} y^{d}}{4}\right)\left(\frac{d^{2} K n^{2}(x y)^{d}}{2}\right)^{3}\left(\frac{d^{2} K n^{2}(x y)^{d}}{2}\right)^{3}$.
$=\frac{d^{14} K^{11} n^{22}}{2^{11}} x^{12 d-2} y^{10 d}$.

## 3. n-dimensional twisted cube

The $n$-dimensional twisted cube [35], is a variant of $n$-dimensional hypercube cube. It is represented by $T Q_{n}$ with $n=2 k+1$. The size, order and degree of $n$-dimensional twisted cube are $n 2^{n-1}, 2^{n}$ and $n$ respectively. To make a twisted cube, we delete some edges from the graph and add some edges in such a way that total number of edges conserved.

Suppose that $n$ is an odd integer with condition $n \geq 3$. We can divide vertices of $T Q_{n}$ into four sets $L^{0,0}$, $L^{0,1}, L^{1,0}, L^{1,1}$ where $L^{i, j}$ consists of those vertices $u$ with $u_{n-1}=i, u_{n-2}=j$ and $(i, j) \in\{(0,0),(0,1),(1,0),(1,1)\}$ and edge set of twisted cube we can define as: vertex $u_{n-1} u_{n-2} u_{n-3}, \ldots, u_{1} u_{0}$ with $P_{n-3}(u)=0$ is connected to $\bar{u}_{n-1} \bar{u}_{n-2} u_{n-3} u_{n-4} \ldots u_{1} u_{0}$ and $\bar{u}_{n-1} u_{n-2} u_{n-3} u_{n-4} \ldots u_{1} u_{0} ; u_{n-1} \bar{u}_{n-2} u_{n-3} u_{n-4} \ldots u_{1} u_{0}$ and $\bar{u}_{n-1} u_{n-2} u_{n-3} u_{n-4} \ldots u_{1} u_{0}$ if $P_{n-3}(u)=1$.
Figure: 2 is drawn for a clearer view of the vertex set of 5-dimensional twisted cube.

This section is about topological indices and M-polynomials of $n$-dimensional twisted cube.

### 3.1. Results for $n$-dimensional twisted cube

Theorem 7. Let $\chi$ be the n-dimensional twisted cube network, then Randić, general and reduced reciprocal Randic indices are

1: $\quad R_{\alpha}(\chi)=2^{n-1} n^{2 \alpha+1}$.
2: $\quad R R_{\alpha}(\chi)=2^{n-1}(n)^{1-2 \alpha}$.
3: $\quad R R R(\chi)=2^{n-1} n(n-1)$.
Proof. As we know that $n$-dimensional twisted cube network has degree $n$ and size is $n 2^{n-1}$, after plugging all the known parameters in to the formulas

1: $\quad R_{\alpha}(\chi)=\sum_{u v \in E(\chi)}\left(d_{u} d_{v}\right)^{\alpha}=n 2^{n-1}(n \times n)^{\alpha}=2^{n-1} n^{2 \alpha+1}$.
2: $\quad R R_{\alpha}(\chi)=\sum_{u v \in E(\chi)} \frac{1}{\left(d_{u} d_{v}\right)^{\alpha}}=n 2^{n-1} \frac{1}{(n \times n)^{\alpha}}=2^{n-1} n^{1-2 \alpha}$.
3: $\quad R R R(\chi)=\sum_{u v \in E(\chi)} \sqrt{\left(d_{u}-1\right)\left(d_{v}-1\right)}=n 2^{n-1} \sqrt{(n-1)(n-1)}=n 2^{n-1} \sqrt{(n-1)^{2}}$.

$$
=n 2^{n-1}(n-1) .
$$

Theorem 8. Let $\chi$ be the n-dimensional twisted cube network, then we have family of Zagreb index are

1: $\quad M_{1}(\chi)=2^{n} n^{2}$.
2: $\quad M_{2}(\chi)=2^{n-1} n^{3}$.
3: $\quad M_{3}(\chi)=0$.
4: $\quad P M_{1}(\chi)=(2 n)^{2^{n-1} n}$.
5: $\quad P M_{2}(\chi)=n^{2^{n} n}$.
6: $\quad \operatorname{AZI}(\chi)=2^{n-1} n\left(\frac{n^{2}}{n^{2}-2}\right)^{3}$.
7: $\quad R M_{2}(\chi)=2^{n-1} n(n-1)^{2}$.
8: ${ }^{m} M_{2}(\chi)=\frac{2^{n-1}}{n}$.
9: $\quad H M(\chi)=2^{n+1} n^{3}$.

Proof. As we know that $n$-dimensional twisted cube network has degree $n$ and size is $n 2^{n-1}$, then following indices can be proven by putting parameters into formulas,

1: $\quad M_{1}(\chi)=\sum_{u v \in E(\chi)}\left(d_{u}+d_{v}\right)=n 2^{n-1}(n+n)=n 2^{n-1}(2 n)=2^{n-1}\left(2 n^{2}\right)=2^{n} n^{2}$.
2: $\quad M_{2}(\chi)=\sum_{u v \in E(\chi)}\left(d_{u} d_{v}\right)=n 2^{n-1}(n \times n)=2^{n-1} n^{3}$.
3: $\quad M_{3}(\chi)=\sum_{u v \in E(\chi)}\left(d_{u}-d_{v}\right)=n 2^{n-1}(n-n)=0$.
4: $P M_{1}(\chi)=\prod_{u v \in E(\chi)}\left(d_{u}+d_{v}\right)=(n+n)^{2^{n-1} n}=(2 n)^{2^{n-1} n}$.
5: $\quad P M_{2}(\chi)=\prod_{u v \in E(\chi)}\left(d_{u} d_{v}\right)=\left(n^{2}\right)^{2^{n-1} n}=n^{2^{n} n}$.
6: $\quad \operatorname{AZI}(\chi)=\sum_{u v \in E(\chi)}\left(\frac{d_{u} d_{v}}{d_{u} d_{v}-2}\right)^{3}=n 2^{n-1}\left(\frac{n^{2}}{n^{2}-2}\right)^{3}$.

7: $\quad R M_{2}(\chi)=\sum_{u v \in E(\chi)}\left(d_{u}-1\right)\left(d_{v}-1\right)=n 2^{n-1}(n-1)(n-1)=2^{n-1} n(n-1)^{2}$.
8: ${ }^{m} M_{2}(\chi)=\sum_{u v \in E(\chi)} \frac{1}{d_{u} d_{v}}=\frac{n 2^{n-1}}{n \times n}=\frac{2^{n-1}}{n}$.
9: $\quad H M(\chi)=\sum_{u v \in E(\chi)}\left(d_{u}+d_{v}\right)^{2}=n 2^{n-1}(n+n)^{2}=2^{n+1} n^{3}$.

Theorem 9. Let $\chi$ be the $n$-dimensional twisted cube network, then sum connectivity and general sum connectivity indices are

1: $\operatorname{SCI}(\chi)=2^{n-\frac{3}{2}} \sqrt{n}$.
2: $\quad \chi_{\alpha}(\chi)=2^{n-1+\alpha} n^{\alpha+1}$.
Proof. As we know that " $n$ "-dimensional twisted cube network has degree $n$ and size is $n 2^{n-1}$ then the following indices can be computed with the same manner,

$$
\begin{aligned}
& 1: \quad \operatorname{SCI}(\chi)=\sum_{u v \in E(\chi)} \frac{1}{\sqrt{d_{u}+d_{v}}}=\frac{n 2^{n-1}}{\sqrt{2 n}}=2^{n-\frac{3}{2}} \sqrt{n .} \\
& 2: \quad \chi_{\alpha}(\chi)=\sum_{u v \in E(\chi)}\left(d_{u}+d_{v}\right)^{\alpha}=n 2^{n-1}(2 n)^{\alpha}=2^{n-1+\alpha} n^{\alpha+1} .
\end{aligned}
$$

Theorem 10. Let $\chi$ be the n-dimensional twisted cube network, then the geometric arithmetic, symmetric division and inverse sum index are
: $\quad G A(\chi)=2^{n-1} n$.
2: $\quad G A_{5}(\chi)=2^{n-1} n$.
3: $\quad S D I(\chi)=2^{n} n$.
4: $\quad I S(\chi)=2^{n-2} n^{2}$.

Proof. The $n$-dimensional twisted cube network has degree $n$ and size is $n 2^{n-1}$,
1: $\quad G A(\chi)=\sum_{u v \in E(\chi)} \frac{2 \sqrt{d_{u} d_{v}}}{d_{u}+d_{v}}=n 2^{n-1} \frac{2 \sqrt{n^{2}}}{2 n}=2^{n-1} n$.
2: $\quad G A_{5}(\chi)=n 2^{n-1} \frac{2 \sqrt{n^{2}}}{2 n}=2^{n-1} n$.
3: $\operatorname{SDI}(\chi)=\sum_{u v \in E(\chi)} \frac{d_{u}^{2}+d_{v}^{2}}{d_{u} d_{v}}=n 2^{n-1} \frac{n^{2}+n^{2}}{n^{2}}=n 2^{n-1} \frac{2 n^{2}}{n^{2}}=2^{n} n$.
4: $\quad I S(\chi)=\sum_{u v \in E(\chi)} \frac{d_{u} d_{v}}{d_{u}+d_{v}}=n 2^{n-1} \frac{n^{2}}{2 n}=2^{n-2} n^{2}$.

Theorem 11. : Let $\chi$ be the n-dimensional twisted cube network, then M-polynomial is

$$
M(\chi ; x, y)=n 2^{n-1} x^{n} y^{n}
$$

Proof. As we know that from the definition of M-polynomials $M(\chi ; x, y)=h(x, y)=\sum_{i \leq j} m_{i j}(\chi) x^{i} y^{j}$, and $m_{i j}(\chi)=|E(\chi)|=2^{n-1} n$ where $\mathrm{i}, \mathrm{j}$ are the degrees of adjacent vertices $(u, v)$, both have same degree $n$. Hence this prove that,

$$
\begin{equation*}
M(\chi ; x, y)=n 2^{n-1} x^{n} y^{n} \tag{14}
\end{equation*}
$$

Figure: 3 shows the graphical interpretation of equation (14), that is M-polynomial Table:1 of different twisted cube networks.

Theorem 12. : Let $\chi$ be the $n$-dimensional twisted cube network, then different $M$-polynomials are:

$$
\begin{align*}
M_{1}(\chi ; x, y) & =2^{n} n^{2}(x y)^{n} \\
M_{2}(\chi ; x, y) & =2^{2 n-2} n^{4}(x y)^{2 n} \\
{ }^{m} M(\chi ; x, y) & =2^{2 n-2}(x y)^{2 n} \\
R_{\alpha}(\chi ; x, y) & =2^{2 \alpha(n-1)} n^{4 \alpha}(x y)^{2 \alpha n} \\
R R_{\alpha}(\chi ; x, y) & =2^{2 \alpha(n-1)}(x y)^{2 \alpha n}  \tag{15}\\
S D I(\chi ; x, y) & =2^{2 n} n^{2}(x y)^{2 n} \\
H(\chi ; x, y) & =2^{2 n-1} n x^{3 n} y^{n} \\
I S(\chi ; x, y) & =2^{4 n-4} n^{5} y^{3 n} x^{5 n} \\
A Z I(\chi ; x, y) & =n^{14} 2^{11 n-12} x^{12 n-2} y^{10 n}
\end{align*}
$$

Proof. To get M-polynomials of different indices computed by " $n$ "-dimensional twisted cube, here are some computed formulas necessary for the definition of M-polynomial

$$
\begin{aligned}
D_{x} & =D_{y}=(x y)^{n} 2^{n-1} n^{2}, \quad J=n 2^{n-1} x^{2 n} . \\
S_{x} & =S_{y}=2^{n-1}(x y)^{n}, \quad Q_{-2}=n 2^{n-1} x^{n-2} y^{n} .
\end{aligned}
$$



Figure 3. M-Polynomials of " $n$ "-dimensional twisted cube network
putting all these junctions into definition of M-polynomials, following are different required M-polynomials for different indices,

$$
\begin{aligned}
& M_{1}(\chi ; x, y)=\left(D_{x}+D_{y}\right) M(\chi ; x, y)=(x y)^{n} 2^{n-1} n^{2}+(x y)^{n} 2^{n-1} n^{2} . \\
& =2^{n} n^{2}(x y)^{n} . \\
& \text { 2: } \quad M_{2}(\chi ; x, y)=\left(D_{x} D_{y}\right) M(x ; x, y)=\left((x y)^{n} 2^{n-1} n^{2}\right)\left((x y)^{n} 2^{n-1} n^{2}\right) \text {. } \\
& =2^{2 n-2} n^{4}(x y)^{2 n} \text {. } \\
& \text { 3: } \quad{ }^{m} M(\chi ; x, y)=\left(S_{x} S_{y}\right) M(\chi ; x, y)=\left(2^{n-1}(x y)^{n}\right)\left(2^{n-1}(x y)^{n}\right) \text {. } \\
& =2^{2 n-2}(x y)^{2 n} \text {. } \\
& \text { 4: } \quad R_{\alpha}(\chi ; x, y)=\left(D_{x}^{\alpha} D_{y}^{\alpha}\right) M(\chi ; x, y)=\left((x y)^{n} 2^{n-1} n^{2}\right)^{\alpha}\left((x y)^{n} 2^{n-1} n^{2}\right)^{\alpha} \\
& =2^{2 \alpha(n-1)} n^{4 \alpha}(x y)^{2 \alpha n} \text {. } \\
& \text { 5: } \quad R R_{\alpha}(x ; x, y)=\left(S_{x}^{\alpha} S_{y}^{\alpha}\right) M(x ; x, y)=\left(2^{n-1}(x y)^{n}\right)^{\alpha}\left(2^{n-1}(x y)^{n}\right)^{\alpha}=2^{2 \alpha(n-1)}(x y)^{2 \alpha n} \text {. } \\
& \text { 6: } S D I(\chi ; x, y)=\left(D_{x} S_{y}+D_{y} S_{x}\right) M(\chi ; x, y)=\left((x y)^{n} 2^{n-1} n^{2}\right)\left(2^{n-1}(x y)^{n}\right)+\left((x y)^{n} 2^{n-1} n^{2}\right)\left(2^{n-1}(x y)^{n}\right) \\
& =2^{2 n} n^{2}(x y)^{2 n} \text {. } \\
& \text { 7: } \quad H(\chi ; x, y)=2 S_{x} J M(\chi ; x, y)=2 \times 2^{n-1}(x y)^{n} n 2^{n-1} x^{2 n}=2^{2 n-1} n x^{3 n} y^{n} \text {. } \\
& \text { 8: } \quad I S(\chi ; x, y)=S_{x} J D_{x} D_{y} M(x ; x, y)=2^{n-1}(x y)^{n}(x y)^{n} 2^{n-1} n^{2}(x y)^{n} 2^{n-1} n^{2} n 2^{n-1} x^{2 n} \text {. } \\
& =2^{4 n-4} n^{5} y^{3 n} x^{5 n} . \\
& \text { 9: } \quad A Z I(\chi ; x, y)=S_{x}^{3} Q_{-2} J D_{x}^{3} D_{y}^{3} M(\chi ; x, y) \\
& =\left(2^{n-1}(x y)^{n}\right)^{3}\left(n 2^{n-1} x^{n-2} y^{n}\right)\left(n 2^{n-1} x^{2 n}\right)\left((x y)^{n} 2^{n-1} n^{2}\right)^{3}\left((x y)^{n} 2^{n-1} n^{2}\right)^{3} . \\
& =n^{14} 2^{11 n-12} x^{12 n-2} y^{10 n} \text {. }
\end{aligned}
$$

Further section is about very famous eye networks in computer networking. In this section basics, indices and some M-polynomials are discussed for eye network.


Figure 4. Eye(2)

## 4. Eye Networks

An eye network graph with $s+1$ layers of concentric cycle [36]. It is represented by Eye (s). The $s+1$ cycles can be written as $l_{0}, l_{1}, l_{2}, \ldots, l_{s-1}$ and $O_{s}$, here $O_{s}$ represents the outermost cycle. The size, order and degree of eye networks are $9\left(2^{s}-1\right), 6\left(2^{s}-1\right)$ and 3 respectively. Vertex and edge set of Eye networks are following

$$
\begin{align*}
V(\text { Eye }(s)) & =\cup_{k=0}^{s-1} V\left(I_{k}\right) \cup V\left(O_{s}\right) . \\
V\left(I_{k}\right) & =\left\{(k, j) \mid 0 \leq j \leq N_{k}-1 ; 0 \leq k \leq s-1\right\} .  \tag{16}\\
V\left(O_{s}\right) & =\left\{(s, j) \mid 0 \leq j \leq N_{s}-1 ; 0 \leq k \leq s-1 \text { and }|j|_{3}=0\right\} .  \tag{17}\\
E(\text { Eye }(s)) & =\cup_{k=0}^{s-1} E\left(I_{k}\right) \cup E\left(O_{s}\right) \cup\left(\cup_{k=0}^{s-1} E_{k}^{k+1}\right) . \tag{18}
\end{align*}
$$

where $N_{k}=9 \times 2^{k-1}$ for positive $k$

$$
\begin{aligned}
& e_{k, j}=\left\{\begin{array}{l}
((k, j),(k, j+1)) ; 0 \leq j \leq N_{k}-1 \text { and } 0 \leq k \leq s-1, \\
((k, j),(k, j+3)) ; k=s \text { and } 0 \leq j \leq N_{s}-1 \text { and }|j|_{3}=0 .
\end{array}\right. \\
& e_{k, j}^{k+1}=\left\{\begin{array}{l}
\left((0, j),\left(1,2 j+|j|_{3}\right)\right) ; k=0 ; j=0,1,2, \ldots, \\
\left((k, j),\left(k+1,2 j+|j|_{3}\right)\right) ;|j|_{3} \neq 0 \text { and } 1 \leq k \leq s-1, \\
\text { and } 0 \leq j \leq N_{k}-1 .
\end{array}\right.
\end{aligned}
$$

The set $\left\{e_{k, j} \mid 0 \leq j \leq N_{k}-1\right\}$ is denoted by $E\left(I_{k}\right)$ if $0 \leq k \leq s-1$, and denoted by $E\left(O_{s}\right)$ if $k=s$. We use $E_{k}^{k+1}$ to denote the set $\left\{e_{k, j}^{K+1} \mid 0 \leq j \leq N_{k}-1\right\}$. For better understanding of the labeling of Eye-networks following Figure is illustrated.

### 4.1. Results for Eye Networks

Theorem 13. Let $\psi$ be the eye network, then Randić, general and reduced reciprocal Randić indices are
1: $\quad R_{\alpha}(\psi)=9^{\alpha+1}\left(2^{s}-1\right)$.
2: $\quad R R_{\alpha}(\psi)=9^{1-\alpha}\left(2^{s}-1\right)$.
3: $\quad \operatorname{RRR}(\psi)=18\left(2^{s}-1\right)$.

Proof. The size and degree of Eye networks are $9\left(2^{s}-1\right), 3$ respectively, after furnishing into formulas we get Randić family of indices,

1: $\quad R_{\alpha}(\psi)=\sum_{u v \in E(\psi)}\left(d_{u} d_{v}\right)^{\alpha}=9\left(2^{s}-1\right)(3 * 3)^{\alpha}=9\left(2^{s}-1\right)(9)^{\alpha}=9^{\alpha+1}\left(2^{s}-1\right)$.
2: $\quad R R_{\alpha}(\psi)=\sum_{u v \in E(\psi)} \frac{1}{\left(d_{u} d_{v}\right)^{\alpha}}=\frac{9\left(2^{s}-1\right)}{(9)^{\alpha}}=9^{1-\alpha}\left(2^{s}-1\right)$.
3: $\quad R R R(\psi)=\sum_{u v \in E(\psi)} \sqrt{\left(d_{u}-1\right)\left(d_{v}-1\right)}=9\left(2^{s}-1\right) \sqrt{4}=18\left(2^{s}-1\right)$.

Theorem 14. Let $\psi$ be the eye network, then we have family of Zagreb indices are

1. $M_{1}(\psi)=54\left(2^{s}-1\right)$,
2. $M_{2}(\psi)=81\left(2^{s}-1\right)$,
3. $M_{3}(\psi)=0$,
4. $P M_{1}(\psi)=6^{9\left(2^{s}-1\right)}$,
5. $P M_{2}(\psi)=9^{9\left(2^{s}-1\right)^{\prime}}$,
6. $A Z I(\psi)=\frac{6561}{343}\left(2^{s}-1\right)$,
7. $R M_{2}(\psi)=36\left(2^{s}-1\right)$,
8. ${ }^{m} M_{2}(\psi)=\left(2^{s}-1\right)$,
9. $H M(\psi)=324\left(2^{s}-1\right)$.

Proof. The size and degree of Eye networks are $9\left(2^{s}-1\right), 3$ respectively, after putting these parameters into formulas we get Zagreb family of indices,

1: $\quad M_{1}(\psi)=\sum_{u v \in E(\psi)}\left(d_{u}+d_{v}\right)=9\left(2^{s}-1\right)(6)=54\left(2^{s}-1\right)$.
2: $\quad M_{2}(\psi)=\sum_{u v \in E(\psi)}\left(d_{u} d_{v}\right)=9\left(2^{s}-1\right)(9)=81\left(2^{s}-1\right)$.
3: $\quad M_{3}(\psi)=\sum_{u v \in E(\psi)}\left(d_{u}-d_{v}\right)=9\left(2^{s}-1\right)(0)=0$.
4: $\quad P M_{1}(\psi)=\prod_{u v \in E(\psi)}\left(d_{u}+d_{v}\right)=6^{9\left(2^{s}-1\right)}$.
5: $\quad P M_{2}(\psi)=\prod_{u v \in E(\psi)}\left(d_{u} d_{v}\right)=9^{9\left(2^{s}-1\right)}$.

6: $\quad \operatorname{AZI}(\psi)=\sum_{u v \in E(\psi)}\left(\frac{d_{u} d_{v}}{d_{u} d_{v}-2}\right)^{3}=9\left(2^{s}-1\right)\left(\frac{9}{7}\right)^{3}=\frac{6561}{343}\left(2^{s}-1\right)$.
7: $\quad R M_{2}(\psi)=\sum_{u v \in E(\psi)}\left(d_{u}-1\right)\left(d_{v}-1\right)=9\left(2^{s}-1\right) 4=36\left(2^{s}-1\right)$.
8: ${ }^{m} M_{2}(\psi)=\sum_{u v \in E(\psi)} \frac{1}{d_{u} d_{v}}=9\left(2^{s}-1\right) \frac{1}{9}=2^{s}-1$.
9: $\quad H M(\psi)=\sum_{u v \in E(\psi)}\left(d_{u}+d_{v}\right)^{2}=9\left(2^{s}-1\right) 6^{2}=324\left(2^{s}-1\right)$.

Theorem 15. Let $\psi$ be the eye network, then sum connectivity and general sum connectivity indices are
1: $\quad \operatorname{SCI}(\psi)=\frac{9}{\sqrt{6}}\left(2^{s}-1\right)$.
2: $\quad \chi_{\alpha}(\psi)=9\left(2^{s}-1\right) 6^{\alpha}$.

Proof. By using size and degree of networks that is $9\left(2^{s}-1\right), 3$ respectively, following indices can be computed like given below,

1: $\operatorname{SCI}(\psi)=\sum_{u v \in E(\psi)} \frac{1}{\sqrt{d_{u}+d_{v}}}=9\left(2^{s}-1\right) \frac{1}{\sqrt{6}}=\frac{9}{\sqrt{6}}\left(2^{s}-1\right)$.
2: $\quad \chi_{\alpha}(\psi)=\sum_{u v \in E(\psi)}\left(d_{u}+d_{v}\right)^{\alpha}=9\left(2^{s}-1\right) 6^{\alpha}$.

Theorem 16. Let $\psi$ be the eye network, then geometric arithmetic, symmetric division and inverse sum indices are
1: $\quad G A(\psi)=G A_{5}(\psi)=9\left(2^{s}-1\right)$.
2: $\operatorname{SDI}(\psi)=18\left(2^{s}-1\right)$.
3: $\quad I S(\psi)=\frac{27}{2}\left(2^{s}-1\right)$
Proof. By using size and degree of networks that is $9\left(2^{s}-1\right), 3$ respectively, following indices geometric arithmetic, symmetric division and inverse sum indices can be computed like given below,

1: $\quad G A(\psi)=\sum_{u v \in E(\psi)} \frac{2 \sqrt{d_{u} d_{v}}}{d_{u}+d_{v}}=9\left(2^{s}-1\right) \frac{2 \sqrt{9}}{6}=9\left(2^{s}-1\right)$.
2: $\operatorname{SDI}(\psi)=\sum_{u v \in E(\psi)} \frac{d_{u}^{2}+d_{v}^{2}}{d_{u} d_{v}}=9\left(2^{s}-1\right) \frac{18}{9}=18\left(2^{s}-1\right)$.
3: $\quad I S(\psi)=\sum_{u v \in E(\psi)} \frac{d_{u} d_{v}}{d_{u}+d_{v}}=9\left(2^{s}-1\right) \frac{9}{6}=\frac{27}{2}\left(2^{s}-1\right)$.

Theorem 17. : Let $\psi$ be the eye network, then different M-polynomials are:

1: $\quad M_{1}(\psi ; x, y)=54\left(2^{s}-1\right)(x y)^{3}$.
2: $\quad M_{2}(\psi ; x, y)=729\left(2^{s}-1\right)^{2}(x y)^{6}$.
3: $\quad m M(\psi ; x, y)=9\left(2^{s}-1\right)^{2}(x y)^{6}$.
4: $\quad R_{\alpha}(\psi ; x, y)=\left(729\left(2^{s}-1\right)^{2}(x y)^{6}\right)^{\alpha}$.
$5: \quad R R_{\alpha}(\psi ; x, y)=9\left(2^{s}-1\right)^{2}(x y)^{6}$.
6: $\operatorname{SDI}(\psi ; x, y)=162\left(2^{s}-1\right)^{2}(x y)^{6}$.
7: $\quad H(\psi ; x, y)=54\left(2^{s}-1\right)^{2} x^{9} y^{3}$.
8: $\quad \operatorname{IS}(\psi ; x, y)=19683\left(2^{s}-1\right)^{3} x^{15} y^{9}$.
Proof. To get the results for the M-polynomials, following few formulas computed by using the definition of M-polynomials given in the form of differential in equation (8).

$$
\begin{aligned}
D_{x} & =D_{y}=27\left(2^{s}-1\right)(x y)^{3}, \quad J=9\left(2^{s}-1\right) x^{6} . \\
S_{x} & =S_{y}=3\left(2^{s}-1\right)(x y)^{3}, \quad Q_{-2}=9\left(2^{s}-1\right) x y^{3} .
\end{aligned}
$$

after plugging all above equations we get required following results.
1 :

$$
\begin{aligned}
& M_{1}(\psi ; x, y)=\left(D_{x}+D_{y}\right) M(\psi ; x, y)=27\left(2^{s}-1\right)(x y)^{3}+27\left(2^{s}-1\right)(x y)^{3} . \\
& =54\left(2^{s}-1\right)(x y)^{3} \text {. } \\
& M_{2}(\psi ; x, y)=\left(D_{x} D_{y}\right) M(\psi ; x, y)=27\left(2^{s}-1\right)(x y)^{3} \times 27\left(2^{s}-1\right)(x y)^{3} . \\
& =729\left(2^{s}-1\right)^{2}(x y)^{6} \text {. } \\
& \text { 3: }{ }^{m} M(\psi ; x, y)=\left(S_{x} S_{y}\right) M(\psi ; x, y)=\left(3\left(2^{s}-1\right)(x y)^{3} * 3\left(2^{s}-1\right)(x y)^{3}\right) \text {. } \\
& =9\left(2^{s}-1\right)^{2}(x y)^{6} \text {. } \\
& \text { 4: } \quad R_{\alpha}(\psi ; x, y)=\left(D_{x}^{\alpha} D_{y}^{\alpha}\right) M(\psi ; x, y)=\left(27\left(2^{s}-1\right)(x y)^{3}\right)^{\alpha}\left(27\left(2^{s}-1\right)(x y)^{3}\right)^{\alpha} . \\
& =\left(729\left(2^{s}-1\right)^{2}(x y)^{6}\right)^{\alpha} \text {. } \\
& \text { 5: } \quad R R_{\alpha}(\psi ; x, y)=\left(S_{x}^{\alpha} S_{y}^{\alpha}\right) M(\psi ; x, y)=\left(3\left(2^{s}-1\right)(x y)^{3}\right)^{\alpha}\left(3\left(2^{s}-1\right)(x y)^{3}\right)^{\alpha} \text {. } \\
& =\left(9\left(2^{s}-1\right)^{2}(x y)^{6}\right)^{\alpha} \text {. } \\
& \text { 6: } \operatorname{SDI}(\psi ; x, y)=\left(D_{x} S_{y}+D_{y} S_{x}\right) M(\psi ; x, y) . \\
& =3\left(2^{s}-1\right)(x y)^{3} 27\left(2^{s}-1\right)(x y)^{3}+27\left(2^{s}-1\right)(x y)^{3} 3\left(2^{s}-1\right)(x y)^{3} \\
& =162\left(2^{s}-1\right)^{2}(x y)^{6} \text {. } \\
& H(\psi ; x, y)=2 S_{x} J M(\psi ; x, y)=2 \times 3\left(2^{s}-1\right)(x y)^{3} \times 9\left(2^{s}-1\right) x^{6}=54\left(2^{s}-1\right)^{2} x^{9} y^{3} . \\
& I S(\psi ; x, y)=S_{x} J D_{x} D_{y} M(\psi ; x, y) . \\
& =27\left(2^{s}-1\right)(x y)^{3} J=9\left(2^{s}-1\right) x^{6} 27\left(2^{s}-1\right)(x y)^{3} 27\left(2^{s}-1\right)(x y)^{3} . \\
& =19683\left(2^{s}-1\right)^{3} x^{15} y^{9} \text {. }
\end{aligned}
$$

$7:$

This completes the proof.

## 5. Discussion

The exploration of computing M-polynomials and degree-based topological indices in different interconnection networks stands as a pivotal endeavor within the realm of network science. This pursuit is underscored by its inherent significance in unraveling the structural intricacies of complex networks, offering a nuanced understanding that extends beyond traditional metrics. The M-polynomials, derived from the characteristic polynomial of graphs, encapsulate essential information about the network's topological features. This mathematical abstraction enables researchers to distill intricate network structures into concise representations, facilitating a more profound exploration of connectivity patterns and graph properties.

The implementation of this computational approach involves navigating the vast landscape of interconnection networks, ranging from classic architectures to modern communication networks. The computational methods utilized in this study transcend mere abstraction, providing a practical means to assess and characterize the connectivity patterns of diverse networks. This implementation allows for a detailed analysis of different interconnection network models, shedding light on their unique topological signatures and the implications of structural variations.

The significance of computing M-polynomials and degree-based topological indices is accentuated by their applications in diverse domains. In the context of communication networks, understanding the inherent connectivity structures becomes paramount for optimizing routing algorithms, enhancing fault tolerance, and improving overall network performance. Furthermore, in the realm of social networks, the insights garnered from these computations could illuminate patterns of influence and information flow, fostering a deeper comprehension of social dynamics.

Beyond theoretical implications, the applications extend to areas such as epidemiology, where the study of contact networks is crucial for modeling the spread of diseases. The ability to compute M-polynomials
and degree-based indices provides a robust foundation for understanding the intricate relationships between individuals and predicting the potential impact of interventions on disease transmission dynamics.

In conclusion, the pursuit of computing M-polynomials and degree-based topological indices in different interconnection networks transcends the theoretical confines of graph theory. It offers a pragmatic lens through which to analyze, model, and optimize complex systems across diverse disciplines, imparting a wealth of knowledge with far-reaching implications in the realms of technology, communication, and societal understanding.

## 6. Conclusions

In this paper, we investigate the topological indices of different interconnection networks. We compute Randić index, general Randić index, sum connectivity index, families of Zagreb index, symmetric division index and geometric arithmetic indices. We also compute the M-polynomials of K-swapped networks of t-regular graphs, $n$-dimension tiwested cube and eye netwoks. Degree-based topological indices and M-polynomials are mathematical tools used in graph theory to analyze the structural properties of graphs, including interconnection networks. Interconnection networks are crucial components in various fields such as computer science, telecommunications, and social network analysis.

## 7. Open Problem

Different new indices are still open for above structures like, reverse-degree-based topological indices, such as the reverse-degree-based topological index of the first, second, and hyper Zagreb, forgotten, geometric arithmetic, atom-bond-connectivity, and the Randic index.

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