Open Journal of
Discrete Applied Mathematics

PSR Press

Article

Covering and 2-degree-packing numbers in graphs

Carlos A. Alfaro[1], Christian Rubio-Montiel[2] and Adrián Vázquez-Ávila[3,*]

1 Banco de México, Ciudad de México, México; carlos.alfaro@banxico.org.mx.
2 División de Matemáticas e Ingeniería, FES Acatlán, Uiversidad Nacional Autónoma de México, Ciudad de México, México; christian.rubio@acatlan.unam.mx.
3 Subdirección de Ingeniería y Posgrado, Universidad Aeronáutica en Querétaro, Querétaro, México; adrian.vazquez@unaq.mx.
* Correspondence: adrian.vazquez@unaq.mx.

Received: 29 September 2023; Accepted: 12 December 2023; Published: 27 February 2024
Communicated by: Muhammad Kamran Jamil

Abstract: In this paper, we give a relationship between the covering number of a simple graph G, $\beta(G)$, and a new parameter associated to G, which is called 2-degree-packing number of G, $\nu_2(G)$. We prove that

$$\lceil \nu_2(G)/2 \rceil \leq \beta(G) \leq \nu_2(G) - 1,$$

for any simple graph G, with $|E(G)| > \nu_2(G)$. Also, we give a characterization of connected graphs that attain the equalities.

Keywords: Covering number, independence number, 2-degree-packing number.

MSC: 05C69, 05C70.

1. Introduction

In this paper, we consider finite undirected simple graphs. For the terminology, notation and missing basic definitions related to graphs we refer the reader to [1]. Let G be a graph. We call $V(G)$ the vertex set of G and we call $E(G)$ the edge set of G. For a subset $A \subseteq V(G)$, $G[A]$ denotes the subgraph of G which is *induced by the vertex set* A. Likewise, for a subset $R \subseteq E(G)$, $G[R]$ denotes the subgraph of G which is *induced by the edge set* R. The distance between two vertices u and v in a graph G is the number $d_G(u,v)$ of edges in any shortest $u-v$ path in G that joins u and v; if u and v are not joined in G, then $d_G(u,v)=\infty$. The *neighborhood* of a vertex $u \in V(G)$, denoted by $N_G(u)$, is the subset of $V(G)$ adjacent to u in G. The set of edges incident to $u \in V(G)$ is denoted by $\mathcal{L}_u$. Hence, the *degree* of u, denoted by $\deg(u)$, is $\deg(x) = |\mathcal{L}_u|$. The minimum and maximum degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let H be a subgraph of G, the *restricted degree* of a vertex $u \in V(H)$, denoted by $\deg_H(u)$, is defined as $\deg_H(u) = |\mathcal{L}_u \cap E(H)|$.

An *independent set* of a graph G is a subset $I \subseteq V(G)$ such that any two vertices of I are not adjacent. The *independence number* of G, denoted by $\alpha(G)$, is the maximum order of an independent set. A *vertex cover* of a graph G is a subset $T \subseteq V(G)$ such that all edges of G has at least one end in T. The *covering number* of G, denoted by $\beta(G)$, is the minimum order of a vertex cover of G. This parameter is well known and intensively studied in a more general context and with different names, see for example [2–8].

A *k-degree-packing set* of a graph G ($k \leq \Delta(G)$), is a subset $R \subseteq E(G)$ such that $\Delta(G[R]) \leq k$. The *k-degree-packing number* of G, denoted by $\nu_k(G)$, is the maximum order of a k-degree-packing set of G. We are interested in this new parameter when $k=2$, since $k=1$ is the *matching number* of G. Hence, the matching number is a particular case of the k-degree-packing number of a graph when $k=1$.

The 2-degree-packing number is studied in [5,9–13] in a more general context, but with a different name, as *2-packing number*. The definition of 2-packing in graphs have a different meaning: A set $X \subseteq V(G)$ is called a 2-packing if $d_G(u,v) > 2$ for any different vertices u and v of X, that is, the 2-packing is a subset $X \subseteq V(G)$ in which all the vertices are in distance at least 3 from each other, see for example [14]. Therefore, we called 2-degree-packing instead of 2-packing only applied for graphs.

As a particular case, Araujo-Pardo el al. proved in [5] any simple graph G satisfies:

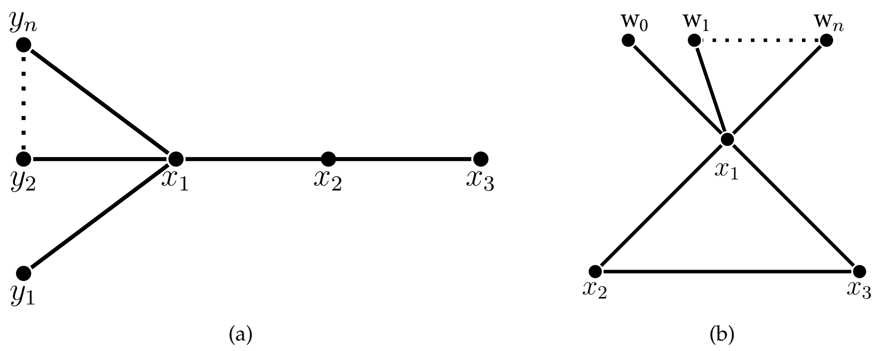

Figure 1. Graphs with $\beta = 2$ and $\nu_2 = 3$.

$$\lceil \nu_2(G)/2 \rceil \leq \beta(G). \tag{1}$$

In this paper, we prove that for any simple graph G, with $|E(G)| > \nu_2(G)$, is such that:

$$\beta(G) \leq \nu_2(G) - 1. \tag{2}$$

Hence, by Equations (1) and (2), we have:

Theorem 1. *If G is a simple connected graph with $|E(G)| > \nu_2(G)$, then*

$$\lceil \nu_2(G)/2 \rceil \leq \beta(G) \leq \nu_2(G) - 1.$$

In this paper, we give a characterization of simple connected graphs that attain the upper and lower bounds in Theorem 1.

2. Some results

Only connected graphs with $|E(G)| > \nu_2(G)$ are considered, since $|E(G)| = \nu_2(G)$ if and only if $\Delta(G) \leq 2$. Moreover, we may assume $\nu_2(G) \geq 4$, since otherwise Araujo-Pardo et al. in [5] proved:

Proposition 2. **[5]** *Let G be a simple graph with $|E(G)| > \nu_2(G)$, then $\nu_2(G) = 2$ if and only if $\beta(G) = 1$.*

Proposition 3. **[5]** *Let G be a simple connected graph with $|E(G)| > \nu_2(G)$. If $\nu_2(G) = 3$, then $\beta(G) = 2$.*

If a graph G satisfies the hypothesis of Proposition 2 with $\nu(G) = 2$, then G is the complete bipartite graph of the form $K_{1,m}$, with $m \geq 2$. If the graph G satisfies the hypothesis of Proposition 3, then G is one of the graphs shown in Figure 1 (see [5]).

The next proposition shows some simple consequences of the definitions given previously. Also, some results are well known.

Proposition 4.

1. *If R is a maximum 2-degree-packing of a graph G, then the components of $G[R]$ are either cycles or paths.*
2. *If G is either a cycle or a path, both of even length, and T is a minimum vertex cover of G, then T is an independent set.*
3. *If G is cycle of length odd and T is a minimum vertex cover of G, then there exists an unique $u \in T$ such that $T \setminus \{u\}$ is an independent set. On the other hand, if G is a path of length odd, then either there exists an unique $u \in T$ such that $T \setminus \{u\}$ is an independent set or T is an independent and $\deg_T(u) = 1$.*
4. *If G is either a path or a cycle of length k, then $\beta(G) = \lceil \frac{k}{2} \rceil$.*
5. $\beta(K_n) = \nu_2(K_n) - 1$.

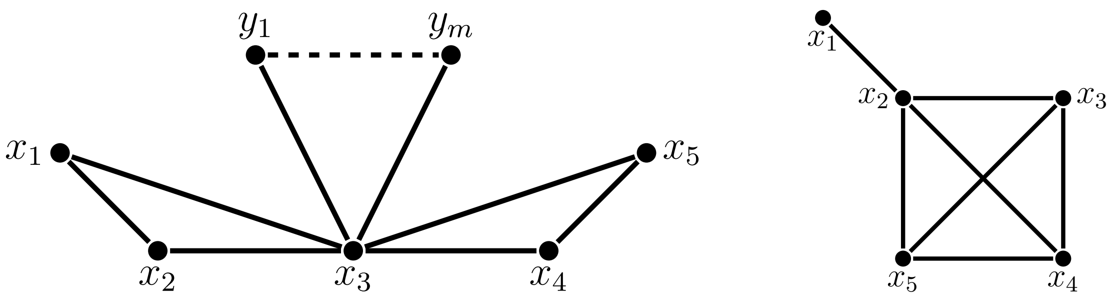

Figure 2. Graphs with $\nu_2(G) = 4$ and $\beta(G) = 3$

Remark 1. Let R be a maximum 2-degree-packing of a simple graph G. It is clear that the number of components of $G[R]$ is at most $\nu_2(G) - 1$. Moreover, if T is a minimum vertex cover of $G[R]$, then $\beta(G) \le k + p$, where k is the number of components of $G[R]$ of a single edge, and $p = |\{v \in V(G[R]) : \deg_R(v) = 2\}|$. Hence, $\beta(G) \le k + p \le \nu_2(G)$.

Proposition 5. *If G is a simple graph with $|E(G)| > \nu_2(G)$, then $\beta(G) \le \nu_2(G) - 1$.*

Proof. By Remark 1, we have $\beta(G) \le k + p \le \nu_2(G)$. It is not hard to see, if $k \ge 1$, then $\beta(G) \le \nu_2(G) - 1$. On the other hand, if $k = 0$, then any component of $G[R]$ is a cycle, since if $G[R]$ has a path (of length at least 2) as a component, then $\beta(G) \le \nu_2(G) - 1$. Hence $p = \nu_2(G)$. We may assume $V(G[R]) = V(G)$, since otherwise if $u \in V(G) \setminus V(G[R])$ and $e_u = uv \in E(G) \setminus R$, where $v \in V(G[R])$, then the following set $(R \setminus \{e_v\}) \cup \{e_u\}$, where $e_v \in R$, is incident to v, is a maximum 2-degree-packing of G with a path as a component, which implies that $\beta(G) \le \nu_2(G) - 1$. Therefore $\{v \in V(G[R]) : \deg_R(v) = 2\} \setminus \{u\}$, for any $u \in V(G[R])$, is a vertex cover of G, implying that $\beta(G) \le \nu_2(G) - 1$. □

Hence, we have:

Theorem 6. *If G is a simple graph with $|E(G)| > \nu_2(G)$, then*

$$\lceil \nu_2(G)/2 \rceil \le \beta(G) \le \nu_2(G) - 1.$$

3. Graphs with $\beta = \nu_2 - 1$

We introduce some terminology in order to simplify the description of simple connected graphs G such that $\beta(G) = \nu_2(G) - 1$.

As a particular case, Araujo-Pardo el al. proved in [5] the following:

Proposition 7. *[5] If G is a simple graph G with $\nu_2(G) = 4$ and $|E(G)| > 4$, then $\beta(G) \le 3$.*

Also, in this paper [5], the authors give all the connected graphs with $\nu_2(G) = 4$ and $\beta(G) = 3$ and they are certain subgraphs of the graphs given in Figure 2. Hence, by Proposition 7, we may assume $\nu_2(G) \ge 5$.

In [15] Vázquez-Ávila constructed the graph $T_{s,t}$, with $s \ge 1$ and $t \ge 2$, (see Figure 3 (a)), where:

$$\begin{aligned} V(T_{s,t}) &= \{p_1, \ldots, p_s\} \cup \{q_1, \ldots, q_s\} \cup \{w_1, \ldots, w_t\}, \\ E(T_{s,t}) &= \{p_i q_i : i = 1, \ldots, s\} \cup \{v p_i : i = 1, \ldots, s\} \cup \{v w_i : i = 1, \ldots, t\}. \end{aligned}$$

Let $G_{s,t}$, with $s \ge 1$ and $t \ge 2$, be the graph constructed from $T_{s,t}$, where (see Figure 3 (b)):

$$\begin{aligned} V(G_{s,t}) &= V(T_{s,t}), \\ E(G_{s,t}) &= E(T_{s,t}) \cup \{v q_i : i = 1, \ldots, s\}. \end{aligned}$$

As a consequence of Corollary 2.1 given in [15], we have:

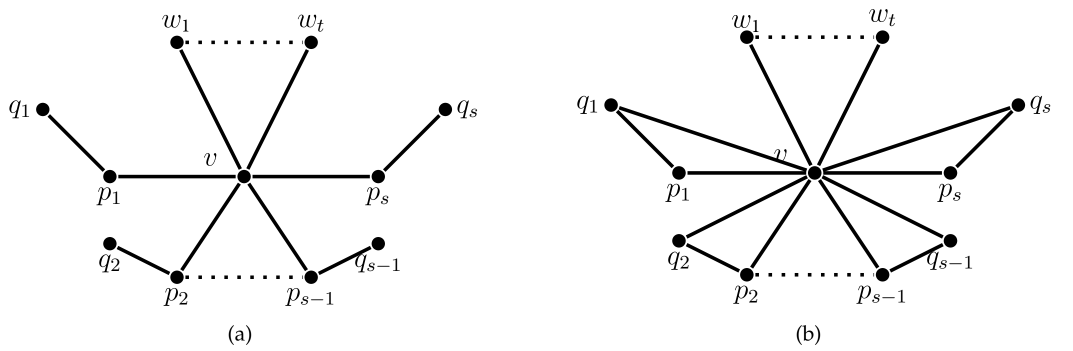

Figure 3. In (a) depict the Graph $T_{s,t}$ and in (b) depict the graph $G_{s,t}$.

Corollary 8. *[15] $\beta(T_{s,t}) = \nu_2(T_{s,t}) - 1 = s + 1$, for every $s \geq 1$ and $t \geq 2$.*

Since the graph $T_{s,t}$ is a spanning graph of $G_{s,t}$ and any minimum vertex cover of $T_{s,t}$ is a vertex covering of $G_{s,t}$, then:

Corollary 9. *$\beta(G_{s,t}) = \nu_2(G_{s,t}) - 1 = s + 1$, for every $s \geq 1$ and $t \geq 2$.*

Corollary 10. *If $T_{s,t}$ is a spanning subgraph of a graph G and G is a spanning subgraph of $G_{s,t}$, then $\beta(G) = \nu_2(G) - 1 = s + 1$.*

Let G be a simple graph with $|E(G)| > \nu_2(G)$ and R be a maximum 2-degree-packing of G. Let $R_1, \ldots, R_s, R_{s+1}, \ldots, R_k$ be the components of $G[R]$, where $|R_i| = 1$, for $i = 1, \ldots, s$ and $|R_j| > 1$, for $j = s + 1, \ldots, k$. It is not difficult to see that $s \leq \nu_2(G) - 2$. If $s = \nu_2(G) - 2$, then $k = \nu_2(G) - 1$ and $|E(G[R_k])| = 2$. Hence, any edge from $E(G) \setminus E(G[R])$ is incident with the unique vertex $v \in V(G[R_k])$ with $\deg_R(v) = 2$. Hence, if $R_i = p_i q_i$, for $i = 1, \ldots, s$, $R_k = w_0 v w_1$, and $V(G) \setminus V(G[R]) = \{w_3, \ldots, w_t\}$ (an independent set), if $t \geq 3$, then $T_{s,t}$ is a spanning subgraph of a graph G and G is a spanning subgraph of $G_{s,t}$. Therefore, $\beta(G) = \nu_2(G) - 1 = s + 1$.

Let $R_1, \ldots, R_s, R_{s+1}, \ldots, R_k$ be the components of a simple connected graph G, with k as small as possible, where $|R_i| = 1$, for $i = 1, \ldots, s$ and $|R_j| > 1$, for $j = s + 1, \ldots, k$. It is clear that $\beta(G) = s + \beta(H)$ and $\nu_2(G) = s + \nu_2(H)$, where H is given by

$$\begin{aligned} V(H) &= V(G) \setminus \bigcup_{i=1}^{s} u_i, \\ E(H) &= E(G) \setminus \bigcup_{i=1}^{s} \mathcal{L}_{u_i}, \end{aligned}$$

where $u_i \in V(G[R_i])$, for $i = 1, \ldots, s$, and deleting those vertices of degree 0 (if any). Therefore, it may be assumed that any simple connected graph G, with $|E(G)| > \nu_2(G)$, has a maximum 2-degree-packing R of G, where each component of $G[R]$ has at least 2 edges; and as a consequence, the set $T = \{u \in V(G[R]) : \deg_{G[R]}(u) = 2\}$ is a vertex cover of G.

Let K_n^1 be the simple connected graph, where

$$\begin{aligned} V(K_n^1) &= \{x_1, \ldots, x_n\} \cup \{u\}, \\ E(K_n^1) &= \{x_i x_j : 1 \leq i < j \leq n\} \cup \{u x_1\}. \end{aligned}$$

The graph K_n^1 is the complete graph of n vertices with one extra edge attached. It is easy to see that $\beta(K_n^1) = \nu_2(K_n^1) - 1 = n - 1$.

Proposition 11. *Let G be a simple graph with $|E(G)| > \nu_2(G)$, $\nu_2(G) \geq 5$ and $\beta(G) = \nu_2(G) - 1$. If R is a maximum 2-degree-packing of G with $V(G[R]) = V(G)$, then either G is the complete graph K_{ν_2} or G is $K_{\nu_2}^1$, where $\nu_2 = \nu_2(G)$.*

Proof. Let R be a maximum 2-degree-packing of G with $V(G[R]) = V(G)$ and $R_1, \ldots, R_k$ be the components of $G[R]$ with k as small as possible. Then:

Case(i) If $k = 1$, then $G[R]$ is either a path or a cycle. Suppose that $R = u_0u_1 \cdots u_{\nu_2-1}u_0$ is a cycle: If there are two non-adjacent vertices $u_i, u_j \in V(G[R]) = V(G)$, then $T = V(G[R]) \setminus \{u_i, u_j\}$ is a vertex cover of G of cardinality $\nu_2(G) - 2$, which is a contradiction. Therefore, any different pair of vertices of G are adjacent. Hence, the graph G is the complete graph with $\nu_2(G)$ vertices.

On the other hand, if $R = u_0u_1 \cdots u_{\nu_2}$ is a path, then $T = \{u_1, \ldots, u_{\nu_2-1}\}$ is a minimum vertex cover of G. We may assume either $u_0u_j \in E(G)$ or $u_{\nu_2}u_j \in E(G)$, for all $u_j \in T^* = T \setminus \{u_1, u_{\nu_2-1}\}$, since otherwise, $T \setminus \{u_j\}$ is a vertex cover of G of cardinality $\nu_2(G) - 2$, which is a contradiction. Without loss of generality, suppose $u_0u_j \in E(G)$, for all $u_j \in T^* = T \setminus \{u_1, u_{\nu_2-1}\}$. If $u_ju_{\nu_2} \in E(G)$, for some $u_j \in T^*$, then $R^* = (R \setminus \{u_ju_{j+1}\}) \cup \{u_ju_{\nu_2}, u_0u_{j+1}\}$ (since $\nu_2(G) \geq 5$) is a 2-degree-packing of size $\nu_2(G) + 1$, a contradiction. Hence $u_ju_{\nu_2} \notin E(G)$, for all $u_j \in T^*$, which implies that $\deg(u_{\nu_2}) = 1$. On the other hand, if there are two vertices $u_i, u_j \in T^*$ non-adjacents, then $(T \setminus \{u_i, u_j\}) \cup \{u_0\}$ is a vertex cover of G of size $\nu_2(G) - 2$, which is a contradiction. Also, $u_1u_j \in E(G)$ and $u_ju_{\nu_2-1} \in E(G)$, for all $u_j \in T^*$, otherwise there exists $u_j \in T^*$ such that either $(T \setminus \{u_1, u_j\}) \cup \{u_0\}$ or $(T \setminus \{u_j, u_{\nu_2-1}\}) \cup \{u_0\}$ is a vertex cover of G of size $\nu_2(G) - 2$, which is a contradiction. Therefore, the graphs G is the graph $K^1_{\nu_2}$.

Case (ii) Suppose $k \geq 2$ and $T = \{v \in V(G[R]) : \deg_R(v) = 2\}$. If there is at least two components as a paths (of length at least 2), say R_1 and R_2, then

$$\begin{aligned}\beta(G) \leq |T| &\leq (|E(R_1)| - 1) + (|E(R_2)| - 1) + \sum_{i=3}^{k} |E(R_i)| \\ &= \sum_{i=1}^{k} |E(R_i)| - 2 = \nu_2(G) - 2,\end{aligned}$$

which is a contradiction. Hence, there are at most one component as a path of length at least 2. Let $u \in V(R_1)$ such that $\deg_R(u) = 1$, then $\deg_G(u) = 1$, otherwise $T \setminus \{v\}$, where u and v are adjacent, is a vertex cover of G of size $\nu(G) - 1$, which is a contradiction. Moreover, if $u \in V(R_1)$ such that $\deg_{R_1}(u) = 2$ and there is $v \in V(G) \setminus V(R_1)$ such that u and v are non-adjacents, then $T \setminus \{v\}$ is a vertex cover of G of size $\nu(G) - 2$, a contradiction. Therefore $k = 1$, which is a contradiction.

□

Theorem 12. *Let G be a simple connected graph with $\nu_2(G) \geq 5$ and $\beta(G) = \nu_2(G) - 1$. Then either G is the complete graph K_{ν_2} or G is $K^1_{\nu_2}$, where $\nu_2 = \nu_2(G)$.*

Proof. Let R be a maximum 2-degree-packing of G and $I = V(G) \setminus V(G[R])$ (independent set of vertices). Then $I \neq \emptyset$, by the Proposition 6.

Case (i): Suppose $G[R]$ is the complete graph of $\nu_2(G)$ vertices. We claim, if $u \in I$, then $\deg(u) = 1$. To verify the claim, we suppose on the contrary, u is incident to at least two vertices of $V(G[R])$, say v and w. If $V(G[R]) = \{u_1, \ldots, u_{\nu_2}\}$, then without loss of generality $u_1 = v$ and $u_j = w$, for some $j \in \{2, \ldots, \nu_2\}$ ($G[R]$ is a complete graph). Then

$$(R \setminus \{u_1u_{\nu_2}, u_{j-1}u_j\}) \cup \{uu_1, uu_j, u_{j-1}u_{\nu_2}\}$$

is a 2-degree-packing of G of size $\nu_2(G) + 1$, which is a contradiction. Hence, if $u \in I$, then $\deg_G(u) = 1$.

On the other hand, if $|I| > 1$, let $u, v \in I$. Without loss of generality, suppose u is adjacent to u_1 and v is adjacent to u_j, for some $j \in \{2, \ldots, \nu_2\}$. Since $G[R]$ is a complete graph, then

$$(R \setminus \{u_1u_{\nu_2}, u_{j-1}u_j\}) \cup \{uu_1, u_{j-1}u_{\nu_2}, vu_j\}$$

is a 2-degree-packing of size $\nu_2(G) + 1$, which is a contradiction. Also, if u and v are adjacent to u_1, then

$$(R \setminus \{u_1u_2, u_1u_{\nu_2}\}) \cup \{uu_1, vu_1, u_2u_{\nu_2}\}$$

is a 2-degree-packing of size $\nu_2(G)+1$, which is contradiction. Hence, $I=\{u\}$ with $\deg(u)=1$, which implies that the graph G is $K^1_{\nu_2}$.

Case (ii): Suppose $G[R]$ is the graph $K^1_{\nu_2}$. Let $v\in V(G)$ such that the $G[R]-v$ is the complete graph of size $\nu_2(G)$. If $u\in I$ is such that $uw\in E(G)$, whit $w\in V(G[R])$, then, there exists a 2-degree-packing of G of size $\nu_2(G)+1$ (see proof of Proposition 6, which is a contradiction. Then $uw\notin E(G)$, for all $w\in V(G[R])\cup\{v\}$, which implies that G is a disconnected graph, unless $I=\varnothing$, and the theorem holds by Proposition 6.

□

4. Graphs with $\beta=\lceil \nu_2/2\rceil$

We introduce some terminology and results in order to simplify the description of the simple connected graphs G which satisfy $\beta(G)=\lceil \nu_2(G)/2\rceil$.

Proposition 13. *Let G be a simple connected graph and R be a maximum 2-degree-packing of G.*

1. *If $\nu_2(G)$ is an even integer and $\beta(G)=\dfrac{\nu_2(G)}{2}$, then the components of R has even length.*
2. *If $\nu_2(G)$ is an odd integer and $\beta(G)=\dfrac{\nu_2(G)+1}{2}$, then there is an unique component of R of odd length.*

Proof. To prove the item 1, let R be a maximum 2-degree-packing of G and let $R_1,\dots,R_k$ be the components of $G[R]$. If T is a minimum vertex cover of G, then

$$\frac{\nu_2(G)}{2}=\beta(G)=|T|=\sum_{i=1}^{k}|T\cap V(R_i)|\geq\sum_{i=1}^{k}\beta(R_i)=\sum_{i=1}^{k}\lceil\nu_2(R_i)/2\rceil.$$

Hence, if R_1 have a odd number of edges, then

$$\sum_{i=1}^{k}\lceil\nu_2(R_i)/2\rceil=\frac{\nu_2(R_1)+1}{2}+\sum_{i=2}^{k}\lceil\nu_2(R_i)/2\rceil\geq\frac{1}{2}+\sum_{i=1}^{k}\frac{\nu_2(R_i)}{2}=\frac{1}{2}+\frac{\nu_2(G)}{2},$$

which is a contradiction. Therefore, each component of $G[R]$ has an even number of edges. To prove the item 2 we use an analogous argument. □

Let A and B be two sets of vertices. The complete graph whose set of vertices is A is denoted by K_A. The graph whose set of vertices is $A\cup B$ and whose set of edges is $\{ab: a\in A, b\in B\}$ is denoted by $K_{A,B}$. On the other hand, let $k\geq 3$ be a positive integer. The cycle of length k and the path of length k are denoted by C^k and P^k, respectively.

If A and B are two sets of vertices from $V(C^k)$ and $V(P^k)$ (not necessarily disjoint) and I be an independent set of vertices different from $V(C^k)$ and $V(P^k)$ then $C^k_{A,B,I}=(V(C^k_{A,B,I}),E(C^k_{A,B,I}))$ and $P^k_{A,B,I}=(V(P^k_{A,B,I}),E(P^k_{A,B,I}))$ are denoted to be the graphs with $V(C^k_{A,B,I})=V(C^k)\cup I$ and $V(P^k_{A,B,I})=V(P^k)\cup I$, respectively, and $E(C^k_{A,B,I})=E(C^k)\cup E(K_A)\cup E(K_{A,B})\cup E(K_{A,I})$ and $E(P^k_{A,B,I})=E(P^k)\cup E(K_A)\cup E(K_{A,B})\cup E(K_{A,I})$, respectively. In an analogous way, we denote by C^k_I to be the graph with $V(C^k_I)=V(C^k)\cup I$ and $E(C^k_I)=E(C^k)$ and we denote by P^k_I to be the graph with $V(P^k_I)=V(P^k)\cup I$ and $E(P^k_I)=E(P^k)$. In Figure 4 are depicted the graphs C^k_I and P^k_I, where $|I|=i$.

We define $\mathcal{C}^k_{A,B,I}$ to be the family of connected graphs G such that C^k_I is a subgraph of G and G is a subgraph of $C^k_{A,B,I}$. Similarly, we define $\mathcal{P}^k_{A,B,I}$ to be the family of connected graphs G such that P^k_I is a subgraph of G and G is a subgraph of $P^k_{A,B,I}$.

That is

$$\mathcal{C}^k_{A,B,I}=\{G: C^k_I\subseteq G\subseteq C^k_{A,B,I}\text{ where } G \text{ is a connected graph}\}$$

$$\mathcal{P}^k_{A,B,I}=\{G: P^k_I\subseteq G\subseteq P^k_{A,B,I}\text{ where } G \text{ is a connected graph}\}$$

Proposition 14. *Let $k\geq 4$ be an even integer, T be a minimum vertex cover of C^k and I be an independent set of vertices different from $V(C^k)$. If $\hat{T}=V(C^k)\setminus T$ and $G\in\mathcal{C}^k_{T,\hat{T},I}$, then $\beta(G)=\frac{k}{2}$ and $\nu_2(G)=k$.*

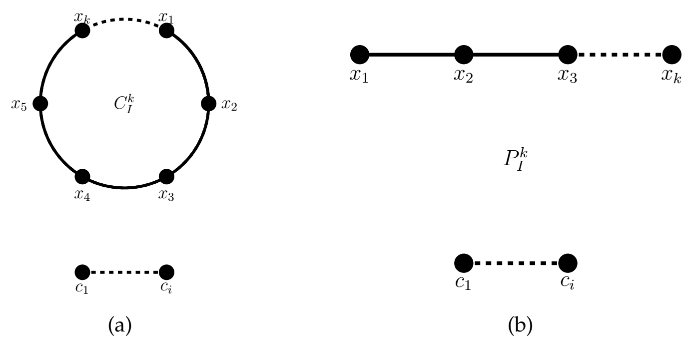

Figure 4. In (a) depict the Graph C_I^k and in (b) depict the graph P_I^k.

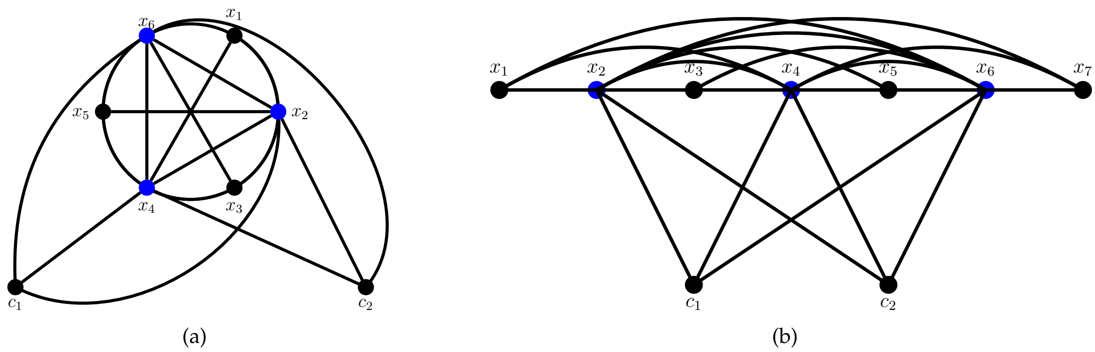

Figure 5. In (a) is depict the Graph $C^6_{T,\hat{T},I}$ and in (b) is depict the graph $P^6_{T,\hat{T},I}$, where $T = \{x_2, x_4, x_6\}$ and $I = \{c_1, c_2\}$.

Proof. It is clear that, if $G \in \mathcal{C}^k_{T,\hat{T},I}$, then $\beta(G) = \frac{k}{2}$. On the other hand, since C^k is a 2-degree-packing of G, then $\nu_2(G) \geq k$. Moreover, since $\lceil \nu_2(G)/2 \rceil \leq \beta(G) = \dfrac{k}{2}$, then $\nu_2(G) = k$. □

In Figure 5 are depicted the graphs $C^6_{T,\hat{T},I}$ and $P^6_{T,\hat{T},I}$, where $T = \{x, x_4, x_6\}$ and $I = \{c_1, c_2\}$.

Corollary 15. *Let $k \geq 4$ be an even integer, T be a minimum vertex cover of P^k and I be an independent set of vertices different from $V(P^k)$. If $\hat{T} = V(P^k) \setminus T$ and $G \in \mathcal{P}^k_{T,\hat{T},I}$, then $\beta(G) = \frac{k}{2}$ and $\nu_2(G) = k$.*

For instance, any connected graph G containing the subgraph of Figure 4 (a) and whose supergraph is the graph of Figure 5 (a) is such that $\tau = 3$ and $\nu_2 = 6$.

Now, let $\hat{\mathcal{C}}^k_{A,B,I}$ be the family of simple connected graphs G with $\nu_2(G) = k$, such that C_I^k is a subgraph of G and G is a subgraph of $C^k_{A,B,I}$. Similarly, let $\hat{\mathcal{P}}^k_{A,B,I}$ be the family of simple connected graphs G with $\nu_2(G) = k$ such that P_I^k is a subgraph of G and G is a subgraph of $P^k_{A,B,I}$. That is

$$\hat{\mathcal{C}}^k_{A,B,I} = \{G : C_I^k \subseteq G \subseteq C^k_{A,B,I} \text{ where } G \text{ is connected and } \nu_2(G) = k\},$$

$$\hat{\mathcal{P}}^k_{A,B,I} = \{G : P_I^k \subseteq G \subseteq P^k_{A,B,I} \text{ where } G \text{ is connected and } \nu_2(G) = k\}.$$

Hence if $k \geq 4$ is an even integer, T is a minimum vertex cover of either C^k or P^k, and I is an independent set different from either $V(C^k)$ or $V(P^k)$, then by Proposition 8 and Corollary 4, we have

$$\hat{\mathcal{C}}^k_{T,\hat{T},I} = \mathcal{C}^k_{T,\hat{T},I} \text{ and } \hat{\mathcal{P}}^k_{T,\hat{T},I} = \mathcal{P}^k_{T,\hat{T},I}.$$

However, if $k \geq 5$ is an odd integer, T is a minimum vertex cover of either C^k or P^k and I is an independent set different from either $V(C^k)$ or $V(P^k)$, then

$$\hat{\mathcal{C}}^k_{T,\hat{T},I} \neq \mathcal{C}^k_{T,\hat{T},I} \text{ and } \hat{\mathcal{P}}^k_{T,\hat{T},I} \neq \mathcal{P}^k_{T,\hat{T},I}.$$

To see this, let R be the cycle of length k and $u, v \in T$ adjacent. Hence, if G is such that $V(G) = V(C^k) \cup \{w\}$, where $w \in I$ and $E(G) = E(C^k) \cup \{uw, vw\}$, then $G \in \mathcal{C}^k_{T,\hat{T},I}$. However, it is clear that $\nu_2(G) = k+1$, which implies that $G \notin \hat{\mathcal{C}}^k_{T,\hat{T},T}$. A similar argument is used to prove that $\hat{\mathcal{P}}^k_{T,\hat{T},I} \neq \mathcal{P}^k_{T,\hat{T},I}$.

Proposition 16. *Let $k \geq 5$ be an odd integer, T be a minimum vertex cover of C^k and I be an independent set of vertices different from $V(C^k)$. If $\hat{T} = V(C^k) \setminus T$ and $G \in \hat{\mathcal{C}}^k_{T,\hat{T},I}$, then $\beta(G) = \frac{k+1}{2}$.*

Proof. It is clear that

$$\frac{k+1}{2} = \left\lceil \nu_2(C^k_I)/2 \right\rceil \leq \lceil \nu_2(G)/2 \rceil \leq \beta(G) \leq |T| = \frac{k+1}{2},$$

which implies that $\beta(G) = \frac{k+1}{2}$. □

Corollary 17. *Let $k \geq 5$ be an odd integer, T be a minimum vertex cover of P^k and I be an independent set of vertices different from $V(P^k)$. If $\hat{T} = V(P^k) \setminus T$ and $G \in \hat{\mathcal{P}}^k_{T,\hat{T},I}$, then $\beta(G) = \frac{k+1}{2}$.*

Proposition 18. *Let G be a connected graph with $|E(G)| > \nu_2(G)$ and $R_1, \ldots, R_k$ be the components of a maximum 2-degree-packing of G. If $\beta(G) = \lceil \nu_2(G)/2 \rceil$, then $\beta(G) = \sum_{i=1}^{k} \beta(R_i)$.*

Proof. Let R be a maximum 2-degree-packing of G and $R_1, \ldots, R_k$ be the components of $G[R]$. Since R_i is a cycle or a path of length $\nu_2(R_i)$, then $\beta(R_i) = \lceil \nu_2(R_i)/2 \rceil$, for $i = 1, \ldots, k$. If $\beta(G) = \lceil \nu_2(G)/2 \rceil$, then by Proposition 7, we have

$$\lceil \nu_2(G)/2 \rceil = \beta(G) \geq \sum_{i=1}^{k} \beta(R_i) = \sum_{i=1}^{k} \lceil \nu_2(R_i)/2 \rceil = \lceil \nu_2(G)/2 \rceil.$$

Therefore $\beta(G) = \sum_{i=1}^{k} \beta(R_i)$. □

By Proposition 13 and Proposition 18, we have:

Theorem 19. *Let G be a connected graph with $|E(G)| > \nu_2(G)$ and $R_1, \ldots, R_k$ be the components of a maximum 2-degree-packing of G.*

Then: $\beta(G) = \lceil \nu_2(G)/2 \rceil$, if and only if, $\beta(G) = \sum_{i=1}^{k} \beta(R_i)$, being:

1. *$|R_i|$ an even integer, for $i = 1, \ldots, k$, if $\nu_2(G)$ an even number.*
2. *$|R_1|$ is an odd integer and $|R_i|$ is an even integer, for $i = 2, \ldots, k$, if $\nu_2(G)$ is an odd number.*

Proposition 20. *Let G be a simple connected graph with $\nu_2(G) \geq 4$, $|E(G)| > \nu_2(G)$ and $R_1, \ldots, R_k$ be the components of a maximum 2-degree-packing R of G, with k as small as possible. If $\beta(G) = \lceil \nu_2(G)/2 \rceil$, then $I = I_1 \cup \cdots \cup I_k = V(G) \setminus V(G[R])$, where either $I_i = \varnothing$ or for every $u \in I_i$ satisfies $N(u) \subseteq V(R_i)$, for $i = 1, \ldots, k$.*

Proof. Suppose there exists $u \in I$, $w_i \in V(R_i)$ and $w_j \in V(R_j)$, for some $i \neq j \in \{1, \ldots, k\}$, such that $uw_i, uw_j \in E(G)$. Hence $(R \setminus \{e_{w_i}, e_{w_j}\}) \cup \{uw_i, uw_j\}$, where $w_i \in e_{w_i} \in E(R_i)$ and $w_j \in e_{w_j} \in E(R_j)$, is a maximum 2-degree-packing with less components than R, which is a contradiction. Therefore $I = I_1 \cup \cdots \cup I_k$, where either $I_i = \varnothing$ or for every $u \in I_i$ satisfies $N(u) \subseteq V(R_i)$, for $i = 1, \ldots, k$. □

Proposition 21. *Let G be a simple connected graph with $\nu_2(G) \geq 4$, $|E(G)| > \nu_2(G)$, $R_1, \ldots, R_k$ be the components of a maximum 2-degree-packing R of G, with k as small as possible, and $I = I_1 \cup \cdots \cup I_k = V(G) \setminus V(G[R])$, where either*

$I_i = \varnothing$ or for every $u \in I_i$ satisfies $N(u) \subseteq V(R_i)$, for $i = 1, \ldots, k$. If $\beta(G) = \lceil \nu_2(G)/2 \rceil$, then $\beta(G[R_i]) = \lceil \nu_2(G[R_i])/2 \rceil$, for $i = 1, \ldots, k$.

Proof. The proof of the proposition is completely analogous to the proof Proposition 20. □

Proposition 22. *Let G be a simple connected graph with $\nu_2(G) \geq 4$, $|E(G)| > \nu_2(G)$ and R be a maximum 2-degree-packingof G, such that $G[R]$ is a connected graph. If $\beta(G) = \lceil \nu_2(G)/2 \rceil$, then either $G \in \hat{\mathcal{C}}^k_{T,\hat{T},I}$ or $G \in \hat{\mathcal{P}}^k_{T,\hat{T},I}$, where T is a minimum vertex cover of either C^k or P^k, $\hat{T} = V(G[R]) \setminus T$ and $I = V(G) \setminus V(G[R])$.*

Proof. By Proposition 13, we have either $\hat{C}^k_I$ is a subgraph of G or P^k_I is a subgraph of G. Let T be a minimum vertex cover of G (hence, a minimum vertex cover of $G[R]$, by Proposition 18). Hence, by definition, if $e \in E(G) \setminus E(G[R]$, then e has an end in T, which implies that G is a subgraph of $\hat{C}^k_{T,\hat{T},I}$. Therefore, either $G \in \hat{\mathcal{C}}^k_{T,\hat{T},I}$ or $G \in \hat{\mathcal{P}}^k_{T,\hat{T},I}$. □

By Proposition 18, Proposition 22 and Corollary 21, we have:

Corollary 23. *Let G be a simple connected graph with $\nu_2(G) \geq 4$, $|E(G)| > \nu_2(G)$, $R_1, \ldots, R_k$ be the components of a maximum 2-degree-packing R of G, with k as small as possible, and $I = I_1 \cup \cdots \cup I_k = V(G) \setminus V(G[R])$, where either $I_i = \varnothing$ or for every $u \in I_i$ satisfies $N(u) \subseteq V(R_i)$, for $i = 1, \ldots, k$. If $\beta(G) = \lceil \nu_2(G)/2 \rceil$, then either $G[V_i] \in \hat{\mathcal{C}}^{k_i}_{T_i,\hat{T}_i,I_i}$ or $G[V_i] \in \hat{\mathcal{P}}^{k_i}_{T_i,\hat{T}_i,I_i}$, where $V_i = V(G[R_i]) \cup I_i$, $k_i = \nu_2(G[R_i])$, T_i is a minimum vertex cover of either C^{k_i} or P^{k_i} and $\hat{T}_i = V(G[R_i]) \setminus T_i$.*

Hence, by Proposition 14, Proposition 22, Corollary 15 and Corollary 23, we have:

Theorem 24. *Let G be a simple connected graph with $\nu_2(G) \geq 4$, $|E(G)| > \nu_2(G)$, $R_1, \ldots, R_k$ be the components of a maximum 2-degree-packing R of G, with k as small as possible, and $I = I_1 \cup \cdots \cup I_k = V(G) \setminus V(G[R])$, where either $I_i = \varnothing$ or for every $u \in I_i$ satisfies $N(u) \subseteq V(R_i)$, for $i = 1, \ldots, k$. Then $\beta(G) = \lceil \nu_2(G)/2 \rceil$, if and only if, either $G[V_i] \in \hat{\mathcal{C}}^{k_i}_{T_i,\hat{T}_i,I_i}$ or $G[V_i] \in \hat{\mathcal{P}}^{k_i}_{T_i,\hat{T}_i,I_i}$, where $V_i = V(G[R_i]) \cup I_i$, $k_i = \nu_2(G[R_i])$, T_i is a minimum vertex cover of either C^{k_i} or P^{k_i} and $\hat{T}_i = V(G[R_i]) \setminus T_i$, being*

1. *$|R_i|$ an even integer, for $i = 1, \ldots, k$, if $\nu_2(G)$ an even number.*
2. *$|R_1|$ is an odd integer and $|R_i|$ is an even integer, for $i = 2, \ldots, k$, if $\nu_2(G)$ is an odd number.*

Acknowledgments: The author would like to thank the referees for careful reading of the manuscript. Research was partially supported by SNI and CONACyT.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] Bondy, J.A. (1976), *Graph theory with applications.* Elsevier Science Ltd., Oxford, UK,.

[2] Alon, N., & Kleitman, D. J. (1992). Piercing convex sets and the Hadwiger-Debrunner (p, q)-problem. *Advances in Mathematics, 96*(1), 103-112.

[3] Alon, N., & Kleitman, D. J. (1992). Piercing convex sets. *Bulletin of the American Mathematical Society, 29*, 252-256.

[4] Alon, N., Kalai, G., Matoušek, J., & Meshulam, R. (2002). Transversal numbers for hypergraphs arising in geometry. *Advances in Applied Mathematics, 29*(1), 79-101.

[5] Araujo-Pardo, G., Montejano, A., Montejano, L. & Vázquez-Ávila, A. (2017). On transversal and 2-packing numbers in straight line systems on $\mathbb{R}^2$, *Utilitas Mathematica, 105*, 317–336.

[6] Huicochea, M., Montejano, L. & Oliveros, D. (2015). About the Piercing Number of a Family of Intervals. *Discrete Mathematics, 338*(12), 2545–2548.

[7] Kynčl, J. & Tancer, M. (2008). The Maximum Piercing Number for some Classes of Convex Sets with the (4,3)-property. *The Electronic Journal of Combinatorics, 15*, #R27.

[8] Montejano, L. & Soberón, P. (2011), Piercing numbers for balanced and unbalanced families. *Discrete & Computational Geometry, 45*(2), 358–364.

[9] Alfaro, C.A., Araujo-Pardo, G., Rubio-Montiel C. & Vázquez-Ávila, A. (2020). On transversal and 2-packing numbers in uniform linear systems. *AKCE International Journal of Graphs and Combinatorics*, *17*(1), 335-3341.

[10] Alfaro, C. & Vázquez-Ávila, A. (2020). A note on a problem of Henning and Yeo about the transversal number of uniform linear systems whose 2-packing number is fixed. *Discrete Mathematics Letters*, *3*, 61–66.

[11] Vázquez-Ávila, A. (2022), On intersecting straight line systems. *Journal of Discrete Mathematical Sciences and Cryptography*, *25*(6), 1931–1936 .

[12] Vázquez-Ávila, A. (2019), A note on domination in intersecting linear systems. *Applied Mathematics E-Notes*, *19*, 310–314.

[13] Vázquez-Ávila, A. (Submitted). On domination and 2-packing numbers in intersecting linear systems.

[14] Topp, J. & Volkmann, L. (1991), On packing and covering numbers of graphs, *Discrete Mathematics*, *96*(3), 229–238.

[15] Vázquez-Ávila, A., (To appear). Domination and 2-degree-packing number in graphs. RAIRO - Operations Research.