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Article

Edge Hub Number of Fuzzy Graphs

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Abstract: Shadi I.K et al. [1] introduced the edge hub number of graphs. This work extends the concept to fuzzy graphs. We derive several properties of edge hub number of fuzzy graphs and establish some relations that connect the new parameter with other fuzzy graph parameters. Also, some bounds of such a parameter are investigated. Moreover, we provide empirical evidence examples to elucidate the behavior and implications of edge hub number of fuzzy graph parameters.

Keywords: fuzzy graph, hub number, edge hub number.

MSC: 05C72, 05C90, 94C15, 90C35.

1. Introduction

Graph theory is used to represent real-life phenomena. Graph theory is the study of relationships between objects, which can be represented as dots or vertices, and their connections as lines or edges. Graphs are mathematical structures used to model pairwise relations between objects, and they are one of the principal objects of study in discrete mathematics [2,3]. Graph theory has expanded beyond mathematics into our everyday life without us even noticing, and it is used today in the world of data science. Graphs can be used to represent and model a wide range of real-world phenomena, including transportation, social networks, computer networks, and more [4,5]. The emergence of giant data sets has forced researchers to expand their toolboxes and develop new types of network model that can find complex structures and signals in the noise of big data. Many real-world phenomena provided motivation to define fuzzy graphs. Fuzzy graphs are more useful than graph structures because they deal with the uncertainty and ambiguity of many real-world phenomena [6]. Kauffman introduced fuzzy graphs using Zadeh's fuzzy relation, and many real-world phenomena provided motivation to define fuzzy graphs [7–10]. Fuzzy graphs were introduced to model real-world phenomena that are not easily represented by crisp graphs [11,12]. The use of fuzzy graphs has been applied to decision-making, distributed coordination, and other real-world applications [13]. The exploration of fuzzy graph theory has been enriched by the diverse contributions of researchers such as [1,8,9,13–21], who have laid the foundation for its theoretical development and explored some of its potential in various real-world applications.

The concept of hub number is a mathematical concept used in graph theory and network science. A hub set in a graph is a set of vertices such that any two vertices outside the set are connected by a path whose internal vertices are in the set. Walsh [15] introduced the concept of hub number of a graph, and H. Ahmed et al. [7] extended the concept to fuzzy graphs. The authors of [7] developed more general concept called the total hub number of fuzzy graph [8]. S. Tobaili et al.[22] analyzed hub number in various fuzzy graphs. In [14] S.I.Khalaf, et al. studied the edge hub number of a graph. In the present study, we introduce and formalize the concept of edge hub number of graphs to fuzzy graphs. We demonstrate its computation through illustrative examples of different fuzzy graphs and establish several properties of edge hub number and find a relation that connects this parameter with other parameters of fuzzy graph parameters through rigorous analysis and mathematical proofs. Both theoretical findings and empirical evidence examples were provided to elucidate the behavior and implications of edge hub number in fuzzy graphs. In particular, we discuss how edge hub number compares to well-established fuzzy graph metrics and how it can offer novel insights into the structural characteristics of fuzzy systems modeled as fuzzy graphs. This pioneering work lays the foundation for further exploration of edge hub number and its applications in diverse domains involving uncertainty, imprecision, or vagueness that can leverage fuzzy graph-theoretic analysis.

2. Preliminaries

In this section, we briefly overview the definitions of fuzzy graphs, hub numbers, and edge hub numbers in graphs.

Definition 1. [10,18] A fuzzy graph $\Gamma = (V, \lambda, \tau)$ is a set of elements V with two functions $\lambda : V \to [0,1]$ and $\tau :$ E → [0,1] such that the value of $\tau\{u,v\}$ is less than or equal to the product of $\lambda(u)$ and $\lambda(v)$ for all $u, v \in V$. The order p and size q of a fuzzy graph $\Gamma = (\lambda, \tau)$ are determined by the sum of $\lambda(u)$ for all $u \in V$ and the sum of $\tau(u,v)$ for all $(u,v) \in E$, respectively.

A path P in a fuzzy graph $\Gamma = (\lambda, \tau)$ is a sequence of distinct vertices $v_0, v_1, v_2, ..., v_n$ (except possibly v_0 and v_n) such that $\lambda(v_{i-1,v_i}) > o$, $\tau(v_{i-1},\ v_i) > 0$, $0 \le i \le n-1$ and $n \ge 1$. The length of the path P is n. A path P is a cycle if $v_0 = v_n$ and $n \ge 3$.

Definition 2. [23] A fuzzy graph $\Gamma = (V, \lambda, \tau)$ is considered a connected fuzzy graph if every pair of vertices in Γ is part of a path. If this condition is not met, then Γ is not connected.

Definition 3. [2,3] The edge cover number is denoted as $\alpha_1(\Gamma)$ and the maximum number of edges in an independent set of edges of Γ is denoted by $\beta_1(\Gamma)$.
The symbols $\alpha(\Gamma)$ and $\beta(\Gamma)$ denote the vertex cover number and the independence number of Γ, respectively.

Definition 4. [19,20] The minimum cardinality taken over all edge dominating sets of Γ is called the edge domination number of Γ and is denoted by $\gamma_e(\Gamma)$. A subset D of V is called the dominating set of Γ if for each $v \in V - D$ there is a $u \in D$ that dominates v. A dominating set D of a fuzzy graph Γ is known as a minimal dominating set if $D - \{v\}$ is not a dominating set of Γ for all $v \in D$. The minimum fuzzy cardinality taken over all minimal dominating sets in a fuzzy graph Γ is called the domination number of Γ and is denoted by $\gamma(\Gamma)$. An edge dominating set of Γ is a set S of edges in a graph Γ such that every edge in $E - S$ is adjacent to some edge in S, and the minimum cardinality taken over all edge dominating sets of Γ is called the edge domination number of Γ and is denoted by $\gamma_e(\Gamma)$.

Definition 5. [1] The edge domination number $\gamma_e(\Gamma)$ of a graph Γ is the minimum cardinality of an edge dominating set in Γ.

Definition 6. [7,8] A $\gamma - set$ D is called a connected dominating set if the fuzzy subgraph $< D >$ induced by D is connected. The connected domination number of a fuzzy graph Γ, denoted by $\gamma_c(\Gamma)$, is the minimum cardinality taken over all connected dominating sets of Γ.

Definition 7. [15,20] Let $S \subseteq V(\Gamma)$ and let $a, b \in V$. An $S - path$ between a and b is a path where all intermediate vertices are from S. A vertex subset $S \subseteq V(\Gamma)$ called the hub set of Γ if it satisfies the property that for any $a, b \in V - S$, there exists an $S - path$ in Γ between a and b. The hub number of a graph Γ is the smallest size of a hub set, denoted by $h(\Gamma)$. A hub set is a vertex subset of G such that for any pair of vertices outside of the hub set, there exists a path between them with all intermediate vertices in the hub set.

The degree of a vertex v in a graph Γ denoted by $d(v)$ is the number of edges of Γ incident with v, where $\Delta(\Gamma) = maxdeg(v), v \in V(\Gamma)$. The degree of an edge uv is defined as $d(u) + d(v) - 2$. Also, $\Delta_e(\Gamma)$ denotes the maximum degree among the edges of Γ for given vertex $v \in V(\Gamma)$. Then the degree of vertex v in a fuzzy graph is defined as $d(v) = \sum_{u \neq v} \rho(u,v)$.

The maximum degree of G is $\Delta(\Gamma) = \vee\{d(v) : v \in V\}$, and the minimum degree of Γ is $\delta(\Gamma) = \wedge\{d(v) : v \in V\}$.

Let $\Gamma = (\mu, \rho)$ be a fuzzy graph and let $v \in V(\Gamma)$. Then, $N(v) = \{u \in V : \rho(u, v) = \mu(u) \wedge \mu(v)\}$ is called the open neighborhood set of v, and $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood set of v.

Let $\Gamma = (\mu, \rho)$ be a fuzzy graph and $v \in V(\Gamma)$. Then the neighborhood degree of v is defined as $d_N(v) = \sum \mu(u) : u \in N(v)$.

The minimum neighborhood degree of a fuzzy graph Γ is $\delta_N(\Gamma) = min\{d_N(v) : v \in V\}$, and the maximum neighborhood degree of Γ is $\Delta_N(\Gamma) = max\{d_N(v) : v \in V\}$.

The effective degree of a vertex v in a fuzzy graph is denoted by $d_E(V) = \sum_{i=1}^{n} \rho(u, v)$.

The maximum degree taken of all effective degrees is denoted by $\Delta_E(\Gamma) = max\{d_E(v) : v \in V(\Gamma)\}$.
The minimum effective degree of Γ is denoted by $\delta_E(\Gamma) = min\{d_E(v) : v \in V\}$.
Also, the effective degree of an edge e in fuzzy graph is denoted by $d_E(e) = \sum_{i=1}^{n} \mu(u) : u \in N(e)$.

The maximum degree taken of all effective degrees is denoted by $\Delta_e(\Gamma) = max\{d_E(e) : e \in E(\Gamma)\}$. The minimum effective degree of Γ is denoted by $\delta_E(\Gamma) = min\{d_E(v) : v \in V\}$.

In a fuzzy graph, a fuzzy hub set is a vertex subset D such that for any pair of vertices outside of D, there exists a fuzzy path between them with all intermediate vertices in D. The hub number of a fuzzy graph is the minimum fuzzy cardinality among all minimal fuzzy hub sets, denoted by $h(\Gamma)$.

Similarly, an edge hub set in a graph Γ is a subset H of edges such that every pair of edges not in H are connected by a path, where all internal edges are from H. The edge hub number of Γ is the minimum cardinality of an edge hub set and denoted by $h_e(\Gamma)$.

3. Discussion and Main Results

The study in this section introduced a new concept of edge hub number of fuzzy graphs, defined edge hub sets, minimal edge hub sets, and edge hub number precisely using fuzzy cardinality. Furthermore, it has provided analytical formulas and bounds for calculating edge hub number of various fuzzy graph structures like complete graphs, bipartite graphs, cycles, trees, etc.

3.1. The Edge Hub Number in Fuzzy Graph

The purpose of this subsection is to introduce and explore the concepts of edge hub numbers in fuzzy graphs.

Definition 8. An edge hub set of a fuzzy graph $\Gamma = (V, \lambda, \tau)$ is a subset H of edges in $E(\Gamma)$ such that every pair of edges in $E(\Gamma)$ is connected by a path where all intermediate edges are in H.

Definition 9. A minimal edge hub set of a fuzzy graph Γ is a fuzzy edge hub set H such that $H - \{e\}$ is not an edge hub set of Γ for any $e \in H$.

Definition 10. The edge hub number $h_e(\Gamma)$ of a fuzzy graph Γ is the minimum fuzzy cardinality among all minimal fuzzy edge hub sets.

Example 1. Let Γ be a fuzzy graph given in Figure 1, where $V = \{\nu_1, \nu_2, \nu_3, \nu_4\}$, $\lambda(\nu_1) = 0.3$, $\lambda(\nu_2) = 0.4$, $\lambda(\nu_3) = 0.5$, $\lambda(\nu_4) = 0.6$ and $\tau(u, \nu) = \lambda(u) \wedge \lambda(\nu)$, $\forall(u, \nu) \in \tau$.

The edge hub set $S = \{e_1\}$ and hence $h_e(\Gamma) = 0.3$.

Below are the results for $h_e(\Gamma)$ of various common fuzzy graphs.

Theorem 11. *If $\Gamma = (V, \lambda, \tau)$ is a complete fuzzy graph of order $p, \Gamma = K_3$, then*
$h_e(\Gamma) = 0$.

Proof. Let $\Gamma = K_\lambda$ be a complete fuzzy graph of order p, p = 3. Then, for every edge $(u, \nu) \in \tau$, $\tau(u, \nu) = \lambda(u) \wedge \lambda(\nu)$. That is, every edge is adjacent to the other edges in Γ and all edges are effective. Thus, $h_e(\Gamma) = 0$. □

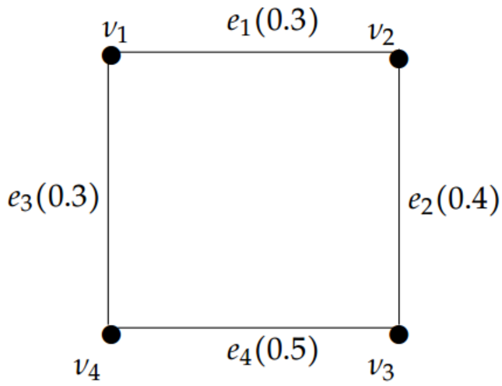

Figure 1

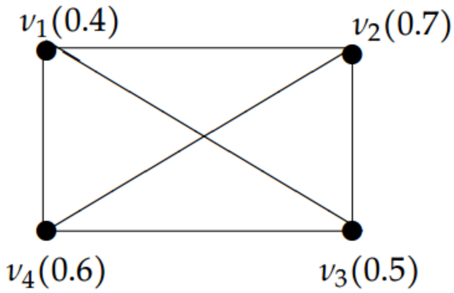

Figure 2

Theorem 12. *If* $\Gamma = (V, \lambda, \tau)$ *is a complete fuzzy graph of order* $p, p \geq 4$. *Then*

$$h_e(\Gamma) = min\{\lambda(e_i) \ \forall \ e_i \ \in E(K_\lambda), where \ \{e_i\} \ is \ a \ hub \ set\}.$$

Proof. Let $\Gamma = K_\lambda$ be a complete fuzzy graph. Then for every edge $(u,v) \in E$, $\tau(u,v) = \lambda(u) \wedge \lambda(v)$. That is every vertex is adjacent to the other vertices in Γ and all edges are effective. Thus, let $S = \{e\}, e$ be any edge in Γ such that e has the smallest membership value in G. Then for any two edges $e_i, e_j \in E(\Gamma)$ there is an $S - path$ between e_i and e_j that is, the e is an intermediate edge of all paths joining e_i and e_j. Hence, $h_e(\Gamma) \leq |S| = min\{\lambda(e) \ \forall \ e \in K_\lambda\}$. □

Example 2. Let $\Gamma = (V, \lambda, \tau)$ be a complete fuzzy graph given in Figure (3.2), where $V = \{v_1, v_2, v_3, v_4\}$, $\lambda(v_1) = 0.4$, $\lambda(v_2) = 0.7$, $\lambda(v_3) = 0.5$, $\lambda(v_4) = 0.6$ and $\tau(u,v) = \lambda(u) \wedge \lambda(v) \ \forall \ u, v \in V$.

We see that $h_e(\Gamma) = 0.4$

Theorem 13. *If* $\Gamma = K_{\lambda_1, \lambda_2}$ *is a complete bipartite fuzzy graph,* $\Gamma = K_{2,3}$ *or* $K_{3,3}$, *then* $h_e(K_{\lambda_1,\lambda_2}) \leq \frac{q}{3}$. *In addition, if equality is maintained, it must be ensured that* $\lambda(v) = 1$ *for all* $v \in V(\Gamma)$.

Example 3. Let $\Gamma = (V, \lambda, \tau)$ be a fuzzy graph such that $\tau(u,v) = min(\lambda(u), \lambda(v))$, for all $u, v \in V$ given in Figure 3.

Clearly that $h_e(\Gamma) = \frac{q}{3} = 2$.

Theorem 14. *If* Γ *is any fuzzy graph without isolated vertices, then*
(i) $h_e(\Gamma) < p$;
(ii) $h_e(\bar{\Gamma}) < p$.

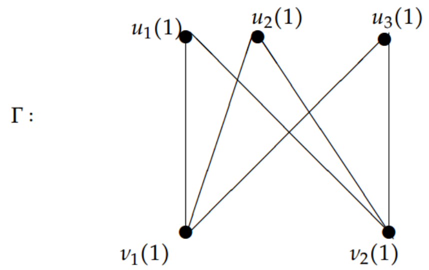

Figure 3

Proof. Let $\Gamma = (V, \lambda, \tau)$ be a fuzzy graph without isolated vertices. Since any fuzzy graph has at least one edge hub set. Hence, the result follows. □

The following theorem provides a result of Nordhaus-Gaddum type for the edge hub number $h_e(\Gamma)$.

Theorem 15. *For any fuzzy graph Γ,*

$$h_e(\Gamma) + h_e(\bar{\Gamma}) < 2p.$$

Proof. Since $h_e(\Gamma) < p$ and $h_e(\bar{\Gamma}) < p$ by (Theorem 3.10).Thus

$$h_e(\Gamma) + h_e(\bar{\Gamma}) < 2p.$$

□

Theorem 16. *For a connected fuzzy graph $\Gamma = (V, \lambda, \tau)$, we have $h_e \leq p - q$, and the equality holds for K_3.*

Proof. Considered connect fuzzy graph, denoted by Γ. There are two cases to be examined. The first case, $p \leq q$, the result holds since $h_e(\Gamma)$ is less than zero. Second case, $p > q$, the graph must be a tree and has at least two nonadjacent edges. In this case, $h_e(\Gamma)$ is greater than zero, since for any tree, $p - q = 1$. Therefore, it follows that $h_e(\Gamma)$ is less than or equal to $p - q$. □

Theorem 17. *For any fuzzy graph Γ,*

$$h_e(\bar{\Gamma}) \leq h_e(\Gamma).$$

Theorem 18. *For any fuzzy graph,*

$$h_e(\Gamma) \leq \alpha_1(\Gamma).$$

Since an edge hub set in fuzzy graphs is an edge covering set, and this completes the proof and the bound hold for $\Gamma = C_4$ *and* K_4.

Corollary 19. *For any fuzzy graph $\Gamma \neq K_3$ or $S_{n,m}$,*

$$h_e(\Gamma) \leq q - \alpha_1(\Gamma).$$

Corollary 20. *For any fuzzy graph $\Gamma \neq K_3$ or $S_{n,m}$,*

$$h_e(\Gamma) \leq q - \beta_1(\Gamma).$$

Theorem 21. *For any fuzzy graph,*

$$h_e(\Gamma) + \alpha_1(\Gamma) \leq q.$$

Theorem 22. *For any tree T of size $q \geq 5$,*

$$h_e(\Gamma) = q - \sqcap,$$

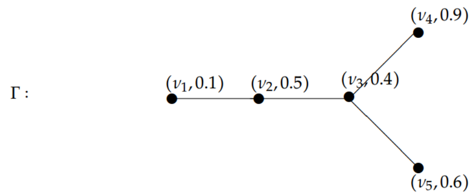

Figure 4

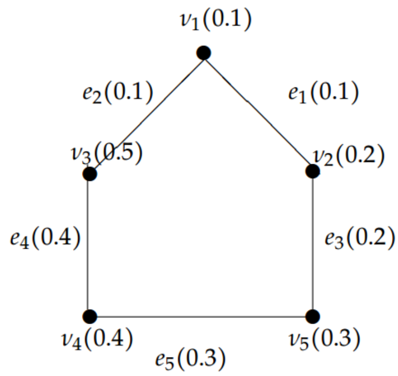

Figure 5

where $\sqcap$ *is the number of pendant vertices.*

Example 4. Suppose that $\Gamma = (V, \lambda, \tau)$ is a fuzzy graph given in Figure 4, where $V = \{v_1, v_2, v_3, v_4, v_5\}$, $\lambda(v_1) = 0.1$, $\lambda(v_2) = 0.5$, $\lambda(v_3) = 0.4$, $\lambda(v_4) = 0.9$, $\lambda(v_5) = 0.6$ and $\tau(u, v) = \lambda(u) \wedge \lambda(v)\ \forall\ (u, v)\ \in E$.

We see that $h_e(\Gamma) = 0.4$.

Proposition 23. *For any cycle fuzzy graph* C_p*, with* $p \geq 4$, $h_e(\Gamma) = min \sum_{i=1}^{n-3} \lambda(e_i)$.

Example 5. Suppose that Γ is a fuzzy graph given in the Figure 5, where $V = \{v_1, v_2, v_3, v_4, v_5\}$, $\lambda(v_1) = 0.1, \lambda(v_2) = 0.2$, $\lambda(v_3) = 0.5$, $\lambda(v_4) = 0.4, \lambda(v_5) = 0.3$, *and* $\tau(u, v) = \lambda(u) \wedge \lambda(v)\ \forall\ u, v \in V$.

We see that $h_e(\Gamma) = 0.2$.

Theorem 24. *For any cycle fuzzy graph* C_p*, with* $p \geq 4$

$$h(\Gamma) = h_e(\Gamma)$$

Proof. It follows by (Proposition 3.20). □

Theorem 25. *Let* $G = (V, \lambda, \tau)$ *be any cycle fuzzy graph such that* $n \geq 4$*. Then*

$$h_e(\Gamma) \leq \gamma_e(\Gamma).$$

Proof. Let Γ be a cycle fuzzy graph and let S be an edge-dominating set. Then S contains $n - 3$ edge. Then for every pair of edge $e_1, e_2 \in E(\Gamma)$ there is an $S - path$ between them. Thus S is an edge hub set. Hence

$$h_e(\Gamma) \leq \gamma_e(\Gamma).$$

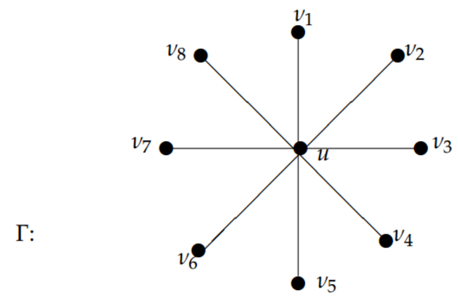

Figure 6

□

Theorem 26. *For any fuzzy graph* Γ, $h_e(\Gamma)+\gamma_e(\Gamma)\leq q$, *and the equality holds for any star fuzzy graph.*

Theorem 27. *For any fuzzy star graph* Γ, $h_e(\Gamma)=0$.

Proof. This can be deduced from the definition of the edge hub number. As all edges in Γ are adjacent, it follows that $h_e(\Gamma)=0$. □

Example 6. Suppose that $\Gamma=(V,\lambda,\tau)$ is a fuzzy graph given in the Figure 6 , where
$V=\{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8\}$, $\lambda(v_1)=\lambda(v_2)=0.6$, $\lambda(v_3)=0.2$, $\lambda(v_4)=0.3$ $\lambda(v_5)=0.8$, $\lambda(v_6)=\lambda(v_7)=0.9$, $\lambda(v_8)=0.4$ and $\tau(u,v)=\lambda(u)\wedge\lambda(v)\ \forall\ (u,v)\in E$.
Clearly that $h_e(\Gamma)=0$.

Proposition 28. (i) *For any path* P_p *with* $p\geq 4, h_e(P_p)=min\sum_{i=1}^{p-3}\lambda(e_i)$;
(ii) *For any wheel fuzzy graph* Γ;

$$h_e(\Gamma)=\begin{cases} min\sum_{i=1}^{\frac{p}{2}}\lambda(e_i) & if\ p\ is\ even \\ min\sum_{i=1}^{\frac{p-1}{2}}\lambda(e_i) & if\ p\ is\ odd\end{cases} \tag{1}$$

(iii) *For the double star* $h_e(\Gamma)=\{\lambda(e), where\ e\ is\ join\ uand\ v\}$.

Proposition 23 shows that for any cycle fuzzy graph C_p, with $p\geq 5, h_e(\Gamma)=min\sum_{i=1}^{n-3}\lambda(e_i)$., but the hub number and edge hub number are not always equivalent for all graphs. For example, the edge hub number of a path is lower than its hub number, as expressed by the inequality:

$$h_e(P_p)=min\sum_{i=1}^{p-3}\lambda(e_i)<h(P_p)=min\sum_{i=1}^{p-2}\lambda(v_i).$$

Conversely, the edge hub number of a wheel is greater than its hub number. Nevertheless, for cycles with p vertices, the two parameters are equivalent, as $h_e(C_p)=h(C_p)$. This raises the question of whether the edge hub number is an appropriate measure of accessibility and if it can differentiate between two graphs, Γ_1 and Γ_2? Numerous examples of graphs suggest that $h_e(\Gamma)$ is indeed a suitable measure of accessibility that can distinguish between graphs.

Example 7. Given the fuzzy graphs $\Gamma_1,\Gamma_2, and\ \Gamma_3$ are in Figure 7.

We have $h(\Gamma_1)=h(\Gamma_2)=h(\Gamma_3)=0.5$, the hub number does not discriminate between graphs Γ_1,Γ_2 and Γ_3. But $h_e(\Gamma_1)=0.5, h_e(\Gamma_2)=0.4$ and $h_e(\Gamma_3)=0.1$, this means $h_e(\Gamma_1)\neq h_e(\Gamma_2)\neq h_e(\Gamma_3)$, so the edge hub number discriminates between graphs Γ_1,Γ_2 and Γ_3.

The hub number for graphs Γ_1, Γ_2, and Γ_3 is equal to 0.5, indicating that there is no discrimination between them. However, the edge hub numbers for these graphs are different, with Γ_1 having a value of 0.5, Γ_2 having

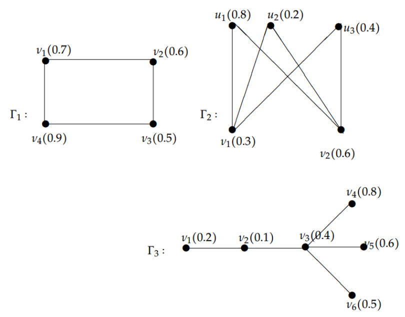

Figure 7

a value of 0.4, and Γ_3 having a value of 0.1. This means that the edge hub number discriminates between the three graphs, unlike the hub number.

Definition 29. A connected edge hub set in a fuzzy graph $\Gamma = (V, \lambda, \tau)$ is called a connected subset of edges H if the induced fuzzy subgraph $< H >$ is connected.

Definition 30. : The connected edge hub number of Γ, denoted by $h_{ce}(\Gamma)$, is the minimum fuzzy cardinality among all sets of connected edge hubs in Γ.

Theorem 31. *For a connected fuzzy graph* Γ*, the following inequalities hold:*
(i) $h_e(\Gamma) \le h_{ce}(\Gamma)$*, and*
(ii) $h_e(\bar{\Gamma}) \le h_{ce}(\Gamma)$.

Proof. Suppose Γ is a connected fuzzy graph and H is a connected set of edges in Γ that acts as a hub. As Γ is connected, every pair of edges in $E(\Gamma)$ has an $S-path$ between them. Since this is a stronger condition for a connected edge hub set, any connected hub edge set is also a hub edge set. Therefore, H is a hub edge set. Hence,

$$h_e(\Gamma) \le h_{ce}(\Gamma).$$

Similarly, (ii) holds. □

Theorem 32. *Let* D *be a* $\gamma-set$ *of a fuzzy graph* Γ. *If there exists an edge* e *in* $E-D$ *adjacent to only edges in* D. *Then*

$$\gamma_e(\Gamma) < h_e(\Gamma) + \lambda(e).$$

Proof. This follows since $D \cup \{e\}$ is an edge-dominating set. □

Theorem 33. *The following statements hold for any fuzzy graph* Γ*, with equality only if* Γ *is complete:*
(i) $h_e(\Gamma) \le q - \triangle_N(\Gamma)$ *and*
(ii) $h_e(\bar{\Gamma}) \le q - \triangle_N(\bar{\Gamma})$.

Proof. Suppose that Γ is a fuzzy graph and let e be an edge in $E(\Gamma)$ such that $d_N(\Gamma) = \Delta_N(\Gamma)$. Then, $E - N(e)$ is an edge hub set of Γ. This means that $h_e(\Gamma) \le |E - N(e)| = q - \Delta_N(\Gamma)$.

If Γ is a complete fuzzy graph, then $|E - N(e)| = |H|$, where H is an edge hub set that contains only one edge. Hence, $h_e(\Gamma) = q - \Delta_N(\Gamma)$.

Since $h_e(\bar{\Gamma}) < h_e(\Gamma)$, then (ii) holds. □

Theorem 34. *For a fuzzy graph* $\Gamma = (V, \lambda, \tau)$ *, we have that*
$h_e(\Gamma) \le q - \delta_N$.

Proof. Assume that there exists an edge $e \in E$ such that its degree in the graph Γ is equal to δ_N, that is $d_N(e) = \delta_N$. Let H be a set of edge hubs in Γ. It is evident that $e \in H$ and $H \subseteq E - N(e)$. Thus, we can conclude that

the size of H, denoted by $h_e(\Gamma)$, is less than or equal to the cardinality of the set of edges not adjacent to e, which is equal to $q - |N(e)|$ where q is the number of edges in Γ. Since δ_N is the minimum degree in Γ, we have $|N(e)| \geq \delta_N$, which implies that
$h_e(\Gamma) = |H| \leq |E - N(e)| = q - |N(e)| \leq q - \delta_N$.

□

Corollary 35. *: For a fuzzy graph* Γ*, we have that*
$h_e(\Gamma) \leq q - \delta_e$.

Proof. It is clear that $\Delta_e \leq \Delta_N$, $\delta_e \leq \delta_N$, and by (Theorem 34), we can conclude that $h_e(\Gamma) \leq p - \delta_e$. □

Theorem 36. *Suppose that* Γ *is a fuzzy graph without isolated vertices. Then* $h_e(\Gamma) \leq \frac{q}{\Delta_e(\Gamma)+1}$.

Theorem 37. *For a connected fuzzy graph* Γ *and its complement* $\bar{\Gamma}$*, we have that:*

$$h_e(\Gamma) + h_e(\bar{\Gamma}) < P + t, \; t = max\{\lambda(e) : \forall \; e \in E(\Gamma)\}$$

Proof. From (Theorem 33), $h_e(\Gamma) < q - \triangle(\Gamma)$ and $h_e(\bar{\Gamma}) < q - \triangle(\bar{\Gamma})$. We have

$$\begin{aligned} h_e(\Gamma) + h_e(\bar{\Gamma}) &< q - \triangle(\Gamma) + q - \triangle(\bar{\Gamma}) \\ &= 2q - (\triangle(\Gamma) + q - t - \delta(\Gamma)) \\ &= q - t - ((\triangle(\Gamma) + \delta(\Gamma)). \end{aligned}$$

Since $((\triangle(\Gamma) + \delta(\Gamma)) \geq 0$, then

$$h_e(\Gamma) + h_e(\bar{\Gamma}) < q + t, \; t = max\{\lambda(e) : \forall \; e \in E(\Gamma)\}.$$

□

4. Conclusion

In this study, we have extended the concept of edge hub number from conventional graphs to fuzzy graphs. By deriving several important properties of edge hub number for fuzzy graphs and establishing relations with other fuzzy graph parameters, we have formalized edge hub number as a meaningful measure for fuzzy graph analysis. Through illustrative examples and mathematical proofs, we have demonstrated the computation and implications of edge hub number. Both theoretical findings and empirical evidence illustrate how edge hub number can characterize structural aspects of fuzzy graphs and compare to established fuzzy graph metrics. This work represents the formalization of edge hub number for fuzzy graphs, laying the groundwork for further research applying this novel parameter across domains involving uncertainty modeled through fuzzy graph theories.

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