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Note: Certain bounds in respect of upper deg-centric graphs

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Abstract: This note presents some upper bounds for the size of the upper deg-centric grapg G_{ud} of a simple connected graph G. Amongst others, a result for graphs for which a compliant graph G has $G_{ud} \cong \overline{G}$ is presented. Finally, results for size minimality in respect upper deg-centrication and minimum size of such graph G are presented.

Keywords: Upper deg-centric graph; upper deg-centrication; equi-eccentric graph.

MSC: Primary 05C12; Secondary 05C45.

1. Introduction

It is assumed that the reader is familiar with the basic notions and notation of graph theory. Where deemed necessary, useful definitions will be recalled from [1–3]. Only finite, undirected and connected simple graphs are considered. Furthermore, since the number of distinct connected graphs on $n = 1, 2, 3$ vertices is respectively given by $1, 1, 2$ this note will, unless stated otherwise, consider graphs of order $n \geq 4$. Results for $n = 1, 2, 3$ can easily be verified. Reference to vertices v_i, v_j will mean that v_i and v_j are distinct vertices. A classical *graph from a graph* G is its complement, $\overline{G}$. The complement of a graph G can be defined in terms of a distance condition i.e. $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{v_iv_j : \text{if and only if } d_G(v_i, v_j) \neq 1\}$. Clearly, a generalized notion of a k-complement of graph G could be that for $k \geq 1$ the k-complement of G is defined as a graph say, $\overline{G}_{(k)}$ where, $V(\overline{G}_{(k)}) = V(G)$ and $E(\overline{G}_{(k)}) = \{v_iv_j : d_G(v_i, v_j) \neq k\}$. If a graphical parameter of a vertex such as its degree, eccentricity, coloring or alike is utilized in a relation condition to obtain a *graph from a graph* the study becomes interesting. Published studies with a distance condition in terms of the vertex eccentricity are found in [4,5]. With the world-wide interest in artificial intelligence, machine learning, deep data mining and alike, the notion of *graphs from a graph* may bring various futuristic applications to the fore. The era of *evolving* graphs has arrived.

2. Preliminaries

In a recently communicated paper the notion of upper degree-centrication has been introduced. This note has relevance to the upper deg-centric graph. See [6].

Definition 1. [6] Let $G = (V(G), E(G))$ be a graph. Then the upper deg-centric graph of G denoted by, G_{ud} has vertices $V(G_{ud}) = V(G)$ and $E(G_{ud}) = \{v_iv_j : d_G(v_i, v_j) \geq deg_G(v_i)\}$.

Clearly, an edge $v_iv_j \in E(G)$ and $v_iv_j \in E(G_{ud})$ if and only if v_i or v_j is a pendant vertex in G. Put differently, an edge $v_iv_j \in E(G)$ and $v_iv_j \in E(G_{ud})$ if and only if $deg_G(v_i) = 1$ or $deg_G(v_j) = 1$. Therefore, $E(G_{ud}) \subseteq E(\overline{G})$ if and only if $\delta(G) \geq 2$.

Theorem 2. *A 2-regular graph G has $G_{ud} \cong \overline{G}$.*

Proof. Assume G is a 2-regular graph. Thus $deg_G(v_i) = 2$, $\forall\ v_i$. So $N_{G_{ud}}(v_i) = V(G)\backslash N_G[v_i]$, $\forall\ i$ which implies that $E(G_{ud}) = E(\overline{G})$. Since $V(G) = V(\overline{G})$ it follows that $G_{ud} \cong \overline{G}$. □

The converse of Theorem 2 does not hold. An example is the windmill graph $Wd(k)$ which is obtained by joining $k \geq 2$ copies of K_3 at a shared central vertex. Clearly, the central vertex has degree equal to $2k > 2$ so $Wd(k)$ is not 2-regular. However, $Wd(k)_{ud} \cong \overline{Wd(k)}$.

Theorem 3. *A graph G has $G_{ud} \cong \overline{G}$ if and only if $V(G)$ can be partitioned into sets*

$$X = \{v_j : deg_G(v_i) = 2\} \text{ and } Y = V(G)\backslash X$$

such that the induced subgraph $\langle Y \rangle$ is complete or empty.

Proof. For any vertex $v_i \in V(G)$ the open neighborhood $N_{G_{ud}}(v_i)$ can be partitioned into three sets that is:

(i) $N^{\rightarrow}_{G_{ud}}(v_i) = \{v_j : deg_G(v_i) \leq d_G(v_i, v_j) \text{ and } deg_G(v_j) > d_G(v_j, v_i)\}$.
(ii) $N^{\leftarrow}_{G_{ud}}(v_i) = \{v_j : deg_G(v_i) > d_G(v_i, v_j) \text{ and } deg_G(v_j) \leq d_G(v_j, v_i)\}$.
(iii) $N^{\leftrightarrow}_{G_{ud}}(v_i) = \{v_j : deg_G(v_i) \leq d_G(v_i, v_j) \text{ and } deg_G(v_j) \leq d_G(v_j, v_i)\}$.

Hence, (iii) represents the commutative initiation of edges.
Part 1: Assume that $V(G)$ can be partitioned into sets

$$X = \{v_j : deg_G(v_i) = 2\} \text{ and } Y = V(G)\backslash X$$

such that the induced subgraph $\langle Y \rangle$ is complete or empty. From Definition 1 it is obvious that each $v_i \in X$ initiates an edge to all vertices v_j if $d_G(v_i, v_j) \geq 2$. All these edges are also obtained in $\overline{G}$. Since each $v_j \in Y$ has $deg_G(v_j) \geq 3$ it cannot initiate all edges in accordance to the definition of $\overline{G}$. Hence, for such v_j the iniation of an edge is prohibited. The aforesaid is in compliance because $\langle Y \rangle$ is complete.
Part 2: Conversely, if $G_{ud} \cong \overline{G}$ then possibly G is 2-regular. In such case $X = V(G)$ and $Y = \emptyset$. Otherwise, any vertex v_i which yields an edge (or edges) in accordance with the definition of $\overline{G}$ has $deg_G(v_i) = 2$ by necessity. Hence a non-empty set X exists. If Y is non-empty then any $v_j \in Y$ has $deg_G(v_j) \geq 3$. A commutative initiated edge from v_j to $v_i \in X$ is in order. However, an intiated edge amongst vertices in Y is prohibited. Such prohibition is only possible if $\langle Y \rangle$ is complete. □

3. Bounds

Recall that the number of edges of a graph G is called the size of G and is denoted by, $\varepsilon(G)$. From Theorem 2 a self-evident corollary follows.

Corollary 4. *For G and $\delta(G) \geq 2$ it follows that,*

$$0 \leq \varepsilon(G_{ud}) \leq \frac{n(n-1)}{2} - \varepsilon(G) = \varepsilon(\overline{G}).$$

There exists a finite number say, γ_n of distinct unlabeled trees on n vertices. The vertices of these distinct trees on n vertices may be labeled v_i, $i = 1, 2, 3, \dots, n$ in any fashion. Let these distinct and labeled trees be T_i, $1 \leq i \leq \gamma_n$ such that:

$$\varepsilon(T_{1_{ud}}) \leq \varepsilon(T_{2_{ud}}) \leq \varepsilon(T_{3_{ud}}) \leq \cdots \leq \varepsilon(T_{\gamma_{n_{ud}}}).$$

Lemma 5. *Amongst all distinct trees T_i, $1 \leq i \leq \gamma_n$ the upper deg-centric graph of a path P_n and a star $S_{1,n-1}$ has respectively, the minimum and maximum size, i.e.*

$$\varepsilon(P_{n_{ud}}) \leq \varepsilon(T_{i_{ud}}) \leq \varepsilon(S_{1,n-1_{ud}}).$$

Proof. The result follows from the fact that for a given n a path has minimum pendants and a star has maximum pendants read together with Definition 1. Indeed, $S_{1,n-1_{ud}} \cong K_n$. □

Recall that $G + e$ means the adding of an edge e to G. If two or more edges say, $e_1, e_2, e_3, \dots, e_k$ are added to G it is denoted by $G + (e_1, e_2, e_3, \dots, e_k)$.

Lemma 6. *For any tree T and $G = T + e$ it follows that,*

$$\varepsilon(G_{ud}) \le \varepsilon(T_{ud}).$$

Proof. The result follows from the fact that if e is added between vertices v_i, v_j then, $deg_T(v_i) < deg_G(v_i)$ and $deg_T(v_j) < deg_G(v_j)$. The aforesaid implies that for each vertes v_t for which $d_T(v_i, v_t) = deg_T(v_i)$ at least the edge $v_iv_t \in E(T_{ud})$ and $v_iv_t \notin E(G_{ud})$. Similar argument follows in respect of vertex v_j. Finally, certain distances between pairs of vertices may have decreased whilst none increased. This settles the result. □

To further this note Lemma 6 has been formulated specifically for trees. A similar result holds for graphs in general. We state it as an axiomatic corollary.

Corollary 7. *For any graph G and $H = G + e$ it follows that,*

$$\varepsilon(H_{ud}) \le \varepsilon(G_{ud}).$$

It is known that any graph G has a finite number of distinct spanning trees. It is also known that a graph G can be reconstructed from any of its spanning trees by adding the required edges (or corresponding edges) needed.

Theorem 8. *Let S be the set of distinct spanning trees of a graph G. Then,*

$$\varepsilon(G_{ud}) \le \varepsilon(T^-_{ud}) \text{ where, } \varepsilon(T^-_{ud}) = min\{\varepsilon(T_{ud}) : T \in S\}.$$

Proof. Through immediate induction on the result of Lemma 6, it follows for any spanning tree T of G that, if $H = T + (e_1, e_2, e_3, \dots, e_k)$ then $\varepsilon(H_{ud}) \le \varepsilon(T_{ud})$. Furthermore, amongst the finite number of distinct spanning trees of G there exists some T^- such that $\varepsilon(T^-_{ud}) = min\{\varepsilon(T_{ud}) : T \in S\}$. That settles the result. □

Recall that a graph G is traceable if G contains a Hamiltonian path.

Proposition 9. *Let G be traceable then,*

$$\varepsilon(G_{ud}) \le \varepsilon(P_{n_{ud}}) = \frac{n^2-3n+6}{2}.$$

Proof. Since G is traceable it contains a Hamilton path. Since $\varepsilon(P_{n_{ud}}) \le \varepsilon(T_{ud})$ where T is a tree of order n and read together with Theorem 8 the result follows. □

Proposition 10. *Let G be Hamiltonian then,*

$$\varepsilon(G_{ud}) \le \varepsilon(C_{n_{ud}}) = \frac{n(n-3)}{2}.$$

Proof. Since G is Hamiltonian it contains a Hamilton cycle. Since a result similar to Lemma 6 holds for cycles and $\varepsilon(C_{n_{ud}}) = \frac{n(n-3)}{2} \le \varepsilon(P_{n_{ud}})$, the result follows. □

Proposition 11. *Let distinct graphs G and H both be of order n.*
(i) If G is a spanning subgraph of H then, $\varepsilon(H_{ud}) \le \varepsilon(G_{ud})$.
(ii) If G is not a spanning subgraph of H but G and H share a common spanning tree as well as $\varepsilon(G) < \varepsilon(H)$ then, $\varepsilon(H_{ud}) \le \varepsilon(G_{ud})$.

Proof. (i) Clearly, the result in Corollary 7 can be applied by iteratively adding appropriate edges to G one at a time to obtain H. For each iteration Corollary 7 remains valid. That settles the result.
(ii) Clearly, the result in Lemma 6 can be applied by iteratively adding appropriate edges to two copies of a common spanning T of G and H, one at a time to first obtain G and thereafter obtain G. For each iteration Lemma 6 remains valid. That settles the result. □

4. Minimum graph size

For $n = 1, 3, 4$ it is easy to verify that the only graphs G for which $G_{ud} = \mathfrak{N}_n$ (alternatively, $G_{ud} = \overline{K}_n$) are the corresponding complete graphs. The graph K_2 is excluded because $K_{2_{ud}} = K_2$. It is obvious that for $n \geq 5$ there exists a non-complete graph G with minimum size such that $G_{ud} = \mathfrak{N}_n$. If $G_{ud} = \mathfrak{N}_n$ and for any edge e the upper deg-centric graph of $G - e$ is not empty then G is said to be a *minimal* graph in respect of upper deg-centrication. For a given $n \geq 5$ let the set of all minimal graphs in respect of upper deg-centrication be $\mathcal{G}(n)$. A graph of order n and of minimum size such that $G_{ud} = \mathfrak{N}_n$ is a graph $G \in \mathcal{G}(n)$ for which $\varepsilon(G) = \min\{\varepsilon(H) : H \in \mathcal{G}(n)\}$. Clearly, such G cannot have an pendant vertex. Hence, for minimality of G such that $G_{ud} = \mathfrak{N}_n$ the graph G must have $diam(G) = 2$ and $\delta(G) = 3$. For a graph G of order $n = 5$ which has $diam(G) = 2$ and has $\delta(G) = 3$ it must have $\varepsilon(G) \geq 8$. Hence, if $\varepsilon(G) = 8$ it represents the minimum size of a graph G of order 5 such that $G_{ud} \cong \mathfrak{N}_5$. The chorded cycle $C_5 + (v_1v_3, v_1v_4, v_2v_5)$ complies. For a graph G of order $n = 6$ which is 2-equi-eccentric (hence, $diam(G) = 2$) and has $\delta(G) = 3$ it must have $\varepsilon(G) \geq 9$. Hence, if $\varepsilon(G) = 9$ it represents the minimum size of a graph G of order 6 such that $G_{ud} \cong \mathfrak{N}_6$. The chorded cycle $C_6 + (v_1v_4, v_2v_5, v_3v_6)$ complies. It is known that the Petersen graph denoted by, $\mathcal{P}$ is both 3-regular and 2-equi-eccentric. Hence $\varepsilon(\mathcal{P}) = 15$ represents the minimum size of a graph G of order 10 such that $G_{ud} = \mathfrak{N}_{10}$. We are left to consider graphs of order $n \geq 7$, $n \neq 10$.

In [1] a graph $G_n = K_m \circ K_1 \bigoplus K_1$, $m \geq 1$ and $n = 2m + 1$ is defined as:
(i) Construct the corona graph $K_m \circ K_1$ and label the vertices of K_m as $v_1, v_2, v_3, \ldots,$ v_m and the m-copies of K_1 vertices as $u_1, u_2, u_3, \ldots, u_m$ and thereafter,
(ii) Join an addition vertex w_1 as a common neighbor to all vertices u_i, $1 \leq i \leq m$.
From Theorem 3 in [1] it is known that such graph G_n is a minimal 2-equi-eccentric graph hence, $diam(G_n) = 2$. Note that this construction yields graphs of odd order. Furthermore, for n is odd the size of G_n is a quadratic function of m where, $m = \frac{n-1}{2}$.

Our first step is to search for graphs which are minimal in respect of 2-equi-eccentricity and have minimum size. Thereafter the minimum number of edges must be added to obtain graphs G of minimum size such that $\delta(G) = 3$. In [1] the *base* graphs K_2 and K_3 were used to construct a graph for a given $n \geq 7$, $n \neq 10$ (our lower bound) by:
(i) Take base graph K_2 or K_3 on vertices v_1, v_2 or v_1, v_2, v_3 respectively.
(ii) Take $t = n - 3$ (for K_2) or $t = n - 4$ (for K_3) isolated vertices u_i, $1 \leq i \leq t, (n - 3$ or $n - 4)$ and attach q_1, q_2 or q_1, q_2, q_3, where $q_i \geq 2$ as pendants to the corresponding v_1, v_2 or v_1, v_2, v_3 where $q_1 + q_2 = n - 3$ or $q_1 + q_2 + q_3 = n - 4$.
(iii) Take an isolated vertex w_1 and add the edges u_iw_1, $\forall\, i$ so that w_1 serves as a common neighbor.

It is known that both graphs obtained above are 2-equi-eccentric and of minimum size. See Theorem 7 in [1]. Note that in both cases the size is given by $2(n-3) + 1 = 2n - 5$ or $2(n-4) + 3 = 2n - 5$. Observe that if K_1 is used as a base graph the size is $2n - 4$ and $2n - 4 > 2n - 5$. Label any of these graphs as M_n^2 (for base graph K_2) and M_n^3 (for base graph K_3). By excluding $n = 10$ and adding the minimum additional edges $u_1u_2, u_3u_4, \ldots, u_{t-1}u_t$ the graphs M_n^{2+}, M_n^{3+} can be obtained. The aforesaid is always possible by selecting the base graph either K_2 or K_3. Clearly, both M_n^{2+}, M_n^{3+} are of minimum size, 2-equi-eccentric with $\delta(M_n^{2+}) = \delta(M_n^{3+}) = 3$. Hence, $M^{2+}_{n_{ud}} = M^{3+}_{n_{ud}} = \mathfrak{N}_n$. We state a theorem.

Theorem 12. *A graph of order $n = 10$ and of minimum size, has $G_{ud} = \mathfrak{N}_n$ if and only if $G = \mathcal{P}$, (the Petersen graph).*

Proof. Firstly, that $\mathcal{P}_{ud} = \mathfrak{N}_{10}$ follows from Definition 1. Since each vertex in the Petersen graph has degree equal to 3 and $diam(\mathcal{P}) = 2$ the size is a minimum. Conversely, the fact that both $\varepsilon(M_{10}^{2+}) > 15$ and $\varepsilon(M_{10}^{3+}) > 15$ whereas, $\varepsilon(\mathcal{P}) = 15$ settles the result. □

Theorem 13. *A graph G of order $n \geq 7$, $n \neq 10$ and of minimum size such that $G_{ud} = \mathfrak{N}_n$ has,*

$$\varepsilon(G) = (2n - 5) + \lceil \tfrac{n-4}{2} \rceil.$$

Proof. Clearly, a graph G of order $n \geq 7$ and of minimum size such that $G_{ud} = \mathfrak{N}_n$ has $\varepsilon(G) = min\{\varepsilon(M_n^{2+}), \varepsilon(M_n^{3+})\}$. It is known that $\varepsilon(M_n^2) = \varepsilon(M_n^3) = 2n-5$ and a minimum size. Furthermore, from the definition of the upper ceiling function it follows that $\lceil \frac{n-4}{2} \rceil = x$ implies that, $\frac{n-4}{2} < x \leq \frac{n-4}{2}$ and $\lceil \frac{n-3}{2} \rceil = y$ implies that, $\frac{n-3}{2} < y \leq \frac{n-3}{2}$. Furthermore, $y \geq x$. Hence, $(2n-5) + x \leq (2n-5) + y$. It means that the minimum size is given by $\varepsilon(M_n^{3+}) = (2n-5) + \lceil \frac{n-4}{2} \rceil$. The ceiling function is required because of the dependency on $n-4$ is odd or even. □

5. Conclusion

The note concludes with a conjecture.

Conjecture 1. *Consider distinct graphs G and H. If* $\varepsilon(G) < \varepsilon(H)$ *then,* $\varepsilon(H_{ud}) \leq \varepsilon(G_{ud})$.

Dedication: This paper is dedicated to late Theresa Bernadette Kok (n*é*e Tomlinson) in acknowledgement of; and with deep gratitude for the profound influence she had on the author's endeavors to become a research mathematician.

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Conflict of interest:

The author declares there is no conflict of interest in respect of this research.

References

[1] Akiyama, J., Ando, K., & Avis, D. (1984). Miscellaneous properties of equi-eccentric graphs. Annals of Discrete Mathematics, 20, 13-23.

[2] Bondy, J. A., & Murty, U. S. R. (1976). Graph Theory with Applications. London: Macmillan Press.

[3] Harary, F. (1969). Graph Theory. Reading, MA: Addison-Wesley.

[4] Raja, M. R., Mangam, T. A., & Naduvath, S. (2022). Eccentric completion of a graph. Communications in Combinatorics and Optimization, 7(2), 193-201.

[5] Raja, M. R., Kok, J., Mangam, T. A., & Naduvath, S. (2023). Cyclic property of iterative eccentrication of a graph. Discrete Mathematics, Algorithms and Applications, 15(7), 2250155(1-13).

[6] Thalavayalil, T. T., Kok, J., & Naduvath, S. A study on upper deg-centric graphs. (Communicated).