Article

Analyzing *b*-color local edge antimagic coloring in graphs

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Abstract: A bijective mapping ς assigns each vertex of a graph G a unique positive integer from 1 to $|V(G)|$, with edge weights defined as the sum of the values at its endpoints. The mapping ensures that no two adjacent edges at a common vertex have the same weight, and each k-color class is connected to every other $k-1$ color class. A graph G possesses b-color local edge antimagic coloring if it satisfies the aforementioned criteria and it corresponds to a maximum graph coloring. This paper extensively studies the bounds, non-existence, and results of b-color local edge antimagic coloring in fundamental graph structures.

1. Introduction

This study analyzes a finite, connected, and undirected graph denoted by G. Here, $V(G)$ and $E(G)$ are the set of vertices and edges. We denote the minimum and maximum degrees of G as $\delta(G)$ and $\Delta(G)$ respectively. An antimagic vertex labeling assigns a unique positive integer to each vertex of a graph. The sum of labels of edges incident to any two vertices is always different. The concept of *b*-coloring was introduced by *Irving and Manlove* [1], while the concept of *b*-chromatic index was introduced by *Carlos Vinícius G.C. Lima et al.* [2] in 2013. In a *b*-coloring problem, the goal is to assign k colors to the edges of the graph. We must ensure that every set of k colors has neighbors in every set of $k-1$ colors. Resolving the local edge antimagic coloring and *b*-chromatic index for a given graph is widely seen as an NP-complete problem [3], leading to the development of many algorithms to tackle this challenge. The local edge antimagic coloring ($\gamma_{lea}(G)$) was studied by *I.H. Agustin* [4], by which a bijective function labels the vertices of G based on the cardinality of vertices of G. In 2018, *Ika et al.* [5] analysed the lower bounds of local edge antimagic coloring and exhibited the results of comb product of path and cycle with path, cycle and star graphs. Further *Rajkumar et al.* [6] extended the study of this coloring for some special graphs.

Arumugam et al. introduced the concept of local antimagic vertex coloring [7] that merges antimagic labeling and vertex coloring. Building upon this idea, *Dafik et al.* [8] prposed a new notion called, Rainbow Antimagic coloring, for which *Septory et al.* [9] provided a general lower bound. Motivated by these developments, we have put forth a novel coloring scheme termed as *b*-color local edge antimagic coloring, which combines the principles of local edge antimagic coloring and *b*-chromatic index. By combining these ideas, our approach not only improves the existing techniques but also lets us explore new problems in graph theory. The maximum number of colors required to achieve *b*-edge coloring through local edge antimagic labeling is denoted as the *b*-color local edge antimagic connetion number, $\varphi'_{ac}(G)$. The *b*-color local edge antimagic connection number of any given graph is bounded between maximum degree and *b*-chromatic index of G, that involves the local edge antimagic coloring. Significant progress in *b*-coloring and *b*-chromatic index is evident in works [10–14].

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Though the *b*-chromatic index can be determined for any graph *G*, the *b*-color local edge antimagic connection number is not applicable universally across all the graphs. Certain graphs have exceptions to the *b*-color local edge antimagic connection number. These include even order cycle C_p, except $p = 6$ and $p = 8$; wheel graph for $p = 3$; fan graph for $p = 3, 4$; complete graph K_p, unless $p \leq 3$ and complete bipartite graph $K_{q,p}$, unless $q = 1$ and $p \geq 2$. These exceptions highlight the nuance of *b*-color local edge antimagic connection numbers that they can be applied only to specific graph structures.

2. Preliminaries

In this section, we provide definitions of the coloring under study and describe some graph structures. For a bijective function $\varsigma : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$ whose corresponding edge weights are defined by, $W_\varsigma^i(xy) = \varsigma(x) + \varsigma(y)$, *G* is said to have *b*-color local edge antimagic coloring if:

(i) For every $i \neq j$, $W_\varsigma^i(e) \neq W_\varsigma^j(e')$ for edges $e, e' \in E(G)$ incident on a common vertex $x \in V(G)$, and
(ii) Each *k*-color class has neighbors in every other $k - 1$ color class.

A simple connected graph obtained by adding a hub vertex and connecting all the *p* vertices of a cycle to the hub vetex is called a *wheel graph* ($W_{1,p}$) [15,16]. It has $p + 1$ vertices and $2p$ edges with a maximum degree of *p*. If the cycle in the wheel graph is replaced by a path and all the *p* vertices of a path are connected to a hub vertex, it forms a *fan graph* ($F_{1,p}$) [17] that has $p + 1$ vertices and $2p - 1$ edges with maximum degree *p*. A *friendship graph* ($\mathcal{F}_p$) [18] is a simple connected graph constructed by taking *p* copies of cycle C_3 and joining one vertex from each copy of C_3 to form a hub vertex. The resulting graph has $2p + 1$ vertices and $3p$ edges with a maximum degree of $2p$.

Vizing [19] proved that for any graph *G*, $\chi'(G)$ lies in the range $\Delta(G)$ and $\Delta(G) + 1$. Graphs whose edge coloring is $\Delta(G)$ are called class I graphs, and those whose edge coloring is $\Delta(G) + 1$ are called class II graphs. Eventually, the *b*-chromatic index of the above graphs have close accordance with its maximum degree $\Delta(G)$, except in the case of cycle C_p when $p = odd$ and wheel graph when $p = 4$, whose *b*-chromatic index is $\Delta(G) + 1$.

Lemma 1. *[4,6] The local edge antimagic chromatic number of certain graphs (path, cycle, wheel, fan, friendship, complete and star graphs) are:*

1. *For $p \geq 3$, $\gamma_{lea}(P_p) = 2$*
2. *For $p \geq 3$, $\gamma_{lea}(C_p) = 3$*
3. *For $p \geq 3$, $\gamma_{lea}(W_{1,p}) = \begin{cases} 5, & \text{for } p = 3, 4 \\ p, & \text{for } p \geq 5 \end{cases}$*
4. *For $p \geq 2$, $\gamma_{lea}(F_{1,p}) = \begin{cases} p + 1, & \text{for } p = 2, 3 \\ p, & \text{for } p \geq 4 \end{cases}$*
5. *For $p \geq 1$, $\gamma_{lea}(\mathcal{F}_p) = \begin{cases} 3, & \text{for } p = 1 \\ 2p, & \text{for } p \geq 2 \end{cases}$*
6. *For $p \geq 3$, $\gamma_{lea}(K_p) = 2p - 3$*
7. *For $p \geq 2$, $\gamma_{lea}(K_{1,p}) = p$*

Lemma 2. *[11] The b-chromatic index of certain graphs (path, cycle, wheel, fan, friendship, complete, star and complete bipartite graphs) are:*

1. *For $p \geq 3$, $\varphi'(P_p) = \begin{cases} 2, & \text{for } 3 \leq p \leq 5 \\ 3, & \text{for } p \geq 6 \end{cases}$*
2. *For $p \geq 3$, $\varphi'(C_p) = \begin{cases} 2, & \text{for } p = 4 \\ 3, & \text{for } p \geq 3; p \neq 4 \end{cases}$*
3. *For $p \geq 3$, $\varphi'(W_{1,p}) = \begin{cases} 5, & \text{for } p = 4 \\ p, & \text{for } p \geq 3; p \neq 4 \end{cases}$*
4. *For $p \geq 3$, $\varphi'(F_{1,p}) = p$*
5. *For $p \geq 2$, $\varphi'(\mathcal{F}_p) = 2p$*
6. *For $3 \leq p \leq 6$, $\varphi'(K_3) = \varphi'(K_4) = 3$, $\varphi'(K_5) = 5$ and $\varphi'(K_6) = 6$*
7. *For $p \geq 2$, $\varphi'(K_{1,p}) = p$*
8. *For $p \leq r \leq p(p-1)$, $\varphi'(K_{p,r}) \leq \min\{p(p-1), p + r - 1\}$*

3. Bounds and Non-Existence of *b*-Color Local Edge Antimagic Coloring

Lemma 3. *For any connected graph G, the bound of b-color local edge antimagic coloring is given by,* $\Delta(G) \leq \gamma_{lea}(G) \leq \varphi'_{ac}(G) \leq \varphi'(G)$.

Proof. By the definition of *b*-chromatic index [11], $\Delta(G) \leq \chi'(G) \leq \varphi'(G) \leq 2\Delta(G) - 1$. Eventually, $\Delta(G) \leq \varphi'(G)$. By [4], $\gamma_{lea}(G) \geq \Delta(G)$. Since, *b*-coloring is a maximum coloring it constitues an upper bound of the *b*-color local edge antimagic coloring, as such we can write, $\Delta(G) \leq \gamma_{lea}(G) \leq \varphi'_{ac}(G) \leq \varphi'(G)$. In case, $\varphi'(G) = \Delta(G)$, then $\varphi'_{ac}(G) \leq \Delta(G)$. $\qquad\square$

Theorem 4. *The b-color local edge antimagic connection number does not exist for:*

  i. *An even order cycle C_p, except $p = 6, 8$.*
  ii. *A complete graph K_p, unless $p \leq 3$.*
  iii. *A wheel graph* $W_{1,p}$*, when $p = 3$.*
  iv. *A fan graph $F_{1,p}$, when $p = 3$.*
  v. *Complete bipartite graph $K_{q,p}$, unless $q = 1; p \geq 2$.*

Proof. The *b*-color local edge antimagic connection number of the following graphs does not exist:

  i. For an even order cycle C_p, for $p = 4$ and $p \geq 10$, assign an antimagic labeling as, $\varsigma(x_1, x_3, x_5, \ldots, x_{p-1}) = \{1, 2, 3, \ldots, \frac{p}{2}\}$ and $\varsigma(x_2, x_4, x_6, \ldots, x_p) = \{p, \ p-1, \ p-2, \ldots, \frac{p}{2}+1\}$. The corresponding edge weights are $\{p+1, \ p+2, \ \frac{p}{2}+2\}$, among which only $p+1$ is adjacent with all the other color classes, whereas other colors fails to satisfy the adjacency.
  ii. For a complete graph K_p, $p > 3$, assign an antimagic labeling as, $\varsigma(x_1, x_2, x_3, \ldots, x_p) = \{1, 2, 3, \ldots, p\}$ whose corresponding edge weights are $\{3, 4, 5, \ldots, 2p-1\}$ in which $p+1$ occuring at an edge $\{x_p x_1\} \in E(K_p)$ is the only color that is adjacent with all the other color classes, whereas other colors fails to satisfy the adjacency.
  iii. Since $W_{1,3} \cong K_4$, following (ii), it is clear that *b*-color local edge antimagic connection number of $W_{1,3}$ does not exist.
  iv. A fan graph $F_{1,3}$ can be obtained either by joining or deleting an edge $e_1 e_3$ or $e_2 e_4$ in C_4 or K_4 respectively. Since *b*-color local edge antimagic connection number of both the cases does not exist, it is easy to show that *b*-color local edge antimagic connection number of $F_{1,3}$ does not exist.
  v. A complete bipartite graph $K_{p,r}$ with $p = 1$ and $r \geq 2$ is a star graph [20] whose $\varphi'_{ac}(K_{1,r}) = r$ for $r \geq 2$. For $p, r \geq 2$, label the vertices antimagically as, $A_{\{x_1, x_2, x_3, \ldots, x_p\}} = i$, for $1 \leq i \leq p$ and $B_{\{y_1, y_2, y_3, \ldots, y_r\}} = p + j$, for $1 \leq j \leq r$. The corresponding edge weights are, $\{p+2, p+3, p+4, \ldots, 2p+r\}$ among which only few colors are adjacent with all the other color classes, whereas other colors fails to satisfy the adjacency.

$\qquad\square$

4. Some Results on Fundamental Graph Structures

Theorem 5. *For any positive integer $p \geq 3$, the b-color local edge antimagic connection number of a path is*

$$\varphi'_{ac}(P_p) = \begin{cases} 2, & \text{for } 3 \leq p \leq 5 \text{ and } p = \text{even}; p \neq 6, 8 \\ 3, & \text{for } p = 6, 8 \text{ and } p = \text{odd}; p \neq 3, 5 \end{cases}$$

Proof. Let P_p be a path graph, whose vertex and edge cardinalities are, $|V(P_p)| = p$ and $|E(P_p)| = p - 1$. The maximum and minimum degrees are, $\Delta(P_p) = 2$ and $\delta(P_p) = 1$.

Case 1: When $3 \leq p \leq 5$, $\varphi'(P_p) = \Delta(P_p)$, so by Lemma 3, $\varphi'(P_p) \leq 2$ is the upper bound. Also, when p = even, except for $p = 6, 8$, using Lemma 1, 2 and 3, the upper bound is, $\varphi'_{ac}(P_p) \leq 3$. In order to prove the lower bound, label the vertices local antimagically as, $\varsigma(x_1, x_2, x_3, \ldots, x_p) = \{1, \ p, \ 2, \ p-1, \ 3, \ p-2, \ \ldots\}$. The corresponding edge weights are: $p+1$ and $p+2$, i.e., $\varphi'_{ac}(P_p) \geq 2$. Thus, $\varphi'_{ac}(P_p) = 2$, for $3 \leq p \leq 5$ and $p = even; p \neq 6, 8$. Figure 1 shows the difference arising in the *b*-chromatic index and *b*-color local edge antimagic coloring of P_{10}.

Case 2: When $p = 6, 8$ and when p = odd, except for $p = 3, 5$, through Lemma 1, 2 and 3, the upper bound is, $\varphi'_{ac}(P_p) \leq 3$. In order to prove the lower bound for $p = 6, 8$, label the vertices as follows:

$$\varsigma(x_t : 1 \leq t \leq 6) = \{1, 4, 5, 2, 3, 6\} \text{ and } \varsigma(x_t : 1 \leq t \leq 8) = \{1, 6, 5, 2, 7, 4, 3, 8\}$$

The corresponding edge weights are: $p - 1, p + 1$ and $p + 3$ then, $\varphi'_{ac}(P_p) \geq 3$. Thus, $\varphi'_{ac}(P_p) = 3$, for $p = 6, 8$. To prove the lower bound, when p = odd; $p \neq 3, 5$, label the vertices local antimagically as,

$$\varsigma(x_1, x_2, x_3, \ldots, x_p) = \{1, \ p, \ 3, \ p - 2, \ 5, \ p - 4, \ \ldots, \ 4, \ p - 1, \ 2\}$$

whose corresponding edge weights are: $p + 1, p + 2$ and $p + 3$ then, $\varphi'_{ac}(P_p) \geq 3$. Thus, $\varphi'_{ac}(P_p) = 3$, when p = odd; $p \neq 3, 5$.

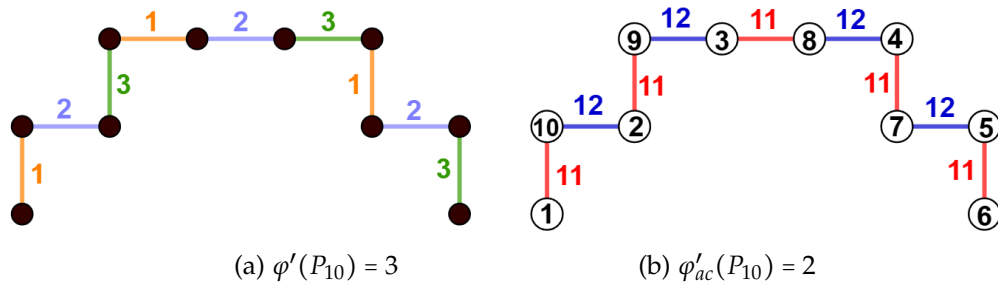(a) $\varphi'(P_{10}) = 3$ (b) $\varphi'_{ac}(P_{10}) = 2$

Figure 1. Difference in *b*-chromatic index and *b*-color local edge antimagic coloring of path P_{10}

$\square$

Theorem 6. *For any odd positive integer $p \geq 3$, the b-color local edge antimagic connection number of cycle is $\varphi'_{ac}(C_p) = 3$.*

Proof. Let C_p be a cycle whose vertex set is $V(C_p) = \{x_t : 1 \leq t \leq p\}$ and edge set is $E(C_p) = \{\{x_t x_{t+1} : 1 \leq t \leq p - 1\} \cup \{x_p x_1\}\}$. Through Lemma 1, 2 and 3, the upper bound is, $\varphi'_{ac}(C_p) \leq 3$. In order to prove the lower bound, define an antimagic labeling $\varsigma_1 : V(C_p) \to \{1, 2, 3, \ldots, p\}$ by,

$$\varsigma_1(x_t) = \begin{cases} \frac{t+1}{2}, & \text{for } t = 1, 3, 5, \ldots, p \\ p - \frac{t}{2} + 1, & \text{for } t = 2, 4, 6, \ldots, p - 1 \end{cases}$$

Based on the labeling, the corresponding edge weights are:

$$W^1_{\varsigma_1}(x_t x_{t+1}) = p + 1 \qquad \text{for } t = 1, 3, 5, \ldots, p - 2$$
$$W^2_{\varsigma_1}(x_t x_{t+1}) = p + 2 \qquad \text{for } t = 2, 4, 6, \ldots, p - 1$$
$$W^3_{\varsigma_1}(x_p x_1) = \frac{p + 3}{2} \qquad \forall p \in N$$

The edge weight calculation shows that, $\varphi'_{ac}(C_p) \geq 3$. On combining the upper and lower bounds, $\varphi'_{ac}(C_p) = 3$. Table 1 shows that each color class has at least one dominating b-edge which is incident with all the other neighboring color classes.

Table 1. b-Edges of cycle.

Color Class	b-Edges
$W^1_{\varsigma_1} = p + 1$	$x_1 x_2$
$W^2_{\varsigma_1} = p + 2$	$x_{p-1} x_p$
$W^3_{\varsigma_1} = \frac{p+3}{2}$	$x_p x_1$

$\square$

Corollary 7. *For any even positive integer p, the b-color local edge antimagic connection number of cycle exists only for $p = 6, 8$.*

Proof. Using Lemma 1, 2 and 3, the upper bound is, $\varphi'_{ac}(C_p) \leq 3$. To prove the lower bound, define a local edge antimagic labeling by,

$$\varsigma(x_t : 1 \leq t \leq 6) = \{1, 4, 5, 2, 3, 6\} \qquad \text{for } p = 6$$

$$\varsigma(x_t : 1 \leq t \leq 8) = \{1, 6, 5, 2, 7, 4, 3, 8\} \qquad \text{for } p = 8$$

The corresponding edge weights are: $p - 1$, $p + 1$ and $p + 3$ and the dominant b-edges are: $x_1 x_2$, $x_p x_1$ and $x_{p-1} x_p$ respectively. The total number of edge weights are $\varphi'_{ac}(C_p) \geq 3$. Thus, $\varphi'_{ac}(C_p) = 3$. In all the other cases of even p, the coloring exceeds and fails to satisfy b-edge coloring. Figure 2 can be used to check the optimal coloring patterns for even order cycles. Moreover, all the coloring pattern fails to satisfy b-edge coloring property for even cycle p, when $p \geq 10$.
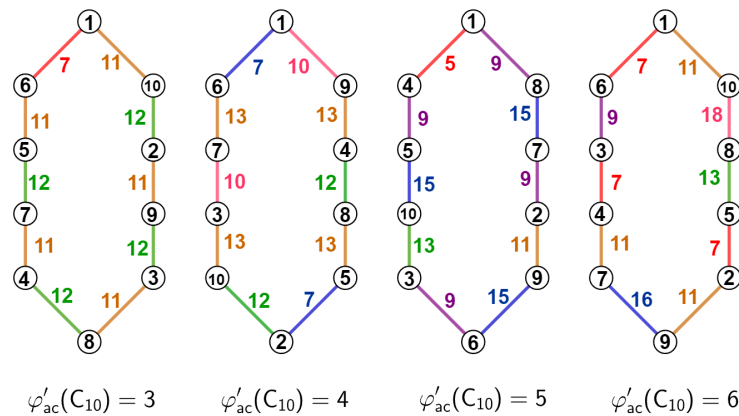

Figure 2. Optimal coloring patterns of *b*-color local edge antimagic coloring for C_{10}

$\square$

Theorem 8. *For any positive integer $p \geq 4$, the b-color local edge antimagic connection number of wheel graph is*

$$p \leq \varphi'_{ac}(W_{1,p}) \leq \begin{cases} 5 & \text{if } p = 4 \\ p & \text{if } p \geq 5 \end{cases}$$

Proof. Let $W_{1,p}$ be a wheel graph with vertex set $V(W_{1,p}) = \{\{h\} \cup \{x_t : 1 \leq t \leq p\}\}$ and edge set $E(W_{1,p}) = \{\{x_t x_{t+1} : 1 \leq t \leq p - 1\} \cup \{x_p x_1\} \cup \{h x_t : 1 \leq t \leq p\}\}$. To prove the theorem, consider the following two cases.
Case 1: For $p = 4$
Using Lemma 1, 2 and 3, the upper bound is, $\varphi'_{ac}(W_{1,4}) \leq p + 1 = 5$. In order to prove the lower bound, label the vertices as follows:

$$\varsigma_2(h) = 3$$

$$\varsigma_2(x_t) = \begin{cases} \frac{p+1}{2}, & t = 1, 3 \\ \frac{p}{2} + 3, & t = 2, 4 \end{cases}$$

Based on the labeling, the corresponding edge weights are:

$$W^1_{\varsigma_2}(hx_1) = 4$$
$$W^2_{\varsigma_2}(hx_3) = W^2_{\varsigma_2}(x_1x_2) = 5$$
$$W^3_{\varsigma_2}(x_2x_3) = W^3_{\varsigma_2}(x_4x_1) = 6$$
$$W^4_{\varsigma_2}(hx_2) = W^4_{\varsigma_2}(x_3x_4) = 7$$
$$W^5_{\varsigma_2}(hx_4) = 8$$

The edge weight calculation shows that, $\varphi'_{ac}(W_{1,p}) \geq 5$. On combining the upper and lower bounds, $\varphi'_{ac}(W_{1,4}) = 5$. Table 2 shows that each color class has at least one dominating b-edge which is incident with all the other neighboring color classes.

Table 2. b-Edges of Wheel $W_{1,4}$.

Color Class	b-Edges
$W^1_{\varsigma_2} = 4$	hx_1
$W^2_{\varsigma_2} = 5$	hx_3
$W^3_{\varsigma_2} = 6$	x_4x_1
$W^4_{\varsigma_2} = 7$	hx_2
$W^5_{\varsigma_2} = 8$	hx_4

Case 2: If $p \geq 5$

Since, $\varphi'(W_{1,p}) = \Delta(W_{1,p})$, by Lemma 3, $\varphi'_{ac}(W_{1,p}) \leq p$ is the upper bound. In order to prove the lower bound, consider the following subcases.

Subcase 1: When $p \equiv 1 \pmod 2$, define a local edge antimagic labeling $\varsigma_3 : V(W_{1,p}) \to \{1, 2, 3, \ldots, p+1\}$ by,

$$\varsigma_3(h) = \frac{p+1}{2}$$

$$\varsigma_3(x_t) = \begin{cases} \frac{t+1}{2} & \text{for } t = 1, 3, 5, \ldots, p-2 \\ p - \frac{t}{2} + 2 & \text{for } t = 2, 4, 6, \ldots, p-1 \\ \frac{p+3}{2} & \text{for } t = p \end{cases}$$

Based on the labeling, the corresponding edge weights are:

$$W^1_{\varsigma_3}(hx_t) = \frac{p+t+2}{2} \qquad \text{for } t = 1, 3, 5, \ldots, p-2$$
$$W^2_{\varsigma_3}(hx_t) = \frac{3p-t+5}{2} \qquad \text{for } t = 2, 4, 6, \ldots, p-1$$
$$W^3_{\varsigma_3}(hx_t) = p + 2 \qquad \text{for } t = p$$
$$W^4_{\varsigma_3}(x_tx_{t+1}) = p + 2 \qquad \text{for } t = 1, 3, 5, \ldots, p-2$$
$$W^5_{\varsigma_3}(x_tx_{t+1}) = p + 3 \qquad \text{for } t = 2, 4, 6, \ldots, p-3$$
$$W^6_{\varsigma_3}(x_tx_{t+1}) = p + 4 \qquad \text{for } t = p-1$$
$$W^7_{\varsigma_3}(x_px_1) = \frac{p+5}{2} \qquad \forall p \in N$$

As the edge weights $W^4_{\varsigma_3}, W^5_{\varsigma_3}, W^6_{\varsigma_3}$ and $W^7_{\varsigma_3}$ has already been included in $W^1_{\varsigma_3}, W^2_{\varsigma_3}$ and $W^3_{\varsigma_3}$, they are deemed superfluous and therefore, it is excluded. The total number of edge weights are obtained using N-term formula:

$$U_N^{W^1_{\varsigma_3}} = a + (N-1)d \longleftrightarrow p = \frac{p+3}{2} + (N-1)(1) \longleftrightarrow N = |W^1_{\varsigma_3}| = \frac{p-1}{2}$$
$$U_N^{W^2_{\varsigma_3}} = a + (N-1)d \longleftrightarrow \frac{2p+6}{2} = \frac{3p+3}{2} + (N-1)(-1) \longleftrightarrow N = |W^2_{\varsigma_3}| = \frac{p-1}{2}$$

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Notably, $|W_{\varsigma_3}^3| = 1$. Hence, $\sum_{i=1}^3 |W_{\varsigma_3}^i| = \frac{p-1}{2} + \frac{p-1}{2} + 1 = p$, implies $\varphi'_{ac}(W_{1,p}) \geq p$. Table 3 shows that each color class has at least one dominating b-edge which is incident with all the other neighboring color classes.

Table 3. b-Edges of Wheel for $p \equiv 1 (mod\,2)$.

Color Class	b-Edges
$W_{\varsigma_3}^1 = \frac{p+t+2}{2}$	$hx_1, hx_3, hx_5, \ldots, hx_{p-2}$
$W_{\varsigma_3}^2 = \frac{3p-t+5}{2}$	$hx_2, hx_4, hx_6, \ldots, hx_{p-1}$
$W_{\varsigma_3}^3 = p + 2$	hx_p

Subcase 2: When $p \equiv 0 (mod\,2); p \neq 4$, define a local edge antimagic labeling $\varsigma_4 : V(W_{1,p}) \to \{1, 2, 3, \ldots, p+1\}$ by,

$$\varsigma_4(h) = \frac{p+2}{2}$$

$$\varsigma_4(x_t) = \begin{cases} \frac{t+1}{2} & \text{for } t = 1, 3, 5, \ldots, p-1 \\ \frac{p+4}{2} & \text{for } t = 2 \\ p - \frac{t}{2} + 3 & \text{for } t = 4, 6, 8, \ldots, p \end{cases}$$

Based on the labeling, the corresponding edge weights are:

$$W_{\varsigma_4}^1(hx_t) = \frac{p+t+3}{2} \qquad \text{for } t = 1, 3, 5, \ldots, p-1$$

$$W_{\varsigma_4}^2(hx_t) = \frac{3p-t}{2} + 4 \qquad \text{for } t = 4, 6, 8, \ldots, p$$

$$W_{\varsigma_4}^3(hx_t) = p + 3 \qquad \text{for } t = 2$$

$$W_{\varsigma_4}^4(x_t x_{t+1}) = \frac{p}{2} + 3 \qquad \text{for } t = 1$$

$$W_{\varsigma_4}^5(x_t x_{t+1}) = W_{\varsigma_4}^5(x_p x_1) = \frac{p}{2} + 4 \qquad \text{for } t = 2$$

$$W_{\varsigma_4}^6(x_t x_{t+1}) = p + 3 \qquad \text{for } t = 3, 5, 7, \ldots, p-1$$

$$W_{\varsigma_4}^7(x_t x_{t+1}) = p + 4 \qquad \text{for } t = 4, 6, 8, \ldots, p-2$$

As the edge weights $W_{\varsigma_4}^4, W_{\varsigma_4}^5, W_{\varsigma_4}^6$ and $W_{\varsigma_4}^7$ has already been included in $W_{\varsigma_4}^1, W_{\varsigma_4}^2$ and $W_{\varsigma_4}^3$, they are deemed superfluous and therefore, it is excluded. The total number of edge weights are obtained using N-term formula:

$$U_N^{W_{\varsigma_4}^1} = a + (N-1)d \quad \longleftrightarrow \quad \frac{2p+2}{2} = \frac{p+4}{2} + (N-1)(1) \quad \longleftrightarrow \quad N = |W_{\varsigma_4}^1| = \frac{p}{2}$$

$$U_N^{W_{\varsigma_4}^2} = a + (N-1)d \quad \longleftrightarrow \quad p + 4 = \frac{3p+4}{2} + (N-1)(-1) \quad \longleftrightarrow \quad N = |W_{\varsigma_4}^2| = \frac{p}{2} - 1$$

Notably, $|W_{\varsigma_4}^3| = 1$. Hence, $\sum_{i=1}^3 |W_{\varsigma_4}^i| = \frac{p}{2} + \frac{p}{2} - 1 + 1 = p$, implies $\varphi'_{ac}(W_{1,p}) \geq p$. Figure 3 illustrates the local edge antimagic labeling of $W_{1,p}$ that satisfies the *b*-edge coloring and Table 4 clearly shows that each color class has at least one dominating b-edge which is incident with all the other neighboring color classes.

Table 4. b-Edges of Wheel for $p \equiv 0 (mod\,2)$.

Color Class	b-Edges
$W_{\varsigma_4}^1 = \frac{p+t+3}{2}$	$hx_1, hx_3, hx_5, \ldots, hx_{p-1}$
$W_{\varsigma_4}^2 = \frac{3p-t}{2} + 4$	$hx_4, hx_6, hx_8, \ldots, hx_p$
$W_{\varsigma_4}^3 = p + 3$	hx_2

From Subcase 1 and Subcase 2, it is evident that, $\varphi'_{ac}(W_{1,p}) \geq p$. On combining both upper and lower bounds, we get, $\varphi'_{ac}(W_{1,p}) = p$, for $p \geq 5$.   □
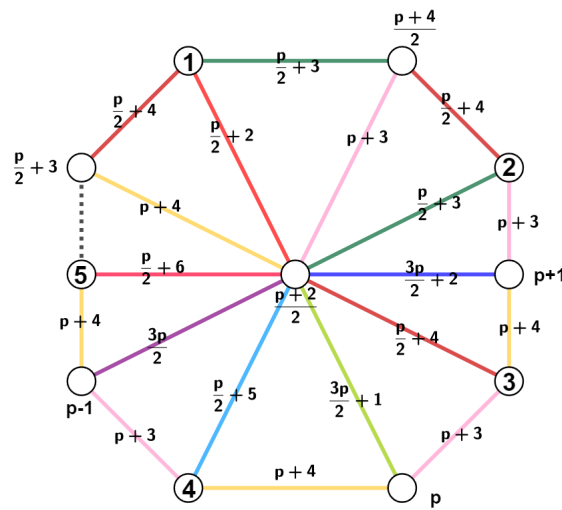

Figure 3. Local edge antimagic labeling of $W_{1,p}$ satisfying *b*-edge coloring

Theorem 9. *For any positive integer $p \geq 4$, the b-color local edge antimagic connection number of fan graph is $\varphi'_{ac}(F_{1,p}) = p$*

Proof. Let $F_{1,p}$ be a fan graph with vertex set $V(F_{1,p}) = \{\{h\} \cup \{x_t : 1 \leq t \leq p\}\}$ and edge set $E(F_{1,p}) = \{\{x_t x_{t+1} : 1 \leq t \leq p - 1\} \cup \{h x_t : 1 \leq t \leq p\}\}$. Since, $\varphi'(F_{1,p}) = \Delta(F_{1,p})$, by Lemma 3, $\varphi'_{ac}(F_{1,p}) \leq p$ is the upper bound. In order to prove the lower bound, consider the following two cases.

Case 1: When $p \equiv 1 (mod\, 2)$, define a local edge antimagic labeling $\varsigma_5 : V(F_{1,p}) \rightarrow \{1, 2, 3, \ldots, p+1\}$ by,
$$\varsigma_5(h) = \frac{p+1}{2}$$

$$\varsigma_5(x_t) = \begin{cases} \frac{t+1}{2} & \text{for } t = 1, 3, 5, \ldots, p-2 \\ p - \frac{t}{2} + 2 & \text{for } t = 2, 4, 6, \ldots, p-1 \\ \frac{p+3}{2} & \text{for } t = p \end{cases}$$

Based on the labeling, the corresponding edge weights are:

$$W^1_{\varsigma_5}(hx_t) = \frac{p+t}{2} + 1 \qquad \text{for } t = 1, 3, 5, \ldots, p-2$$

$$W^2_{\varsigma_5}(hx_t) = \frac{3p-t+5}{2} \qquad \text{for } t = 2, 4, 6, \ldots, p-1$$

$$W^3_{\varsigma_5}(hx_t) = p+2 \qquad \text{for } t = p$$

$$W^4_{\varsigma_5}(x_t x_{t+1}) = p+2 \qquad \text{for } t = 1, 3, 5, \ldots, p-2$$

$$W^5_{\varsigma_5}(x_t x_{t+1}) = p+3 \qquad \text{for } t = 2, 4, 6, \ldots, p-1$$

$$W^6_{\varsigma_5}(x_{p-1} x_p) = p+4 \qquad \text{for } t = 3, 5, 7, \ldots, p-1$$

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As the edge weights $W_{\varsigma_5}^4$, $W_{\varsigma_5}^5$ and $W_{\varsigma_5}^6$ has already been included in $W_{\varsigma_5}^2$ and $W_{\varsigma_5}^3$, they are deemed superfluous and therefore, it is excluded. The total number of edge weights are obtained using N-term formula:

$$U_N^{W_{\varsigma_5}^1} = a + (N-1)d \quad \longleftrightarrow \quad p = \frac{p+3}{2} + (N-1)(1) \quad \longleftrightarrow \quad N = |W_{\varsigma_5}^1| = \frac{p-1}{2}$$
$$U_N^{W_{\varsigma_5}^2} = a + (N-1)d \quad \longleftrightarrow \quad \frac{2p+6}{2} = \frac{3p+3}{2} + (N-1)(-1) \quad \longleftrightarrow \quad N = |W_{\varsigma_5}^2| = \frac{p-1}{2}$$

Notably, $|W_{\varsigma_5}^3| = 1$. Hence, $\sum_{i=1}^3 |W_{\varsigma_5}^i| = \frac{p-1}{2} + \frac{p-1}{2} + 1 = p$, implies $\varphi_{ac}'(F_{1,p}) \geq p$. Table 5 shows that each color class has at least one dominating b-edge which is incident with all the other neighboring color classes.

Table 5. b-Edges of Fan for $p \equiv 1(mod\,2)$.

Color Class	b-Edges
$W_{\varsigma_4}^1 = \frac{p+t+3}{2}$	$hx_1, hx_3, hx_5, \ldots, hx_{p-1}$
$W_{\varsigma_4}^2 = \frac{3p-t}{2} + 4$	$hx_4, hx_6, hx_8, \ldots, hx_p$
$W_{\varsigma_4}^3 = p + 3$	hx_2

Case 2: When $p = 4$, the antimagic labeling of vertices are, $\varsigma(h) = \{4\}$ and $\varsigma(x_1, x_2, x_3, x_4) = \{1, 5, 2, 3\}$, whose corresponding edge weights are, $W_\varsigma = \{5, 6, 7, 9\}$. Hence, $|W_\varsigma| = 4$. Further, when $p \equiv 0(mod\,2); p \neq 4$, define a local edge antimagic labeling $\varsigma_6 : V(F_{1,p}) \to \{1, 2, 3, \ldots, p+1\}$ by,

$$\varsigma_6(h) = \frac{p}{2}$$

$$\varsigma_6(x_t) = \begin{cases} \frac{t+1}{2} & \text{for } t = 1, 3, 5, \ldots, p-3 \\ p - \frac{t}{2} + 2 & \text{for } t = 2, 4, 6, \ldots, p \\ \frac{p+2}{2} & \text{for } t = p-1 \end{cases}$$

Based on the labeling, the corresponding edge weights are:

$$W_{\varsigma_6}^1(hx_t) = \frac{p+t+1}{2} \qquad \text{for } t = 1, 3, 5, \ldots, p-3$$
$$W_{\varsigma_6}^2(hx_t) = \frac{3p-t}{2} + 2 \qquad \text{for } t = 2, 4, 6, \ldots, p$$
$$W_{\varsigma_6}^3(hx_t) = p + 1 \qquad \text{for } t = p-1$$
$$W_{\varsigma_6}^4(x_t x_{t+1}) = p + 2 \qquad \text{for } t = 1, 3, 5, \ldots, p-3$$
$$W_{\varsigma_6}^5(x_t x_{t+1}) = p + 3 \qquad \text{for } t = 2, 4, 6, \ldots, p \text{ and } t = p-1$$
$$W_{\varsigma_6}^6(x_t x_{t+1}) = p + 4 \qquad \text{for } t = p-2$$

As the edge weights $W_{\varsigma_6}^4$, $W_{\varsigma_6}^5$ and $W_{\varsigma_6}^6$ has already been included in $W_{\varsigma_6}^2$, they are deemed superfluous and therefore, it is excluded. The total number of edge weights are obtained using N-term formula:

$$U_N^{W_{\varsigma_6}^1} = a + (N-1)d \quad \longleftrightarrow \quad \frac{2p-2}{2} = \frac{p+2}{2} + (N-1)(1) \quad \longleftrightarrow \quad N = |W_{\varsigma_6}^1| = \frac{p}{2} - 1$$
$$U_N^{W_{\varsigma_6}^2} = a + (N-1)d \quad \longleftrightarrow \quad p + 2 = \frac{3p+2}{2} + (N-1)(-1) \quad \longleftrightarrow \quad N = |W_{\varsigma_6}^2| = \frac{p}{2}$$

Notably, $|W_{\varsigma_6}^3| = 1$. Hence, $\sum_{i=1}^3 |W_{\varsigma_6}^i| = \frac{p}{2} - 1 + \frac{p}{2} + 1 = p$, implies $\varphi_{ac}'(F_{1,p}) \geq p$. Table 6 shows that each color class has at least one dominating b-edge which is incident with all the other neighboring color classes.

Table 6. b-Edges of Fan for $p \equiv 0 \pmod 2$.

Color Class	b-Edges
$W^1_{\varsigma_6} = \frac{p+t+1}{2}$	$hx_1, hx_3, hx_5, \ldots, hx_{p-3}$
$W^2_{\varsigma_6} = \frac{3p-t}{2} + 2$	$hx_2, hx_4, hx_6, \ldots, hx_p$
$W^3_{\varsigma_6} = p + 2$	hx_{p-1}

From Case 1 and Case 2, it is evident that $\varphi'_{ac}(F_{1,p}) \geq p$. On combining both the upper and lower bounds, we get, $\varphi'_{ac}(F_{1,p}) = p$. □

Theorem 10. *For any positive integer $p \geq 2$, the b-color local edge antimagic connection number of friendship graph is $\varphi'_{ac}(\mathcal{F}_p) = 2p$.*

Proof. Let $\mathcal{F}_p$ be a friendship graph whose vertex set $V(\mathcal{F}_p) = \{h\} \cup \{x_t : 1 \leq t \leq p\}$ and edge set $E(\mathcal{F}_p) = \{hx_t, hy_t, x_t y_t : 1 \leq t \leq p\}$. Since, $\varphi'(\mathcal{F}_p) = \Delta(\mathcal{F}_p)$, by Lemma 3, $\varphi'_{ac}(\mathcal{F}_p) \leq 2p$ is the upper bound. In order to prove the lower bound, define a local edge antimagic labeling $\varsigma_7 : V(\mathcal{F}_p) \to \{1, 2, 3, \ldots, 2p+1\}$ by,

$$\varsigma_7(h) = p$$

$$\varsigma_7(x_t) = \begin{cases} t & \text{for } 1 \leq t \leq p-1 \\ t+1 & \text{for } t = p \end{cases}$$

$$\varsigma_7(y_t) = \begin{cases} 2p-t+2 & \text{for } 1 \leq t \leq p-1 \\ t+2 & \text{for } t = p \end{cases}$$

Based on the labeling, the corresponding edge weights are:

$$\begin{aligned} W^1_{\varsigma_7}(hx_t) &= p + t & \text{for } 1 \leq t \leq p-1 \\ W^2_{\varsigma_7}(hx_t) &= 2(p+1) & \text{for } t = p \\ W^3_{\varsigma_7}(hy_t) &= 3p - t + 2 & \text{for } 1 \leq t \leq p-1 \\ W^4_{\varsigma_7}(hy_t) &= 2p + 1 & \text{for } t = p \\ W^5_{\varsigma_7}(x_t y_t) &= 2(p+1) & \text{for } 1 \leq t \leq p-1 \\ W^6_{\varsigma_7}(x_t y_t) &= 2p + 3 & \text{for } t = p \end{aligned}$$

Since the edge weights $W^5_{\varsigma_7}$ and $W^6_{\varsigma_7}$ are already included in $W^2_{\varsigma_7}$ and $W^3_{\varsigma_7}$, they are deemed superfluous and therefore, it is excluded. The total number of edge weights can be obtained using N-term formula:

$$\begin{aligned} U_N^{W^1_{\varsigma_7}} = a + (N-1)d \longleftrightarrow & \quad 2p - 1 = p + 1 + (N-1)(1) & \longleftrightarrow & \quad N = |W^1_{\varsigma_7}| = p - 1 \\ U_N^{W^3_{\varsigma_7}} = a + (N-1)d \longleftrightarrow & \quad 2p + 3 = 3p + 1 + (N-1)(-1) & \longleftrightarrow & \quad N = |W^3_{\varsigma_7}| = p - 1 \end{aligned}$$

Notably, $|W^2_{\varsigma_7}| = |W^4_{\varsigma_7}| = 1$. Hence, $\sum_{i=1}^4 |W^i_{\varsigma_7}| = p - 1 + 1 + p - 1 + 1 = 2p$, implies $\varphi'_{ac}(\mathcal{F}_p) \geq 2p$ is the lower bound. On combining the upper and lower bounds, $\varphi'_{ac}(\mathcal{F}_p) = 2p$. Table 7 shows that each color class has at least one dominating b-edge which is incident with all the other neighboring color classes.

□

Table 7. b-Edges of Friendship Graph.

Color Class	b-Edges
$W^1_{\varsigma_7} = p + t$	$hx_1, hx_2, hx_3, \ldots, hx_{p-1}$
$W^2_{\varsigma_7} = 2(p + 1)$	hx_p
$W^3_{\varsigma_7} = 3p - t + 2$	$hy_1, hy_2, hy_3, \ldots, hy_{p-1}$
$W^4_{\varsigma_7} = 2p + 1$	hy_p

Theorem 11. *For any positive integer $p \geq 2$, the b-color local edge antimagic connection number of star graph is* $\varphi'_{ac}(K_{1,p}) = p$.

Proof. Since, $\varphi'(K_{1,p}) = \Delta(K_{1,p})$, by Lemma 3, $\varphi'_{ac}(K_{1,p}) \leq p$ is the upper bound. Since all the edges of a star graph are incident to a single hub vertex h, each edge has a unique edge weight and every color class has neighbors in every other classes. Hence the total number of edge weights are p, i.e., $\varphi'_{ac}(K_{1,p}) \geq p$. On combining both the upper and lower bounds, $\varphi'_{ac}(K_{1,p}) = p$. □

5. Conclusion

The proposed *b*-color local edge antimagic coloring of graphs represents a new and innovative approach to this field of research, which is currently wide open for exploration. This innovative coloring technique takes inspiration from the successful implementation of rainbow antimagic coloring [8,9] and *b*-chromatic index in diverse graph structures. The paper offers meticulous proofs that establish the optimality of the proposed coloring approach for specific graphs, with future endeavors aimed at extending these findings to other captivating graph families.

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