Article

Philosophical note: The principle of transmitting the definition component-wise

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Abstract: This note addresses impracticalities or possible absurdities with regards to the definition corresponding of some graph parameters. To remedy the impracticalities the *principle of transmitting the definition* is put forward. The latter principle justifies a comprehensive review of many known graph parameters, the results related thereto, as well as the methodology of applications which draw a distinction between connected versus disconnected simple graphs. To illustrate the notion of transmitting the definition, various parameters are re-examined such as, connected domination number, graph diameter, girth, vertex-cut, edge-cut, chromatic number, irregularity index and quite extensively, the hub number of a graph. Ideas around *undefined* viz-a-viz *permissibility* viz-a-viz *non-permissibility* are also discussed.

Keywords: Graph parameters, parameter impracticalities, transmitting definition.

MSC: 05C40, 05C69, 05C70.

1. Introduction

Good reference to important concepts, notation and graph parameters are found in [1–3]. All graphs under consideration are finite simple graphs.

The length of a shortest path having end-vertices u and v is denoted by, $d_G(u,v)$ and is called the distance between u and v in G. Clearly, $d_G(u,u) = 0$. If a shortest uv-path exists then, $d_G(u,v) \geq 0$ and finite. If a shortest uv-path does not exist then, $d_G(u,v) = \infty$. From the aforesaid it is evident that the notion of distance between a pair of vertices of a graph is well-defined for both, connected and disconnected graphs. If $d_G(u,v) \geq 2$ and finite then a vertex w on a uv-path, $w \neq u$, $w \neq v$ is called an internal vertex. Let the set $X(u,v)$ be the set of internal vertices of a shortest path between u and v. Then $|X(u,v)| = d_G(u,v) - 1$. Note that $X(u,u) = \varnothing$ as well as, if $d_G(u,v) = 1$ then $X(u,v) = \varnothing$. It follows naturally that if $d_G(u,v) \geq d_G(w,z)$ and both are finite then, $|X(u,v)| \geq |X(w,z)|$. So what can be said about $X(u,v)$ if $d_G(u,v) = \infty$? In the latter case intuition provides that $X_G(u,v) = \varnothing$. However, if $d_G(w,z) \geq 2$ and finite an absurdity comes to the fore in view of $d_G(u,v) >>> d_G(w,z)$. Hence, it is sensible to state that $X_G(u,v)$ is undefined if $d_G(u,v) = \infty$. Though not in all cases, it appears to be popular to define a graph parameter say, $p(G)$ to be ∞ or to state it is undefined if G is disconnected.

Motivation for this note: Classical graph theory originated in an era where the real world was deemed to be physically linked. Hence, the distinction between a connected and a disconnected graph was a straightforward fact. Networks in the modern world often present a cluster (disjoint elements) of *hard* connected graphs which amongst themselves have *soft connectivity* (or transmitting connectivity). The modern world soft connectivity justifies a comprehensive review of many known graph parameters, the results related thereto, as well as the methodology of applications which draw a distinction between connected versus disconnected simple graphs.

2. Discussion

Recall that $D \subseteq V(G)$ is a dominating set of G if and only if each vertex $v \in V(G) \backslash D$ is adjacent to some vertex $u \in D$. The minimum cardinality of some D is called the domination number, $\gamma(G)$ of G. The classical

definition is well-defined for both connected and disconnected graphs. Conceptually, the domination number for a disconnected graph G is obtained as if *transmitting the definition component-wise*. Therefore, if:

$$G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l, l \geq 1 \text{ then, } \gamma(G) = \sum_{i=1}^{l} \gamma(G_i).$$

Consider a cluster of locally connected networks spread over continents. The decision is that a minimum number of subsystems have to be implemented to dominate the cluster of connected networks. Clearly, the result for the domination number of a disconnected graph will apply. If the principle of transmitting the definition component-wise is applied in respect of a graph parameter say, $p(G)$, let $c\text{-}p(G)$ and $c\text{-}p_c(G)$ denote the non-necessarily connected and necessarily connected parameters of graph G, respectively. Note that inherently $\gamma(G) = c\text{-}\gamma(G)$. Similarly, it is inherent that for the independence number of G, $\alpha(G) = c\text{-}\alpha(G)$.

The notion of connected domination number was introduced in [4]. The notion is restricted to connected graph because of the classical interpretation is that a disconnected graph must be viewed as a *singular whole* rather than a cluster of connected graphs (or components). Let D be a minimum domination set of a disconnected graph G which has $l \geq 1$ components, G_i, $i = 1, 2, 3, \ldots, l$. Then certainly $D = \bigcup_{i=1}^{l} D_i$ where $|D_i| = \gamma(G_i)$. Thus, if each $\langle D_i \rangle$ is connected per component the definition can be relaxed to view D as a derivative connected dominating set of G. For the purposes of real world applications the author views this as a sensible alternative. Therefore, $c\text{-}\gamma_c(G) = \sum_{i=1}^{l} \gamma_c(G_i)$.

By analogy, assume that a maximum number of subsystems have to implemented at vertices of G by some condition with regards to the diameter of a graph G. Let the maximum number of subsystems $s(G)$ be bounded by $s(G) \leq diam(G)$. If G is disconnected an absurdity comes to the fore. Note that if G is disconnected a subsystem is required at each vertex of G because $diam(G) = \infty$. The principle of transmitting the definition component-wise yields a new parameter with regards to the diameter of any graph. Let,

$$G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l, l \geq 1 \text{ then, } c\text{-}diam(G) = \sum_{i=1}^{l} diam(G_i).$$

Requiring that $s(G_i) \leq diam(G_i)$ implies that, $c\text{-}s(G) = \sum_{i=1}^{l} s(G_i)$. This approach applies to both connected and disconnected graphs. Furthermore, $c\text{-}s(G) \leq c\text{-}diam(G)$.

The notion of doubly connected domination number of a graph was introduced in [5]. In [5] it is stated that, "For any connected graph G of order $p \geq 2$, $\gamma_{cc}(G) \leq p - \kappa(G) + 1$." An interesting question that comes to the fore is, "Is it valid to state that, $c\text{-}\gamma_{cc}(G) \leq p - c\text{-}\kappa(G) + l$ if $G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l, l \geq 2$?"

In [6] the irregularity index of a graph G (connected by assumption in [6]) is the number of distinct terms in the degree sequence of G. The irregularity index is denoted by, $t(G)$. The definition of the irregularity index does inherently apply to a disconnected graph. Hence, $t(G) = c\text{-}t(G)$. However, the note relies on connectivity for its results such as the first proposition in Section 2 [6].

Proposition 1. *[6] Let G be a connected graph of order n. The diameter of G satisfies the inequality*

$$diam(G) \leq n - t + 1,$$

where t is the irregularity index of G. Moreover, this inequality is sharp.

The weakest equivalent result by the principle of transmitting definition component-wise is:

Proposition 2. *Let $G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l, l \geq 1$ each of respectively order, n_i, $1 \leq i \leq l$. Then,*

$$c\text{-}diam(G) \leq n - t + l,$$

where $t = \sum_{i=1}^{l} t(G_i)$, $n = \sum_{i=1}^{l} n_i$ and the inequality is sharp.

Other, improved results are valid for certain configurations of disconnected graphs. The derivative results are:

(i) c-$diam(G) \leq n - t + 1$,

(ii) c-$diam(G) \leq n - c$-$t(G) + l$, where c-$t(G) \leq t$,

(iii) c-$diam(G) \leq n - c$-$t(G) + 1$, where c-$t(G) \leq t$.

Seeking characterization of the configurations of disconnected graphs for which each of the above is valid is of interest.

Conjecture 3. *For any graph G, c-$diam(G) \leq n - c$-$t(G) + q$, where c-$t(G) \leq t$ and $1 \leq q \leq l$.*

Example 1. For a connected graph G of order n Erd*ós* et al. presented a classical upper bound in [7] namely, $diam(G) \leq \frac{3n}{\delta(G)+1} - 1$. This result can easily be generalized for disconnected graphs in that,

$$c\text{-}diam(G) \leq \sum_{i=1}^{l} \frac{3n_i}{\delta(G_i)+1} - l.$$

The girth of a graph G is defined as the length of a shortest cycle C_k, $k \geq 3$ in G. This definition holds for almost all simple graphs be it, connected or disconnected. The proviso is that at least one component of a disconnected graph G must contain a cycle else, $g(G) = \infty$. However, if the non-proper cycle graphs K_1 (a point-cycle) and K_2 (a flat-cycle) are included as cycles then the next alternative definition may be considered.

Definition 4. Let G be a graph with $l \geq 1$ components, $G_1, G_2, G_3, \ldots, G_l$. The girth $g(G)$ is defined to be the length of the shortest cycle in G with the proviso that:

$$g(G) = \begin{cases} 1, & \text{if } G \text{ has some component(s), } K_1; \\ 2, & \text{if } G \text{ does not have some component(s), } K_1 \text{ but has some tree;} \\ min\{g(G_i) : 1 \leq i \leq l\}, & \text{if } \forall\, i, G_i \not\cong K_1 \text{ and } G \text{ has no tree.} \end{cases}$$

Note that, if G is a graph with $l \geq 1$ components, $G_1, G_2, G_3, \ldots, G_l$ of which at least one has a cycle, the classical definition provides for $g(G) = min\{g(G_i) : 1 \leq i \leq l\}$. The salient feature of the latter is that the principle of transmitting the definition comparatively over all components (component-wise) is inherent. Definition 4 invalidates a previously valid statement such as: "A connected graph G has a finite girth $g(G)$ if and only if G contains at least one cycle." Instead the valid statement is now: "The girth $g(G)$ of a graph G is finite." Much research have been publish on the domination number vis-a-vis the girth of graphs. By the classical definition of girth such research is restricted to connected graphs. By Definition 4 a new avenue of comparative research is open. A similar remark applies to research on the independence number vis-a-vis the girth of a graph. See [8,9] with references thereto.

Recall that a cut-vertex v of a connected graph $G \not\cong K_n$ is such that, the induced graph $\langle V(G) \backslash \{v\} \rangle$ has two or more components. A vertex-cut is a set $X \subseteq V(G)$ such that, the induced graph $\langle V(G) \backslash X \rangle$ has two or more components. The cardinality of a smallest vertex-cut of a connected graph G is called the vertex connectivity of G and is denoted by, $\kappa(G)$. To deal with complete graph such that most general results remain valid the convention is that, $\kappa(K_n) = n - 1$, $n \geq 2$ and $\kappa(K_1) = 1$. By utilizing the principle of transmitting the definition component-wise for $G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l$, $l \geq 1$ it is proposed that, c-$\kappa(G) = min\{\kappa(G_i) : 1 \leq i \leq l\}$. Analogous notions can be defined with regards to the edge set of a connected graph. The cardinality of a smallest edge-cut of a connected graph G is called the edge connectivity of G and is denoted by, $\lambda(G)$. It is proposed that for $G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l$, $l \geq 1$ the edge connectivity is, c-$\lambda(G) = min\{\lambda(G_i) : 1 \leq i \leq l\}$. Whereas, for a connected graph G it is known that, $\kappa(G) \leq \lambda(G)$ no such relationship exists between c-$\kappa(G)$ and c-$\lambda(G)$ if $G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l$, $l \geq 2$.

Recall that the chromatic number $\chi(G)$ of a graph G is the minimum number of distinct colors with which the vertices can be colored such that no pair of adjacent vertices (distinct) has the same color. Such coloring is called a χ-coloring of G. The classical definition provides for disconnected graphs in that, if $G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l$, $l \geq 1$ then, $\chi(G) = max\{\chi(G_i) : 1 \leq i \leq l\}$. Researchers typically argue that the study in respect of connected graphs is sufficient. However, certain results did not necessarily translate as valid for

disconnected graphs. If a connected graph G has a cut-vertex v and $G - v$ has the components $G_1, G_2, G_3, \ldots, G_l$ then it is known that,

$$max\{\chi(G_i) : 1 \leq i \leq l\} \leq \chi(G) \leq max\{\chi(G_i) : 1 \leq i \leq l\} + 1.$$

Hence, if in the first instance $G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l$, $l \geq 1$ then by the principle of transmitting the definition component-wise it follows that, $c\text{-}\chi(G) = max\{\chi(G_i) : 1 \leq i \leq l\}$. For connected graphs the famous Brooks' Theorem [10] states that, $\chi(G) \leq \Delta(G) + 1$. It is easy to verify that for $G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l$, $l \geq 1$ it follows that, $c\text{-}\chi(G) \leq c\text{-}\Delta(G) + 1$ where $c\text{-}\Delta(G) = \max\{\Delta(G_i) : 1 \leq i \leq l\}$. Consider a graph G (connected or disconnected) and let the set of colors $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_{\chi(G)}\}$ which yields a minimum proper coloring. Let $\eta(c_i)$ be the number of times c_i has been allocated. It is always possible to color a graph such that, without loss of generality the color c_1 is allocated a minimum times. For example, the odd cycle C_n has $\chi(C_n) = 3$. It is easy to see that the color c_1 can be allocated once or more times in a proper coloring of C_n. Assume that in a disconnected graph, subsystems have to be placed at the vertices colored c_1 subject to a minimum in total. Then clearly, the solution is given by some $c\text{-}\chi$-coloring such that, $min\{\sum\limits_{i=1}^{l} \eta(c_1) : \text{over all}$ $c\text{-}\chi\text{-colorings}\}$ is obtained.

The parameter called the hub number of a graph G and denoted by $h(G)$ was introduced in [11]. The hub number and derivative parameters such as the connected hub number $h_c(G)$, and the doubly connected hub number $h_{cc}(G)$ (see [12]) enjoy noticeable research interest. A reader can do an easy Google search to find an abundance of published work.

See page 1 (journal page 117), paragraph 2, of [11] for a motivation of a study of the hub number linked to a possible real world application. A salient feature of a hub-vertex in a hub set is that it has (or should have) *utility value*. It is then stated that, "if G is disconnected, any hub set must contain all of the vertices in all but one of the components, as well as a hub set in the remaining component." Hence, if $G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l$, $l \geq 2$ and say, $h(G) = h(G_j) + \sum\limits_{k=1,2,3,\ldots,j-1,j+1,\ldots,l} |V(G_k)|$, the aforesaid signals a possible absurdity in practical terms. For example if G_j is complete the vast RTS network is redundant because easy walks in G_j do not require traversing a hub-vertex. Furthermore, if the cluster of connected graphs represents different networks in metropolises or on continents and all but one say, G_j are complete graphs then only G_j requires the RTS. The stated impracticality can be overcome by utilizing the principle of transmitting the definition component-wise. Author proposes that in the latter case the definition should provide that, $c\text{-}h(G) = h(G_j)$. Hence, let $c\text{-}h(G) = \sum\limits_{i=1}^{l} h(G_j)$.

Example 2. Lemma 4.1 [11] states that, "For any graph G, $\gamma(G) \leq h(G) + 1$." The aforesaid is certainly valid for any connected graph. Under the definition in [11] it also holds for disconnected graphs because of the vast and mostly redundant hub set which is defined. If the principle of transmitting the definition component-wise is applied to $G = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_l$, $l \geq 1$ the result will be:

Lemma 5. *For any graph G with $l \geq 1$ components, $\gamma(G) \leq \sum\limits_{i=1}^{l} h(G_i) + l$.*

Note that the upper bound improves on the stated upper bound in [11].

Alternatively, let Y_i, $i = 1, 2, 3, \ldots, l$ be the respective minimum hub sets of G_i. Lemma 5 will then state that, "For any graph G with $l \geq 1$ components, $c\text{-}\gamma(G) \leq |\bigcup\limits_{i=1}^{l} Y_i| + l$." Similarly, Lemma 4.2 in [11] will state: "For any graph G with $l \geq 1$ components, $c\text{-}h_c(G) \leq c\text{-}\gamma_c(G)$." Also, Lemma 4.3 in [11] will state: "For any graph G with $l \geq 1$ components, $c\text{-}h(G) \geq c\text{-}diam(G) - l$ and the inequality is sharp.

2.1. Permissibility viz-a-viz non-permissibility

From the definition of the hub number of a graph read together with the fact that adjacent vertices say u, v if such exist, represent an easy walk along the uv edge it follows that, $h(K_n) = 0$, $n \geq 1$. The aforesaid also implies that the empty set is a permissible hub set. In fact, $h(G) = 0$ if and only if G is complete. Inherently, the

only interest one has with a hub set $\mathcal{H}(G)$ of a connected graph G is whether it is possible to traverse between two non-adjacent vertices u, v in $V(G)\backslash\mathcal{H}(G)$ such that all internal vertices are in $\mathcal{H}(G)$. There is no interest in how to traverse between any two vertices in $\mathcal{H}(G)$. However, the conditions on traversability changes with the introduction of the hubtic number of a graph. The hubtic number denoted by $\xi(G)$ is defined as the maximum order of a partition of the vertex set into hub sets, [13]. The salient feature of the aforesaid definition is that the partition hub sets are not required to be minimum. A further salient feature is that adjacent vertices must be able to find a path through the hub set. The aforesaid explains why Observation 2.2(1) in [13] correctly yields $\xi(K_p) = p$, $p \geq 1$. To clarify the definition note that if $\xi(G) = 2$ and $V(G) = \mathcal{H}_1(G) \cup \mathcal{H}_2(G)$ then for each pair $u, v \in \mathcal{H}_1(G)$ there must be a uv-path with all internal vertices in $\mathcal{H}_2(G)$ and vice versa. The aforesaid property is called *bi-traversability*. For $\xi(G) \geq 3$ it means that for each pair of subsets, $\mathcal{H}_j$ and $V(G)\backslash\mathcal{H}_j$, $1 \leq j \leq \xi(G)$ bi-traversability must be possible. We called such partition a *hubtic partition*. It means that if $\xi(G) = 2$ then it is possible to write $V(G) = \mathcal{H}_1 \cup \mathcal{H}_2$ and by the definition of a partition both $\mathcal{H}_1, \mathcal{H}_2$ are non-empty. If $\xi(G) = 1$ or put differently, the hubtic partition is $V(G) \equiv V(G) \cup \varnothing$ (for sake of argument $\varnothing$ is permitted). Furthermore, either G is possibly complete (for sake of argument) or a non-permissible case is at hand. The latter follows from the argument that, if G is non-complete then in some cases at least one pair of non-adjacent vertices u, v exists with no defined (or permissible) path which has all internal vertices in the empty set. Observation 2.2(2) in [13] presents such a case for C_p, $p \geq 7$. Similarly, for paths P_p, $p \geq 5$. Coming back to the argument that G is possibly complete another fallacy comes to the fore. Note that Observation 2.2(1) in [13] correctly yields $\xi(K_p) = p$, $p \geq 1$. However, for $p \geq 2$, $\xi(K_p) \neq 1$. Hence, for a non-complete connected graph G, $\xi(G) \neq 1$ or put differently, a hubtic partition is not permitted. By the aforesaid Theorem 2.4 in [13] is false as well. For a connected graph we put forward the following conjecture.

Conjecture 6. *(i) If G is connected let $\mathcal{H}$ be a minimum hub set of G then, G permits a hubtic partition if and only if $V(G)\backslash\mathcal{H}$ is a hub set.*
(ii) If G of order n is connected then G permits a hubtic partition if and only if $0 \leq h(G) \leq \lfloor \frac{n}{2} \rfloor$.

Observation 1. *Let $\xi(G) = l \geq 2$. There exists a hubtic partition $V(G) = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3 \cup \cdots \cup \mathcal{H}_l$ such that,*

$$h(G) = min\{|\mathcal{H}_i| : 1 \leq i \leq l\}.$$

The analysis worsens if a disconnected graph is considered. In Proposition 2.14 of [13] it is stated that, for any two connected graphs G_1 and G_2,

$$\xi(G_1 \cup G_2) = \begin{cases} 1, & \text{if } G_1 \text{ or } G_2 \text{ is non-complete;} \\ 2, & \text{if } G_1 \text{ and } G_2 \text{ are complete.} \end{cases}$$

In the first case i.e. $\xi(G_1 \cup G_2) = 1$ the reasoning in the proof suggests that the hub set is say, $V(G_1) \cup \mathcal{H}(G_2)$. The aforesaid means that, $\xi(G_1 \cup G_2) = 2 \neq 1$. However, the set,

$$(V(G_1) \cup V(G_2))\backslash(V(G_1) \cup \mathcal{H}(G_2))$$

cannot serve as a hub for non-adjacent vertices in $V(G_1)$. Hence, the first case ($\xi = 1$ or 2) is false. Similarly, the second case is false. A valid statement would be that, for any two connected graphs G_1 and G_2 of order $n_1 \geq 2$, $n_2 \geq 2$ respectively, and $\xi(G_1) \geq 2$, $\xi(G_2) \geq 2$ it follows that,

$$\xi(G_1 \cup G_2) = \begin{cases} min\{\xi(G_1), \xi(G_2)\}, & \text{if } G_1 \text{ or } G_2 \text{ is non-complete;} \\ min\{n_1, n_2\}, & \text{if } G_1 \text{ and } G_2 \text{ are complete.} \end{cases}$$

In the conclusion of [13] the characterization in 2. has been answered i.e. "A graph G has $\xi(G) = 1$ if and only if $G = K_1$."

The notion of hub number has evolved to the notion of the *edge hub number* of a graph. Essentially the edge hub number is equivalent to the hub number of the line graph $L(G)$ of G. Furthermore, the ideas have been extended to fuzzy graphs. See [14,15] and related research. It will be worthy to revisit the results in an attempt to eliminate absurdities with regards to real world applications.

3. Conclusion

Through this philosophical note the observation is that all graph parameters which are defined with the restriction that the graph G must be connected can be revisited and possibly relaxed sensibly for application in modern day network structures. Few examples have been highlighted to show that existing results can either be extended or require reconsideration. To provide for classical (or historical) results the alternative notation i.e. $c\text{-}p(G)$ and $c\text{-}p_c(G)$ where $p(G)$ is some graph parameter is proposed. It permits alternative derivative parameters such as the *upper minimum degree* of G (connected or disconnected) denoted by, $c\text{-}\delta(G) = \sum\limits_{i=1}^{l} \delta(G_i)$ (opposed to the classical definition i.e. $\delta(G) = min\{\delta(G_i) : 1 \leq i \leq l\}$). Note that, if $l = 1$ hence, G is connected then $\delta(G) = c\text{-}\delta(G)$. Similarly, the *upper maximum degree* of G is denoted by, $c\text{-}\Delta(G) = \sum\limits_{i=1}^{l} \Delta(G_i)$. Theorem 4.7 in [11] remains valid for any graph G of order n in that,

$$c\text{-}h(G) \leq n - c\text{-}\Delta(G).$$

Author argues that the principle of transmitting the definition component-wise together with the notion of permissibility will greatly enhance the revised results. Author believes that this philosophical note has opened a wide avenue for at least, Bachelor Thesis researchers.

Dedication

This philosophical note is dedicated to Johann Swiegelaar in acknowledgement of; and with deep gratitude for the profound influence he had on the author's endeavors to delve into the philosophy of mathematics. Johann as philosopher himself, greatly appreciates the philosophical reasoning of amongst others, Albert Camus, Immanuel Kant, Friedrich Nietzsche and Arthur Schopenhauer. Finally, this philosophical note has been submitted on 28 July 2024 in memory of late Severn Swiegelaar (son of Johann) who tragically died on 28 July 2019.

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