

Article

The novel resolvability parameter, local edge partition dimension of graphs

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Abstract: In this paper, we introduce a new resolvability parameter named as the local edge partition dimension (*LEPD*) of graphs. The local edge partition dimension (*LEPD*) makes a specialty of partitioning the vertex set of a graph into awesome instructions based totally on localized resolving properties. Our findings offer a fresh angle on graph resolvability, offering capability insights for optimizing network overall performance and structural analysis. Let $G = (V, E)$ be a connected graph with vertex set V and edge set E . A partition set $R_p = \{R_{p1}, R_{p2}, R_{p3} \dots, R_{pn}\}$ contain subsets of vertices of G . If for every pair of adjacent edges p and q in G , then $d(p, R_p) \neq d(q, R_p)$ and if p and q are non-adjacent then not necessary $d(p, R_p) \neq d(q, R_p)$ then R_p is called a local edge resolving partition set and minimum cardinality of such set is called local edge partition dimension. We discussed local metric, local edge metric, metric, edge metric dimension, local partition, local edge partition, partition dimension, and edge partition dimension of the Petersen graph.

Keywords: Local resolving set, Local edge resolving set, Local metric, Local edge metric, Metric, Edge metric dimension, Local partition, Local edge partition, Partition dimension, Edge partition dimension, Petersen graph, Path graph, Cycle graph, Complete bipartite graph

MSC: 05C10, 05C50, 05C70, 05C90, 15A18, 15A90, 74E40, 74F25

1. Introduction

Graph theory is a branch of mathematics that studies the properties and systems of graphs, which include vertices (or nodes) related via edges (or links). It performs a crucial role in modeling relationships and systems in numerous fields, such as computer science, biology, chemistry, social sciences, and engineering. A graph can be directed, where edges have a particular route, or undirected, wherein connections among vertices are bidirectional. Key standards in graph concepts include paths, cycles, connectivity, and graph coloring. Specialized topics like metric size, side metric size, and partitioning have wide packages, from community safety and verbal exchange structures to molecular chemistry and computational biology. The versatility of the graph principle permits it to be an effective device for studying each theoretical and practical troubles in complicated networks [1].

Resolvability parameters in graph theory refer to certain properties that allow for the distinction between vertices in a graph based on their distances to a selected subset of vertices, known as a resolving set. The most basic of these parameters is the metric dimension, which is the minimum number of vertices in a resolving set. Every vertex in the graph can be uniquely identified by its distance to the vertices in the set. It is introduced by P. J. Slater in 1975 [2] and later on by [3]. Blumenthal had already described the metric dimension and the rotating set [4]. After this the extension of this work is done by G. Chartrand in 2000 that is partition dimension [5] Another related parameter is the edge metric dimension, which focuses on uniquely distinguishing edges rather than vertices by using similar principles. It is introduced by A Kelenc in 2018 [6]. Extensions of these concepts include the mixed metric dimension, which simultaneously considers both vertices and edges in the graph and it is also introduced by A. Kelenc in 2017 [7]. These parameters are critical in understanding the structural complexity and uniqueness of graphs, as they provide a way to quantify how well the graph

can be resolved or "mapped" by using a minimal set of references within the graph. Other parameters, Latter on the edge partition dimension introduced by I.G. Yero et. al in [8] and after this, in 2024 a novel resolvability parameter is introduced by Sikander Ali et. al [9], take a different approach by grouping vertices or edges into partitions that can still uniquely identify other elements in the graph based on distances. These resolvability parameters are key tools in the theoretical study of graphs, offering insight into their structure and distinguishability.

In this article, we discuss about local metric dimension and introduce a new parameter known as the local edge partition dimension. The local metric dimension of a graph is a variation of the classic metric dimension that focuses on resolving the neighborhoods of vertices, rather than the entire graph. Specifically, for a given vertex, the aim is to distinguish vertices within its local neighborhood based on their distances to a specific subset of vertices known as a local resolving set. A local resolving set for a vertex v is a set of vertices such that for any two distinct vertices in the neighborhood of v , their distances to the vertices in the set uniquely identify them. The local metric dimension of the graph is the minimum number of vertices required in a resolving set to locally resolve all vertices in the graph. This concept is useful in scenarios where the focus is on distinguishing vertices within localized regions of the graph rather than the entire structure. It is introduced by F. Okamoto et, al [10]. A location partition dimension of a graph G is a partition of its vertex set into subsets $P_1, P_2, P_3, \dots, P_k$ such that for any two distinct vertices u and v in the graph, the distance from u and v to each subset is distinct. This means that for any pair of distinct vertices u and v , the ordered tuple of distances from u to each subset P_i (and similarly for v) should be different [11].

The local edge metric dimension of a graph is a variation of the edge metric dimension that focuses on distinguishing edges within the local neighborhood of a vertex. In this context, the goal is to uniquely identify the edges connected to a given vertex by using a subset of vertices, called a local edge resolving set [12]. The local edge partition dimension is the extension of the local edge partition dimension. Let $R = \{R_{p1}, R_{p2}, R_{p3} \dots, R_{pi}\}$ be the set of subsets of vertices of G , and $R_{p1} \cup R_{p2} \cup R_{p3} \dots \cup R_{pi} = V(G)$, $R_{p1} \cap R_{p2} \cap R_{p3} \dots \cap R_{pi} = \emptyset$ then R is said to be partition set of G . Let R show the unique representation with every pair (adjacent) of edges of G then R is called the local edge resolving partition set and its minimum cardinality is known as the local edge partition dimension. It is introduced in this article and we apply the definition of path, cycle, and Petersen graph.

The concept of metric dimension finds applications in various fields and has been the subject of extensive research. The researchers have employed it in a wide range of applications, showcasing its versatility and importance. For instance, metric dimension has been used to determine similarities between different medications [13]. Metamaterials refer to artificially designed materials and structures [14], which is crucial in the pharmaceutical industry. Other applications include solving combinatorial optimization problems [15], facilitating robot navigation [16], addressing pharmaceutical chemistry issues [17], optimizing computer networks [18], and canonically labeling graphs [19]. The utility of metric dimension extends to solving location problems, aiding in the operation of sonar and coast guard Loran systems [20], facilitating image processing, and solving complex weighing problems [21]. Additionally, it has found applications in coding and decoding strategies for games like Mastermind [22]. The use of metric dimension in the development of city [23].

During the past 40 years, there has been a lot of research done on the metric dimension of various types of graphs. For instance constant time calculation of the metric dimension of the join of path graphs discussed in [24], the double resolvability parameters of Fosmidomycin anti-malaria drug and exchange property is studied in [25], metric dimensions of bicyclic graphs with potential applications in supply chain logistics studied in [26], double resolving set and exchange property in nanotube discussed in [27], metric dimensions of bicyclic graphs discussed by [28], the double edge resolving set and exchange property for nanosheet discussed in [29]. The concept of the metric dimension is applied to resolve a wide range of challenging issues due to its diversity. We mention [30–32] for the resolvability criteria of various chemical structures. Structural analysis of octagonal nanotubes via double edge-resolving partitions [33]. Mixed metric dimension and exchange property of hexagonal nano-network discussed in [34] and fault-tolerant basis of generalized fat trees and perfect binary tree derived architectures [35], fault-tolerance and unique identification of vertices and edges in a graph, the fault-tolerant mixed metric dimension studied by [36], and Redefining fractal cubic networks and determining their metric dimension and fault-tolerant metric dimension discussed in [37] and in other field see [38]. Fault-tolerant metric dimensions of leafless cacti graphs with application in supply chain management [39]. The concept of metric dimension has been applied to solve challenging problems in various

contexts. The resolving set of silicate stars was explored in a publication by George [40]. Cellulose networks' upper bounds for metric dimension were determined in research conducted by [41]. The discussion on the graph of convex polytopes was presented in a work by [42].

Definition 1. Let $N(v)$ be the neighborhood of a vertex $v \in V$, i.e., the set of all vertices adjacent to v , formally:

$$N(v) = \{u \in V : uv \in E\}.$$

A set of vertices $S \subseteq V$ is called a **local resolving set** for v if for any two distinct vertices $u, w \in N(v)$, there exists a vertex $s \in S$ such that the distance from u to s , denoted $d(u, s)$, is different from the distance from w to s [10], i.e.,

$$d(u, s) \neq d(w, s).$$

Definition 2. The **local partition dimension** [11] of a graph $G = (V, E)$ is defined as follows:

Let $N(v)$ be the **open neighborhood** of a vertex $v \in V$, defined as the set of vertices adjacent to v :

$$N(v) = \{u \in V : uv \in E\}.$$

A partition $\Pi = \{P_1, P_2, \dots, P_k\}$ of the vertex set V is called a **local resolving partition** for a vertex $v \in V$ if for any two distinct vertices $u, w \in N(v)$, the distance between u and each partition set P_i differs from the distance between w and each partition set P_i . That is, for each pair of distinct vertices $u, w \in N(v)$, there exists a partition $P_i \in \Pi$ such that:

$$d(u, P_i) \neq d(w, P_i),$$

where $d(x, P_i) = \min\{d(x, z) : z \in P_i\}$ is the minimum distance from vertex x to any vertex in P_i .

The **local partition dimension** of the graph G , denoted as $LPD(G)$, is the minimum number of partitions required to form a local resolving partition for each vertex $v \in V$. Formally:

$$LPD(G) = \min \{k : \Pi = \{P_1, P_2, \dots, P_k\} \text{ is a local resolving partition for each } v \in V\}.$$

Definition 3. The **local edge metric dimension** [43] of a graph $G = (V, E)$ is defined as follows:

Let $N(v)$ be the **open neighborhood** of a vertex $v \in V$, defined as the set of vertices adjacent to v :

$$N(v) = \{u \in V : uv \in E\}.$$

A set of vertices $S \subseteq V$ is called a **local edge resolving set** for v if for any two distinct edges uv, vw incident to v , there exists a vertex $s \in S$ such that the distance from the endpoints of the edges to s differs. That is, for the edges uv and vw , we have:

$$d(u, s) \neq d(v, s) \quad \text{and} \quad d(v, s) \neq d(w, s).$$

The **local edge metric dimension** of the graph G , denoted as $LEMD(G)$, is the minimum size of a local edge resolving set that can distinguish all incident edges at each vertex $v \in V$. Formally:

$$LEMD(G) = \min \{|S| : S \subseteq V \text{ is a local edge resolving set for each vertex } v \in V\}.$$

Definition 4. The **local edge partition dimension** of a graph $G = (V, E)$ is defined as follows:

Let $N(v)$ be the neighborhood of a vertex $v \in V$, defined as the set of vertices adjacent to v :

$$N(v) = \{u \in V : uv \in E\}.$$

A partition $\Pi = \{P_1, P_2, \dots, P_k\}$ of the vertex set V is called a **local resolving partition** for a vertex $v \in V$ if for any two distinct edges $u, w \in N(E)$, the distance between u and each partition set P_i differs from the distance between w and each partition set P_i . That is, for each pair of distinct vertices $u, w \in N(v)$, there exists a partition $P_i \in \Pi$ such that:

$$d(u, P_i) \neq d(w, P_i),$$

where $d(x, P_i) = \min\{d(x, z) : z \in P_i\}$ is the minimum distance from edge x to any vertex in P_i .

The **local edge partition dimension** of the graph G , denoted as $LEPD(G)$, is the minimum number of partitions required to form a local edge resolving partition for each vertex $v \in V$. Formally:

$$LEPD(G) = \min \{k : \Pi = \{P_1, P_2, \dots, P_k\} \text{ is a local edge resolving partition for each } v \in V\}.$$

2. Main Results

In this section, we discussed novel resolvability parameters, the local edge partition dimension of the path, cycle, and Petersen graph, the local metric dimension of the cycle for even and odd lengths, and other resolvability parameters of the Petersen graph.



Figure 1. Path graph

Theorem 5. *The local edge partition dimension of a graph G is 2 if and only if $G = P_n$.*

Proof. We begin by noting that $lepd(P_n) = 2$, which can be demonstrated by dividing the vertices of P_n into two groups: one set containing a single leaf, and the other set containing the remaining vertices. Now, let's consider a graph G where $lepd(G) = 2$, and assume V_1, V_2 forms an local edge-resolving partition set. If there exist two edges, $m = pq$ and $n = st$, such that without loss of generality, $p, s \in V_1$ and $q, t \in V_2$, then the distances $d(m, V_1) = d(m, V_2) = d(n, V_1) = d(n, V_2)$ would all be zero, which is a contradiction. Therefore, since G is a connected graph, there must be exactly one edge, say $h = q_1q_2$, where $q_1 \in V_1$ and $q_2 \in V_2$.

Next, assume the maximum degree of G is at least 3. Let w be the closest vertex to q_1 with the highest degree, and suppose $u \in V_1$. Consider two vertices adjacent to u , labeled u_1 and u'_1 , that also belong to V_1 and are not part of the shortest path between V_1 and u . Under these conditions, the edges uu_1 and uu'_1 would result in $d(uu_1, V_2) = d(uu_1, V_1) + 1$ and $d(uu'_1, V_2) = d(uu'_1, V_1) + 1$. This creates a contradiction, implying that the maximum degree of G must be no greater than 2. If G were a cycle, any partition of its vertices into two sets would generate at least two edges with one endpoint in each set, which is not feasible. Consequently, G must be a path, thus completing the proof. \square

Cycle Graph

A cycle graph is also a simple graph in graph theory it is denoted by C_n where $n \geq 3$. It is defined as, a closed walk in which all the vertices are different except $(a_n = a_1)$ is called a cycle graph. The total number of vertices is n and the total number of edges is also n and the vertex and edge set is defined as

$$V(C_n) = \{a_{i+1}; 1 \leq i \leq n\}$$

$$E(C_n) = \{a_i a_{i+1}, a_n a_1; 1 \leq i \leq n, \}$$

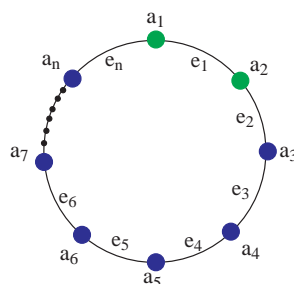


Figure 2. Cycle graph

Theorem 6. *The local edge partition dimension of cycle graph C_n is 3, where $n \geq 3$.*

Proof. Assume that a subset of the vertices of C_n is $R = \{R_{p1}, R_{p2}, R_{p3}\}$ where $R_{p1} = \{a_1\}$, $R_{p2} = \{a_2\}$, and $R_{p3} = V(C_n) \setminus \{a_1, a_2\}$. We shall demonstrate that R is a local edge partition resolving set to demonstrate that $lep d(C_n) \leq 3$. The distinct representations of all the edges of C_n in Table 1 are shown below.

Table 1. Edges representation for local edge partition dimension

The representation of $R = \{a_1, a_2, a_n\}$ for $n \geq 3$ present in Table 1

$r(\cdot R)$	(R_{p1}, R_{p2}, R_{p3})
e_1	(0, 0, 1)
e_2	(1, 0, 0)
e_3	(2, 1, 0)
e_4	(3, 2, 0)
e_5	(4, 3, 0)
\cdot	\cdot
\cdot	\cdot
\cdot	\cdot
e_{n-1}	(1, 2, 0)
e_n	(0, 1, 0)

From Table 1 it is clear that the R has unique representation with all edges of C_n so R is local edge partition resolving set of cardinality 3.

2.1. Generalize Formulas of distance of Edge of Cycle Graph

The generalized formulas for all edges of the cycle graph show that the local edge partition dimensions are 3 because all adjacent edges are distinct. Let η be the distance’s symbol. Let $\eta(e_i, R_{p1}) = w'_1$, $\eta(e_i, R_{p2}) = w'_2$, $\eta(e_i, R_{p3}) = w'_3$, and $r(a_i | R) = (w'_1, w'_2, w'_3)$

$$w'_1 = \begin{cases} i - 1 & \text{for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\ n - i & \text{for } \lfloor \frac{n}{2} \rfloor < i \leq n, \end{cases}$$

$$w'_2 = \begin{cases} 0 & \text{for } i = 1, \\ i - 2 & \text{for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1, \\ n - i + 1 & \text{for } \lfloor \frac{n}{2} \rfloor + 1 < i \leq n, \end{cases}$$

$$w'_3 = \begin{cases} 1 & \text{for } e_1 \\ 0 & \text{otherwise} \end{cases}$$

The minimum cardinality of R is the local edge partition dimension so the local edge partition dimension of C_n is 3. From all the above discussion it is clear that the metric dimension of C_n is less or equal to 3. Now we are going to prove that the local edge partition dimension of C_n is not less than 3. The local edge partition dimension 2 is possible only in the case of a path graph see Theorem 5. Hence the the local edge partition dimension of C_n is 3. □

Theorem 7. *The local metric dimension of cycle graph C_n is 2 where n is odd.*

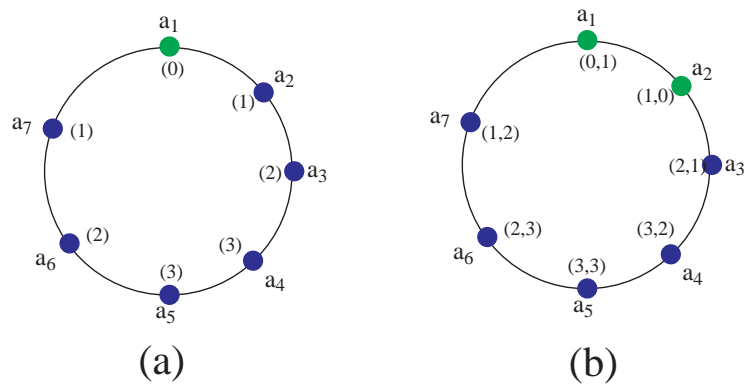


Figure 3. Cycles of Odd length

Proof. Let $R = \{a_1\}$ be the subset of the vertices of C_n in Figure 3(a). We note that the distance of a_4 and a_5 are the same and both are adjacent so R is not a local resolving set. We want to prove that the R is a local resolving so R must have a unique representation with all adjacent vertices of C_n . In general we can say that if we take the cardinality of R is one then the representation of two adjacent vertices $a_{\frac{n+1}{2}}$ and $a_{\frac{n+3}{2}}$ remain same. In Figure 3(b) we note that $a_{\frac{n+1}{2}}$ and $a_{\frac{n+3}{2}}$ have unique representation with $R = \{a_1, a_2\}$ and all other adjacent vertices of odd cycle when R has cardinality 2. So R is the local resolving set and the minimum cardinality of the local resolving set is the local metric dimension. Hence the local metric dimension of cycle graph C_n is two when n is odd. Let η be the distance's symbol. Let $\eta(a_i, a_1) = w_1, \eta(a_i, a_2) = w_2, \eta(a_i, a_n) = w_3,$ and $r(a_i | R) = (w_1, w_2, w_3).$

$$w_1 = \begin{cases} i - 1 & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ n - i + 1 & \text{for } \left\lceil \frac{n}{2} \right\rceil < i \leq n, \end{cases}$$

$$w_2 = \begin{cases} 1 & \text{for } i = 1, \\ i - 2 & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1, \\ n - i + 2 & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 < i \leq n, \end{cases}$$

□

Theorem 8. The local metric dimension of a complete bipartite graph also is 1.

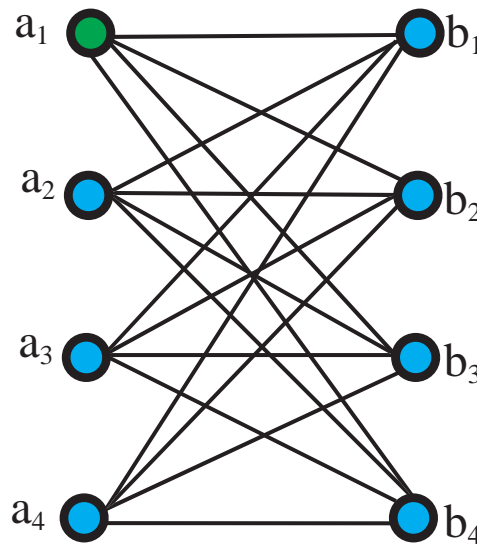


Figure 4. Bipartite graph

Proof. Suppose $R = \{a_1\}$ be the subset of the vertices of $K_{m,n}$. We want to prove that the R is a local resolving set so R must have a unique representation with all adjacent vertices of $K_{m,n}$. In Figure 4 we note that all adjacent vertices have unique distance from a_1 because all a_i vertices have distance 2 with a_1 but all are non-adjacent. All b_i have 1 distance from a_1 because all are attached with a_1 but these are also non-adjacent and $R = \{a_1\}$ so R has unique representation. Hence R is the local resolving set and the minimum cardinality of the local resolving set is the local metric dimension. Hence the local metric dimension of the bipartite graph is one. \square

Theorem 9. The local metric dimension of the Petersen graph is 3.

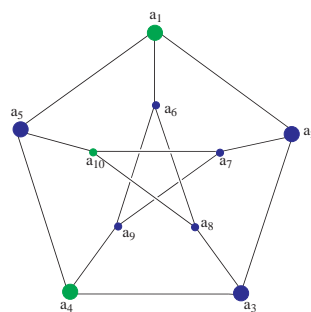


Figure 5. Petersen graph for LMD

Proof. We want to prove that $R_l = \{a_1, a_4, a_{10}\}$ is the local resolving set of minimum cardinality. Let R_l be the subset of the vertices of the Petersen graph and the representation of all vertices of the graph is present in Table 2.

Table 2. Representation of adjacent vertices w.r.t R_l of Figure 5

	$R_l = \{a_1, a_4, a_{10}\}$				
vertex	a_1	a_2	a_3	a_4	a_5
$r(\cdot R_l)$	(0,2,2)	(1,2,2)	(2,2,1)	(2,2,0)	(1,1,1)
vertex	a_6	a_7	a_8	a_9	a_{10}
$r(\cdot R_l)$	(1,2,2)	(2,1,2)	(2,1,2)	(2,2,1)	(2,0,2)

Table 2 is the avoidance that the representation of adjacent vertices is unique so R_l is a local resolving set of cardinality 3 and R_l is the set of minimum cardinality that show unique coding for each vertex. Hence the local metric dimension of the Petersen graph is 3. \square

Theorem 10. *The local edge metric dimension of the Petersen graph is 4.*

Proof. Let $R_{le} = \{a_1, a_4, a_9, a_{10}\}$ be the subset of the vertices of Petersen graph and we want to show that it is an edge resolving set. The representation of all edges of the graph is present in Table 5.

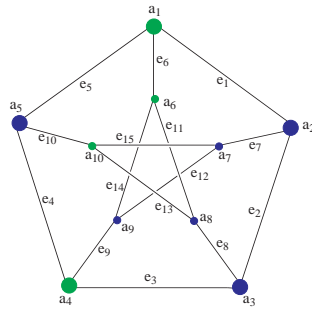


Figure 6. Petersen graph for LEMD

Table 3. Representation of adjacent edges w.r.t R_{le} of Figure 6

$R_{le} = \{a_1, a_4, a_9, a_{10}\}$					
Edges	e_1	e_2	e_3	e_4	e_5
$r(\cdot R_{le})$	(0,2,1,2)	(1,1,2,2)	(2,0,2,2)	(1,0,2,1)	(0,1,1,1)
Edges	e_6	e_7	e_8	e_9	e_{10}
$r(\cdot R_{le})$	(0,2,0,2)	(1,2,2,1)	(2,1,1,1)	(2,0,1,2)	(1,1,2,0)
Edges	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
$r(\cdot R_{le})$	(1,2,0,1)	(1,2,0,1)	(2,2,1,0)	(1,1,0,2)	(2,2,2,0)

From 3 it is clear that the representation of adjacent edges is unique so R_{le} is the local edge resolving set of cardinality 4 and R_{le} is the set of minimum cardinality that show unique coding for adjacent edges. Hence the local edge metric dimension of the Petersen graph is 4. \square

Theorem 11. *The metric dimension of the Petersen graph is 4.*

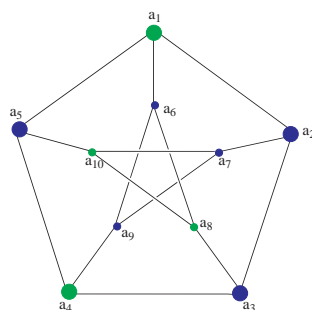


Figure 7. Petersen graph for MD

Proof. Suppose $R = \{a_1, a_4, a_8, a_{10}\}$ be the subset of the vertices of Petersen graph and we want to show that it is a resolving set. The representation of all vertices of the graph is present in Table 4.

Table 4. Representation of vertices w.r.t R of Figure 7

$$R = \{a_1, a_4, a_{10}, a_8\}$$

vertex	a_1	a_2	a_3	a_4	a_5
$r(\cdot R)$	(0,2,2,2)	(1,2,2,2)	(2,1,1,2)	(2,0,2,2)	(1,1,2,2)
vertex	a_6	a_7	a_8	a_9	a_{10}
$r(\cdot R)$	(1,2,1,2)	(2,2,2,1)	(2,2,1,0)	(2,1,2,2)	(2,2,0,1)

From Table 4 it is clear that the representation is unique so R is resolving set of cardinality 4 and R is the set of minimum cardinality that shows unique coding for each vertex. Hence the metric dimension of the Petersen graph is 4. □

Theorem 12. The edge metric dimension of the Petersen graph is 5.

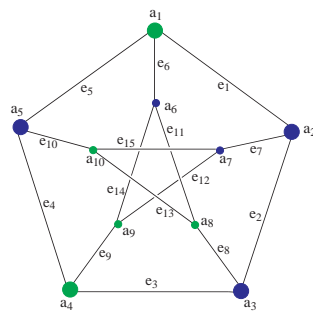


Figure 8. Petersen graph for EMD

Proof. Let $R_e = \{a_1, a_4, a_8, a_9, a_{10}\}$ be the subset of the vertices of Petersen graph and we want to show that it is an edge resolving set. The representation of all edges of the graph is present in Table 5.

Table 5. Representation of edges w.r.t R_e of Figure 8

$$R_e = \{a_1, a_4, a_8, a_9, a_{10}\}$$

Edges	e_1	e_2	e_3	e_4	e_5
$r(\cdot R_e)$	(0,2,2,1,2)	(1,1,1,2,2)	(2,0,1,1,2)	(1,0,2,1,1)	(0,1,2,2,1)
Edges	e_6	e_7	e_8	e_9	e_{10}
$r(\cdot R_e)$	(0,2,1,1,2)	(1,2,2,2,1)	(2,1,0,2,1)	(2,0,2,1,2)	(1,1,1,2,0)
Edges	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
$r(\cdot R_e)$	(1,2,0,1,1)	(2,1,2,0,1)	(2,2,0,2,0)	(1,1,1,0,2)	(2,2,1,2,0)

Table 5 shows that the representation of edges is unique so R_e is edge resolving set of cardinality 5 and R_e is the set of minimum cardinality that show unique coding for each edge. Hence the edge metric dimension of the Petersen graph is 5. □

Theorem 13. The local partition dimension of the Petersen graph is 4.

Proof. $R_{p1} = a_1, R_{p2} = a_4, R_{p3} = a_{10}, R_{p4} = \{a_2, a_3, a_5, a_6, a_7, a_8, a_9\}$ and $R_p = \{R_{p1}, R_{p2}, R_{p3}, R_{p4}\}$. R_p is the set of the subset of the vertices of the Petersen graph and we want to show that it is a local partition resolving set. The representations of all adjacent vertices of the graph are present in Table 6.

Table 6. Representation of adjacent vertices w.r.t R_l of Figure 5

$$R_{lp} = \{R_{p1}, R_{p2}, R_{p3}, R_{p4}\}$$

<i>vertex</i>	a_1	a_2	a_3	a_4	a_5
$r(\cdot R_{lp})$	(0,2,2,1)	(1,2,2,0)	(2,2,0)	(2,2,0,1)	(1,1,1,0)
<i>vertex</i>	a_6	a_7	a_8	a_9	a_{10}
$r(\cdot R_{lp})$	(1,2,2,0)	(2,1,2,0)	(2,1,2,0)	(2,2,1,0)	(2,0,2,1)

Table 6 shows that the representation of adjacent vertices is unique so R_p is the local partition resolving set of cardinality 4 and R_p is the set of minimum cardinality that show unique coding for each adjacent vertices. Hence the partition metric dimension of the Petersen graph is 4. \square

Theorem 14. *The local edge partition dimension of the Petersen graph is 5.*

Proof. $R_{p1} = a_1, R_{p2} = a_4, R_{p3} = a_9, R_{p4} = a_{10}, R_{p5} = \{a_2, a_3, a_5, a_6, a_7, a_8\}$ and $R_{lep} = \{R_{p1}, R_{p2}, R_{p3}, R_{p4}, R_{p5}\}$. R_p is the set of the subset of the vertices of the Petersen graph and we want to show that it is a local partition edge resolving set. The representations of all edges of the graph are present in Table 7.

Table 7. Representation of adjacent edges w.r.t R_{le} of Figure 6

$$R_{lep} = \{R_{p1}, R_{p2}, R_{p3}, R_{p4}, R_{p5}\}$$

<i>Edges</i>	e_1	e_2	e_3	e_4	e_5
$r(\cdot R_{lep})$	(0,2,1,2,0)	(1,1,2,2,0)	(2,0,2,2,0)	(1,0,2,1,0)	(0,1,1,1,0)
<i>Edges</i>	e_6	e_7	e_8	e_9	e_{10}
$r(\cdot R_{lep})$	(0,2,0,2,1)	(1,2,2,1,0)	(2,1,1,1,0)	(2,0,1,2,0)	(1,1,2,0)
<i>Edes</i>	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
$r(\cdot R_{lep})$	(1,2,0,1,0)	(1,2,0,1,0)	(2,2,1,0,0)	(1,1,0,2,0)	(2,2,2,0,0)

From 7 it is clear that the representation of adjacent edges is unique so R_{lep} is the local edge resolving set of cardinality 5 and R_{lep} is the set of minimum cardinality that show unique coding for adjacent edges. Hence the local edge metric dimension of the Petersen graph is 5. \square

Theorem 15. *The partition dimension of the Petersen graph is 5.*

Proof. $R_{p1} = a_1, R_{p2} = a_4, R_{p3} = a_8, R_{p4} = a_{10}, R_{p5} = \{a_2, a_3, a_5, a_7, a_8, a_9\}$ and $R_p = \{R_{p1}, R_{p2}, R_{p3}, R_{p4}, R_{p5}\}$. R_p is the set of a subset of the vertices of the Petersen graph and we want to show that it is a partition-resolving set. The representations of all vertices of the graph are present in Table 8.

Table 8. Representation of vertices w.r.t R of Figure 7

$$R_p = \{R_{p1}, R_{p2}, R_{p3}, R_{p4}, R_{p5}\}$$

<i>vertex</i>	a_1	a_2	a_3	a_4	a_5
$r(\cdot R)$	(0,2,2,2,1)	(1,2,2,2,0)	(2,1,1,2,0)	(2,0,2,2,1)	(1,1,2,2,0)
<i>vertex</i>	a_6	a_7	a_8	a_9	a_{10}
$r(\cdot R)$	(1,2,1,2,1)	(2,2,2,1,0)	(2,2,1,0,0)	(2,1,2,2,9)	(2,2,0,1,1)

From Table 8 it is clear that the representation is unique so R_p is the partition resolving set of cardinality 5 and R_p is the set of minimum cardinality that show unique coding for each vertex. Hence the partition dimension of the Petersen graph is 5. \square

Theorem 16. *The edge partition dimension of the Petersen graph is 6.*

Proof. suppose $R_{p1} = a_1, R_{p2} = a_4, R_{p3} = a_8, R_{p4} = a_9, R_{p5} = a_{10}, R_{p6} = \{a_2, a_3, a_5, a_7, a_8\}$ and $R_{pe} = \{R_{p1}, R_{p2}, R_{p3}, R_{p4}, R_{p5}, R_{p6}\}$. R_{pe} be the set of subset of the vertices of Petersen graph and we want to show that it is a partition edge resolving set. The representations of all edges of the graph are present in Table 9.

Table 9. Representation of edges w.r.t R_{pe} of Figure 8

$$R_{pe} = \{R_{p1}, R_{p2}, R_{p3}, R_{p4}, R_{p5}, R_{p6}\}$$

Edges	e_1	e_2	e_3	e_4	e_5
$r(\cdot R_{pe})$	(0,2,2,1,2,0)	(1,1,1,2,2,0)	(2,0,1,1,2)	(1,0,2,1,1,0)	(0,1,2,2,1)
Edges	e_6	e_7	e_8	e_9	e_{10}
$r(\cdot R_{pe})$	(0,2,1,1,2,0)	(1,2,2,2,1)	(2,1,0,2,1,0)	(2,0,2,1,2,0)	(1,1,1,2,0)
Edges	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
$r(\cdot R_{pe})$	(1,2,0,1,1,0)	(2,1,2,0,1)	(2,2,0,2,0,0)	(1,1,1,0,2,0)	(2,2,1,2,0)

Table 9 shows that the representation of edges is unique so R_{pe} is edge resolving set of cardinality 5 and R_{pe} is the set of minimum cardinality that show unique coding for each edge. Hence the edge metric dimension of the Petersen graph is 5. □

Data Availability

All the data supporting the results are included in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

3. Conclusion

This article presents a comprehensive analysis of the Petersen graph through the lens of various resolvability parameters, including the novel local edge partition dimension. By calculating and comparing dimensions such as local metric, local edge metric, metric, edge metric, local partition, local edge partition, partition, and edge partition dimensions, we identified distinct structural characteristics that highlight the versatility of the Petersen graph as a model for complex networks. The inclusion of the local edge partition dimension offers a refined perspective on graph resolvability, contributing to a more nuanced understanding of connectivity and distance-based measures within graph theory. These insights not only enhance the theoretical foundation of resolvability parameters.

Table 10. Representation of adjacent edges w.r.t R_{le} of Figure 6

Resolvability parameters	Graphs	Values
Local metric dimension	Petersen	3
Local edge metric dimension	Petersen	4
Metric dimension	Petersen	4
Edge metric dimension	Petersen	5
Local partition dimension	Petersen	4
Local edge partition dimension	Petersen	5
Partition dimension	Petersen	5
Edge partition dimension	Petersen	6
Local partition dimension	Petersen	4
Local metric dimension	cycle (even order)	1
Local metric dimension	cycle (odd order)	2
Local metric dimension	complete bipartite	1
Local edge partition dimension	cycle	3

3.1. Open problem

Is it feasible to define the mixed local partition dimension for a simple, undirected, connected graph?

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References

- [1] Bondy, J. A., & Murty, U. S. R. (2008). *Graph theory*. Springer. <https://link.springer.com/book/10.1007/978-1-84628-970-5>
- [2] Slater, P. J. (1975). Leaves of trees. In *Proceedings of the 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Congressus Numerantium* (Vol. 14, pp. 549–559).
- [3] Harary, F., & Melter, R. A. (1976). On the metric dimension of graphs. *Ars Combinatoria*, 2, 191–195.
- [4] Blumenthal, L. M. (1953). *Theory and applications of distance geometry*. Clarendon Press.
- [5] Chartrand, G., Salehi, E., & Zhang, P. (2000). The partition dimension of a graph. *Aequationes Mathematicae*, 59, 45–54. <https://doi.org/10.1007/PL00000127>
- [6] Kelenc, A., Tratnik, N., & Yero, I. G. (2016). Uniquely identifying the edges of a graph: The edge metric dimension. *Discrete Applied Mathematics*. <https://doi.org/10.1016/j.dam.2018.05.052>
- [7] Kelenc, A., Kuziak, D., Taranenko, A., & Yero, I. G. (2017). Mixed metric dimension of graphs. *Applied Mathematics and Computation*, 314, 429–438. <https://doi.org/10.1016/j.amc.2017.07.027>
- [8] Kuziak, D., Maritz, E., Vetrík, T., & Yero, I. G. (2023). The edge partition dimension of graphs. *Discrete Mathematics Letters*, 12, 34–39. <https://doi.org/10.47443/dml.2023.010>
- [9] Ali, S., Azeem, M., Zahid, M. A., Usman, M., & Pal, M. (2024). Novel resolvability parameter of some well-known graphs and exchange properties with applications. *Journal of Applied Mathematics and Computation*. <https://doi.org/10.1007/s12190-024-02137-w>
- [10] Okamoto, F., Phinezy, B., & Zhang, P. (2010). The local metric dimension of a graph. *Mathematica Bohemica*, 135(3), 239–255. <https://doi.org/10.21136/MB.2010.140702>
- [11] Alfarisi, R., Kristiana, A. I., & Dafik, D. (2021). The local partition dimension of graphs. *Discrete Mathematics, Algorithms and Applications*, 3. <https://doi.org/10.1142/S1793830921500282>
- [12] Adawiyah, R., Dafik, D., Alfarisi, R., Prihandini, R. M., Agustin, I. H., & Venkatachalam, M. (2020). The local edge metric dimension of a graph. *Journal of Physics: Conference Series*, 1543. <https://doi.org/10.1088/1742-6596/1543/1/012009>
- [13] Johnson, M. A. (1993). Structure-activity maps for visualizing the graph variables arising in drug design. *Journal of Biopharmaceutical Statistics*, 3, 203–236.
- [14] Wang, W., Jin, Y., Mu, Y., Zhang, M., & Du, J. (2023). A novel tubular structure with negative Poisson's ratio based on gyroid-type triply periodic minimal surfaces. *Virtual and Physical Prototyping*, 18. <https://doi.org/10.1080/17452759.2023.2203701>
- [15] Ahmad, A., Koam, A. N. A., Siddiqui, M. H. F., & Azeem, M. (2021). Resolvability of the starphene structure and applications in electronics. *Ain Shams Engineering Journal*. <https://doi.org/10.1016/j.asej.2021.09.014>
- [16] Khuller, S., Raghavachari, B., & Rosenfeld, A. (1996). Landmarks in graphs. *Discrete Applied Mathematics*, 70, 217–229.
- [17] Chartrand, G., Eroh, L., Johnson, M. A. O., & Ortrud, R. (2000). Resolvability in graphs and the metric dimension of a graph. *Discrete Applied Mathematics*, 105, 99–113.
- [18] Manuel, P., Bharati, R., Rajasingh, I., & Monica, M. C. (2008). On minimum metric dimension of honeycomb networks. *Journal of Discrete Algorithms*, 6, 20–27.
- [19] Piperno, A. (2008). Search space contraction in canonical labeling of graphs. *arXiv preprint*. <https://arxiv.org/abs/0804.4881>
- [20] Slater, P. J. (1975). Leaves of trees. In *Proceedings of the 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Congressus Numerantium* (Vol. 14, pp. 549–559).
- [21] Söderberg, S., & Shapiro, H. (1963). A combinatorial detection problem. *American Mathematical Monthly*, 70, 1066–1070.
- [22] Chvatal, V. (1983). Mastermind. *Combinatorica*, 3, 125–129.
- [23] Ali, S., & Jamil, M. K. (2024). Exchange property in double edge resolving partition sets and its use in city development. *Spectrum of Decision Making Applications*, 9, 14. <https://doi.org/10.31181/sdmap1120246>
- [24] Zhang, C., Haidar, G., Khan, M. U. I., Yousafzai, F., Hila, K., & Khan, A. U. I. (2023). Constant time calculation of the metric dimension of the join of path graphs. *Symmetry*, 15(3), 708. <https://doi.org/10.3390/sym15030708>
- [25] Ismail, R., Ali, S., Azeem, M., & Zahid, M. A. (2024). Double resolvability parameters of Fosmidomycin anti-malaria drug and exchange property. *Heliyon*. <https://doi.org/10.1016/j.heliyon.2024.e33211>
- [26] Wang, M., Haidar, G., Yousafzai, F., Khan, M. U. I., Sikandar, W., & Khan, A. U. I. (2024). Metric dimensions of bicyclic graphs with potential applications in supply chain logistics *arXiv preprint*. <https://arxiv.org/abs/2409.02947>

- [27] Koam, A. N. A., Ali, S., Ahmad, A., Azeem, M., & Jamil, M. K. (2023). Double resolving set and exchange property in the nanotube. *AIMS Mathematics*, 8(9), 20305–20323. <https://doi.org/10.3934/math.20231035>
- [28] Khan, A., Haidar, G., Abbas, N., Khan, M. U. I., Niazi, A. U. K., & Khan, A. U. I. (2023). Metric dimensions of bicyclic graphs. *Mathematics*, 11(4), 869. <https://doi.org/10.3390/math11040869>
- [29] Koam, A. N. A., Ahmad, A., Ali, S., Jamil, M. K., & Azeem, M. (2024). Double edge resolving set and exchange property for nanosheet. *Heliyon Open Access*, 5(10), e26992. <https://doi.org/10.1016/j.heliyon.2024.e26992>
- [30] Hussain, Z., Munir, M., Choudhary, M., & Kang, S. M. (2018). Computing metric dimension and metric basis of the 2D lattice of alpha-boron nanotubes. *Symmetry*, 10(8), 300. <https://doi.org/10.3390/sym10080300>
- [31] Krishnan, S., & Rajan, B. (2016). Fault-tolerant resolvability of specific crystal structures. *Applied Mathematics*, 7, 599–604.
- [32] Ahmad, A., Bača, M., & Sultan, S. (2020). Computing the metric dimension of kayak paddle graph and cycles with chord. *Proyecciones Journal of Mathematics*, 39(2), 287–300.
- [33] Alali, A. S., Ali, S., & Jamil, M. K. (2024). Structural analysis of octagonal nanotubes via double edge-resolving partitions. *Processes*, 12(9). <https://doi.org/10.3390/pr12091920>
- [34] Liu, P., Ali, S., Azeem, M., Jamil, M. K., Zahid, M. A., & Almohsen, B. (2024). Mixed metric dimension and exchange property of hexagonal nano-network. *Scientific Reports*, 14(1), 26536. <https://doi.org/10.1038/s41598-024-77697-9>
- [35] Prabhu, S., Manimozhi, V., Davoodi, A., et al. (2024). Fault-tolerant basis of generalized fat trees and perfect binary tree-derived architectures. *Journal of Supercomputing*. <https://doi.org/10.1007/s11227-024-06053-5>
- [36] Khan, A., Ali, S., Hayat, S., Azeem, M., Zhong, Y., & Zahid, M. A. (2025). Fault-tolerance and unique identification of vertices and edges in a graph: The fault-tolerant mixed metric dimension. *Journal of Parallel and Distributed Computing*, 197, 105024. <https://doi.org/10.1016/j.jpdc.2024.105024>
- [37] Arulperumjothi, M., Klavžar, S., & Prabhu, S. (2023). Redefining fractal cubic networks and determining their metric dimension and fault-tolerant metric dimension. *Applied Mathematics and Computation*, 452, 128037. <https://doi.org/10.1016/j.amc.2023.128037>
- [38] Liu, P., Asim, M. H., Ali, S., Azeem, M., & Almohsen, B. (2024). Utilizing lexicographic max product of picture fuzzy graph in human trafficking. *Ain Shams Engineering Journal*, 15(11), Article 103009. <https://doi.org/10.1016/j.asej.2024.103009>
- [39] Asif, T., Haidar, G., Yousafzai, F., Khan, M. U. I., Khan, Q., & Fatima, R. (2024). Fault-tolerant metric dimensions of leafless cacti graphs with application in supply chain management. *arXiv preprint*. <https://arxiv.org/abs/2409.12199>
- [40] Simonraj, F., & George, A. (2015). On the metric dimension of silicate stars. *ARPN Journal of Engineering and Applied Science*, 5, 2187–2192.
- [41] Imran, S., Siddique, M. K., & Hussain, M. (2019). Computing the upper bounds for metric dimension of cellulose network. *Applied Mathematics*, 19, 585–605.
- [42] Ahsan, M., Zahid, Z., Zafar, S., Rafiq, A., Sindhu, M. S., & Umar, M. (2020). Computing the metric dimension of convex polytopes related graphs. *Mathematics and Computer Science*, 22, 174–188.
- [43] Amini, F., Henning, M. A., & Tavakoli, M. (2024). Local edge metric dimensions via corona products and integer linear programming. *Computational and Applied Mathematics*, 358. <https://doi.org/10.1007/s40314-024-02879-0>



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