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$(-1,1)$-incidence matrix of a graph

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Abstract: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set and $E(G) = \{e_1, e_2, \ldots, e_m\}$ be the edge set of a graph G. The Seidel adjacency matrix of a graph G is defined as $S(G) = [s_{ij}]$ of order $n \times n$, in which $s_{ij} = -1$ if v_i is adjacent to v_j, $s_{ij} = 1$ if v_i is not adjacent to v_j and $s_{ii} = 0$. We introduce here the $(-1,1)$-incidence matrix of G as $B_S(G) = [c_{ij}]$ of order $n \times m$, in which $c_{ij} = -1$ if v_i is incident to e_j and $c_{ij} = 1$ if v_i is not incident to e_j. Further we explore properties of $B_S(G)$ and of its transpose.

Keywords: adjacency matrix, incidence matrix, Seidel adjacency matrix

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1. Introduction

Graph theory is utilized to describe a variety of real world phenomena such as to study the properties of molecules [1–3], social networks [4], nano-networks [5,6], ladder networks [7] etc. The matrices of graphs are used to study the spectral and structural properties of graphs [8,9]. There are several matrices of graphs exists, such as adjacency matrix, incidence matrix, Seidel matrix, Laplacian matrix, distance matrix etc. The Seidel adjacency matrix of a graph has been studied in the literature [10]. The elements of the Seidel adjacency matrix are either -1 or 1 or zero. In this paper we introduce a new matrix, which we call $(-1,1)$-incidence matrix of a graph, whose elements are either -1 or 1 and explore some of its properties.

Let G be a finite, simple graph with $n \geq 2$ vertices and $m \geq 1$ edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G and $E(G) = \{e_1, e_2, \ldots, e_m\}$ be the edge set of G. The degree of a vertex v_i is the number of edges incident to it and is denoted by $d(v_i)$. The degree of an edge e_i whose end points are u and v is $d(e_i) = d(u) + d(v) - 2$. The line graph of G is a graph $L(G)$, whose vertex set has one-to-one correspondence with the edge set of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G [11]. Let I_n denotes the identity matrix of order n, $J_{p\times q}$ be the matrix of order $p \times q$, whose all elements are equal to 1 and M^T be the transpose of the matrix M.

The adjacency matrix [12] of a graph G is a matrix $A(G) = [a_{ij}]$ of order $n \times n$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

The incidence matrix [12] of a graph G is a matrix $B(G) = [b_{ij}]$ of order $n \times m$, where

$$b_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ is incident to an edge } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

The degree matrix of a graph G is a diagonal matrix $D(G) = \mathrm{diag}[d(v_1), d(v_2), \ldots, d(v_n)]$.

Lemma 1. [12] *For any graph G with n vertices and m edges,*
(i) $B(G)B(G)^T = A(G) + D(G)$ *and*
(ii) $B(G)^T B(G) = A(L(G)) + 2I_m$.

The Seidel adjacency matrix of a graph G, introduced by van Lint and Seidel [13], is a matrix $S(G) = [s_{ij}]$ of order $n \times n$, where

$$s_{ij} = \begin{cases} -1 & \text{if } v_i \text{ is adjacent to } v_j, \\ 1 & \text{if } v_i \text{ is not adjacent to } v_j, \\ 0 & \text{if } i = j. \end{cases}$$

Properties of Seidel adjacency matrix and its eigenvalues can be found in [14–17].

Recent studies on graph, including the fault-tolerant mixed metric dimension [18], Cycle-super magic labeling of polyomino linear and zig-zag chains [19], and mixed partition dimension [20], provide valuable insights into the structural properties of the structures.

2. $(-1,1)$-incidence matrix

Analogous to the Seidel adjacency matrix of a graph, we define here the $(-1,1)$-incidence matrix of G as $n \times m$ matrix $B_S(G) = [c_{ij}]$, where

$$c_{ij} = \begin{cases} -1 & \text{if the vertex } v_i \text{ is incident to the edge } e_j, \\ 1 & \text{if the vertex } v_i \text{ is not incident to the edge } e_j. \end{cases}$$

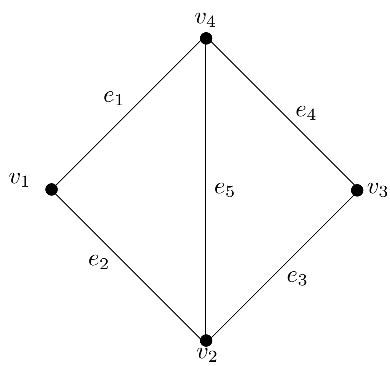

Figure 1. Graph G

The $(-1,1)$-incidence matrix of a graph given in Figure 1 is

$$B_S(G) = \begin{bmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 \end{bmatrix}.$$

Let graph G has two components G_1 and G_2. If G_i has n_i vertices and $m_i \geq 1$ edges, $i = 1,2$. Then the vertices and edges of G can be labeled in such a way that its $(-1,1)$-incidence matrix is of the form

$$\begin{bmatrix} B_S(G_1) & J_{n_1 \times m_2} \\ J_{n_2 \times m_1} & B_S(G_2) \end{bmatrix}.$$

Proposition 2. *If G is a graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges, then*

(i) -1 *appeares* $d(v_i)$ *times and* 1 *appeares* $m - d(v_i)$ *times in the* i*-th row of* $B_S(G)$.

(ii) -1 *appeares* 2 *times and* 1 *appeares* $n - 2$ *times in each column of* $B_S(G)$.

Proof. (i) Vertex v_i is incident to $d(v_i)$ edges. Hence -1 appeares $d(v_i)$ times and 1 appeares $m - d(v_i)$ times in the i-th row of $B_S(G)$.

(ii) Each edge is incident to its two end points. Hence -1 appeares 2 times and 1 appeares $n - 2$ times in each column of $B_S(G)$. □

Proposition 3. *If G is a graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges, then*

(i) $B_S(G) = J_{n\times m} - 2B(G)$.
(ii) $B_S(G)^T = J_{m\times n} - 2B(G)^T$.
(iii) *sum of the elements of i-th row in* $B_S(G)$ *is* $m - 2d(v_i)$.
(iv) *sum of the elements of each column in* $B_S(G)$ *is* $n - 4$.

Proof. (i) By the definition of $(-1,1)$-incidence matrix, we have $B_S(G) = -B(G) + J_{n\times m} - B(G) = J_{n\times m} - 2B(G)$.

(ii) By above (i), $B_S(G)^T = (J_{n\times m} - 2B(G))^T = J_{m\times n} - 2B(G)^T$.

(iii) By the first result of Proposition 2, the sum of the elements of the i-th row in $B_S(G)$ is $-d(v_i) + m - d(v_i) = m - 2d(v_i)$.

(iv) By the second result of Proposition 2, the sum of the elements of each column in $B_S(G)$ is $-2 + n - 2 = n - 4$. □

Corollary 4. *If G is a graph with n vertices* $v_1, v_2, \ldots, v_n$ *and m edges, then*
(i) *sum of the elements of i-th row in* $B_S(G)$ *is zero if* $d(v_i) = m/2$.
(ii) *sum of the elements of i-th row in* $B_S(G)$ *is positive if* $d(v_i) < m/2$.
(iii) *sum of the elements of i-th row in* $B_S(G)$ *is negative if* $d(v_i) > m/2$.

Corollary 5. *For a graph G with n vertices and m edges,*
(i) *the sum of the elements of any column in* $B_S(G)$ *is zero if* $n = 4$.
(ii) *the sum of the elements of any column in* $B_S(G)$ *is positive if* $n > 4$.
(iii) *the sum of the elements of any column in* $B_S(G)$ *is negative if* $n < 4$.

Since $d(v_i)$ is a non-negative integer, by Corollary 4(i), we note that there is no graph with odd number of edges so that the sum of the elements of at least one row in $B_S(G)$ is zero. Also by Corollary 5(i), there is no graph with n vertices ($n \neq 4$) so that the sum of the elements of any column in $B_S(G)$ is zero.

Analogous to Lemma 1, we give Proposition 6.

For this we define the matrix $D_S(G) = [d_{ij}]$ of order $n \times n$, where

$$d_{ij} = \begin{cases} d(v_i) + d(v_j) & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Proposition 6. *For a graph G with n vertices and m edges,*
(i) $B_S(G)B_S(G)^T = mJ_{n\times n} + 4A(G) - 2D_S(G)$.
(ii) $B_S(G)^T B_S(G) = (n-8)J_{m\times m} + 4A(L(G)) + 8I_m$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. By Proposition 3 we have,

$$B_S(G) = J_{n\times m} - 2B(G) \quad \text{and} \quad B_S(G)^T = J_{m\times n} - 2B(G)^T.$$

$$\begin{aligned}
\text{(i)} \quad B_S(G)B_S(G)^T &= (J_{n\times m} - 2B(G))(J_{m\times n} - 2B(G)^T) \\
&= J_{n\times m}J_{m\times n} - 2B(G)J_{m\times n} - 2J_{n\times m}B(G)^T + 4B(G)B(G)^T \\
&= mJ_{n\times n} - 2\begin{bmatrix} d(v_1) & d(v_1) & \cdots & d(v_1) \\ d(v_2) & d(v_2) & \cdots & d(v_2) \\ \vdots & & \vdots & \\ d(v_n) & d(v_n) & \cdots & d(v_n) \end{bmatrix} \\
&\quad -2\begin{bmatrix} d(v_1) & d(v_2) & \cdots & d(v_n) \\ d(v_1) & d(v_2) & \cdots & d(v_n) \\ \vdots & & \vdots & \\ d(v_1) & d(v_2) & \cdots & d(v_n) \end{bmatrix} + 4(A(G) + D(G))
\end{aligned}$$

$$
\begin{aligned}
&= mJ_{n\times n} + 4A(G) - 2\begin{bmatrix} 0 & d(v_1)+d(v_2) & \cdots & d(v_1)+d(v_n) \\ d(v_2)+d(v_1) & 0 & \cdots & d(v_2)+d(v_n) \\ \vdots & & \vdots & \\ d(v_n)+d(v_1) & d(v_n)+d(v_2) & \cdots & 0 \end{bmatrix} \\
&= mJ_{n\times n} + 4A(G) - 2D_S(G).
\end{aligned}
$$

(ii)
$$
\begin{aligned}
B_S(G)^T B_S(G) &= (J_{m\times n} - 2B(G)^T)(J_{n\times m} - 2B(G)) \\
&= J_{m\times n}J_{n\times m} - 2B(G)^T J_{n\times m} - 2J_{m\times n}B(G) + 4B(G)^T B(G) \\
&= nJ_{m\times m} - 4J_{m\times m} - 4J_{m\times m} + 4(A(L(G)) + 2I_m) \\
&= (n-8)J_{m\times m} + 4A(L(G)) + 8I_m.
\end{aligned}
$$

□

Proposition 7. *If G is a graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges, then*
(i) *(ij)-th element of $B_S(G)B_S(G)^T$ is $m + 4 - 2(d(v_i) + d(v_j))$ if v_i is adjacent to v_j.*
(ii) *(ij)-th element of $B_S(G)B_S(G)^T$ is $m - 2(d(v_i) + d(v_j))$ if v_i is not adjacent to v_j.*
(iii) *(ii)-th element of $B_S(G)B_S(G)^T$ is m.*

Proof. (i) Let v_i be adjacent to v_j. Then the (ij)-th element of $A(G)$ is 1 and (ij)-th element of $D_S(G)$ is $d(v_i) + d(v_j)$. Therefore by the first result of Proposition 6, the (ij)-th element of $B_S(G)B_S(G)^T$ is $m + 4 - 2(d(v_i) + d(v_j))$.

(ii) Let v_i be not adjacent to v_j. Then the (ij)-th element of $A(G)$ is 0 and (ij)-th element of $D_S(G)$ is $d(v_i) + d(v_j)$. Therefore by the first result of Proposition 6, the (ij)-th element of $B_S(G)B_S(G)^T$ is $m + 0 - 2(d(v_i) + d(v_j)) = m - 2(d(v_i) + d(v_j))$.

(iii) Diagonal elements of $A(G)$ and $D_S(G)$ are zeros. Therefore by the first result of Proposition 6, the (ii)-th element of $B_S(G)B_S(G)^T$ is m. □

Proposition 8. *If G is a graph with n vertices and m edges $e_1, e_2, \ldots, e_m$, then*
(i) *(ij)-th element of $B_S(G)^T B_S(G)$ is $n - 4$ if e_i is adjacent to e_j.*
(ii) *(ij)-th element of $B_S(G)^T B_S(G)$ is $n - 8$ if e_i is not adjacent to e_j.*
(iii) *(ii)-th element of $B_S(G)^T B_S(G)$ is n.*

Proof. (i) Let e_i be adjacent to e_j. Then the (ij)-th element of $A(L(G))$ is 1 and (ij)-th element of I_m is zero $(i \neq j)$. Therefore by the second result of Proposition 6, the (ij)-th element of $B_S(G)^T B_S(G)$ is $n - 8 + 4 + 0 = n - 4$.

(ii) Let e_i be not adjacent to e_j. Then the (ij)-th element of $A(L(G))$ and of I_m is zero $(i \neq j)$. Therefore by the second result of Proposition 6, the (ij)-th element of $B_S(G)^T B_S(G)$ is $n - 8 + 0 + 0 = n - 8$.

(iii) Diagonal elements of $A(L(G))$ are zeros. Therefore by the second result of Proposition 6, the (ii)-th element of $B_S(G)^T B_S(G)$ is $n - 8 + 0 + 8 = n$. □

Proposition 9. *For any graph G, the matrices $B_S(G)B_S(G)^T$ and $B_S(G)^T B_S(G)$ are symmetric.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of G and $e_1, e_2, \ldots, e_m$ be the edges of G.

By Proposition 7, if v_i is adjacent to v_j, then the (ij)-th element and (ji)-th element of $B_S(G)B_S(G)^T$ is $m + 4 - 2(d(v_i) + d(v_j))$. Also if v_i is not adjacent to v_j, then the (ij)-th element and (ji)-th element of $B_S(G)B_S(G)^T$ is $m - 2(d(v_i) + d(v_j))$. Further (ii)-th element of $B_S(G)B_S(G)^T$ is m. Hence $B_S(G)B_S(G)^T$ is a symmetric matrix.

Similarly by Proposition 8 we can show that $B_S(G)^T B_S(G)$ is also symmetric matrix. □

Proposition 10. *If G is a graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges $e_1, e_2, \ldots, e_m$, then*
(i) *the sum of the elements of i-th row (or i-th column) in $B_S(G)B_S(G)^T$ is $(n-4)(m - 2d(v_i))$.*
(ii) *the sum of the elements of i-th row (or i-th column) in $B_S(G)^T B_S(G)$ is $(n-8)m + 4d(e_i) + 8$.*

Proof. (i) By Proposition 6, $B_S(G)B_S(G)^T = mJ_{n\times n} + 4A(G) - 2D_S(G)$ and it is symmetric by the Proposition 9. Therefore sum of the elements of i-th row (or i-th column) in $B_S(G)B_S(G)^T$ is

$$\begin{aligned} mn + 4d(v_i) - 2\sum_{j=1;\, i\neq j}^{n}[d(v_i)+d(v_j)] &= mn + 4d(v_i) - 2[(n-1)d(v_i) + 2m - d(v_i)] \quad \text{since } \sum_{j=1}^{n} d(v_j) = 2m \\ &= (n-4)(m-2d(v_i)). \end{aligned}$$

(ii) By Proposition 6, $B_S(G)^T B_S(G) = (n-8)J_{m\times m} + 4A(L(G)) + 8I_m$ and it is symmetric by the Proposition 9. Therefore sum of the elements of i-th row (or i-th column) in $B_S(G)^T B_S(G)$ is $(n-8)m + 4d(e_i) + 8$. □

Corollary 11. *If G is a graph with n vertices $v_1, v_2, \dots, v_n$ and m edges, then*
(i) *the sum of the elements of i-th row (or i-th column) in $B_S(G)B_S(G)^T$ is zero if $n = 4$ or $d(v_i) = m/2$.*
(ii) *the sum of the elements of i-th row (or i-th column) in $B_S(G)B_S(G)^T$ is positive if $n > 4$ and $d(v_i) < m/2$.*
(iii) *the sum of the elements of i-th row (or i-th column) in $B_S(G)B_S(G)^T$ is negative if $n < 4$ and $d(v_i) > m/2$.*

Corollary 12. *If G is a graph with n vertices and m edges $e_1, e_2, \dots, e_m$, then*
(i) *the sum of the elements of i-th row (or i-th column) in $B_S(G)^T B_S(G)$ is zero if $d(e_i) = (8m - mn - 8)/4$.*
(ii) *the sum of the elements of i-th row (or i-th column) in $B_S(G)^T B_S(G)$ is positive if $d(e_i) > (8m - mn - 8)/4$.*
(iii) *the sum of the elements of i-th row (or i-th column) in $B_S(G)^T B_S(G)$ is negative if $d(e_i) < (8m - mn - 8)/4$.*

Example 1. For a graph given in Figure 1,

$$B_S(G)B_S(G)^T = \begin{bmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & -3 & -1 \\ -1 & 5 & -1 & -3 \\ -3 & -1 & 5 & -1 \\ -1 & -3 & -1 & 5 \end{bmatrix},$$

and

$$5J + 4A(G) - 2D_S(G) = 5\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 5 & 4 & 5 \\ 5 & 0 & 5 & 6 \\ 4 & 5 & 0 & 5 \\ 5 & 6 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & -3 & -1 \\ -1 & 5 & -1 & -3 \\ -3 & -1 & 5 & -1 \\ -1 & -3 & -1 & 5 \end{bmatrix}.$$

Therefore $B_S(G)B_S(G)^T = 5J + 4A(G) - 2D_S(G)$.
Also

$$B_S(G)^T B_S(G) = \begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -4 & 0 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ -4 & 0 & 4 & 0 & 0 \\ 0 & -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

and

$$(4-8)J + 4A(L(G)) + 8I = -4\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & -4 & 0 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ -4 & 0 & 4 & 0 & 0 \\ 0 & -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Therefore $B_S(G)^T B_S(G) = -4J + 4A(G) + 8I$.

Corollary 13. *For all graphs G with $n \geq 8$ vertices, the sum of the elements of any row (or column) in $B_S(G)^T B_S(G)$ is positive.*

3. Conclusion

In this article we have introduced the $(-1,1)$-incidence matrix $B_S(G)$ of a graph G and explored some properties of it and its transpose. This matrix further may be studied to explore the spectral and structural properties of a graph. Particularly, the study of singular values of $B_S(G)$.

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Data Availability Statement

All the data supporting the results are included in the manuscript.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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