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Research note: Ruv*é* numbers of a graph

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Abstract: A finite, connected simple graph G is a geodetic graph if and only if for each pair of vertices v_i, v_j there exists a unique distance path (or unique shortest v_iv_j-path). The insertion of vertices in an edge or edges of a non-geodetic graph G to, if possible, obtain a resultant geodetic graph is called geodetication of the graph G. The paper introduces two new graph parameters generally called the Ruv*é* numbers of a graph. The Ruv*é* numbers of G are denoted by $\rho_1(G)$ and $\rho_2(G)$ respectively, and $\rho_1(G) = \rho_2(G) = 0$ if and only if G is geodetic. Furthermore, for some graphs the parameter, $\rho_1(G) \to \infty$. The latter graphs G do *not permit* geodetication in respect of $\rho_1(G)$. It is evident that geodetication presents various challenging minimization problems. The core field of application will be, restricting graphs to *distance path uniqueness*. Intuitive applications are foreseen in military science, IT anti-hacking coding and predictive flow through networks.

Keywords: Ruvé number; geodetication; geodetication set; anti-Ruv*é* graph.

MSC: Primary 05C12; Secondary 05C38, 05C69

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1. Introduction

It is assumed that the reader has good knowledge of the basic notions and notation in graph theory. For reference reading see [1,2]. For this introductory study a graph G will be a finite, connected simple graph. Reference to vertices v_i, v_j (or u, v) in a particular graph G will mean that v_i and v_j (or u and v) are distinct vertices in G. If a vertex (or more) is added to an edge (or edges) of a graph the operation is called *vertex insertion* or *insert a vertex* (or insert vertices). Recall that a graph G is a geodetic graph if and only if for each pair of vertices v_i, v_j there exists a unique distance path (or unique shortest v_iv_j-path). The length of such shortest path is denoted by $d_G(v_i, v_j)$.

Definition 1. (Ruv*é*$_1$ number): Consider a non-geodetic graph $G = (V, E)$ and let $X = \{w_j : j \in \mathbb{N}\}$($j$ sufficiently large) be a separate set of vertices. The Ruv*é*$_1$ number of G denoted by $\rho_1(G)$ is the minimum number of vertices $\rho_1(G) = |Y|$, $w_i \in Y \subseteq X$, to be inserted in an edge or edges of G to, if possible, obtain a geodetic graph G' with $V(G') = V(G) \bigcup Y$. If obtaining a finite $\rho_1(G)$ is impossible then, $\rho_1(G) \to \infty$ or for purposes of bounds, $\rho_1(G) \geq \ell$ where, $\ell \to \infty$. The latter graphs G do *not permit* geodetication in respect of $\rho_1(G)$.

Definition 2. (Ruv*é*$_2$ number): Consider a non-geodetic graph $G = (V, E)$ and let $X = \{w_j : j \in \mathbb{N}\}$($j$ sufficiently large) be a separate set of vertices. The Ruv*é*$_2$ number of G denoted by $\rho_2(G)$ is the minimum number of vertices $\rho_2(G) = |Y|$, $w_i \in Y \subseteq X$, to be inserted in an edge or edges of G to obtain a graph G'' such that, $\forall\ v_i, v_j \in V(G)$ the shortest (v_i, v_j)-path in G'' is unique.

Note that both the Ruv*é* numbers are well-defined because,
(i) A graph G is well-defined,
(ii) It is permissible that $|X| \to \infty$,
(iii) $0 \leq \rho_1(G) \leq k$, $k \in \mathbb{N}_0$ or $\ell \to \infty$ is permissible,

(iv) Non-permissibility in respect of $\rho_1(G)$ is unambiguous,
(v) $0 \leq \rho_2(G) \leq k, k \in \mathbb{N}_0$,
(vi) Both G' and G'' are well-defined.

Three classical families of graphs i.e. complete graphs K_n, $n \geq 1$, odd cycles C_n, $n \geq 3$ and trees T will be of importance in this study. Note that a graph G from any of the three said families of graphs is geodetic thus has, $\rho_1(G) = \rho_2(G) = 0$. Hereafter, unless mentioned otherwise, all graphs will be undirected, simple and connected *non-geodetic* graphs of order $n \geq 4$. The operation to yield the graph G' or G'' from G as per Definition 1 or Definition 2 respectively, is called *geodetication* of G in respective of $\rho_1(G)$, $\rho_2(G)$ respectively. If the context is clear we simply refer to geodetication. Note that geodetication in respect of $\rho_1(G)$ or $\rho_2(G)$ is an iterative graph operation. Hence in both cases, following the insertion of say, w_1 in an edge of G a graph G_1 is obtained to be considered and so on. The number of iterations required is exactly, $\rho_1(G)$ or $\rho_2(G)$ meaning, $G_{\rho_1(G)} = G'$ and $G_{\rho_2(G)} = G''$.

Various real world applications may require geodetication of a graph or a network. Whereas in G, generic flow between v_i and v_j could be along different shortest paths such flow, following geodetication, is restricted to a unique distance path. For the Ruvé$_1$ number the vertices $w_i \in Y$ are *functional* vertices whilst those for the Ruvé$_2$ number are *blockage* vertices. Conceptually similar studies have been published with regards to the notion of *forbidden transitions* in graphs. See [3–5]. The latter observation motivates this introductory study. The core field of application will be, restricting graphs to *distance path uniqueness*. Intuitive applications are foreseen in military science, IT anti-hacking coding and predictive flow through networks.

Sections 2 and 3 will provide preliminary results and concepts related to the Ruvé$_1$ number. Thereafter, a discussion of the Ruvé$_2$ number will follow.

2. Introductory results on the Ruvé$_1$ number

Clearly, if G is geodetic then $\rho_1(G) = 0$. It is well known that a cycle graph (or cycle) C_n, $n \geq 4$ and n is even, is non-geodetic. By inserting one $w_i \in X$ to any edge of C_n the cycle C'_{n+1}, $n+1$ is odd is obtained. Since C'_{n+1} is geodetic it follows that for C_n, $n \geq 4$ and n is even, $\rho_1(C_n) = 1$.

The $J9$-graphs were first defined in [6]. These graphs were independently conceptualized by Scott and Seymour in [7]. In [7] these graphs are called *bananas*. A revised though equivalent definition is provided below.

Definition 3. Take $k \geq 1$ copies of a path P_n, $n \geq 3$. Let $j = 1, 2, 3, \dots, k$. Merge the respective, k origin vertices and the k terminus vertices of the paths and label the vertices consecutively as follows:

$$u_1, v_{1,j}, v_{2,j}, \dots, v_{n-2,j}, u_2, \; n \geq 3 \quad \text{for} \quad j = 1, 2, 3, \dots, k, \; k \geq 1.$$

The family of graphs is called the $J9$-graphs. A member of the $J9$-graphs is denoted by, $P_n^{(k)}$, and is called a Joost graph.

For this study we consider Joost graphs for $k \geq 3$. Note that each such Joost graph has $\binom{k}{2} \geq 3$ distinct C_n cycles each of even order. Hence, any Joost graph $P_n^{(k)}$, $k \geq 3$ is non-geodetic.

Proposition 1. *A Joost graph $G = P_n^{(k)}$, $k \geq 3$ does not permit geodetication in respect of $\rho_1(G)$.*

Proof. Consider any Joost graph $P_n^{(k)}$, $k \geq 3$. Without loss of generality let step 1 of geodetication be inserting the first vertex $w_1 \in X$ in any edge of the path labeled $u_1, v_{1,1}, v_{2,1}, \dots, v_{n-2,1}, u_2$. Exactly $k-1$ distinct C_{n+1} cycles are of odd order are obtained. The remaining $\binom{k}{2} - (k-1)$ distinct C_n cycles are of even order. At step 2 of geodetication that is, inserting vertex $w_2 \in X$ either in an edge of $u_1, v_{1,1}, v_{2,1}, \dots, v_{n-2,1}, u_2$ or otherwise then, either $(k-1)$ distinct C_{n+2} cycles and $\binom{k-1}{2}$ distinct C_n cycles are of even order or, exactly one C_{n+2} cycle is of even order. Note that other distinct C_n cycles of even order may exist as well. Clearly this dilemma perpetuates infinitely. Therefore, if $k \geq 3$ then $\rho(P_n^{(k)}) \to \infty$. Hence, a Joost graph $P_n^{(k)}$, $k \geq 3$ does not permit geodetication. □

Lemma 1. *Consider two cycle C_n, $n \geq 3$ and C_m, $m \geq 3$. Obtain G by merging a vertex of C_n with a vertex of C_m. This is called the Type I simple cycle link.*

(i) If both n, m are odd then $\rho_1(G) = 0$.

(ii) If say, n is odd and m is even then $\rho_1(G) = 1$.

(iii) If both n, m are even then $\rho_1(G) = 2$.

Proof. Result (i) follows from the fact the all odd cycles are geodetic.

Result (ii) follows from the fact that an even cycle is non-geodetic and requires the insertion of one vertex in any one edge of cycle C_m.

Result (iii) follow from the fact that all even cycles are non-geodetic and require the insertion of one vertex in any one edge of each cycle. □

By immediate induction it follows cycles $C_n, C_m, \ldots, C_q$ can pairwise be merged by a common vertex in "treelike" fashion to construct multiple (clustered or chained) Type I simple cycle links. If a graph G has an induced cycle C_k such cycle is called a simple cycle (or unchorded cycle) in G. A graph G is called a cactus graph if any two simple cycles in G, if such exist, share at most one common vertex (Type I simple cycle link). Note that any tree and any cycle may be considered to be cactus graphs.

Proposition 2. *Let G be a cactus graph which has $t_1 \geq 0$ even simple cycles and $t_2 \geq 0$ odd simple cycles. Then $\rho_1(G) = t_1$.*

Proof. If $t_1 = 0$ then G is geodetic hence, $\rho_1(G) = 0$. If $t_1 > 0$ then G is non-geodetic. Then the result follows through the application of Lemma 1. □

3. Characterization of graphs G which do not permit geodetication in respect of $\rho_1(G)$

Consider two cycle C_n, $n \geq 3$ and C_m, $m \geq 3$. Obtain G by merging a path section of C_n with a path section of C_m. This is called the *Type II simple cycle link*. By immediate induction it follows cycles $C_n, C_m, \ldots, C_q$ can be merged in "treelike" fashion to construct multiple (clustered or chained) Type II simple cycle links. This family of graphs is called the anti-Ruv$\acute{e}_1$ graphs (for brevity, $(a\text{-}R)$-graphs).

Lemma 2. *An anti-Ruv$\acute{e}_1$ graph G does not permit geodetication in respect of $\rho_1(G)$.*

Proof. Clearly, G is non-geodetic. The proof follows by similar reasoning found in the proof of Proposition 1. □

Recall from [1] that a closed trail in a graph G is called a *cycle in G*. The latter cycle in G may or may not be a simple cycle in G. A graph without any cycle is a *tree*. It is axiomatically true that if a vertex (or more vertices) is inserted in an edge (or more edges) of a tree T only the distance between some pairs of vertices will increase. No cycle can be constructed in tree T. In cyclic graphs, vertex insertion can only change the order of a cycle in G to switch between even and odd. It is axiomatically true that the insertion of a single vertex on an edge e of a particular cycle can switch the order of another cycle if and only if both cycles share e as a common edge. Hence, the latter two cycles induce a $(a\text{-}R)$-graphs). Let,

$$\mathcal{G} = \{G : G \text{ has some } (a\text{-}R)\text{-subgraph(s) and for each pair } u,v \text{ of vertices of a } (a\text{-}R)\text{-subgraph say, } H \text{ there exist a shortest } (u,v)\text{-path of length } d_G(u,v) \text{ in } H\}.$$

Theorem 1. *A graph G permits geodetication in respect of $\rho_1(G)$ if and only if G is $(G \in \mathcal{G})$-free.*

Proof. It is obvious that if G has an induced $(G \in \mathcal{G})$-subgraph then G does not permit geodetication. If G is $(G \in \mathcal{G})$-free then it is trivially possible to insert a sufficiently large number say, ℓ vertices into some edges of G to yield a graph H which besides minimization (or minimum minimality), satisfies Definition 1. The aforesaid is possible because vertex insertion cannot construct a $(G \in \mathcal{G})$-subgraph. Thus $\rho_1(G) \leq \ell$ and ℓ is finite. The aforesaid implies that graph G permits geodetication. □

Proposition 3. *A complete bipartite graph of the form $K_{n,m}$, $n \geq m \geq 2$ does not permit geodetication in respect of $\rho_1(G)$.*

Proof. Let $K_{n,m}$ have the independent vertex sets $X = \{v_i : 1 \leq i \leq n\}$ and $Y = \{u_i : 1 \leq i \leq m\}$. The $K_{m,m}$ induced subgraph of $K_{n,m}$ can be presented diagrammatically as a chorded cycle $C^{\oplus}_{2m}$. However, each vertex v_{m+1},$v_{m+2}, \ldots, v_n$ when added with corresponding edges constructs a $(G \in \mathcal{G})$-graph. By Theorem 1 the complete bipartite graph $K_{n,m}$, $n \geq m \geq 2$ does not permit geodetication. □

4. On the Ruvé$_2$ number

Clearly, if G is geodetic then $\rho_2(G) = 0$. By inserting one $w_i \in X$ to any edge of C_n, $n \geq 4$ and n is even the cycle C'_{n+1}, $n+1$ is odd is obtained. Since C'_{n+1} is geodetic it follows that for C_n, $n \geq 4$ and n is even, $\rho_2(C_n) = 1$.

Lemma 3. *A graph G always permit geodetication in respect of $\rho_2(G)$.*

Proof. It is known that if G is geodetic then $\rho_2(G) = 0$. Hence, although geodetication is not required it is permitted. The aforesaid is equivalent to inserting the empty set $\emptyset \subseteq X$ to some edges of G. Assume G is non-geodectic. Then it is axiomatically valid that a sufficient number of vertices from the set X can be inserted in some edges (at most $\lceil \frac{\varepsilon(G)}{2} \rceil$) such that all induced subgraphs in G'' restricted to vertices in $V(G)$ will be trees. Since $w_i \in Y$ is of no concern the vertex insertions can be done such that, $\forall\, v_i, v_j \in V(G)$ the shortest (v_i, v_j)-path in G'' is unique. This settles the result. □

Theorem 2. *For a graph G it follows that, $\rho_1(G) \geq \rho_2(G)$.*

Proof. If G is geodetic then $\rho_1(G) = \rho_2(G) = 0$. Assume G is non-geodectic and does not permit geodetication in respect of $\rho_1(G)$. From Lemma 3 G permits geodetication in respect of $\rho_2(G)$. Hence, $\rho_2(G)$ is finite say, $\rho_2(G) = k$. Surely, $\rho_1(G) \geq \ell > k$, $\ell \to \infty$.

Finally assume G is non-geodectic and permits geodetication in respect of $\rho_1(G)$. Let $\rho_1(G) = t$. If distance paths are tested which exclude $w_i \in Y$ then, since G'' is geodetic it implies that, $\rho_1(G) \leq \rho_2(G)$. The latter is true because if $\rho_1(G) > \rho_2(G)$ then, $\rho_1(G)$ was not a minimum. It thus implies that $\rho_1(G) = \rho_2(G)$. □

The next corollary is a direct consequence of the proof of Theorem 2.

Corollary 1. *If a graph G permits geodetication in respect of $\rho_1(G)$ then, $\rho_1(G) = \rho_1(G)$.*

Lemma 4. *Consider two cycle C_n, $n \geq 3$ and C_m, $m \geq 3$. Obtain G by merging a section of length l of C_n with a path section of length l of C_m.*

(i) If both n, m are odd then $\rho_2(G) = 2$.

(ii) If say, n is odd and m is even then $\rho_1(G) = 1$.

(iii) If both n, m are even then $\rho_1(G) = 2$.

Proof. (i) Insert a vertex $w_i \in X$ in any edge of the common path section to obtain G''. Clearly, the induced graph $\langle V(G) \rangle$ in G'' is a cactus with one even cycle. Hence, from Proposition 2 it follows that $\rho_2(G) = 2$.

(ii) Insert a vertex $w_i \in X$ in any edge of the common path section to obtain G''. Clearly, the induced graph $\langle V(G) \rangle$ in G'' is a cactus with one odd cycle. Hence, from Proposition 2 it follows that $\rho_2(G) = 1$.

(iii) Follow similar to result (i). □

5. On graphs from graph

It is assumed that the reader is familiar with the definitions of the line graph $L(G)$, the middle graph $M(G)$ and the total graph $T(G)$ respectively, of a graph G. Various results from [8] have relevance with regards to the study of the Ruvé number. We recall four useful results from [8].

Lemma 5. *(Lemma 2 in [8]). If a graph G is non-geodetic, then its line graph $L(G)$ is non-geodetic.*

Theorem 3. *(Theorem 3 in [8]). Let G be a connected graph with at least one edge. Then the line graph $L(G)$ is geodetic if and only if G is a tree or an odd cycle.*

Corollary 2. *(Corollary 2 in [8]). The middle graph $M(G)$ of a connected graph G is geodetic if and only if G is a tree.*

Theorem 4. *(Theorem 4 in [8]). The total graph $T(G)$ of a graph G is geodetic if and only if every connected component of G has at most one edge.*

Without further proof we can deduce the following useful results with regards to the study of the Ruv*é* number.

Theorem 5. *(i) For a graph G the line graph $L(G)$ is non-geodetic if G is non-geodetic or, not a tree nor an odd cycle.*
(ii) Let G be, not a tree then the middle graph $M(G)$ is non-geodetic.
(iii) If G has more than one edge then the total graph $T(G)$ is non-geodetic.

Theorem 5 read together with Theorem 1 distinguishes which of the line, middle and total graphs of a graph, permit (or not permit) geodetication.

Recall that the corona of the two graphs G of order n with H of order m and denoted by $G \circ H$ is the operation whereby we take n copies of H labeled H_i, $i = 1, 2, 3, \dots, n$ and attach the vertex $v_i \in V(G)$ to each vertex $u_{i,l} \in V(H_i)$, $l = 1, 2, 3, \dots, m$. Recall that $K_1 + H$ is obtained by attaching the vertex K_1 to each vertex in $V(H)$.

Theorem 6. *For $G \circ H$ where both G and H permits geodetication in respect of $\rho_1(G)$, $\rho_2(H)$ respectively, let $diam(H) = t$. Then,*

$$\rho_1(G \circ H) \leq \rho_1(G) + min\{2n\varepsilon(H), n((m-1)(t-1) + \rho_1(H))\},$$

and,

$$\rho_2(G \circ H) \leq \rho_2(G) + min\{2n\varepsilon(H), n((m-1)(t-1) + \rho_2(H))\}.$$

Proof. Part 1: Requiring the term $\rho_1(G)$ is obvious. For each of the n subgraphs which are isomorphic to $K_1 + H$ at least two options are considered. Option 1 is to insert two vertices from X in each edge of each H_i. The latter option will for each pair $u_{i,j}, u_{i,k} \in V(H_i)$ result in a unique distance path of length two. For each $v_j \in V(G)$ the second option is to insert $(t-1)$ vertices from X in all but one of the edges $v_j u_{i,k}$, $u_{i,k} \in V(H_i)$ and to geodeticate each H_i in respect of $\rho_1(H_i)$. Any of the two options will yield a unique distance path for any pair $v_i, u_{j,l}$ and any pair $u_{i,j}, u_{i,k}$. Clearly, the minimum of the two options i.e. $min\{2n\varepsilon(H), n((m-1)(t-1) + \rho_2(H))\} \geq 0$ yields an upper bound.

Part 2: The result follows by similar reasoning to that in Part 1. □

the next corollary requires no further proof.

Corollary 3. *For graphs G and H in Theorem 6 both, $\rho_1(G) \geq 0$ and $\rho_1(H) \geq 0$ and both be finite. Furthermore, Theorem 6 settles all the possible cases i.e.:*
(i) $\rho_1(G) > 0$ and $\rho_1(H) > 0$,
(ii) $\rho_1(G) = 0$ and $\rho_1(H) = 0$,
(iii) $\rho_1(G) > 0$ and $\rho_1(H) = 0$,
(iv) $\rho_1(G) = 0$ and $\rho_1(H) > 0$.
Finally, the same applies to $\rho_2(G)$ and $\rho_2(H)$.

6. Conclusion

Finding an efficient algorithm to obtain $\rho_1(G)$ and $\rho_2(G)$ is of importance. The complexity of an exhaustive method lies in the fact that geodetication is an iterative graph operation. It requires the evaluation of,

$$\prod_{i=0}^{\rho_1(G)\ or\ \rho_2(G)} (n+i-1)$$

possibe vertex insertions. On completion we have that,

$$\text{(i) } \varepsilon(G') = \varepsilon(G) + \rho_1(G),$$
$$\text{(ii) } \varepsilon(G'') = \varepsilon(G) + \rho_2(G).$$

In any graph the distance path between any pair of adjacent vertices is unique. It can be said that any graph is *partially 1-geodetic*. A cycle C_n, $n \geq 4$ and n is even is partially $(\frac{n}{2}-1)$-geodetic in that all distance paths of length $1 \leq l \leq \frac{n}{2}-1$ are unique. Therefore, any graph G is partially l-geodetic for some $1 \leq l \leq diam(G)-1$. Note that if $l = diam(G)$ that G is geodetic. Author proposes that the notion of partially l-geodetic graphs is worthy of further research. More specifically in the context of research related to the Ruv*é* numbers of graphs.

For a graph G define the total distance weight as,

$$\psi(G) = \sum_{\forall v_i, v_j \in V(G)} d_G(v_i, v_j).$$

In respect of G' (geodetication in respect of $\rho_1(G)$) the total distance weight is restricted to,

$$\psi(G') = \sum_{\forall v_i, v_j \in V(G')} d_{G'}(v_i, v_j).$$

Clearly, after geodetication of G we have that, $\psi(G) \leq \psi(G')$. Let the edge $e_q = v_iv_j$ and let the string $s_q = [e_q : w_k, w_l, \ldots, w_t]$ denote the vertices inserted in e_q. Let a geodetication set of G be $Y = \{s_q : 1 \leq q \leq \varepsilon(G)\}$.

Problem 1: Find a geodetication set Y of G such that $\psi(G') - \psi(G)$ is a minimum.

If a graph G has say, one induced $M = (G \in \mathcal{G})$-subgraph then a minimum number of edges can be added to complete M, to obtain a graph $G^\star$ which will permit geodetication in respect of $\rho_1(G^\star)$. Clearly, G may have more than one $(G \in \mathcal{G})$-subgraph. The minimum number of edges to be added to G to obtain $G^\star$ is called the edge-Ruv*é* number of a G. The edge-Ruv*é* number is denoted by $\rho^e(G)$. Further research on $\rho^e(G)$ remains open.

Another interesting observation is that the Cartesian product $P_n \times P_2$, $n \geq 2$ yields a ladder graph L_n. Note that although both P_n and P_2 are geodetic hence, $\rho_{1,2}(P_n) = \rho_{1,2}(L_2) = 0$, the ladder is a $(a\text{-}R)$-graph. The transition between geodeticability was from one extreme to the other extreme. In other words, whereas neither P_n nor P_2 requires geodetication in respect of ρ_1, the ladder does not permit geodetication in respect of $\rho_1(L_n)$. This observation leads to a trivial theorem.

Theorem 7. *For any graph G of order $n \geq 3$ let $H = G \times P_2$. The graph G does not permit geodetication in respect of $\rho_1(H)$.*

Proof. Because $G \times P_2$ is a prism or prism-like it has either an induced ladder subgraph or an induced circular ladder subgraph. Thus it contains a $(G \in \mathcal{G})$-subgraph. Hence, by Theorem 1 the graph H does not permit geodetication in respect of $\rho_1(H)$. □

Researching the Cartesian product to gain an understanding in respect of geodeticability in respect of $\rho_1(H)$ is deemed a worthy avenue. Numerous other graph products can be studied.

Geodetication in respect of $\rho_1(G)$ or $\rho_2(G)$ may change certain graph parameters. One example, is that the chromatic number of an even cycle is given by $\chi(C_n) = 2$. After inserting one vertex into any edge an odd cycle is obtained and $\chi(C'_{n+1}) = 3$. Let t be even then a cycle C_{3t} has domination number $\gamma(C_n) = \lceil \frac{3t}{3} \rceil = t$. The cycle C_{3t+1} has $\gamma(C_n) = t+1$. Some parameters for some graphs will remain unchanged. For example, let n be even then the independence number of C_n is $\alpha(C_n) = \frac{n}{2} = t$. However, $\alpha(C_{n+1}) = \lfloor \frac{n+1}{2} \rfloor = t$. In the case of $\rho_2(G)$ it is suggested to insert the $\rho_2(G)$ vertices as multiples into the minimum number of edges of $E(G)$ to be optimal. In more advanced graphs the notion of geodetication sets come into play. Since various graph parameters are possibly changed for some graphs through geodetication, numerous minimization problems similar to Problem 1 come to the fore.

Finding an efficient algorithm to test whether or not a graph G is $(a\text{-}R)$-free is of importance. Adaption of Brent's algorithm [9] or other can be considered. Complexity studies with regards to finding $\rho_1(G)$ will be insightfull.

Dedication

This paper is dedicated to Ruv*é* de Beer. Ruv*é* is a very special family member of the author.

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Conflict of interest:

The author declares there is no conflict of interest in respect of this research.

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