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Universal updates of Dyck-nest signatures

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Abstract: Let $0 < k \in \mathbb{Z}$. The anchored Dyck words of length $n = 2k + 1$ (obtained by prefixing a 0-bit to each Dyck word of length $2k$ and used to reinterpret the Hamilton cycles in the odd graph O_k and the middle-levels graph M_k found by Mütze et al.) represent in O_k (resp., M_k) the cycles of an n- (resp., $2n$-) 2-factor and its cyclic (resp., dihedral) vertex classes, and are equivalent to Dyck-nest signatures. A sequence is obtained by updating these signatures according to the depth-first order of a tree of restricted growth strings (RGS's), reducing the RGS-generation of Dyck words by collapsing to a single update the time-consuming i-nested castling used to reach each non-root Dyck word or Dyck nest. This update is universal, for it does not depend on k.

Keywords: Dyck words, Hamilton cycles, Dihedral, Cyclic group, Odd graph

MSC: 05C15, 05C38, 05C75, 68R15

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1. Introduction odd and middle-levels graphs

Let $0 < k \in \mathbb{Z}$, let $n = 2k + 1$ and let O_k be the *k-odd graph* [1], namely the graph whose vertices are the k-subsets of the cyclic group $\mathbb{Z}_n$ over the set $[0, 2k] = \{0, 1, \ldots, 2k\}$ and having an edge uv for each two vertices u, v if and only if $u \cap v = \emptyset$. The *characteristic vectors* of such subsets u, v of $[0, 2k]$ are the n-vectors $\vec{u}, \vec{v}$ over Z_2 whose *supports* (i.e, the subsets of $[0, 2k]$ composed by all nonzero entries of u, v), are exactly u, v, respectively. We may write $\vec{u}, \vec{v} \in V(O_k)$, instead of $u, v \in V(O_k)$. The set $V(O_k)$ of vertices of O_k admits a partition into *cyclic classes* mod n, where two vertices $\vec{u}, \vec{v}$ are in the same *class* if and only if they are related by a translation mod n, e.g., if $\vec{u} = u_0 \cdots u_{2k}$, then $\vec{v} = u_i u_{i+1} \cdots u_{2k} u_0 u_1 \cdots u_{i-1}$, for some $i \in \mathbb{Z}_n = [0, 2k]$. This is a translation that we denote by $i \in \mathbb{Z}_n$. The said cyclic classes mod n are to be optionally used in our final result, Corollary 5.

We also consider the double covering graph M_k of O_k, where M_k, referred to as *middle-levels graph*, is the subgraph of the Boolean lattice of subsets of $[0, 2k]$ induced by the *levels* L_k $(= V(O_k))$ and L_{k+1}, formed by the binary n-strings of weight k and $k + 1$, respectively [2–4]. Two vertices $u \in L_k$ and $v \in L_{k+1}$ of M_k are adjacent in M_k if and only if $u \subset v$, with u and v taken as subsets of $[0, 2k]$. The *double-covering graph map* $\Theta : M_k \quad O_k$ restricts to the identity map over L_k and to the *reversed complement* bijection $\aleph$ over L_{k+1}, that is: if $v \in L_{k+1}$ has characteristic vector $\vec{v} = v_0 v_1 \cdots v_{2k-1} v_{2k}$, then $\Theta(v) = \aleph(v)$ has characteristic vector $\bar{v}_{2k} \bar{v}_{2k-1} \cdots \bar{v}_1 \bar{v}_0$ in $V(O_k)$, where $\bar{0} = 1$ and $\bar{1} = 0$. To the partition of $V(O_k)$ into cyclic classes mod n, or $\mathbb{Z}_n$-classes, corresponds a partition of $V(M_k) = L_k \cup L_{k+1}$ into *dihedral classes*, or *$\mathbb{D}_n$-classes*, where $\mathbb{D}_n \supset \mathbb{Z}_n$ is the dihedral group of order $2n$.

An n-string $\Psi = 0\psi_1 \cdots \psi_{2k}$ in the alphabet $[0, n]$ in which each nonzero entry appears exactly twice is seen as a concatenation $W^i|X|Y|Z^i$ of substrings W^i, X, Y and Z^i, where W^i and Z^i have length i, for some $0 < i < k$. In that case, the n-string $W^i|Y|X|Z^i$ is said to be a *i-nested castling* of Ψ (time-consuming as it swaps parts of Ψ, with many position changes).

A *k-factor* of a graph G is a spanning k-regular subgraph. A *k-factorization* is a partition of $E(G)$ into disjoint k-factors. A 2-factor (or *cycle factor* [5]) in O_k formed by n-cycles, with a pullback 2-factor in M_k of $2n$-cycles via Θ^{-1}, and used in constructing Hamilton cycles [6] and optionally in Corollary 5 below, was analyzed in [4] from the viewpoint of restricted growth strings (RGS's [7, p. 325]), which form the *RGS-tree* $\mathcal{T}$ of Lemma 1, below.

In Section 2, a modification of the arguments of [4] shows that such RGS's exert control over the Dyck paths of length n, that represent bijectively the cyclic (resp., dihedral) classes of vertices of O_k (resp., M_k). These paths, viewed as *Dyck nests*, defined in Subsection 4.1, were related via the (time-consuming) i-nested castling operation controlled by the RGS-tree $\mathcal{T}$ that yields each non-root Dyck nest from its parent nest ([2–4], or Theorem 1) in the reinterpretation of the Hamilton cycle constructions in O_k [6] and M_k [8,9].

Such RGS-control will be reduced below, first by viewing each Dyck nest as its *signature*, defined in Subsection 5.2 and shown to be equivalent to that Dyck nest in Theorem 4, and second by collapsing each i-nested castling to a *universal* single (one-step) update of the signature of each non-root Dyck nest from the signature of its parent nest in the RGS-tree $\mathcal{T}$. The term *universal*, introduced in Theorem 5, is taken in the sense that the integers representing such updates do not depend on the values of k, so that those integers are valid and unique for all concerned O_k's and M_k's. The sequence formed by all such updates, controlled by the RGS-tree $\mathcal{T}$, is presented in Theorem 9, accompanied by the sequence of their corresponding locations in Corollary 5, leading to its asymptotic analysis (Subsection 6.4).

2. Restricted growth strings and i-nested castling

The *k-th Catalan number* [10] A000108 is given by $C_k = \frac{(2k)!}{k!(k+1)!}$. Let $\mathcal{S}$ be the *sequence of RGS's* [10] A239903. It was shown in [2,3] that the first C_k terms of $\mathcal{S}$ represent both the Dyck words of length $2k$ and the *extended* Dyck words of length n, obtained by prefixing a 0-bit to each Dyck word, and yielding a sole corresponding Dyck path (Subsection 4.1).

The sequence $\mathcal{S} = (\beta(i))_{0\leq i\in\mathbb{Z}}$ starts as

$$\begin{aligned}\mathcal{S} =&(\beta(0),\dots,\beta(17),\dots)\\ =&(0,1,10,11,12,100,101,110,111,112,120,121,122,123,1000,1001,1010,1011,\dots),\end{aligned}$$

and has the lengths of any two contiguous terms $\beta(m-1)$ and $\beta(m)$, $(1 \leq m \in \mathbb{Z})$, constant unless $m = C_k$, for some $k > 1$, in which case $\beta(m-1) = \beta(C_k - 1) = 12\cdots k$ has length k, and $\beta(m) = \beta(C_k) = 10^k = 10\cdots 0$ has length $k+1$.

To work in middle-levels and odd graphs in relation to their Hamilton cycles [6,8,9], RGS's were tailored as *germs* in [2–4]. A *k-germ* ($k > 1$) is a $(k-1)$-string $\alpha = a_{k-1}a_{k-2}\cdots a_2a_1$ such that:

(a) the leftmost position of α, namely position $k-1$, contains the entry $a_{k-1} \in \{0,1\}$;

(b) given $1 < i < k$, the entry a_{i-1} at position $i-1$ satisfies $0 \leq a_{i-1} \leq a_i + 1$.

Each RGS $\beta = \beta(m)$, where $0 \leq m \in \mathbb{Z}$, is transformed, for every $k \in \mathbb{Z}$ such that $k \geq \text{length}(\beta)$, into a k-germ $\alpha = \alpha(\beta,k) = \alpha(\beta(m),k)$ by prefixing $k-$ length(β) zeros to β.

Every k-germ $a_{k-1}a_{k-2}\cdots a_2a_1$ yields the $(k+1)$-germ $0a_{k-1}a_{k-2}\cdots a_2a_1$. A *non-null* RGS is obtained by stripping a k-germ $\alpha = a_{k-1}a_{k-2}\cdots a_2a_1 \neq 00\cdots 0$ of all the zeros to the left of its leftmost position containing a 1. We denote such an RGS still by α, say that the *null* RGS $\alpha = 0$ represents all null k-germs α, $(0 < k \in \mathbb{Z})$, and use $\alpha = \alpha(m)$, or $\beta = \beta(m)$, both for a k-germ and for its corresponding RGS. In fact, $\alpha = \alpha(m)$, or $\beta = \beta(m)$, will be considered to be the RGS representing all the k-germs $\alpha = \alpha(m)$, or $\beta = \beta(m)$, respectively, $(0 < k \in \mathbb{Z})$ leading to α, or β, as an RGS, by stripping their zeros as indicated.

If $a, b \in \mathbb{Z}$, then let

(1) $[a,b] = \{j \in \mathbb{Z}; a \leq j \leq b\}$; **(2)** $[a,b[= \{j \in \mathbb{Z}; a \leq j < b\}$;

(3) $]a,b] = \{j \in \mathbb{Z}; a < j \leq b\}$; **(4)** $]a,b[= \{j \in \mathbb{Z}; a < j < b\}$.

Given two k-germs $\alpha = a_{k-1}\cdots a_1$ and $\beta = b_{k-1}\cdots b_1$, where $\alpha \neq \beta$, we say that α precedes β, written $\alpha < \beta$, whenever either

(i) $0 = a_{k-1} < b_{k-1} = 1$ or

(ii) $\exists i \in [1,k[$ such that $a_i < b_i$ with $a_j = b_j, \forall j \in]i,k[$.

The resulting order of k-germs yields a bijection from $[0,C_k[$ onto the set of k-germs that assigns each $m \in [0,C_k[$ to a corresponding k-germ $\alpha = \alpha(m)$. In fact, there are exactly C_k k-germs $\alpha = \alpha(m) < 10^k$, $\forall k > 0$. Moreover, we have the following trees $\mathcal{T}_k$, correspondences $F(\cdot)$ and RGS-tree $\mathcal{T}$ (this one, partially exemplified in display (1) via its section for $k \leq 5$).

3. Ordered trees of k-germs and Dyck words

We recall from [2, Theorem 3.1] or [3, Theorem 1] that the k-germs are the nodes of an ordered tree $\mathcal{T}_k$ rooted at 0^{k-1} and such that each k-germ $\alpha = a_{k-1} \cdots a_2 a_1 \neq 0^{k-1}$ with rightmost nonzero entry a_i ($1 \leq i = i(\alpha) < k$) has parent $\beta(\alpha) = b_{k-1} \cdots b_2 b_1 < \alpha$ in $\mathcal{T}_k$ with $b_i = a_i - 1$ and $a_j = b_j$, for every $j \neq i$ in $[1, k-1]$.

Lemma 1. *By considering k-germs as RGS's, an infinite chain $\mathcal{T}_2 \subset \mathcal{T}_3 \subset \cdots \subset \mathcal{T}_k \subset \cdots$ of finite trees converges to their union, the RGS-tree $\mathcal{T}$.*

Proof. Iterative inclusion of the successive trees $\mathcal{T}_k$ tends to the RGS-tree, as k converges to infinity, where the original k-germs are considered as RGS as indicated. □

Theorem 1. *To each k-germ $\alpha = a_{k-1} \cdots a_1$ corresponds an n-string $F(\alpha)$ with initial entry 0 and having each $j \in [1, k]$ as an entry exactly twice. Moreover,*

$$F(0^{k-1}) = 012 \cdots (k-2)(k-1)kk(k-1) \cdots 21, \ (e.g., \ F((0) = 011, \ F(00) = 01221).$$

Furthermore, if $\alpha \neq 0^{k-1}$, let

1. *W^i and Z^i be the leftmost and rightmost, respectively, substrings of length $i = i(\alpha)$ in $F(\beta)$, where β is the parent of α in $\mathcal{T}_k$;*
2. *$c > 0$ be the leftmost entry of $F(\beta) \setminus (W^i \cup Z^i)$, and*
3. *$F(\beta) \setminus (W^i \cup Z^i)$ be the concatenation $X|Y$, where Y starts at the entry $c+1$ of $F(\beta)$.*

Then $F(\alpha) = W^i|Y|X|Z^i$ is the i-nested castling of $F(\beta) = W^i|X|Y|Z^i$. In addition, W^i is an ascending i-substring, Z^i is a descending i-substring, and kk is a substring of $F(\alpha)$.

Proof. The proof is a slight modification of that of [2, Theorem 3.2] or [3, Theorems 2], where the rightmost appearances of each integer of $[1, k]$ in every $F(\alpha)$ as in the statement were given as asterisks, *, or in [4, Theorem 2] as equal signs, =. □

The disposition of RGS's in an initial section of the RGS-tree of Lemma 1 (for $k \leq 5$) is shown in display (1), where the children of an RGS α at any level are disposed from left to right in the subsequent level, starting just below α:

0														
1	10	100			1000									
	11	101	110		1001	1010	1100							
	12		111	120		1011	1101	1110		1200				
			112	121		1012		1111	1120	1201	1210			
				122				1112	1121		1211	1220		(1)
				123					1122		1212	1221	1230	
									1123			1222	1231	
												1223	1232	
													1233	
													1234	

4. Dyck words, k-germs and 1-factorizations

A *binary k-string* (or *k-bitstring* [6,8,9]) is a sequence of length k whose terms are the digits 0, called 0-*bits*, and/or 1, called 1-*bits*, respectively. The *weight* of a binary k-string is its number of 1-bits.

In this work, a *Dick word of length* $2k$ is defined as a binary $2k$-string of weight k such that in every prefix the number of 0-bits is at least equal to the number of 1-bits (differing from the Dyck words of [6] in which the number of 1-bits is at least the number of 0-bits).

The concept of *empty Dyck word*, denoted ϵ, whose weight is 0, also makes sense in this context. We will present each Dyck word as its associated *anchored Dyck word*, obtained by prefixing a 0-bit to it. In particular, ϵ is represented by the anchored Dyck word 0.

For each k-germ α, where $k > 1$, we define the binary string form $f(\alpha)$ of $F(\alpha)$ by replacing each first appearance of an integer $j \in [0,k]$ as an entry of $F(\alpha)$ by a 0-bit and the second appearance of j, in case $j \in [1,k]$, by a 1-bit (where 0-bits and 1-bits correspond respectively to the 1-bits and 0-bits used in [6]). Such $f(\alpha)$ is a binary n-string of weight k, namely an anchored Dyck word of length n whose *support* $\text{supp}(f(\alpha))$ is a vertex of O_k and an element of L_k, while $\aleph(f(\alpha))$ is an element of L_{k+1}. Note that the pair $\{f(\alpha), \aleph(f(\alpha))\}$ together with the $\mathbb{Z}_n$-class of $f(\alpha)$ in L_k $(= V(O_k))$ generate the $\mathbb{D}_n$-class of $f(\alpha)$ in $V(M_k)$. Thus, $f(\alpha)$ represents both a $\mathbb{Z}_n$-class of $V(O_k)$ and a $\mathbb{D}_n$-class of $V(M_k)$, which has Hamilton cycles lifted from those in O_k [4,6], or independently, as in [2,3,8,9]

4.1. Dyck paths

Each anchored Dyck word $f(\alpha)$ yields a *Dyck path* [4] obtained as a curve $\rho(\alpha)$ that grows from $(0,0)$ in the Cartesian plane Π via the successive replacement of the 0-bits and 1-bits of $f(\alpha)$, from left to right, by *up-steps* and *down-steps*, namely segments $(x,y)(x+1,y+1)$ and $(x,y)(x+1,y-1)$, respectively. We assign the integers of the interval $[0,k]$ in decreasing order (from k to 0) to the up-steps of $\rho(\alpha)$, from the top unit layer intersecting $\rho(\alpha)$ to the bottom one and from left to right at each concerning unit layer between contiguous lines $y, y+1 \in \mathbb{Z}$, where $0 \leq y \in \mathbb{Z}$. These assigned integers correspond to their leftmost appearances as entries of $F(\alpha)$. Each leftmost appearance j' of an integer $j \in [1,k]$ in $F(\alpha)$ corresponds to the starting entry of a Dyck subword $0u1v$ in $f(\alpha)$, where u, v are Dyck subwords (possibly ϵ). The Dyck subword $0u1v$ corresponds in $F(\alpha)$ to a substring $j'Uj''V$, where U and V correspond to u and v, respectively, and $j'' = j' \in [1,k]$.

α	F(α)	B(α)	A(α)	i(α)	o(α)	A_i(α)
0	01221		1	|/	0	
1	02211	1	0	1110	1	0

α	F(α)	B(α)	A(α)	i(α)	o(α)	A_i(α)
00	0123321		12	|/	0	
01	0233211	12	10	1120	1	0
10	0133221	12	02	2110	2	0
11	0221331	02	01	1121	3	k-2
12	0332211	01	00	1210	4	0

α	F(α)	B(α)	A(α)	i(α)	o(α)	A_i(α)
000	012344321		123	|/	0	
001	023443211	123	120	1130	1	0
010	013443221	123	103	2120	2	0
011	022134431	103	102	1132	3	k-2
012	034432211	102	100	1220	4	0
100	012443321	123	023	3110	5	0
101	024433211	023	020	1130	6	0
110	013324421	023	013	2121	7	k-3
111	024421331	013	011	1131	8	1
112	033244211	011	010	1210	9	0
120	014433221	013	003	2210	10	0
121	022144331	003	002	1132	11	k-2
122	033221441	002	001	1221	12	k-3
123	044332211	001	000	1310	13	0

α	F(α)	B(α)	A(α)	i(α)	o(α)	A_i(α)
0000	01234554321		1234	|/	0	
0001	02345543211	1234	1230	1140	1	0
0010	01345543221	1234	1204	2130	2	0
0011	02213455431	1204	1203	1143	3	k-2
0012	03455432211	1203	1200	1230	4	0
0100	01245543321	1234	1034	3120	5	0
0101	02455433211	1034	1030	1140	6	0
0110	01332455421	1034	1024	2132	7	k-3
0111	02455421331	1024	1021	1141	8	1
0112	03324554211	1021	1020	1210	9	0
0120	01455433221	1024	1004	2220	10	0
0121	02214554331	1004	1003	1143	11	k-2
0122	03322145541	1003	1002	1232	12	k-3
0123	04554332211	1002	1000	1320	13	0
1000	01235544321	1234	0234	4110	14	0
1001	02355443211	0234	0230	1140	15	0
1010	01355443221	0234	0204	2130	16	0
1011	02213554431	0204	0203	1143	17	k-2
1012	03554432211	0203	0200	1230	18	0
1100	01244355321	0234	0134	3121	19	k-4
1101	02443553211	0134	0130	1140	20	0
1110	01355324421	0134	0114	2131	21	1
1111	02442135531	0114	0112	1142	22	k-3
1112	03553244211	0112	0110	1220	23	0
1120	01443553221	0114	0104	2210	24	0
1121	02214435531	0104	0103	1143	25	k-2
1122	03553221441	0103	0101	1231	26	1
1123	04435532211	0101	0100	1310	27	0
1200	01255443321	0134	0034	3210	28	0
1201	02554433211	0034	0030	1140	29	0
1210	01332554421	0034	0024	2132	30	k-3
1211	02554421331	0024	0021	1141	31	1
1212	03325544211	0021	0020	1210	32	0
1220	01443325521	0024	0014	2221	33	k-4
1221	02552144331	0014	0012	1142	34	2
1222	03325521441	0012	0011	1221	35	1
1223	04433255211	0011	0010	1310	36	0
1230	01554433221	0014	0004	2310	37	0
1231	02215544331	0004	0003	1143	38	k-2
1232	03322155441	0003	0002	1232	39	k-3
1233	04433221551	0002	0001	1321	40	k-4
1234	05544332211	0001	0000	1410	41	0

Figure 1. List of k-germs α, n-nests $F(\alpha)$, signatures and update entries, for $k = 2,3,4,5$.

Each edge uv of O_k is taken as the union of a pair of *arcs* $\vec{uv}$ and $\vec{vu}$, that is a pair of oriented edges with *sources* u and v and *targets* v and u, respectively. Let us see that each first appearance of an integer $i \in [0,k]$ in $F(\alpha)$ (that we refer to as *color* i) determines uniquely an arc of O_k and two edges of M_k. The n-strings $F(\alpha)$ of Theorem 1 will be said to be *Dyck nests* of length n, or *n-nests*. Say $u \in V(O_k)$ belongs to a Dyck nest $F(\alpha)$, seen as a $\mathbb{Z}_n$-class of O_k, and that $i' \in [0,k]$ is the first appearance of an integer i in $F(\alpha)$. Then, there is a unique vertex v in a $\mathbb{Z}_n$-class of O_k corresponding to a Dyck nest $F(\alpha')$ such that uv is an edge of O_k and u has its i-colored entry i' in the same position as the entry with color $k-i$ in v, so we say that the *color of the arc* $\vec{uv}$ *is* i. In that case, the arc $\vec{vu}$ has color $k-i$, allowing to recover u from v as the unique vertex of O_k such that to the entry of v with color $k-i$ corresponds the entry in the same position in u with color $k-(k-i) = i$. Thus, we say $\vec{uv}$ has color i and $\vec{vu}$ has color $k-i$, this being the *supplementary color* of i in $[0,k]$. The inverse images Θ^{-1}

of $\vec{uv}$ and $\vec{vu}$ are formed by an arc from L_k to L_{k+1} and another arc from L_{k+1} onto L_k (see Example 1); they end up yielding a pair of edges in M_k.

Example 1. The translations $j \in \mathbb{Z}_n$ act on any anchored Dyck word $f(\alpha)$, yielding binary n-strings $f(\alpha).j$, so $f(\alpha).0 = f(\alpha)$ itself. This notation is also used for n-nests $F(\alpha)$. Given $u = f(000).0 = 000001111 \in O_4$, the arc color $i = 3 \in [0,4]$ determines an arc $\vec{uv}$ with source u and target $v = f(001).5 = 111010000$. This information can be arranged as follows:

α	j	$F(\alpha).j$	$f(\alpha).j$	O_k	$L_4 \searrow L_5$	$\aleph$	$L_5 \searrow L_4$
000	0	$012\mathbf{3}44321$	$000\mathbf{0}01111$	$u = 5678$	$000\mathbf{0}01111$	$\leftrightarrow$	$0000\mathbf{1}1111$
001	5	$432\mathbf{1}10234$	$111\mathbf{0}10000$	$v = 0124$	$000\mathbf{1}01111$	$\leftrightarrow$	$00001\mathbf{0}111$

(2)

Display (2) shows from left to right: the 4-germs α for the source u and target v (columnwise) of the arc $\vec{uv}$; the corresponding translations $j \in \mathbb{Z}_9$; the $\mathbb{Z}_9$-translated Dyck nests $F(\alpha).j$, where the i-th entries are shown in bold trace; the $\mathbb{Z}_9$-translated anchored Dyck words $f(\alpha).j$, where the i-th entries are again shown in bold trace; and the two edges in the double covering M_4 of O_4 projecting onto $\vec{uv}$, which are related via $\aleph$.

4.2. Arc coloring and 1-factorizations

Note that there is a coloring (or partition) of the set of arcs of O_k resulting from Subsection 4.1 and exemplified in Example 1. It induces a 1-factorization of M_k into $(k+1)$ 1-factors, each formed by the edges whose arcs from L_k to L_{k+1} are colored with a corresponding integer of $[0,k]$. This factorization is known as the *modular* 1-factorization of M_k [4]. In contrast, a different 1-factorization known as the *lexical* 1-factorization of M_k [8] exists. This is presented and exemplified in Example 2.

Example 2. Continuing as in Example 1 but with M_k rather than O_k, we modify and, instead of coloring with $k-i \in [0,k]$ the arc $\vec{uv}$ determined by the first appearance of $i \in [0,k]$ in the Dyck nest $F(\alpha)$ of each vertex u of M_k in L_k, we now color $\vec{uv}$ with $i \in [0,k]$, so that a 1-factorization of M_k is determined, namely the lexical one [8] mentioned above, with $\vec{vu}$ also colored with i. This is exemplified as follows, where $k = 4$, color $i = 3 \in [0,4]$, and $\alpha = 000$, so that $u = f(\alpha).0 = f(\alpha) = 000\mathbf{0}01111$ (with the i-th entry in bold trace) is sent by $\aleph$ onto $\aleph(u) = 0000\mathbf{1}1111 \in L_5$:

$V(M_4)$	α	j	$F(\alpha).j$	$f(\alpha).j$	$\aleph$	$\aleph(f(\alpha).j)$
L_4	000	0	$012\mathbf{3}44321$	$000\mathbf{0}01111$	$\leftrightarrow$	$0000\mathbf{1}1111 \in L_5$
L_5	100	8	$123\mathbf{3}44210$	$000\mathbf{1}01111$	$\leftrightarrow$	$00001\mathbf{0}111 \in L_4$

(3)

In display (3), the corresponding edges from u and $\aleph(u)$ end up onto $v = \aleph(w) = 000\mathbf{1}01111 \in L_5$ and $w = \aleph^{-1}(v) = f(100).8 = 00001\mathbf{0}111 \in L_4$. These are the edges $uv = u\aleph(w)$ and $\aleph(u)v$ with both oppositely oriented arcs in each case having the same (lexical) color i, which differs with the modular-color situation in Subsection 4.1 and Example 1 (that is: with the colors i and $k-i$ of the arcs of each edge differing as supplementary colors in $[0,k]$).

5. Dyck nests and signatures

Theorem 2. *Each anchored Dyck word w of length n is the binary string $f(\alpha)$ associated to an n-nest $F(\alpha)$ obtained via the procedure of Theorem 1 from a specific k-germ $\alpha = \alpha(w)$.*

Proof. The Lexical Procedure [2, Section 7], [3, Section 7] restores the positive integer entries of $F(\alpha)$ corresponding to the k non-initial 0-bits of $w = f(\alpha)$. These are the first appearances j' of each integer $j \in [1,k]$ in $F(\alpha)$. By forming the Dyck word $0u1v$ of $f(\alpha)$, the second appearance j'' of j is found by replacing its corresponding 1-bit in $f(\alpha)$ by $j = j''$ in $F(\alpha)$. □

00000	0123456654321		12345		0		11120	0146643553221	01125	01105	2220	66	0
00001	0234566543211	12345	12340	1150	1	0	11121	0221466435531	01105	01104	1154	67	k-2
00010	0134566543221	12345	12305	2140	2	0	11122	0355322146641	01104	01102	1242	68	k-4
00011	0221345665431	12305	12304	1154	3	k-2	11123	0466435532211	01102	01100	1320	69	0
00012	0345665432211	12304	12300	1240	4	0	11200	0125546643321	01145	01045	3210	70	0
00100	0124566543321	12345	12045	3130	5	0	11201	0255466433211	01045	01040	1150	71	0
00101	0245665433211	12045	12040	1150	6	0	11210	0133255466421	01045	01035	2150	72	k-3
00110	0133245665421	12045	12035	2143	7	k-3	11211	0255466421331	01035	01031	1151	73	1
00111	0245665421331	12035	12031	1151	8	1	11212	0332554664211	01031	01030	1210	74	0
00112	0332456654211	12031	12030	1210	9	0	11220	0146643325521	01035	01015	2231	75	1
00120	0145665433221	12035	12005	2230	10	0	11221	0255214664331	01015	01013	1153	76	k-3
00121	0221456654331	12005	12004	1154	11	k-2	11222	0332552146641	01013	01012	1232	77	k-4
00122	0332214566541	12004	12003	1243	12	k-3	11223	0466433255211	01012	01010	1320	78	0
00123	0456654332211	12003	12000	1330	13	0	11230	0155466433221	01015	01005	2310	79	0
01000	0123566544321	12345	10345	4120	14	0	11231	0221554664331	01005	01004	1154	80	k-2
01001	0235665443211	10345	10340	1150	15	0	11232	0332215546641	01004	01003	1243	81	k-3
01010	0135665443221	10345	10305	2150	16	0	11233	0466433221551	01003	01001	1331	82	1
01011	0221356654431	10305	10304	1154	17	k-2	11234	0554664332211	01001	01000	1410	83	0
01012	0356654432211	10304	10300	1240	18	0	12000	0123665544321	01345	00345	4210	84	0
01100	0124435665321	10345	10245	3132	19	k-4	12001	0236655443211	00345	00340	1150	85	0
01101	0244356653211	10245	10240	1150	20	0	12010	0136655443221	00345	00305	2140	86	0
01110	0135665324421	10245	10215	2141	21	1	12011	0221366554431	00305	00304	1154	87	k-2
01111	0244213566531	10215	10213	1153	22	k-3	12012	0366554432211	00304	00300	1240	88	0
01112	0356653244211	10213	10210	1230	23	0	12100	0124436655321	00345	00245	3132	89	k-4
01120	0144356653221	10215	10205	2210	24	0	12101	0244366553211	00245	00240	1150	90	0
01121	0221443566531	10205	10204	1154	25	k-2	12110	0136655324421	00245	00215	2141	91	1
01122	0356653221441	10204	10201	1241	26	1	12111	0244213665531	00215	00213	1153	92	k-3
01123	0443566532211	10201	10200	1310	27	0	12112	0366553244211	00213	00210	1230	93	0
01200	0125665443321	10245	10045	3220	28	0	12120	0144366553221	00215	00205	2210	94	0
01201	0256654433211	10045	10040	1150	29	0	12121	0221443665531	00205	00204	1154	95	k-2
01210	0133256654421	10045	10035	2143	30	k-3	12122	0366553221441	00204	00201	1241	96	1
01211	0256654421331	10035	10031	1151	31	1	12123	0443665532211	00201	00200	1310	97	0
01212	0332566544211	10031	10030	1210	32	0	12200	0125544366321	00245	00145	3221	98	k-5
01220	0144332566521	10035	10025	2232	33	k-4	12201	0255443663211	00145	00140	1150	99	0
01221	0256652144331	10025	10022	1152	34	2	12210	0136632554421	00145	00125	2142	100	2
01222	0332566521441	10022	10021	1221	35	1	12211	0255442136631	00125	00122	1152	101	k-4
01223	0443325665211	10021	10020	1310	36	0	12212	0366325544211	00122	00120	1220	102	0
01230	0156654433221	10025	10005	2320	37	0	12220	0144366325521	00125	00115	2221	103	1
01231	0221566544331	10005	10004	1154	38	k-2	12221	0255214436631	00115	00113	1153	104	k-3
01232	0332215665441	10004	10003	1243	39	k-3	12222	0366325521441	00113	00111	1231	105	1
01233	0443322156651	10003	10002	1332	40	k-4	12223	0443663255211	00111	00110	1310	106	0
01234	0566544332211	10001	10000	1410	41	0	12230	0155443663221	00115	00105	2310	107	0
10000	0123466554321	12345	02345	5110	42	0	12231	0221554436631	00105	00104	1154	108	k-2
10001	0234665543211	02345	02340	1150	43	0	12232	0366322155441	00104	00102	1242	109	2
10010	0134665543221	02345	02305	2140	44	0	12233	0443663221551	00102	00101	1321	110	1
10011	0221346655431	02305	02304	1154	45	k-2	12234	0554436632211	00101	00100	1410	111	0
10012	0346655432211	02304	02300	1240	46	0	12300	0126655443321	00145	00045	3310	112	0
10100	0124665543321	02345	02045	3130	47	0	12301	0266554433211	00045	00040	1150	113	0
10101	0246655433211	02045	02040	1150	48	0	12310	0133266554421	00045	00035	2143	114	k-3
10110	0133246655421	02045	02035	2143	49	k-3	12311	0266554421331	00035	00031	1151	115	1
10111	0246655421331	02035	02031	1151	50	1	12312	0332665544211	00031	00030	1210	116	0
10112	0332466554211	02031	02030	1210	51	0	12320	0144332665521	00035	00025	2232	117	k-4
10120	0146655433221	02035	02005	2230	52	0	12321	0266552144331	00025	00022	1152	118	2
10121	0221466554331	02005	02004	1154	53	k-2	12322	0332665521441	00022	00021	1221	119	1
10122	0332214665541	02004	02003	1243	54	k-3	12323	0443326655211	00021	00020	1310	120	0
10123	0466554332211	02003	02000	1330	55	0	12330	0155443326621	00025	00015	2321	121	k-5
11000	0123554664321	02345	01345	4150	56	k-5	12331	0266215544331	00015	00013	1153	122	3
11001	0235546643211	01345	01340	1150	57	0	12332	0332662155441	00013	00012	1232	123	2
11010	0135546643221	01345	01305	2140	58	0	12333	0443326621551	00012	00011	1321	124	1
11011	0221355466431	01305	01304	1154	59	k-2	12334	0554433266211	00011	00010	1410	125	0
11012	0355466432211	01304	01300	1240	60	0	12340	0166554433221	00015	00005	2410	126	0
11100	0124664355321	01345	01145	3131	61	1	12341	0221665544331	00005	00004	1154	127	k-2
11101	0246643553211	01145	01140	1150	62	0	12342	0332216655441	00004	00003	1243	128	k-3
11110	0135532466421	01145	01125	2142	63	k-4	12343	0443322166551	00003	00002	1332	129	k-4
11111	0246642135531	01125	01122	1152	64	2	12344	0554433221661	00002	00001	1421	130	k-5
11112	0355324664211	01122	01120	1220	65	0	12345	0665544332211	00001	00000	1510	131	0

Figure 2. List of k-germs α, n-nests $F(\alpha)$, signatures and update entries, for $k = 6$.

5.1. Dyck nests

Our calling the strings $F(\alpha)$ by the name of *Dyck nests*, or *n-nests*, was suggested by the sets of nested intervals formed by the projections on the x-axis of the two appearances j' and j'' of each integer $j \in [1,k]$ as numbers assigned to the respective up- and down-steps of each Dyck path $\rho(\alpha)$.

We take the tree $\mathcal{T}_k$ whose nodes were originally denoted via the k-germs α, and denote them, further, via the n-nests $F(\alpha)$, in representation of the corresponding anchored Dyck words $f(\alpha)$. With this nest notation, $\mathcal{T}_k$ will be now said to be a *tree of Dyck nests*.

Corollary 1. *The set of n-nests $F(\alpha)$ is in one-to-one correspondence with the set of anchored Dyck words $f(\alpha)$ of length n.*

5.2. Signatures

Each n-nest $F(\alpha)$ is encoded by its *signature* $A(\alpha) = (A_{k-1}(\alpha), \ldots, A_2(\alpha), a_1)alpha)$, defined as the vector of halfway-distance floors $A_j(\alpha)$ between the first (j') and second (j'') appearances of each integer j assigned to the respective up- and down-steps of the path $\rho(\alpha)$, where $k > j > 0$. We write For example, if $j'k'k''j''$ (resp., $j'(k-1)'k'k''(k-1)''j''$) is a substring of $F(\alpha_1)$ (resp., $F(\alpha_2)$), then the halfway-distance floor of j is $\lfloor d(j', j'') \rfloor = \lfloor 3/2 \rfloor = 1$ (resp. $\lfloor d(j', j'') \rfloor = \lfloor 5/2 \rfloor = 2$), engaged as the j-th entry of $A(\alpha_1)$ (resp., $A(\alpha_2)$).

Claim 1. Using the equivalence of n-nests $F(\alpha)$ and signatures $A(\alpha)$ provided by Theorem 4, below, construction of the tree $\mathcal{T}_k$ of Dyck nests $F(\alpha)$ is simplified by updating just one entry of $A(\beta)$ to get $A(\alpha)$, instead of using the procedure in Theorem 1 to get $F(\alpha)$ from $F(\beta)$.

Example 3. Claim 1 is exemplified in Figures 1–2 for $k = 2, 3, 4, 5, 6$. In these figures, the first column for each such k shows the k-germs $\alpha = a_{k-1} \cdots a_1$ in depth-first order of the node set of $\mathcal{T}_k$, in black except for $a_{i(\alpha)}$, which is in red; the second column shows the corresponding n-nests $F(\alpha)$ initialized in the top row as $F(0^{k-1}) =$

$$012 \cdots (k-2)(k-1)kk(k-1)(k-2) \cdots 21 = 01'2' \cdots (k-2)'(k-1)'k'k''(k-1)''(k-2)'' \cdots 2''1''),$$

(with the "prime" notation after the equal sign in accordance to Subsection 4.1) and continued from the second row on as $F(\alpha) = W^i|Y|X|Z^i$, (as in Theorem 1), where W^i and Z^i are in black, Y is in red and X is in green, and the parent β of α in $\mathcal{T}_k$ having $F(\beta) = W^i|X|Y|Z^i$; this second column has the red-green numbers underlined; the third and fourth columns have their rows as the *signatures* $B(\alpha) = B_{k-1}B_{k-2} \cdots B_2B_1$ of β (starting at the second row) and $A(\alpha) = A_{k-1}A_{k-2} \cdots A_2A_1$ of α, specified by having $B_j = B_j(\alpha)$ and $A_j = A_j(\alpha)$, for each $j \in [1, k[$, as the numbers of pairs formed by the two appearances of each integer between the two appearances of j in $F(\beta)$ and $F(\alpha)$, respectively; these third and fourth columns are determined by the black-red-green second column at each row; the fifth column, starting at the second row, is formed by four single-digit columns:

1. the value $i = i(\alpha)$ in the current application of Theorem 1; (i in red if and only $i > 1$);
2. the corresponding value of $a_i = a_{i(\alpha)}$ in $\alpha = a_{k-1}a_{k-2} \cdots a_2a_1$;
3. the corresponding value of $B_i(\alpha) = B_{i(\alpha)}(\alpha)$ in the third column;
4. the value of $A_i(\alpha) = A_{i(\alpha)}(\alpha)$ in the fourth column, with A_i in red if and if $A_i > 0$;

the sixth column is the depth-first order $o(\alpha)$ of α in $\mathcal{T}_k$; all rows of the second column, below the first row, have the substring kk (that is, $k'k''$, in terms of the appearances k' and k'' of k) either in Y (red) or in X (green); after the initial black row $F(\alpha) = F(0^{k-1})$, the substring kk is red in the two subsequent rows and becomes green in the fourth row; this corresponds to the red value $k - \ell = k - 2$ of the seventh column. For all columns but for the second one in Figures 1 and 2, each row which in the first column has k-germ $\alpha = a_{k-1} \cdots a_1$ with a_1 a local maximum (so that the following k-germ, say $\gamma = c_{k-1} \cdots c_1$, in the same first column, if any, has $c_1 = 0$) appears underlined.

5.3. Role of substrings kk in Dyck nests

Each value in the seventh column of Figures 1 and 2 equals the corresponding value of item (4) in the fifth column, expressed in terms of the number c of Theorem 1, item 2, as:

- **(a)** ℓ, if kk is red, where ℓ is the number of green pairs (j', j'') with $j > c$;
- **(b)** $k - \ell$, if kk is green, where ℓ is the sum of $c + 1$ and the number d of red pairs (j', j'') with $j > c + 1$.

For example, all cases with $d > 0$ (item (b)) in Figure 2 happen precisely for

$$(\alpha, c, d) = (01111, 2, 1),\ (11110, 3, 1),\ (11122, 3, 1),\ (11221, 2, 1),\ (12111, 2, 1),\ (12211, 2, 2),\ (12221, 2, 1).$$

Let g be the correspondence that assigns the values $A_{i(\alpha)}(\alpha)$, (in the seventh column of Figures 1 and 2), to the orders $o(\alpha)$, (in the sixth column), where α refers to k-germs.

Theorem 3. *For each k-germ $\alpha \neq 0^{k-1}$, the signatures $B(\alpha)$ and $A(\alpha)$ of the parent β (of α in $\mathcal{T}_k$), and α, respectively, differ solely at the $i(\alpha)$-th entry, that is:*

$$B_i(\alpha) = B_{i(\alpha)}(\alpha) \neq A_{i(\alpha)}(\alpha) = A_i(\alpha), \text{ while } B_j(\alpha) = A_j(\alpha),\ \forall j \neq i = i(\alpha).$$

Proof. There is a sole difference between the parent $\beta = b_{k-1} \cdots b_1$ of $\alpha = a_{k-1} \cdots a_1$ and α itself, occurring at the $i(\alpha)$-th position, whose entry is increased in one unit from β to α, that is: $a_{i(\alpha)} = b_{i(\alpha)} + 1$. The effect of this on $F(\alpha)$, namely the i-nested castling of the inner strings Y and Z of $F(\beta) = X^i|Y|Z|W^i$ into $F(\alpha) = X^i|Z|Y|W^i$, modifies just one of the halfway-distance floors $A_j = \lfloor d(j', j'')/2 \rfloor$ between the first appearance j' of the corresponding $j \in [0, k[$ in $F(\alpha)$ and its second appearance, j'', namely $A_i = \lfloor d(i', i'')/2 \rfloor$, where $i = i(\alpha)$. □

Theorem 4. *The correspondence that assigns each n-nest to its signature is a bijection.*

Proof. Let $\alpha = a_{k-1} \ldots a_2 a_1$ be a k-germ. The n-nest $F(\alpha) = c_0 c_1 \ldots c_{2k}$ has rightmost entry $c_{2k} = 1''$, so $A_1(\alpha)$ determines the position of $1'$. For example, if $A_1(\alpha) = 0$, then $c_{2k-1} = 1'$, so a_1 is a local maximum (indicated in Figures 1 and 2 by having α, $B(\alpha)$, $A(\alpha)$, $\cdots$, $o(\alpha)$, $A_{i(\alpha)}(\alpha)$ underlined). To obtain $F(\alpha)$ from $A(\alpha)$, we initialize $F(\alpha)$ as the n-string $F^0 = 00 \cdots 0$. Setting the positions of $1'', 1', 2'', 2', \ldots, (k-1)'', (k-1)'$ successively in place of the zeros of F^0 in their places from right to left according to the indications $A_1(\alpha)$, $A_2(\alpha)$, …, $A_{k-1}(\alpha)$, is done in stages: first setting the pairs (i', i'') as outermost pairs from right to left; when reaching the initial 0, we restart if necessary on the right again with the replacement of the remaining zeros by the remaining pairs (i', i'') in ascending order from right to left. Thus, given $A(\alpha)$, we recover $F(\alpha)$. □

Example 4. With $k = 6$, $A(11111) = 01122$, (resp., $A(12122) = 00201$), we go from F^0 to

0200002100001 to	(resp.,)	0300003221001 to
0240042130031 to		0366553221441
0236642135531		

,

the last row yielding four (resp., two) entries separating the two appearances $1'$ and $1''$ of $1 \in [0, k]$, namely $3', 5', 5''$ and $3''$, (resp., $4'$ and $4''$).

Theorem 4 provides a fashion of counting Catalan numbers via RGS's [2,3] different from that of [11, item (u), p. 224]. Both fashions, which are compared in [2], accompany the counting list of RGS's in reversed order. In both cases (namely Theorem 4 and item (u)), the null root RGS, 0, corresponds to the signatures $12 \cdots k$, for all $0 < k \in \mathbb{Z}$; and the last RGS for every such k corresponds to the signatures 0^k. Thus, these initial (resp., terminal) terms coincide. However, these two counting lists with same initial (resp., terminal) terms differ in general.

Theorem 5. *(1) The correspondence g, whose definition precedes Theorem 3, is extended uniquely for each $k > 1$ and k-germ α, so that in terms of α seen as an RGS, the value of $g(o(\alpha)) = A_{i(\alpha)}(\alpha)$ is expressible either as ℓ or as $k - \ell$, as in Subsection 5.3.*

(2) Registration of the value ℓ (resp., $-\ell$) at each stage in $\mathcal{S} \setminus \beta(0)$ for which $g(o(\alpha))$ is expressible as ℓ (resp., $k - \ell$) as in item (1), is performed independently of k, so it constitutes a universal single update of Dyck-nest signatures, just controlled by the RGS tree. This yields an integer sequence accompanying the natural order of RGS's in $\mathcal{S}$.

The updates mentioned in Theorem 5, item (2), will be expressed in terms of the function in display (4), to be employed in Theorems 7 and 8, respectively.

Proof. The options in item (1) depend on whether the substring $k'k''$ lies in Y (red) or in X (green). In the first case, $g(o(\alpha))$ is of the form ℓ. Otherwise, it is of the form $k-\ell$, for if k is increased to $k+1$, then the substring $(k+1)'(k+1)''$ separates k' and k'', thus adding one unit to $g(o(\alpha))$, so that $k-\ell$ becomes $(k+1)-\ell$. This happens independently of the values of k, yielding item (2). □

Example 5. The nonzero values $g(k)$ are initially as follows: $g(3)=k-2$, $g(7)=k-3$, $k(8)=1$, $g(11)=k-2$, $g(12)=k-3$, $g(17)=k-2$, $g(19)=k-4$, $g(21)=1$, $g(22)=k-3$, $g(25)=k-2$, $g(26)=1$, $g(30)=k-3$, $g(31)=1$, $g(33)=k-4$, $g(34)=2$, $g(35)=1$, $g(38)=k-2$, $g(39)=k-3$, $g(40)=k-4$, etc.

Corollary 2. *The following items hold:*

(A) *The leftmost entry in the substring W^i of $F(\alpha)=X^i|Z|Y|W^i$ is i''.*

(B) *If the substring $k'k''$ of $F(\alpha)$ appears to the left of i' in $F(\alpha)$, then $g(o(\alpha))$ equals the number of pairs (j',j'') in the interval $]i',i''[$, for all pertaining integers $j\in[1,k[$. In particular, $F(\alpha)$ ends at the substring $1'1''$ if and only if $g(o(\alpha))=0$.*

(C) *If $k'k''$ lies in $]i',i''[$ then $k'k''$ is contained in X (green substring in $F(\alpha)$, Figures 1–2) and $g(o(\alpha))=k-j$, where $j=j(\alpha)$ is determined as follows: since $i(\beta)=1+i(\alpha)$, where $\beta=\beta(\alpha)$ is the parent of α, then j is the sum of $g(o(\beta))$ (which is as in item (B)) plus the leftmost red number of $F(\alpha)$.*

Proof. The statement follows from Subsection 5.3 and Theorems 4 and 5. In particular, items (B) and (C) are equivalent to items 1 and 2 of Subsection 5.3, respectively. □

Example 6. Let $k=5$. Then, $g(21)=g(o(1110))=1$, as $]i',i''[=]2',2''[$ contains just the pair $(4',4'')$, accounting for one pair by Corollary 2(B). For $\alpha=1111$, $k'k''$ is green and $g(22)=g(o(1111))=g(o(\alpha))=k-j=k-3$, where $j=3$ is the sum of $g(o(\beta))=g(o(1110))=g(21)=1$ and the leftmost red number of $F(\alpha)$, namely 2. In addition, $g(28)=g(o(1200))=0$ has child $\alpha=1210$ with $g(o(\alpha))=g(30)=k-3$, because the leftmost red entry of $F(\alpha)$ is 3. The child $\alpha'=1220$ of α has $g(o(\alpha'))=g(33)=k-(3+1)=k-4$. However, the child $\alpha''=1230$ of α' has $g(o(\alpha''))=g(37)=0$. Now, the child 1211 of α has $g(o(1211))=1$, because $1'$ is the leftmost number of W^1 and there is only one pair of appearances of a member of $[1,k-1]=[1,4]$, namely $3'3''$, between $1'$ and $1''$.

6. Universal single updates

Now, we introduce strings A_i^j, for all pairs $(i,j)\in\mathbb{Z}^2$ with $1<i\leq j$. The entries of each A_i^j are integer pairs (ι,ζ), denoted ι_ζ, starting with 1_1, initial case of the more general notation 1_j, for $j\geq 1$. The strings A_i^j are conceived as shown in Table 1. The components ι in the entries ι_ζ represent the indices $i=i(\alpha)$ of Theorem 1 in their order of appearance in $\mathcal{S}$, and ζ is an indicator to distinguish different entries ι_ζ while ι is locally constant.

Recalling items (B) and (C) of Corollary 2, we define the updating integers $h(\alpha)$ by:

$$h(\alpha)=\begin{cases} g(o(\alpha)), & \text{if } g(o(\alpha)) \text{ is as in (B);}\\ g(o(\alpha))-k, & \text{if } g(o(\alpha)) \text{ is as in (C).}\end{cases} \quad (4)$$

Next, consider the infinite string A of integer pairs i_ζ formed as the concatenation

$$A=A_1^1|A_2^2|\cdots|A_j^j|\cdots=*1_1|A_2^2|\cdots|A_j^j|\cdots, \quad (5)$$

with $A_1^1=*|1_1=*1_1$ standing for the first two lines in tables as in Figures 1–2, where $*$, standing for the root of $\mathcal{T}_k$, represents the first such line, and A_1^1 represents the second one.

Example 7. Illustrating (5), Table 2 has its double-line heading formed by the subsequent terms of a suffix of A. The third heading line is formed first by the root $*$ of all trees $\mathcal{T}_k$ and then by the successive parameters $i=i(\alpha)>1$ initiating the substrings in the second line. The fourth line contains the values $h(\alpha)$ for the

Table 1. Introduction of strings A_i^j, for all pairs $(i, j) \in \mathbb{Z}^2$ such that $1 < i \leq j$.

$A_2^2 = 2_1\|1_1\|1_2;$
$A_2^3 = 2_2\|1_1\|1_2\|1_3;$
$A_2^4 = 2_3\|1_1\|1_2\|1_3\|1_4;$
$A_2^5 = 2_4\|1_1\|1_2\|1_3\|1_4\|1_5;$
$\cdots$
$A_3^3 = 3_1\|1_1\|A_2^2\|A_2^3 = 3_1\|1_1\|2_1 1_1 1_2\|2_2 1_1 1_2 1_3;$
$A_3^4 = 3_2\|1_1\|A_2^2\|A_2^3\|A_2^4 = 3_2\|1_1\|2_1 1_1 1_2\|2_2 1_1 1_2 1_3\|2_3 1_1 1_2 1_3 1_4;$
$A_3^5 = 3_3\|1_1\|A_2^2\|A_2^3\|A_2^4\|A_2^5 = 3_2\|1_1\|2_1 1_1 1_2\|2_2 1_1 1_2 1_3\|2_3 1_1 1_2 1_3 1_4\|2_4 1_1 1_2 1_3 1_4 1_5;$
$\cdots$
$A_4^4 = 4_1\|1_1\|A_2^2\|A_3^3\|A_3^4 = 4_1\|1_1\|2_1 1_1 1_2\|3_1 1_1 2_1 1_1 1_2 2_2 1_1 1_2 1_3\|3_2 1_1 2_1 1_1 1_2\|2_2 1_1 1_2 1_3\|2_3 1_1 1_2 1_3 1_4;$
$A_4^5 = 4_2\|1_1\|A_2^2\|A_3^3\|A_3^4\|A_3^5;$
$\cdots$
$A_5^5 = 5_1\|1_1\|A_2^2\|A_3^3\|A_4^4\|A_4^5;$
$A_5^6 = 5_2\|1_1\|A_2^2\|A_3^3\|A_4^4\|A_4^5\|A_4^6;$
$\cdots$
$A_i^{i+j} = i_{i+j}\|1_1\|A_2^2\| \cdots \|A_{i-1}^{i-1}\|A_{i-1}^i\| \cdots \|A_{i-1}^{i+j}, \forall 0 < i \in \mathbb{Z}, \forall 0 < j \in \mathbb{Z}.$

parameters $i(\alpha) > 1$ of the third line. In every column, the values below that line are the values $h(\alpha)$ for RGS's α of the successive k-germs α with $i = i(\alpha) = 1$. Thus, below the third heading line, the values of each column represent the updates $h(\alpha)$ corresponding to all the maximal paths of trees $\mathcal{T}_k$ that, after its first node α, has all other nodes α with $i = i(\alpha) = 1$. Note that A_1^1 is represented as $[{}^{*}_{1_1}]$. In the same way, we use notations $[{}^{3_1}_{1_1}]$ and $[{}^{4_1}_{1_1}]$, that could be generalized to $[{}^{j_1}_{1_1}]$.

Each prefix of A corresponds to all k-germs representing a specific RGS α for increasing values of $k > 1$, and is assigned the value $h(\alpha)$ to be its updating integer, in accordance to Corollary 2 but for the initial position, that is assigned an asterisk * to represent all the roots of the trees $\mathcal{T}_k$, for all $k > 1$. More specifically, all prefixes of A with Catalan-number lengths C_k are the strings formed by locations $i = i(\alpha)$ in the natural order of the corresponding trees $\mathcal{T}_k$, while the values $h(\alpha)$ of the participating RGS's α occupy the subsequent positions down below the heading lines.

Table 2. Exemplification of $A = A_1^1|A_2^2| \cdots |A_j^j| \cdots = *1_1|A_2^2| \cdots |A_j^j| \cdots$

A_1^1	A_2^2	A_3^3			A_4^4								
$[{}^{*}_{1_1}]$	A_2^2	$[{}^{3_1}_{1_1}]$	A_2^2	A_2^3	$[{}^{4_1}_{1_1}]$	A_2^2	A_3^3			A_3^4			
$*$	2	3	2	2	4	2	3	2	2	3	2	2	2
$*$	0	0	-3	0	0	0	−4	1	0	0	−3	−4	0
0	−2	0	1	−2	0	−2	0	−3	−2	0	1	2	−2
	0		0	−3		0		0	1		0	1	−3
				0					0			0	−4
													0

In Figure 3, the heading line of the top layer extends and continues the third heading line of Table 2, its entries leading corresponding columns of values $h(\alpha)$, for $k < 7$. This setting can be also seen as a left-to-right list representation of $\mathcal{T}_6$ in Table 3, whose nodes are pairs $(i(\alpha), h(\alpha))$ for the successive RGS's α in $\mathcal{S}$, where if some $h(\alpha)$ equals a negative integer $-\eta < 0$, then is shown as $\bar{\eta}$, with the minus sign preceding η shown as a bar over η. With such notation, the leftmost column of Table 3 shows the children of the root $(*, *)$ of $\mathcal{T}_6$. The adequately indented subsequent columns show the remaining descendant nodes at increasing distances from $(*, *)$. Also in Table 3, horizontal lines separate the node sets of $\mathcal{T}_3 - (*, *)$, $\mathcal{T}_4 - \mathcal{T}_3$, $\mathcal{T}_5 - \mathcal{T}_4$ and $\mathcal{T}_6 - \mathcal{T}_5$.

By reading the entries of the successive columns of Table 2, and more extensively in Figure 3, etc., and then writing them from left to right, we obtain the integer sequence $h(\mathcal{S})$ formed by the values $h(\alpha)$ associated to the RGS's α of $\mathcal{S}$. For example, starting with Table 2, we have that

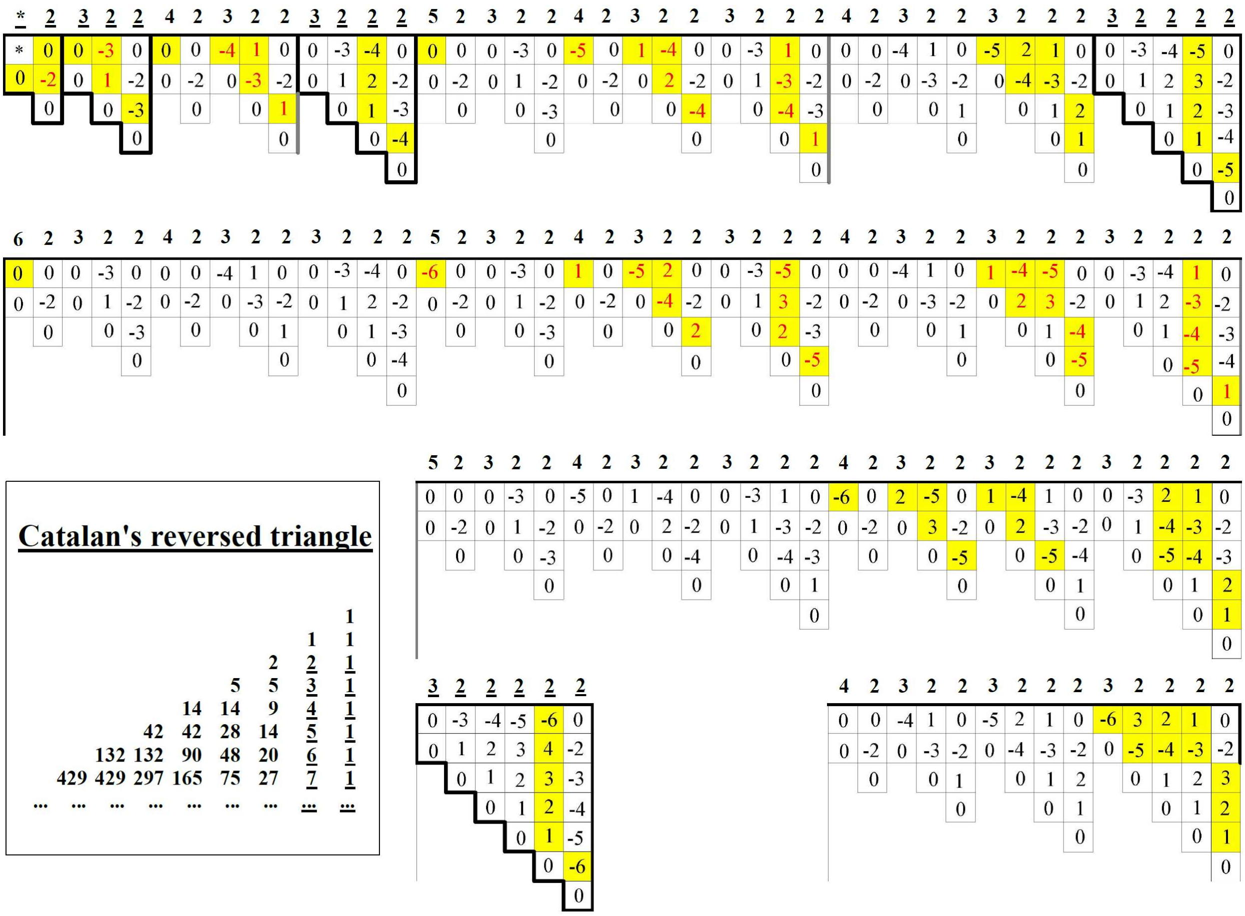

Figure 3. Extension of Table 2 and partial view of Δ', for $k = 2, 3, 4, 5, 6, 7$

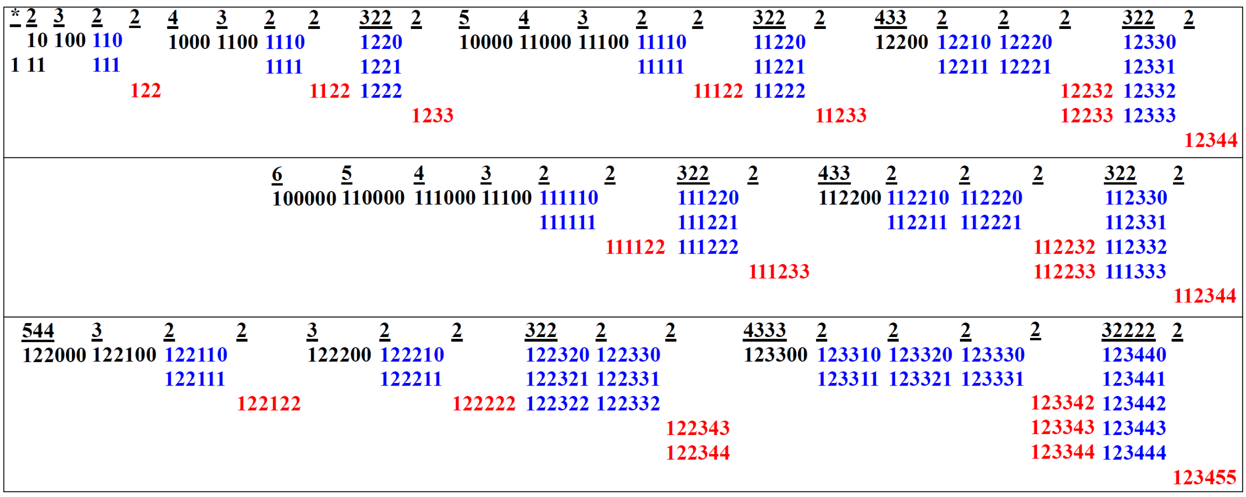

Figure 4. Members of Φ_1, for $k = 2, 3, 4, 5, 6, 7$

$$\begin{aligned} h(\mathcal{S}) &= (h(0), \ldots, h(41), \ldots) \\ &= (*, 0, 0, -2, 0, 0, 0, -3, 1, 0, 0, -2, -3, 0, 0, 0, 0, -2, 0, -4, 0, 1, \\ &\quad -3, 0, 0, -2, 1, 0, 0, 0, -3, 1, 0, -4, 2, 1, 0, 0, -2, -3, -4, 0, \ldots). \end{aligned}$$

Table 3. Left-to-right list representation of $\mathcal{T}_6$ whose nodes are pairs $(i(\alpha), h(\alpha))$ for the subsequent RGS's α in $\mathcal{S}$, and if some $h(\alpha)$ equals a negative integer $-\eta < 0$, then it is shown as $\bar{\eta}$. The leftmost column shows the children of the root $(*,*)$ of $\mathcal{T}_6$.

$(1,0)$														
$(2,0)$	$(1,\bar{2})$	$(1,0)$												
$(3,0)$	$(1,0)$													
	$(2,\bar{3})$	$(1,1)$	$(1,0)$											
		$(2,0)$	$(1,\bar{2})$	$(1,\bar{3})$	$(1,0)$									
$(4,0)$	$(1,0)$													
	$(2,0)$	$(1,\bar{2})$	$(1,0)$											
	$(3,\bar{4})$	$(1,0)$												
		$(2,1)$	$(1,\bar{3})$	$(1,0)$										
			$(2,0)$	$(1,\bar{2})$	$(1,1)$	$(1,0)$								
$(5,0)$	$(1,0)$													
	$(2,0)$	$(1,\bar{2})$	$(1,0)$											
	$(3,0)$	$(1,0)$												
		$(2,\bar{3})$	$(1,1)$	$(1,0)$										
			$(2,0)$	$(1,\bar{2})$	$(1,\bar{3})$	$(1,0)$								
	$(4,\bar{5})$	$(1,0)$												
		$(2,0)$	$(1,\bar{2})$	$(1,0)$										
		$(3,1)$	$(1,0)$											
			$(2,\bar{4})$	$(1,2)$	$(1,0)$									
				$(2,0)$	$(1,\bar{2})$	$(1,\bar{4})$	$(1,0)$							
			$(3,0)$	$(1,0)$										
				$(2,\bar{3})$	$(1,1)$	$(1,0)$								
					$(2,1)$	$(1,\bar{3})$	$(1,\bar{4})$	$(1,0)$						
						$(2,0)$	$(1,\bar{2})$	$(1,\bar{3})$	$(1,1)$	$(1,0)$				
		$(4,0)$	$(1,0)$											
			$(2,0)$	$(1,\bar{2})$	$(1,0)$									
			$(3,\bar{4})$	$(1,0)$										
				$(2,1)$	$(1,\bar{3})$	$(1,0)$								
					$(2,0)$	$(1,\bar{2})$	$(1,1)$	$(1,0)$						
				$(3,\bar{5})$	$(1,0)$									
					$(2,2)$	$(1,\bar{4})$	$(1,0)$							
						$(2,1)$	$(1,\bar{3})$	$(1,1)$	$(1,0)$					
							$(2,0)$	$(1,\bar{2})$	$(1,2)$	$(1,1)$	$(1,0)$			
					$(3,0)$	$(1,0)$								
						$(2,\bar{3})$	$(1,1)$	$(1,0)$						
							$(2,\bar{4})$	$(1,2)$	$(1,1)$	$(1,0)$				
								$(2,\bar{5})$	$(1,3)$	$(1,2)$	$(1,1)$	$(1,0)$		
									$(2,0)$	$(1,\bar{2})$	$(1,\bar{3})$	$(1,\bar{4})$	$(1,\bar{5})$	$(1,0)$

6.1. Sequence of updates of Dyck-nest signatures

The numbers in Italics in Table 2 initiate the subsequence $h(\Phi_1)$ of h-values of a subsequence Φ_1 of $\mathcal{S}$, that will allow the continuation of the sequence of updates of the Dyck-nest signatures. These numbers reappear and are extended, in yellow squares in Figure 3. Expressing $h(\Phi_1)$ with its initial terms as in Table 2, we may write $h(\Phi_1) = (h(j); j = 1, 2, 3, 5, 7, 8, 12, 14, 19, 21, 22, 27, 34, 35, 36, 41, \ldots) = (0, 0, -2, 0, -3, 1, -3, 0, -4, 1, -3, 1, -4, 2, 1, -4, \ldots)$.

In order to use Φ_1, we recur to *Catalan's reversed triangle* Δ', whose initial lines, for $k = 0, 1, \ldots, 7$, are shown on the lower left enclosure of Figure 3 and is obtained in general from Catalan's triangle Δ [2] by reversing its lines, so that with notation from [2], the portion of Δ' shown in Figure 3 may be written as in Table 4.

6.2. Formations

Both in Table 2 and at the top layer of Figure 3, we have the representations (to be called *formations*) of:
(i) (A_1^1), namely the leftmost column, (just $C_1 = \tau_1^1 = 1$ columns), with $C_2 = 2$ entries;

Table 4. An initial detailed portion of Catalan's reversed triangle Δ'.

								$\tau_0^0 = 1$
							$\tau_1^1 = 1$	$\tau_0^1 = 1$
						$\tau_2^2 = 2$	$\tau_1^2 = 2$	$\tau_0^2 = 1$
					$\tau_3^3 = 5$	$\tau_2^3 = 5$	$\tau_1^3 = 3$	$\tau_0^3 = 1$
				$\tau_4^4 = 14$	$\tau_3^4 = 14$	$\tau_2^4 = 9$	$\tau_1^4 = 4$	$\tau_0^4 = 1$
			$\tau_5^5 = 42$	$\tau_4^5 = 42$	$\tau_3^5 = 28$	$\tau_2^5 = 14$	$\tau_1^5 = 5$	$\tau_0^5 = 1$
		$\tau_6^6 = 132$	$\tau_5^6 = 132$	$\tau_4^6 = 90$	$\tau_3^6 = 48$	$\tau_2^6 = 20$	$\tau_1^6 = 6$	$\tau_0^6 = 1$
	$\tau_7^7 = 429$	$\tau_6^7 = 429$	$\tau_5^7 = 297$	$\tau_4^7 = 165$	$\tau_3^7 = 75$	$\tau_2^7 = 27$	$\tau_1^7 = 7$	$\tau_0^7 = 1$
......	......	......	......	......	......	......	......	......

(ii) $(A_1^1|A_2^2)$, namely the $C_2 = \tau_2^2 = \tau_1^2 = 2$ leftmost columns, with a total of $C_3 = 5$ entries;

(iii) $(A_1^1|A_2^2|A_3^3)$, namely the $C_3 = \tau_3^3 = \tau_2^3 = 5$ leftmost columns, with $C_4 = 14$ entries;

(iv) $(A_1^1|A_2^2|A_3^3|A_4^4)$, namely the $C_4 = \tau_4^4 = \tau_3^4 = 14$ columns in Table 2 or the $C_4 = 14$ leftmost columns in Figure 3, with a total of $C_5 = 42$ entries;

and

(v) $(A_1^1|A_2^2|A_3^3|A_4^4|A_5^5)$, namely the top $C_5 = \tau_5^5 = \tau_4^5 = 42$ columns in Figure 3, with a total of $C_6 = 132$ entries.

These five formations correspond respectively to the trees $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$, $\mathcal{T}_5$ and $\mathcal{T}_6$. We subdivide the sets of respective columns according to the corresponding lines of Δ' considered as integer partitions Δ'_{k-2}, namely: $\Delta'_0 = (1)$, $\Delta_1 = (1,1)$, $\Delta'_2 = (2,2,1)$, $\Delta'_3 = (5,5,3,1)$, $\Delta'_4 = (14,14,9,4,1)$, and $\Delta'_5 = (42,42,28,14,5,1)$ to be discussed subsequently.

Figure 3 contains the continuation for $k = 7$ of the commented formations, extending the mentioned top layer of $\tau_5^5 = 42$ columns with a second and third layers (having $\tau_4^5 = 42$ and $\tau_3^5 = 28$ columns, respectively) and then with two additional parts in the fourth layer (having $\tau_2^5 = 14$ on the right, and $\tau_1^5 + \tau_0^5 = 5 + 1$ columns on the left, respectively), and representing all of $\mathcal{T}_7$. These numbers of columns, namely (42,42,28,14,5,1), correspond to the sixth line Δ'_5 of Δ', namely $\Delta'_5 = (\tau_5^5, \tau_4^5, \tau_3^5, \tau_2^5, \tau_1^5, \tau_0^5)$.

Still in Figure 3 for $\mathcal{T}_7$, the first $\tau_5^5 = 42$ columns (top layer) have lengths correspondingly equal to the lengths of the subsequent $\tau_4^5 = 42$ columns (second layer, delimited on the right by a thick gray vertical segment). Of these, the final 28 columns have lengths correspondingly equal to the lengths of the subsequent $\tau_3^5 = 28$ columns (third layer). Of these, the final 14 columns have lengths correspondingly equal to the lengths of the subsequent $\tau_2^5 = 14$ columns (fourth right layer). Of these, the final 5 columns have lengths correspondingly equal to the lengths of the subsequent $\tau_1^5 = 5$ columns (in the fourth left layer). It remains just $\tau_0^5 = 1$ column, formed by $k = 7$ values of $h(\alpha)$. The said numbers of columns account for the partition $\Delta'_5 = (42,42,28,14,5,1)$, representing all the columns associated with the maximal paths of $\mathcal{T}_7$ formed by nodes associated with RGS's α with $i(\alpha) = 1$. Similar cases are easy to obtain in relation to $\mathcal{T}_k$, for $k < 7$, where thick gray vertical segments delimit on the right the 14 (resp., 5) columns next to the first 14 (resp., 5) columns; (the same could have been done for the two columns next to the first two columns). A similar observation holds for every other row of Δ'.

Some of the heading numbers in Figure 3 appear underlined, corresponding to the final $k - 1 = \tau_1^{k-2} + \tau_0^{k-2} = (k-2) + 1$ columns for each exemplified $\mathcal{T}_k$. The resulting column sets appear encased with a thicker border.

6.3. Main results

The subsequence Φ_1 of $\mathcal{S}$, a member of a family of subsequences $\{\Phi_j; 1 \leq j \in \mathbb{Z}\}$ satisfying for $j > 1$ the rules 1–3 below, is such that $i(\Phi_1)$ is the subsequence of $i(\mathcal{S})$ formed by all indices $i(\alpha)$ larger than 1, exemplified in the heading line of Figure 3. The mentioned rules 1–3 are as follows:

1. the first term of Φ_j is

$$\phi_1 = \begin{cases} \text{the RGS } 1, & \text{if } j = 1; \\ \text{the smallest RGS with suffix } (j-1)(j-1), & \text{if } j > 1; \end{cases}$$

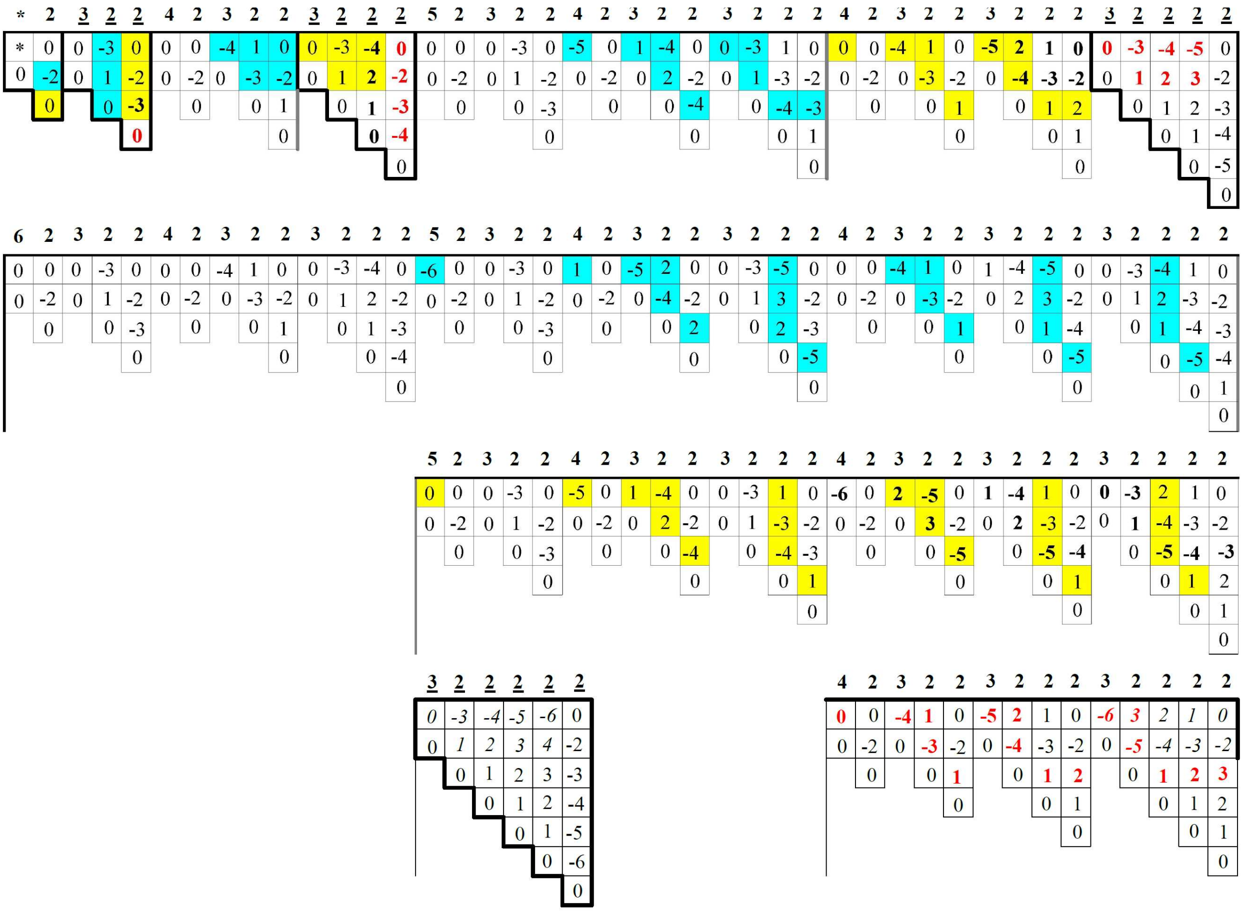

Figure 5. Information for $\Phi_2, \Phi_3, \Phi_4, \Phi_5$

Table 5. Example for Theorem 9, where the lists corresponding to $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_4$ are represented according to the respective pairs $(\alpha, h(\alpha))$ indicating column pairs $(\alpha, h(\alpha))$ and $(\alpha_j, h(\alpha_j))$, for $j = 1,2,3,4,5$, as shown in the heading line of the table.

α	$h(\alpha)$	α_1	$h(\alpha_1)$	α_2	$h(\alpha_2)$	α_3	$h(\alpha_3)$	α_4	$h(\alpha_4)$	α_5	$h(\alpha_5)$
0	$*$	00	$*$	10	0						
1	0	01	**0**	11	−2	12	**0**				
0	$*$	000	0	100	0						
1	0	001	0	101	0						
10	0	010	**0**	110	−3	120	**0**				
11	−2	011	**−2**	111	*1*	121	**−2**				
12	0	012	0	112	0	122	1	123	0		
0	$*$	0000	$*$	1000	0						
1	0	0001	0	1001	0						
10	0	0010	0	1010	0						
11	−2	0011	−2	1011	−2						
12	0	0012	0	1012	0						
100	0	0100	**0**	1100	*−4*	1200	**0**				
101	0	0101	0	1101	0	1201	0				
110	−3	0110	**−3**	1110	*1*	1210	*−3*				
111	1	0111	**1**	1111	−3	1211	**1**				
112	0	0112	0	1112	0	1212	0				
120	0	0120	0	1120	0	1220	−4	1230	0		
121	−2	0121	−2	1121	**−2**	1221	2	1231	**−2**		
122	−3	0122	**−3**	1122	*1*	1222	**1**	1232	*−3*		
123	0	0123	0	1123	0	1223	**0**	1233	*−4*	1234	**0**

2. if $\alpha = a_{k-1}\cdots a_1 \in \Phi_j$ and either $a_1 = 0$ or $a_{k-1}\cdots a_2a_1' \notin \Phi_j$ for every $a_1' < a_1$, then $\alpha|j \in \Phi_j$ for $j \in [0, a_1]$; in that case, if $\alpha_{j'} \in \Phi_j$ with $\alpha_{j'} = a_{k-1}\cdots a_2(a_1 + j')$, for $1 \le j' \in \mathbb{Z}$, then $\alpha_{j'}|j \in \Phi_j$;
3. for each maximal subsequence $S = (\iota, 2, \ldots, 2)$ of $i(\mathcal{S})$ ($\iota > 2$), if there are z penultimate terms $i = 2$ of S ($z > 0$) heading maximal vertical prefixes of a fixed length y in $h(\Phi_j)$ ($y > 0$) and ending at $h(\alpha_j) = h(a_{k-1}\cdots a_3(y+j)y)$ ($j \in [0, z[$), then $\alpha_{j'} = a_{k-1}\cdots a_3(y+z)j' \in \Phi_j$, for $j' \in]y, y+z]$, yielding a vertical suffix $\{h(\alpha_{j'}); j' \in]y, y+z]\}$.

Example 8. The three rectangular enclosures of Figure 4 contain in left-to-right columnwise form (only showing those columns with yellow squares in Figure 3) the subsequence Φ_1 of $\mathcal{S}$ in Subsection 6.1, (of RGS's α in yellow squares). Such enclosures contain in red the RGS's for the prefixes in item 3 above, and in blue the RGS's for the suffixes.

The columns in the formations of Subsection 6.2, as in Figure 3, end up with null values $h(\alpha) = 0$, which correspond to the terminal nodes α of maximal paths that after their initial nodes β with $i(\beta) > 1$, have the remaining nodes β' with $i = i(\beta') = 1$. Clearly, the associated nodes α have degree 1 in the pertaining trees $\mathcal{T}_k$.

Theorem 6. *Let α be a node of $\mathcal{T}_k$. Then,*

1. *if α is a terminal node of a maximal path of $\mathcal{T}_k$ whose initial node β has $i(\beta) > 1$ and whose remaining nodes γ have $i(\gamma) = 1$, then $g(\alpha) = 0$;*
2. *if $\alpha = a_{k-1}\cdots a_1$ with $a_{k-1} = 1$ and $a_j = 0$, for $j = 1, \ldots, k-2$, then $g(\alpha) = 0$.*

Proof. Item 1 in the statement arises because of the presence of the substring $1'1''$ in $F(\alpha)$. Item 2 arises because of the presence of all substrings $j'j''$ in $F(\alpha)$, for $j = 1, \ldots, k-1$. □

Theorem 7. *Let α_1 be a node of $\mathcal{T}_k$. Then, $\alpha_1' = 1|\alpha_1$ is a node of $\mathcal{T}_{k+1}$ and*

1. *if $h(\alpha_1) \in \Phi_1$, then $h(\alpha_1') \in \Phi_1$ and $h(\alpha_1') = k - h(\alpha_1)$;*
2. *if $h(\alpha_1) \notin \Phi_1$, then $h(\alpha_1') \notin \Phi_1$ and $h(\alpha_1') = h(\alpha_1)$.*

Proof. Item 1 in the statement occurs exactly when the substring $k'k''$ in $F(\alpha)$ changes position from one side of $1'$ to the opposite side in the procedure of Theorem 1 starting at the parent β of α and ending at α. Item 2 occurs exactly when that is not the case. □

Example 9. Since $\alpha_1 = 1$ is a node of $\mathcal{T}_2$ as in Theorem 6 item 1, then $\alpha_1' = 1|\alpha_1 = 11$ is a node of $\mathcal{T}_3$ with $h(\alpha_1') = h(11) = h(1) - k = 0 - 2 = -2 \in \Phi_1$, by Theorem 7 item 1. This is indicated by $h(1) = 0$ in the upper leftmost yellow square in Figure 3 and its accompanying $h(11) = -2$ as the upper leftmost red integer in the figure. Note that this pattern is continued by associating each yellow square in Figure 3 to a corresponding red integer for all $k < 7$. We can annotate this via the successive pairs $(\alpha_1, h(\alpha_1))$ taken by reading the data in Figure 3 from left to right and from top downward:

$$(1(0),11(-2)),(10(0),110(-3)),11(-2),111(1)),(100(0),1100(-4)),(110(-3),1110(1)),(111(1),1111(-3)),(122(-3),1122(1)).$$

The last pair here arises from $h(122) = -3$, which follows from Corollary 3, below.

Theorem 8. *Let $1 < j \le k \in \mathbb{Z}$. Let $\alpha_j = 1\cdots(j-1)(j-1)a_{k-j-1}\cdots a_1$ be a node of $\mathcal{T}_k$. Then, $\alpha_j' = 1\cdots(j-1)ja_{k-j-1}\cdots a_1$ is a node of $\mathcal{T}_k$ and*

1. *if $h(\alpha_j) \in \Phi_j$, then $h(\alpha_j') = k - h(\alpha_j)$;*
2. *if $h(\alpha_j) \notin \Phi_j$, then $h(\alpha_j') = h(\alpha_j)$.*

Proof. Similar to the proof of Theorem 7. □

Corollary 3. *Let $1 \le k \in \mathbb{Z}$. Let $\alpha_2 = 11a_{k-3}\cdots a_1$ be a node in $\mathcal{T}_k$. Then, $\alpha_2' = 12a_{k-3}\cdots a_1$ is a node of $\mathcal{T}_k$ and*

1. *if $h(\alpha_2) \in \Phi_2$, then $h(\alpha_2') = k - h(\alpha_2)$;*

2. *if $h(\alpha_2) \notin \Phi_2$, then $h(\alpha'_2) = h(\alpha_2)$.*

Example 10. Applying Corollary 3 to $\alpha_2 = 11, 110, 111, 112$, with respective $h(\alpha_2) = -2, -3, 1, 0 \in \Phi_2$ yields $\alpha'_2 = 12, 120, 121, 122$ with respective $h(\alpha'_2) = 0, 0, -2, -3$. In Figure 5, the RGS's α_2 are shown in light-blue squares while the corresponding RGS's α'_2 are shown in yellow squares. Figure 5 extends this coloring for $k \le 7$.

Corollary 4. *Let $1 \le k \in \mathbb{Z}$. Let $\alpha_3 = 122a_{k-4} \cdots a_1$ be a node of $\mathcal{T}_k$. Then, $\alpha'_3 = 123a_{k-4} \cdots a_1$ is a node of $\mathcal{T}_k$ and*

1. *if $h(\alpha_3) \in \Phi_3$, then $h(\alpha'_3) = k - h(\alpha_3)$;*
2. *if $h(\alpha_3) \notin \Phi_3$, then $h(\alpha'_3) = h(\alpha_3)$.*

Example 11. Applying Corollary 4 to $\alpha_3 = 122, 1220, 1221, 1222, 1223$ with respective $h(\alpha_3) = -3, -4, 2, 1, 0 \in \Phi_3$ yields $\alpha'_3 = 123, 1230, 1231, 1232, 1233$ with respective $h(\alpha'_3) = 0, 0, -2, -3, -4$. In Figure 5, the RGS's α_3 are shown in thick black while the corresponding RGS's α'_3 are shown in thick red. Moreover, Figure 5 extends this font treatment for $k \le 7$. For $k = 7$, numbers in Italics in Figure 5 corresponds to members of Φ_4.

Both the integer-valued functions $i = i(\alpha)$ of Theorem 1 and $h = h(\alpha)$ of display (4) have the same domain, $\mathcal{S} \setminus \beta(0)$. A *partition of a string A* is a sequence of substrings $A_1, A_2, \ldots, A_n$ whose concatenation $A_1|A_2|\cdots|A_n$ is equal to A.

Theorem 9. *The following items hold.*

(A) *The node set of $\mathcal{T}_{k+1}$ is given by the string $A^k_k = A^1_1|A^2_2|\cdots|A^{k-1}_{k-1}|\ A^k_{k-1}$, with partition $\{A^1_1, A^2_2, \ldots, A^{k-1}_{k-1}, A^k_{k-1}\}$, each A^j_i as a column set as in Table 2 and Figures 3–5, refined by splitting the last column A^k_{k-2} of A^k_{k-1} into the set B^k_{k-2} of its first $k-1$ entries and the set C^k_{k-2} of its last entry, $a_{k-1}a_{k-2}\cdots a_1 = 12\cdots(k-1)$. The sizes $|A^1_1|$, $|A^2_2|$, …, $|A^{k-1}_{k-1}|$, $|B^k_{k-2}|$, $|C^k_{k-2}|$ form the line Δ'_{k-1} of Δ'.*

(B) *The sequence $h(\mathcal{S} \setminus \beta(0))$ is generated by stepwise consideration of the trees $\mathcal{T}_{k+1}$, ($1 \le k \in \mathbb{Z}$). In the k-th step, the determinations in Theorems 7 and 8 are to be performed in the natural order of the $(k+1)$-germs α_j. More specifically, the k-step completes those determinations, namely $(\alpha_j, h(\alpha_j))\ \ (\alpha'_j, h(\alpha'_j))$, for the lines of Δ' corresponding to the sets A^j_j ($j = 1, \ldots, k-1$), and ends up with the determinations $(\alpha_k, h(\alpha_k))\ \ (\alpha'_k, h(\alpha'_k))$ in the line corresponding to B^k_{k-2} and $(\alpha_{k+1}, h(\alpha_{k+1}))\ \ (\alpha'_{k+1}, h(\alpha'_{k+1}))$ in the final line, corresponding to C^k_{k-2}.*

Proof. Item (A) represents the set of nodes of $\mathcal{T}_{k+1}$ via A^k_k and Δ'_{k-1}. This is used in item (B) to express the stepwise nature of the generation of the sequence $h(\mathcal{S} \setminus \beta(0))$. The methodology in the statement is obtained by integrating steps applying Theorems 7 and 8 in the way prescribed, that yields the correspondence with the lines of Δ'. □

Example 12. Theorem 9 is exemplified via Table 5, where the lists corresponding to $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_4$ are represented according to the respective pairs $(\alpha, h(\alpha))$ indicating column pairs $(\alpha, h(\alpha))$ and $(\alpha_j, h(\alpha_j))$, for $j = 1, 2, 3, 4, 5$, as shown in the heading line of the table.

The first pair, $(\alpha, h(\alpha))$ shows RGS's α in each case and their corresponding $h(\alpha)$. The following pair, $(\alpha_1, h(\alpha_1))$, shows the k-germs α_1 corresponding to the RGS's α of the first column and $h(\alpha_1) = h(\alpha)$ but in bold trace if corresponding to a yellow square as in Figure 3; in that case, the subsequent determinations $(\alpha_1, h(\alpha_1))\ \ (\alpha'_1, h(\alpha'_1))$ have the corresponding $h(\alpha'_1)$ in Italics. This is the case of $h(01) = 0$ in bold trace and $h(11) = -2$ in Italics, that we may indicate "$h(01) = 0\ _1\ (11) = -2$". If a determination $(\alpha_2, h(\alpha_2))\ \ (\alpha'_2, h(\alpha'_2))$ happens, then the numbers in Italics are assigned on their right to numbers in bold trace, again. The cases with bold trace and Italics in Table 5 can then be summarized as follows:

$h(01) = 0\ _1\ h(11) = -2\ _2\ h(12) = 0$, $h(011) = -2\ _1\ h(111) = 1\ _2\ h(121) = -2$,
$h(010) = 0\ _1\ h(110) = -3\ _2\ h(120) = 0$, $h(1000) = 0\ _1\ h(1100) = -4\ _2\ h(1200) = 0$,
$h(0110) = -3\ _1\ h(1110) = 1\ _2\ h(1210) = -3$, $h(1121) = -2\ _2\ h(1221) = 2\ _3\ h(1231) = -2$,
$h(0111) = 1\ _1\ h(1111) = -3\ _2\ h(1211) = 1$, $h(0122) = -3\ _1\ h(1122) = 1\ _2\ h1222) = 1\ _3\ h(1232) = -3$,
$h(1223) = 0\ _3\ h(1233) = -4\ _4\ h(1234) = 0$.

Corollary 5. *The sequence of pairs $(a(S\backslash\beta(0)), h(S\backslash\beta(0)))$ allows to retrieve any vertex v in O_k (resp., M_k) by locating its oriented n- (resp., $2n$-) cycle in the cycle-factor of [4,5] or in the $\mathbb{Z}_n$- (resp., $\mathbb{D}_n$-) classes as in Section 1, and then locating v departing from the anchored Dyck word in such cycle or class; the sequence also allows to enlist all such vertices v by ordering their cycles (resp., classes), including all vertices in each such cycle (resp., class), starting with the corresponding anchored Dyck word.*

Proof. The function $a(S \setminus b(0))$, arising from Theorem 1, yields the required update locations, while the function $h(\mathcal{S}\backslash\beta(0))$ yields the specific updates, as determined in Theorem 9. This produces the corresponding signatures. Then, Theorem 3 allows to recover the original Dyck words from those signatures, and thus the vertices of O_k (resp., M_k) by local translation in their containing cycles in the mentioned cycle-factors, or cyclic (resp., dihedral) classes. □

6.4. Asymptotic behavior

It is known that asymptotically the Catalan numbers C_k grow as $\frac{4^k}{k^{\frac{3}{2}}\sqrt{\pi}}$, which is the limit of the single-update process that takes to the determination of all Dyck words of length $n = 2k+1$, as k tends to infinity. By Corollary 5, an orderly determination of all the vertices of O_k, resp., M_k, is then asymptotically $\frac{4^k}{k^{\frac{3}{2}}\sqrt{\pi}}(2k+1)$, resp., $\frac{4^k}{k^{\frac{3}{2}}\sqrt{\pi}}(4k+2)$.

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