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*Article*

# On some topological indices of R-graphs via graph operations

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**Abstract:** The topological index is a molecular property that is determined from a chemical compound's molecular graph. Topological indices are numerical graph parameters that inform us about the topology of the graph and are generally graph invariants. In this paper, we consider some topological indices based on the second distance of each vertex of the graph $\alpha$ and the number of unordered pairs of vertices $\{s, q\} \subseteq V(\alpha)$ which are at distance 3 in $\alpha$. These indices are called the leap Zagreb index and the Wiener polarity index, respectively. we compute these indices of $R$-vertex join and $R$-edge join of graphs.

**Keywords:** leap Zagreb index, Wiener polarity index, R-vertex and R-edge join

**MSC:** 05C10, 05C12, 05C90.

## 1. Introduction

In terms of graph theory, a molecular graph is a representation of a chemical compound's structural formula. The vertices of this graph represent the compound's atoms, while the edges represent chemical bonds. Let a molecular graph $\alpha$ have vertex set $V(\alpha)$ and edge set $E(\alpha)$. Several vertices show the order, and several edges show the size of the graph $\alpha$. We denote the order and size of graph $\alpha$ by $|V(\alpha)| = n$ and $|E(\alpha)| = m$, respectively. If edge connects vertices $q$ and $s$, they are said to be adjacent and can be written as $e = sq \in E(\alpha)$. A path $P_n$, is a graph on $n$ vertices $s_1, s_2, s_3, \ldots, s_n$ with edge set $\{s_j s_{j+1} \mid 1 \leq j \leq n_1\}$. The number of edges in a path $P_n$ determines its length. If a path connects any pair of vertices in the graph, then the graph $\alpha$ is connected, and it is disconnected if at least one pair of vertices is not linked by a path. A graph $\alpha$ having vertex set $\{s_j \mid 1 \leq j \leq n\}$ and edge set $\{s_i s_{j+1} \mid 1 \leq j \leq n-1\} \cup \{s_n s_1\}$ is called cyclic $C_n$ graph. The shortest path between two vertices $q$ and $s$ in graph $\alpha$ is called the distance between those vertices and is denoted as $d_\alpha(q, s)$. The degree of vertex $q$ in graph $\alpha$ is denoted as $d_\alpha(q|K) = N_\alpha(q|k)$. The term $N_\alpha(q|k)$ is defined as the number of vertices lying at distance $k$ from vertex $q$. The terms $p_\alpha(s|2)$ and $t(s)$ are defined as the number of distinct paths of distance 2 in graph $\alpha$ from vertex $s$ to other vertices and the number of triangles at $s$ in graph $\alpha$, respectively.

In chemistry, topology offers a way within the constraints of three-dimensional space to explain and predict the molecular structure. Considering the chemical bonding determinants and the chemical properties of the atoms. Topology offers a way to understand how the ethereal wave functions of atoms ought to fit together. The topology of molecules is a part of mathematical chemistry that deals with the algebraic description of chemical compounds to allow them to be described in a specific and simple way.

The topological index is a molecular property that is determined from a chemical compound's molecular graph [1,2]. Topological indices are numerical graph parameters that inform us about the topology of the graph and are generally graph invariants.

Classification models used in chemical, biological, and engineering sciences are known as quantitative structure-activity relationship models. Quantitative structure activity relationship regression models, like other regression models, relate a collection of predictor variables $(W)$ to the potency of the response variable $(Z)$, while classification quantitative structure activity relationship regression models relate the predictor variables to the response variable's categorical significance [3].

The predictors in quantitative structure-activity relationship modelling could be physico-chemical properties or theoretical molecular descriptors of chemicals, whereas the quantitative structure-activity relationship response variable could be the chemicals' biological activity. In a data collection of chemicals, quantitative structure-activity relationship models summarise a supposed relationship between chemical structures and biological activity. Second, quantitative structure-activity relationship models can predict how new chemicals will behave [4,5]. When chemical properties are used as response variables, similar concepts include quantitative structure-property relationships [6,7]. The quantitative structure-property relationships field has discovered various properties of chemical molecules. Relationships between quantitative structure and reactivity, quantitative structure and chromatography, quantitative structure and toxicity, quantitative structure and electrochemistry, and quantitative structure and biodegradability are a few examples [8,9].

Recently, Naji et al. [10] introduced a new topological index called the Leap Zagreb index, based on vertices' second degree and defined as:

$$LM_1(\alpha) = \sum_{s\in V(\alpha)} d_\alpha 2^2(s|2), \tag{1}$$

$$LM_2(\alpha) = \sum_{sq\in E(\alpha)} d_\alpha(s|2)d_\alpha(q|2), \tag{2}$$

$$LM_3(\alpha) = \sum_{s\in V(\alpha)} d_\alpha(s)d_\alpha(s|2). \tag{3}$$

Wiener polarity index $W_p(\alpha)$ is another topological index of graph $\alpha$ introduced by Wiener [11]. Its definition is the quantity of unordered pairs of vertices $\{s,q\} \subseteq V(\alpha)$ that are located 3 away from each other in $\alpha$.

$$W_p(\alpha) = \left|\{\{s,q\} \subseteq V(\alpha) \mid d_\alpha(s,q) = 3\}\right| = \frac{1}{2}\sum_{q\in V(\alpha)} d_\alpha(v\,|\,3). \tag{4}$$

An American chemist, H. Wiener [11] in 1947, introduced an index named the Wiener index. Wiener index is the total of all distances between every pair of vertices in the graph $\alpha$. Concerning Wiener index senbagamalar [12] find different algebraic properties of regular, trees, and unicyclic graphs. The Wiener index of edge complement of complete subgraphs, stars, and cyclics in $k_n$ calculated by Durgietal [13]. Shaohui et al. [14] determined the Hosoya polynomial and Wiener index for all integer numbers $m \geq 3$ for the Jahangir graphs $J_{5,m}$ in 2016. The relationship between the line graph and the Wiener index of the graph was studied by Nathann Cohen et al. [15].

Our approach is also motivated by recent developments in graph resolvability and metric-based parameters. Notably, the work on mixed metric dimension and exchange properties in hexagonal nano-networks provides a foundational framework for characterizing complex structures through minimal identifiers [16,17]. Similarly, fault-tolerant resolvability concepts [18,19] and newly introduced parameters like the local edge partition dimension [20] and some novel parameters are also introduced, like the mixed partition dimension and face metric dimension [21,22] highlight the importance of nuanced graph invariants. Our analysis also draws from studies on graph labeling [23], bridging the gap between theoretical constructs and practical applications. Topological indices can be discussed on these structures.

By using the second distances in graphs, Naji et al. [10] created the leap Zagreb indices. They compared these indices, Zagreb indices, and also found the properties of these indices. By replacing the vertices' degree with their second degree, Ali and Trijnastic [24] obtained the same indices.

In the present work, we extend the current study of Wiener polarity index and the leap Zagreb indices to the graph of $R$-vertex and edge join graphs. In the next Section, we compute the Wiener polarity and leap Zagreb indices of $R$-vertex and edge join graphs.

## 2. $R$ graphs

From a graph $\alpha$, we obtained a new graph called $R$-graph [25] is shown by $R(\alpha)$ is the graph got by adding a new vertex $w_e \notin V(\alpha)$ corresponding to every vertex $e = sq \in E(\alpha)$ and joining new vertex $w_e$ to the end vertices for each edge $e = sq \in E(\alpha)$. Let $|N(\alpha)|$ denote the set of all new vertices in $R(\alpha)$. Where,

$$|N(\alpha)| = \{w_e \mid e = sq \in E(\alpha) \text{ where } s, q \in V(\alpha)\}.$$

Similarly, let $J(\alpha)$ denote the new edges in $R(\alpha)$. Where,

$$J(\alpha) = \{w_e s, w_e q \mid e = sq \in E(\alpha) \text{ and } s, q \in V(\alpha)\}.$$

Then we have

$$V(R(\alpha)) = V(\alpha) \cup |N(\alpha)|,$$

and

$$E(R(\alpha)) = E(\alpha) \cup J(\alpha).$$

The significance of $R$-graphs is that they help to generate interesting models and analyze complex networks, such as social networks, internet connections, and biological systems. We can obtain two new graph operations from two arbitrary graphs $\alpha$ and $\beta$ based on the graph $R(\alpha)$ [26]. We called them $R$-vertex join and $R$-edge join of $\alpha$ and $\beta$ and denoted by $\alpha\langle v\rangle\beta$ and $\alpha\langle e\rangle\beta$ respectively. For two vertex disjoint graphs $\alpha$ and $\beta$, the $R$-vertex join of $\alpha$ and $\beta$ is obtained from $R(\alpha)$ and $\beta$ by connect every vertex $q$ of $V(\alpha)$ and $V(\beta)$ by an edge $e$, similarly, we obtained $R$-edge join of $\alpha$ and $\beta$ from $R(\alpha)$ and $\beta$ by connects every vertex of $V(\beta)$ by every vertex of $|N(\alpha)|$ by an edge. The graphs $C_5\langle v\rangle C_3$ and $C_5\langle e\rangle C_3$ are shown in Figure 1.

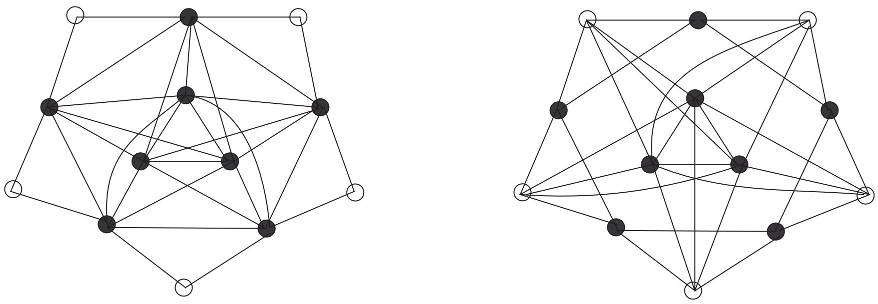

**Figure 1.** The $R$-vertex join (left) and $R$-edge join (right) of the graphs $C_5$ and $C_3$

## 3. Structural properties of $R$-vertex and $R$-edge join of graphs

Let two graphs $\alpha$ and $\beta$ of order $n_\alpha$ and $n_\beta$ and of size $m_\alpha$ and $m_\beta$, respectively. Let $\alpha_q$ and $\alpha_e$ are represents the graphs $\alpha\langle v\rangle\beta$ and $\alpha\langle e\rangle\beta$, respectively. The edge and vertex sets of the graphs $\alpha_q$ and $\alpha_e$ are given as follows.

$$V(\alpha\langle v\rangle\beta) = V(\alpha\langle e\rangle\beta) = V(\alpha) \cup V(\beta) \cup |N(\alpha)|,$$

$$E\alpha_q = E(\alpha) \cup E(\beta) \cup J(\alpha) \cup K_q,$$

where $K_q = \{uv \mid s \in V(\alpha), q \in V(\beta)\}$. Similarly,

$$E(\alpha_e) = E(\alpha) \cup E(\beta) \cup J(\alpha) \cup K_e,$$

where $K_e = \{w_e s \mid w_e \in |N(\alpha)|, s \in V(\beta)\}$. Then the order $n_q$ and size $m_q$ of the graph $\alpha_q$ can be observed as $|V(\alpha_q)| = n_q = n_\alpha + n_\beta + m_\alpha$ and $|E(\alpha_q)| = m_q = m_\alpha + m_\beta + n_\alpha n_\beta + 2m_\alpha$. Similarly, the order $n_e$ and size $m_e$ of the graph $\alpha_e$ is given by $|V(\alpha_e)| = n_e = n_\alpha + n_\beta + m_\alpha$ and $|E(\alpha_e)| = m_e = m_\alpha + m_\beta + 2m_\alpha + m_\alpha n_\beta$.

The degree of a vertex in $\alpha_q$ and $\alpha_e$ can be obtained from the following expressions. (Article - Azhar Iqbal)

$$d_{\alpha_q}(w) = \begin{cases} 2d_\alpha(w) + n_\beta, & s \in V(\alpha) \\ d_\beta(w) + n_\alpha, & s \in V(\beta) \\ 2, & s \in |N(\alpha)| \end{cases}$$

$$d_{\alpha_e}(w) = \begin{cases} 2d_\alpha(w), & s \in V(\alpha) \\ d_\beta(w) + m_\alpha, & s \in V(\beta) \\ 2 + n_\beta, & s \in |N(\alpha)| \end{cases}$$

The second distance of a vertex in $\alpha_q$ and $\alpha_e$ can be obtained from the following expressions.

$$d_{\alpha_q}(s|2) = \begin{cases} n_\alpha + p_\alpha(s|2) + t_\alpha(s) - d_\alpha(s) - 1, & s \in V(\alpha) \\ n_\beta - d_\beta(s) + |N(\alpha)| - 1, & s \in V(\beta) \\ d_{\alpha_q}(x) + d_{\alpha_q}(y) - 3 - |N_{\alpha_q}(x) \cap N_{\alpha_q}(y)| & s_{xy} \in |N(\alpha)|,\ x, y \in V(\alpha). \end{cases}$$

$$d_{\alpha_e}(s|2) = \begin{cases} n_\beta + 2d_\alpha(s|2) + t_\alpha(s), & s \in V(\alpha) \\ n_\beta - 1 - d_\beta(s) + n_\alpha, & s \in V(\beta) \\ |N(\alpha)| + d_\alpha(x) + d_\alpha(y) - 3 - |N_\alpha(x) \cap N_\alpha(y)| & s_{xy} \in |N(\alpha)|,\ x, y \in V(\alpha). \end{cases}$$

The third distance of a vertex in $\alpha_q$ and $\alpha_e$ can be obtained from the following expressions.

$$W_{\alpha_q}(s|p) = \begin{cases} |N(\alpha)| - d_\alpha(s) - p_\alpha(s|2) - t(s), & s \in V(\alpha) \\ 0, & s \in V(\beta) \\ |N_{\alpha_q}(x|2) \cup N_{\alpha_q}(y|2)| - |(N_\alpha(x) \cap N_\alpha(y|2)) \cup (|N_\alpha(x|2) \cap N_\alpha(y))| & s_{xy} \in |N(\alpha)|,\ x, y \in V(\alpha). \end{cases}$$

$$W_{\alpha_e}(s|p) = \begin{cases} d_\alpha(s|3) + |N(\alpha)| - d_\alpha(s) - p_\alpha(s|2) - t(s), & s \in V(\alpha) \\ 0, & s \in V(\beta) \\ n_\alpha + m_\alpha - |N_\alpha(x) \cap N_\alpha(y)| - |s_{x'y'}|x' \ or & \\ y' \in \{x, y\}| - 1 & s_{xy} \in N(\alpha),\ x, y \in V(\alpha). \end{cases}$$

## 4. Leap Zagreb indices of $R$-vertex and edge join of graphs

In this section, we will discuss some results based on the second distances of $R$-vertex and edge join of graphs.

**Theorem 1.** *The first leap Zagreb index of $\alpha_q$ is as follows:*

$$\begin{aligned} LM_1(\alpha_q) = {} & n_\alpha^3 + p^2 + 2n_\alpha p + t^2 - 11M_1(\alpha) - 2t^c + n_\alpha + 2n_\alpha t - 4m_\alpha n_\alpha + 2t^p - 2p^c \\ & -2t + 13m_\alpha - 2n_\alpha^2 - 2p + n_H^3 + M_1(\beta) - 4m_\beta n_\beta + m_\alpha^2 n_\beta + n_\beta - 14m_\alpha n_\beta \\ & +6m_\alpha n_H^2 - 2n_H^2 - 4m_\alpha m_\beta + 4m_\beta + 4F(\alpha) + 8n_\beta M_1(\alpha) + 8M_2(\alpha) + B^2 + 6B - 2A. \end{aligned}$$

*where*
$p = \sum_{q \in V(\alpha)} p_\alpha(q|2)^2,$

$t = \sum_{q\in V(\alpha)} t^2(q),$
$t^c = \sum_{q\in V(\alpha)} t(q)d_\alpha(q),$
$t^p = \sum_{q\in V(\alpha)} t(q)p_\alpha(q|2),$
$p^c = \sum_{q\in V(\alpha)} p_\alpha(q|2)d_\alpha(q),$
$B = \sum_{xy\in E(\alpha)} \left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|$ *and*
$A = \sum_{xy\in E(\alpha)} \left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right| (d_{\alpha_q}(x)+d_{\alpha_q}(y)).$

**Proof.** Using (1) and the vertex set of $\alpha_q$, we obtain the following.

$$\begin{aligned}
LM_1(\alpha_q) &= \sum_{q\in V(\alpha_q)} d(q|2)^2 \\
&= \sum_{q\in V(\alpha)} [(n_\alpha + p_\alpha(q|2)) + (t(q)-d_\alpha(q)) + (-1)]^2 \\
&\quad + \sum_{q\in V(\beta)} [(n_\beta - d_\beta(q)) + (|N(\alpha)|-1)]^2 + \sum_{q_{x,y}\in|N(\alpha)|} [(d_{\alpha_q}(x)+d_{\alpha_q}(y)) - (3+\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|)]^2 \\
&= \sum_{q\in V(\alpha)} [n_\alpha^2 + p_\alpha(q|2)^2 + 2n_\alpha p_\alpha(q|2) + t^2(q) + d_\alpha(q)^2 - 2t(q)d_\alpha(q) + 1 + 2n_\alpha t(q) - 2n_\alpha d_\alpha(q) \\
&\quad +2t(q)p_\alpha(q|2) - 2p_\alpha(q|2)d_\alpha(q) - 2t(q) + 2d_\alpha(q) - 2n_\alpha - 2p_\alpha(q|2)] + \sum_{q\in V(\beta)} [n_\beta^2 d_\beta(q)^2 - 2n_\beta d_\beta(q) \\
&\quad + |N(\alpha)|^2 + 1 - 2|N(\alpha)| + 2n_\beta|N(\alpha)| - 2n_\beta - 2d_\beta(q)|N(\alpha)| + 2d_\beta(q)] + \sum_{q_{x,y}\in|N(\alpha)|} [d_{\alpha_q}(x)^2 \\
&\quad +d_{\alpha_q}(y)^2 + 2d_{\alpha_q}(x)d_{\alpha_q}(y) + 9 + \left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|^2 + 6\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right| - 6d_{\alpha_q}(x) \\
&\quad -2d_{\alpha_q}(x)\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right| - 6d_{\alpha_q}(y) - 2d_{\alpha_q}(y)\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|] \\
&= n_\alpha^3 + p^2 + 2n_\alpha^2 p + t^2 + \sum_{q\in V(\alpha)} d_\alpha(q)^2 - 2t^c + n_\alpha + 2n_\alpha t - 2n_\alpha \sum_{q\in V(\alpha)} d_\alpha(q) + 2t^p - 2p^c - 2t \\
&\quad +2\sum_{q\in V(\alpha)} d_\alpha(q) - 2n_\alpha^2 - 2p + n_\beta^3 + \sum_{q\in V(\beta)} d(q)^2 - 2n_\beta \sum_{q\in V(\beta)} d(q) + \sum_{q\in V(\beta)} |N(\alpha)|^2 + n_\beta \\
&\quad -2\sum_{q\in V(\beta)} |N(\alpha)| + 2n_\beta \sum_{q\in V(\beta)} |N(\alpha)| - 2n_\beta^2 - 2\sum_{q\in V(\beta)} d(q)|N(\alpha)| + 2\sum_{q\in V(\beta)} d(q) \\
&\quad + \sum_{xy\in E(\alpha)} [d_{\alpha_q}(x)^2 + d_{\alpha_q}(y)^2] + 2\sum_{xy\in E(\alpha)} d_{\alpha_q}(x)d_{\alpha_q}(y) + 9m_\alpha - 6\sum_{xy\in E(\alpha)} [d_{\alpha_q}(x)+d_{\alpha_q}(y)] \\
&\quad + \sum_{xy\in E(\alpha)} \left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|^2 + 6\sum_{xy\in E(\alpha)} \left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right| \\
&\quad -2\sum_{xy\in E(\alpha)} \left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right| (d_{\alpha_q}(x)+d_{\alpha_q}(y)) \\
&= n_\alpha^3 + p^2 + 2n_\alpha p + t^2 + M_1(\alpha) - 2t^c + n_\alpha + 2n_\alpha t - 4n_\alpha m_\alpha + 2t^p - 2p^c - 2t + 4m_\alpha - 2n_\alpha^2 - 2p \\
&\quad +n_\beta^3 + M_1(\beta) - 4m_\beta n_\beta + m_\alpha^2 n_\beta + n_\beta - 2m_\alpha n_\beta 2m_\alpha n_\beta^2 - 2n_\beta^2 - 4m_\alpha m_\beta + 4m_\beta \\
&\quad + \sum_{xy\in E(\alpha)} [(2d_\alpha(x)+n_\beta)^2 + (2d_\alpha(y)+n_\beta)^2] + 2\sum_{xy\in E(\alpha)} (2d_\alpha(x)+n_\beta)(2d_\alpha(y)+n_\beta) + 9m_\alpha \\
&\quad -6\sum_{xy\in E(\alpha)} [2d_\alpha(x)+n_\beta+2d_\alpha(y)+n_\beta] + B^2 + 6B - 2A \\
&= n_\alpha^3 + p^2 + 2n_\alpha p + t^2 + M_1(\alpha) - 2t^c + n_\alpha + 2n_\alpha t - 4n_\alpha m_\alpha + 2t^p - 2p^c - 2t + 4m_\alpha - 2n_\alpha^2 - 2p + n_\beta^3 \\
&\quad +M_1(\beta) - 4m_\beta n_\beta + m_\alpha^2 n_\beta + n_\beta - 2m_\alpha n_\beta 2m_\alpha n_\beta^2 - 2n_\beta^2 - 4m_\alpha m_\beta + 4m_\beta + 4\sum_{xy\in E(\alpha)} (d(x)^2 + d(y)^2)
\end{aligned}$$

$$
\begin{aligned}
& +4n_\beta \sum_{xy\in E(\alpha)} (d(x)d(y)) + 2m_\alpha n_\beta^2 + 8 \sum_{xy\in E(\alpha)} d(x)d(y) + 4n_\beta \sum_{xy\in E(\alpha)} (d(x)+d(y)) \\
& +2m_\alpha n_\beta^2 + 9m_\alpha - 12 \sum_{xy\in E(\alpha)} (d(x)+d(y)) - 12m_\alpha n_\beta + B^2 + 6B - 2A \\
= \;& n_\alpha^3 + p^2 + 2n_\alpha p + t^2 + M_1(\alpha) - 2t^c + n_\alpha + 2n_\alpha t - 4n_\alpha m_\alpha + 2t^p - 2p^c \\
& -2t + 4m_\alpha - 2n_\alpha^2 - 2p + n_\beta^3 + M_1(\beta) - 4m_\beta n_\beta + m_\alpha^2 n_\beta + n_\beta - 2m_\alpha n_\beta \\
& 2m_\alpha n_\beta^2 - 2n_\beta^2 - 4m_\alpha m_\beta + 4m_\beta + 4F(\alpha) + 8n_\beta M_1(\alpha) + 4m_\alpha n_\beta^2 + 8M_2(\alpha) \\
& +9m_\alpha - 12M_1(\alpha) - 12m_\alpha n_\beta + B^2 + 6B - 2A \\
= \;& n_\alpha^3 + p^2 + 2n_\alpha p + t^2 - 11M_1(\alpha) - 2t^c + n_\alpha + 2n_\alpha t - 4m_\alpha n_\alpha + 2t^p - 2p^c \\
& -2t + 13m_\alpha - 2n_\alpha^2 - 2p + n_\beta^3 + M_1(\beta) - 4m_\beta n_\beta + m_\alpha^2 n_\beta + n_\beta - 14m_\alpha n_\beta \\
& +6m_\alpha n_\beta^2 - 2n_\beta^2 - 4m_\alpha m_\beta + 4m_\beta + 4F(\alpha) + 8n_\beta M_1(\alpha) + 8M_2(\alpha) \\
& +B^2 + 6B - 2A.
\end{aligned}
$$

□

**Theorem 2.** *The second leap Zagreb index of $\alpha_q$ is as follows:*

$$
\begin{aligned}
LM_2(\alpha_q) = \;& m_\alpha n_\alpha^2 - 8m_\alpha n_\alpha + 3n_\alpha M_1(\alpha) - 3M_1(\alpha) - M_2(\alpha) + (n_\alpha - 1)P_1 + P_2 - P_3 - P_4 + P_5 \\
& +P_6 + (n_\alpha - 1)T_1 + T_2 - T_3 + 7m_\alpha + m_\beta n_\beta^2 - 2m_\beta n_\beta - n_\beta M_1(\beta) + M_2(\beta) \\
& -m_\alpha M_1(\beta) + M_1(\beta) + 2m_\alpha m_\beta + 2m_\alpha m_\beta n_\beta - m_\alpha^2 m_\beta + m_\beta + n_\alpha^2 n_\beta^2 - 2m_\beta n_\alpha^2 \\
& -2m_\alpha n_\beta^2 + m_\alpha n_\alpha^2 n_\beta - n_\alpha^2 n_\beta + (n_\beta - 1)A_1 - A_2 + m_\alpha A_1 + (n_\beta - 1)A_3 - A_4 + m_\alpha A_3 \\
& +2m_\alpha n_\beta + 2m_\beta n_\alpha - n_\alpha n_\beta^2 - m_\alpha n_\alpha n_\beta + n_\alpha n_\beta + 4m_\alpha n_\alpha n_\beta + (2 - 2n_\alpha)N_1 + 2D_1 \\
& +(2m_\alpha n_\beta - 3m_\alpha)D_2 - N_2 + 2B_1 + (2m_\alpha n_\beta - 3m_\alpha)B_2 - B_2 - 4m_\alpha M_1(\alpha) + 6m_\alpha^2 \\
& -4m_\alpha^2 n_\beta + 2m_\alpha N_1 + 2N_1,
\end{aligned}
$$

*where*

$P_1 = \sum_{sq\in E(\alpha)} (p_\alpha(s|2), p_\alpha(q|2))$,

$P_2 = \sum_{sq\in E(\alpha)} p_\alpha(s|2)p_\alpha(q|2)$,

$P_3 = \sum_{sq\in E(\alpha)} (p_\alpha(s|2)d_\alpha(q))$,

$P_4 = \sum_{sq\in E(\alpha)} p_\alpha(q)d_\alpha(s)$,

$P_5 = \sum_{sq\in E(\alpha)} p_\alpha(s|2)t_\alpha(q)$,

$P_6 = \sum_{sq\in E(\alpha)} p_\alpha(q)t_\alpha(s)$,

$T_1 = \sum_{sq\in E(\alpha)} (t_\alpha(s) + t_\alpha(q))$,

$T_2 = \sum_{sq\in E(\alpha)} t_\alpha(s)t_\alpha(q)$,

$T_3 = \sum_{sq\in E(\alpha)} (d_\alpha(s)t_\alpha(q) + d_\alpha(q)t_\alpha(s))$,

$A_1 = \sum_{sq\in k_q} p_\alpha(s|2)$,

$A_2 = \sum_{sq\in k_q} p_\alpha(s|2)d_\beta(q)$,

$A_3 = \sum_{sq\in k_q} t_\alpha(s)$,

$A_4 = \sum_{sq\in k_q} t_\alpha(s)d_\beta(q)$,

$N_1 = \sum_{xy\in E(\alpha)} (\left|N_{\alpha_q}(x) \cap N_{\alpha_q}(y)\right|)$.

**Proof.** Using (2) and edge set of $\alpha_q$, we obtain

$$
\begin{aligned}
LM_2(\alpha_q) &= \sum_{sq\in E(\alpha_q)} d_2(s)d_2(q)\\
&= \sum_{sq\in E(\alpha)} (n_\alpha + p_\alpha(s|2) + t_\alpha(s) - d_\alpha(s) - 1)(n_\alpha + p_\alpha(q|2) + t_\alpha(q) - d_\alpha(q) - 1)\\
&\quad + \sum_{sq\in E(\beta)} (n_\beta - d_\beta(s) + |N(\alpha)| - 1)(n_\beta - d_\beta(q) + |N(\alpha)| - 1)\\
&\quad + \sum_{sq\in K_q} (n_\alpha + p_\alpha(s|2) + t_\alpha(s) - d_\alpha(s) - 1)(n_\beta - d_\beta(q) + |N(\alpha)| - 1) +\\
&\quad \sum_{sq\in J(\alpha)} (n_\alpha + p_\alpha(s|2) + t_\alpha(s) - d_\alpha(s) - 1)(d_{\alpha_q}(x) + d_{\alpha_q}(y) - 3 - \left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|)\\
&= \sum_{sq\in E(\alpha)} [n_\alpha^2 + n_\alpha p_\alpha(q|2) + n_\alpha t_\alpha(q) - n_\alpha d_\alpha(q) - n_\alpha + n_\alpha p_\alpha(s|2) + p_\alpha(s|2)p_\alpha(q|2)\\
&\quad + p_\alpha(s|2)t_\alpha(q) - p_\alpha(s|2)d_\alpha(q) - p_\alpha(s|2) + n_\alpha t_\alpha(s) + p_\alpha(q|2)t_\alpha(s) + t_\alpha(s)t_\alpha(q)\\
&\quad - d_\alpha(q)t_\alpha(s) - t_\alpha(s) - n_\alpha d_\alpha(s) - d_\alpha(s)p_\alpha(q) - d_\alpha(s)t_\alpha(q) - d_\alpha(s)d_(q) + d_\alpha(s)\\
&\quad - n_\alpha - p_\alpha(q) - t_\alpha(q) + d_\alpha(q) + 1] + \sum_{sq\in E(\beta)} [n_\beta^2 - n_\beta d_\beta(q) + n_\beta\,|N(\alpha)| - n_\beta\\
&\quad - n_\beta d_\beta(s) + d_\beta(s)d_\beta(q) - |N(\alpha)|\,d_\beta(s) + d_\beta(s) + n_\beta\,|N(\alpha)| - d_\beta(q)\,|N(\alpha)|\\
&\quad + |N(\alpha)|^2 - |N(\alpha)| - n_\beta + d_\beta(q) - |N(\alpha)| + 1] + \sum_{sq\in K_q} [n_\alpha n_\beta - n_\alpha d_\beta(q) + n_\alpha\,|N(\alpha)|\\
&\quad - n_\alpha + n_\beta p_\alpha(s|2) - p_\alpha(s|2)d_\beta(q) + |N(\alpha)|\,p_\alpha(s|2) - p_\alpha(s|2) + n_\beta t_\alpha(s) - t_\alpha(s)d_\beta(q)\\
&\quad + |N(\alpha)|\,t_\alpha(s) - t_\alpha(s) - n_\beta d_\alpha(s) + d_\alpha(s)d_\beta(q) - |N(\alpha)|\,d_\alpha(s) + d_\alpha(s) - n_\beta + d_\beta(q)\\
&\quad - |N(\alpha)| + 1] + \sum_{sq\in J(\alpha)} (n_\alpha + p_\alpha(s|2) + t_\alpha(s) - d_\alpha(s) - 1)(2d_\alpha(x) + 2n_\beta + 2d_\alpha(y) - 3\\
&\quad - \left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|)\\
&= \sum_{sq\in E(\alpha)} [n_\alpha^2 - 2n_\alpha - n_\alpha(d_\alpha(s) + d_\alpha(q)) + (d_\alpha(s) + d_\alpha(q)) - d_\alpha(s)d_\alpha(q)\\
&\quad + n_\alpha(p_\alpha(s|2) + p_\alpha(q|2)) + p_\alpha(s|2)p_\alpha(q|2) - (p_\alpha(s|2) + p_\alpha(q|2))\\
&\quad - (p_\alpha(s|2)d_\alpha(q) + p_\alpha(q|2)d_\alpha(s)) + (p_\alpha(s|2)t_\alpha(q) + p_\alpha(q|2)t_\alpha(s)) + n_\alpha(t_\alpha(s) + t_\alpha(q))\\
&\quad + t_\alpha(s)t_\alpha(q) - (t_\alpha(s) + t_\alpha(q)) - (d_\alpha(q)t_\alpha(s) + d_\alpha(s)t_\alpha(q)) + 1]\\
&\quad + \sum_{sq\in E(\beta)} [n_\beta^2 - 2n_\beta - n_\beta(d_\beta(s) + d_\beta(q)) + d_\beta(s)d_\beta(q) - |N(\alpha)|\,(d_\beta(s) + d_\beta(q))\\
&\quad + (d_\beta(s) + d_\beta(q)) - 2\,|N(\alpha)| + 2n_\beta\,|N(\alpha)| + 2\left|N(\alpha)^2\right| + 1\\
&\quad + \sum_{sq\in K_q} [n_\alpha n_\beta - n_\alpha d_\beta(q) - n_\beta d_\alpha(s) + n_\alpha\,|N(\alpha)| - n_\alpha + (n_\beta - 1)p_\alpha(s|2)\\
&\quad - p_\alpha(s|2)d_\beta(q) + |N(\alpha)|\,p_\alpha(s|2) + (n_\beta - 1)t_\alpha(s) - t_\alpha(s)d_\beta(q) + |N(\alpha)|\,t_\alpha(s)\\
&\quad + d_\alpha(s)d_\beta(q) - |N(\alpha)|\,d_\alpha(s) + (d_\alpha(s) + d_\beta(q)) - n_\beta - |N(\alpha)| + 1]\\
&\quad + \sum_{q_{xy}\in|N(\alpha)|}\sum_{s\in\{x,y\}} [2n_\alpha d_\alpha(x) + 2n_\alpha n_\beta + 2n_\alpha d_\alpha(y) - 3n_\alpha - n_\alpha\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|\\
&\quad + 2p_\alpha(s|2)d_\alpha(x) + 2n_\beta p_\alpha(s|2) + 2p_\alpha(s|2)d_\alpha(y) - 3p_\alpha(s|2) - p_\alpha(s|2)\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|\\
&\quad + 2t_\alpha(s)d_\alpha(x) + 2n_\beta t_\alpha(s) - 3t_\alpha(s) + 2t_\alpha(s)d_\alpha(y) - t_\alpha(s)\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|\\
&\quad - 2d_\alpha(s)d_\alpha(x) - 2n_\beta d_\alpha(s) - 2d_\alpha(s)d_\alpha(y) + 3d_\alpha(s) + d_\alpha(s)\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right| - 2d_\alpha(x)\\
&\quad - 2n_\beta - 2d_\alpha(y) + 3 + \left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|]\\
&= m_\alpha n_\alpha^2 - 2m_\alpha n_\alpha - n_\alpha\sum_{sq\in E(\alpha)}(d_\alpha(s) + d_\alpha(q)) + \sum_{sq\in E(\alpha)}(d_\alpha(s) + d_\alpha(q)) - \sum_{sq\in E(\alpha)} d_\alpha(s)d_\alpha(q)
\end{aligned}
$$

$$
\begin{aligned}
&+(n_\alpha-1)\sum_{sq\in E(\alpha)}(p_\alpha(s|2)+p_\alpha(q|2))+\sum_{sq\in E(\alpha)}p_\alpha(s|2)p_\alpha(q|2)-\sum_{sq\in E(\alpha)}(p_\alpha(s|2)d_\alpha(q))\\
&-\sum_{sq\in E(\alpha)}p_\alpha(q)d_\alpha(s)+\sum_{sq\in E(\alpha)}p_\alpha(s|2)t_\alpha(q)+\sum_{sq\in E(\alpha)}p_\alpha(q)t_\alpha(s)+(n_\alpha-1)\sum_{sq\in E(\alpha)}(t_\alpha(s)+t_\alpha)\\
&+\sum_{sq\in E(\alpha)}t_\alpha(s)t_\alpha(q)-\sum_{sq\in E(\alpha)}(d_\alpha(s)t_\alpha(q)+d_\alpha(q)t_\alpha(s))+m_\alpha+m_\beta n_\beta^2-2m_\beta n_\beta\\
&-n_\beta\sum_{sq\in E(\beta)}(d_\beta(s)+d_\beta(q))+\sum_{sq\in E(\beta)}d_\beta(s)d_\beta(q)-|N(\alpha)|\sum_{sq\in E(\beta)}(d_\beta(s)+d_\beta(q))\\
&+\sum_{sq\in E(\beta)}(d_\beta(s)+d_\beta(q))-2m_\alpha m_\beta+2m_\alpha m_\beta n_\beta+m_\alpha^2m_\beta+m_\beta+n_\alpha^2n_\beta^2\\
&-n_\alpha\sum_{sq\in K_q}d_\beta(q)-n_\beta\sum_{sq\in k_q}d_\alpha(s)+m_\alpha n_\alpha^2n_\beta-n_\alpha^2n_\beta+(n_\beta-1)\sum_{sq\in k_q}p_\alpha(s|2)\\
&-\sum_{sq\in k_q}p_\alpha(s|2)d_\beta(q)+|N(\alpha)|\sum_{sq\in k_q}p_\alpha(s|2)+(n_\beta-1)\sum_{sq\in k_q}t_\alpha(s)-\sum_{sq\in k_q}t_\alpha(s)d_\beta(q)\\
&+|N(\alpha)|\sum_{sq\in k_q}t_\alpha(s)+\sum_{sq\in k_q}d_\alpha(s)d_\beta(q)-|N(\alpha)|\,d_\alpha(s)+\sum_{sq\in k_q}d_\alpha(s)+\sum_{sq\in k_q}d_\beta(q)\\
&-n_\alpha n_\beta^2-m_\alpha n_\alpha n_\beta+n_\alpha n_\beta+4n_\alpha\sum_{xy\in E(\alpha)}(d_\alpha(x)+d_\alpha(y))+4m_\alpha n_\alpha n_\beta-6m_\alpha n_\alpha\\
&+2(1-n_\alpha)\sum_{xy\in E(\alpha)}(\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|)+2\sum_{xy\in E(\alpha)}(p_\alpha(x|2)+p_\alpha(y|2))(d_\alpha(x)+d_\alpha(y))\\
&+(2m_\alpha n_\beta-3m_\alpha)\sum_{xy\in E(\alpha)}(p_\alpha(x|2)+p_\alpha(y|2))-\sum_{xy\in E(\alpha)}(p_\alpha(x|2)+p_\alpha(y|2))(\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|)\\
&+2\sum_{xy\in E(\alpha)}(t_\alpha(x)+t_\alpha(y))(d_\alpha(x)+d_\alpha(y))+(2m_\alpha n_\beta-3m_\alpha)\sum_{xy\in E(\alpha)}(t_\alpha(x)+t_\alpha(y))\\
&-\sum_{xy\in E(\alpha)}(t_\alpha(x)+t_\alpha(y))(\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|)-4m_\alpha\sum_{xy\in E(\alpha)}(d_\alpha(x)+d_\alpha(y))+6m_\alpha^2\\
&-4m_\alpha^2n_\beta+2m_\alpha\sum_{xy\in E(\alpha)}(\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|)-\sum_{xy\in E(\alpha)}(d_\alpha(x)+d_\alpha(y))+6m_\alpha\\
&+2\sum_{xy\in E(\alpha)}(\left|N_{\alpha_q}(x)\cap N_{\alpha_q}(y)\right|)\\
=\ &m_\alpha n_\alpha^2-2m_\alpha n_\alpha-n_\alpha M_1(\alpha)+M_1(\alpha)-M_2(\alpha)+(n_\alpha-1)P_1+P_2-P_3-P_4+P_5+P_6\\
&+(n_\alpha-1)T_1+T_2-T_3+m_\alpha+m_\beta n_\beta^2-2m_\beta n_\beta-n_\beta M_1(\beta)+M_2(\beta)-m_\alpha M_1(\beta)\\
&+M_1(\beta)-2m_\alpha m_\beta+2m_\alpha m_\beta n_\beta+m_\alpha^2m_\beta+m_\beta+n_\alpha^2n_\beta^2-2m_\beta n_\alpha^2-2m_\alpha n_\beta^2+m_\alpha n_\alpha^2n_\beta\\
&-n_\alpha^2n_\beta+(n_\beta-1)A_1-A_2+m_\alpha A_1+(n_\beta-1)A_3-A_4+m_\alpha A_3+4m_\alpha m_\beta-2m_\alpha^2m_\beta\\
&+2m_\alpha n_\beta+2m_\beta n_\alpha-n_\alpha n_\beta^2-m_\alpha n_\alpha n_\beta+n_\alpha n_\beta+4n_\alpha M_1(\alpha)+4m_\alpha n_\alpha n_\beta-6m_\alpha n_\alpha\\
&+(2-2n_\alpha)N_1+2D_1+(2m_\alpha n_\beta-3m_\alpha)D_2-N_2+2B_1+(2m_\alpha n_\beta-3m_\alpha)B_2-B_2\\
&-4m_\alpha M_1(\alpha)+6m_\alpha^2-4m_\alpha^2n_\beta+2m_\alpha N_1-4M_1(\alpha)+6m_\alpha+2N_1\\
=\ &m_\alpha n_\alpha^2-8m_\alpha n_\alpha+3n_\alpha M_1(\alpha)-3M_1(\alpha)-M_2(\alpha)+(n_\alpha-1)P_1+P_2-P_3-P_4+P_5+P_6\\
&+(n_\alpha-1)T_1+T_2-T_3+7m_\alpha+m_\beta n_\beta^2-2m_\beta n_\beta-n_\beta M_1(\beta)+M_2(\beta)-m_\alpha M_1(\beta)\\
&+M_1(\beta)+2m_\alpha m_\beta+2m_\alpha m_\beta n_\beta-m_\alpha^2m_\beta+m_\beta+n_\alpha^2n_\beta^2-2m_\beta n_\alpha^2-2m_\alpha n_\beta^2+m_\alpha n_\alpha^2n_\beta\\
&-n_\alpha^2n_\beta+(n_\beta-1)A_1-A_2+m_\alpha A_1+(n_\beta-1)A_3-A_4+m_\alpha A_3+2m_\alpha n_\beta+2m_\beta n_\alpha-n_\alpha n_\beta^2\\
&-m_\alpha n_\alpha n_\beta+n_\alpha n_\beta+4m_\alpha n_\alpha n_\beta+(2-2n_\alpha)N_1+2D_1+(2m_\alpha n_\beta-3m_\alpha)D_2-N_2+2B_1\\
&+(2m_\alpha n_\beta-3m_\alpha)B_2-B_2-4m_\alpha M_1(\alpha)+6m_\alpha^2-4m_\alpha^2n_\beta+2m_\alpha N_1+2N_1
\end{aligned}
$$

□

**Theorem 3.** *The third leap Zagreb index of $\alpha_q$ is as follows:*

$$\begin{aligned} LM_3(\alpha_q) &= 4m_\alpha n_\alpha + 2p^c + 2t^c - 2M_1(\alpha) - 4m_\alpha + n_\alpha^2 n_\beta + n_\beta p + n_\beta t \\ &\quad -3m_\alpha n_\beta + 2m_\beta n_\beta - M_1(\beta) + 2m_\alpha m_\beta - 2m_\beta + n_\alpha n_\beta^2 - 2n_\alpha m_\beta \\ &\quad +m_\alpha n_\alpha n_\beta - n_\alpha n_\beta + 4M_1(\alpha) + 4m_\alpha n_\beta - 6m_\alpha - 2B, \end{aligned}$$

*where* $B = \sum_{xy\in E(\alpha)} \left| N_{\alpha_q}(x) \cap N_{\alpha_q}(y) \right|$.

**Proof.** Using (3) and vertex set of $\alpha_q$, we obtain

$$\begin{aligned} LM_3(\alpha_q) &= \sum_{s\in V(\alpha_q)} d(s).d(s|2) \\ &= \sum_{s\in V(\alpha)} (2d_\alpha(s) + n_\beta)(n_\alpha + p_\alpha(s|2) + t(q) - d_\alpha(s) - 1) \\ &\quad + \sum_{s\in V(\beta)} (d_\beta(s) + n_\alpha)(n_\beta - d_\beta(s) + |N(\alpha)| - 1) \\ &\quad + \sum_{s_{x,y}\in |N(\alpha)|} (2)(d_{\alpha_q}(x) + d_{\alpha_q}(y) - 3 - \left| N_{\alpha_q}(x) \cap N_{\alpha_q}(y) \right|) \\ &= \sum_{s\in V(\alpha)} [2n_\alpha d_\alpha(s) + 2p_\alpha(s|2)d_\alpha(s) + 2d_\alpha(s).t(q) \\ &\quad -2d_\alpha(s)^2 - 2d_\alpha(s) + n_\alpha n_\beta + n_\beta p_\alpha(s|2) + n_\beta.t(q) - n_\beta d_\alpha(s) - n_\beta] \\ &\quad + \sum_{s\in V(\beta)} [n_\beta d_\beta(s) - d_\beta(s)^2 + d_\beta(s)\,|N(\alpha)| - d_\beta(s) \\ &\quad +n_\alpha n_\beta - n_\alpha d_\beta(s) + n_\alpha\,|N(\alpha)| - n_\alpha] \\ &\quad + \sum_{s_{x,y}\in |N(\alpha)|} [2d_\alpha(x) + 2d_\alpha(y) - 6 - 2\left| N_{\alpha_q}(x) \cap N_{\alpha_q}(y) \right|] \\ &= 2n_\alpha \sum_{s\in V(\alpha)} d_\alpha(s) + 2\sum_{s\in V(\alpha)} p_\alpha(s|2)d_\alpha(s) + 2\sum_{s\in V(\alpha)} d_\alpha(s).t(q) \\ &\quad -2\sum_{s\in V(\alpha)} d_\alpha(s)^2 - 2\sum_{s\in V(\alpha)} d_\alpha(s) + n_\alpha^2 n_\beta + n_\beta \sum_{s\in V(\alpha)} p_\alpha(s|2) \\ &\quad +n_\beta \sum_{s\in V(\alpha)} t(q) - n_\beta \sum_{s\in V(\alpha)} d_\alpha(s) - n_\alpha n_\beta + n_\beta \sum_{s\in V(\beta)} d_\beta(s) \\ &\quad - \sum_{s\in V(\beta)} d_\beta(s)^2 + \sum_{s\in V(\beta)} d_\beta(s)\,|N(\alpha)| - \sum_{s\in V(\beta)} d_\beta(s) \\ &\quad +n_\alpha n_\beta^2 - n_\alpha \sum_{s\in V(\beta)} d_\beta(s) + n_\alpha \sum_{s\in V(\beta)} |N(\alpha)| - n_\alpha n_\beta \\ &\quad +2\sum_{s_{x,y}\in |N(\alpha)|} d_{\alpha_q}(x) + \sum_{s_{x,y}\in |N(\alpha)|} d_{\alpha_q}(y) - 6m_\alpha - 2\sum_{s_{x,y}\in |N(\alpha)|} \left| N_{\alpha_q}(x) \cap N_{\alpha_q}(y) \right| \\ &= 4m_\alpha n_\alpha + 2p^c + 2t^c - 2M_1(\alpha) - 4m_\alpha + n_\beta n_\alpha^2 + n_\beta p + n_\beta t \\ &\quad -2m_\alpha n_\beta - n_\alpha n_\beta + 2m_\beta n_\beta - M_1(\beta) + 2m_\alpha m_\beta - 2m_\beta \\ &\quad +n_\alpha n_\beta^2 - 2m_\beta n_\alpha + m_\alpha n_\alpha n_\beta - n_\alpha n_\beta + 2\sum_{xy\in E(\alpha)} [2d_\alpha(x) + n_\beta + 2d_\alpha(y) + n_\beta] \\ &\quad -6m_\alpha - 2\sum_{xy\in E(\alpha)} \left| N_{\alpha_q}(x) \cap N_{\alpha_q}(y) \right| \\ &= 4m_\alpha n_\alpha + 2p^c + 2t^c - 2M_1(\alpha) - 4m_\alpha + n_\beta n_\alpha^2 \\ &\quad +n_\beta p + n_\beta t - 2m_\alpha n_\beta - 2n_\alpha n_\beta + 2m_\beta n_\beta \\ &\quad -M_1(\beta) + 2m_\alpha m_\beta - 2m_\beta + n_\alpha n_\beta^2 - 2m_\beta n_\alpha \\ &\quad +m_\alpha n_\alpha n_\beta + 4\sum_{xy\in E(\alpha)} (d_\alpha(x) + d_\alpha(y)) + 4m_\alpha n_\beta - 6m_\alpha - 2B \end{aligned}$$

$$\begin{aligned}&= 4m_\alpha n_\alpha + 2p^c + 2t^c - 2M_1(\alpha) - 4m_\alpha + n_\beta n_\alpha^2\\ &\quad + n_\beta p + n_\beta t - 2m_\alpha n_\beta - 2n_\alpha n_\beta + 2m_\beta n_\beta\\ &\quad - M_1(\beta) + 2m_\alpha m_\beta - 2m_\beta + n_\alpha n_\beta^2\\ &\quad - 2m_\beta n_\alpha + m_\alpha n_\alpha n_\beta + 4M_1(\alpha) + 4m_\alpha n_\beta - 6m_\alpha - 2B.\end{aligned}$$

□

**Corollary 4.** *Let $G = H = S_n$, $n \geq 3$ be a star graph with n vertices. Then*

$$LM_3(S_n\langle v\rangle S_n) = 4n^3 + 5n^2 - 26n + 10.$$

**Theorem 5.** *The first leap Zagreb index of $\alpha_e$ is as follows:*

$$\begin{aligned}LM_1(\alpha_e) &= 3n_\alpha n_\beta^2 + 4LM_1(\alpha) + t^2 + 4n_\beta X + 4t^e + 2n_\beta t + n_\beta^3 + n_\alpha^2 n_\beta\\ &\quad + M_1(\beta) + n_\beta + 4m_\beta - 4m_\beta n_\beta - 2n_\beta^2 - 4m_\beta n_\alpha - 2n_\alpha n_\beta\\ &\quad + m_\alpha^3 + F(\alpha) + 2m_\alpha M_1(\alpha) + 2M_2(\alpha) + 9m_\alpha + C^2 + 6C - 6m_\alpha^2\\ &\quad - 2m_\alpha C - 6M_1(\alpha) - 2D,\end{aligned}$$

*where*
$t = \sum_{q\in V(\alpha)} t^2(q)$,
$t^e = t(q)d_\alpha(q|2)$,
$C = \sum_{xy\in E(\alpha)} |N_\alpha(x) \cap N_\alpha(y)|$,
$X = \sum_{q\in V(\alpha)} d_\alpha(q|2)$ *and*
$D = \sum_{xy\in E(\alpha)} |N_\alpha(x) \cap N_\alpha(y)| \,(d_\alpha(x) + d_\alpha(y))$.

**Proof.** Using (1) and vertex set of $\alpha_e$, we obtain

$$\begin{aligned}LM_1(\alpha_e) &= \sum_{q\in V(\alpha_e)} d(q|2)^2\\ &= \sum_{q\in V(\alpha)} [n_\beta + 2d_\alpha(q|2) + t(q)]^2 + \sum_{q\in V(\beta)} [n_\beta - 1 - d_\beta(q) + n_\alpha]^2\\ &\quad \sum_{q_{xy}\in|N(\alpha)|} [|N(\alpha)| + d_\alpha(x) + d_\alpha(y) - 3 - |N_\alpha(x) \cap N_\alpha(y)|]^2\\ &= \sum_{q\in V(\alpha)} [n_\beta^2 + 4d_\alpha^2(q|2) + t^2(q) + 4n_\beta d_\alpha(q|2) + 4t(q)d_\alpha(q|2) + 2n_\beta t(q)]\\ &\quad \sum_{q\in V(\beta)} [n_\beta^2 + n_\alpha^2 + 2n_\beta n_\alpha + d_\beta^2(q) + 1 + 2d_\beta(q) - 2n_\beta d_\beta(q) - 2n_\beta\\ &\quad - 2n_\alpha d_\beta(q) - 2n_\alpha] + \sum_{q_{xy}\in|N(\alpha)|} [|N(\alpha)|^2 + d_\alpha^2(x) + d_\alpha^2(y) + 2\,|N(\alpha)|\,d_\alpha(x)\\ &\quad + 2d_\alpha(x)d_\alpha(y) + 2d_\alpha(y)\,|N(\alpha)| + 9 + |N_\alpha(x) \cap N_\alpha(\alpha)|^2 + 6|N_\alpha(x) \cap N_\alpha(y)|\\ &\quad - 6\,|N(\alpha)| - 2\,|N(\alpha)|\,|N_\alpha(x) \cap N_\alpha(y)| - 6d_\alpha(x) - 2d_\alpha(x)|N_\alpha(x) \cap N_\alpha(y)|\\ &\quad - 6d_\alpha(y) - 2d_\alpha(y)|N_\alpha(x) \cap N_\alpha(y)|]\\ &= \sum_{q\in V(\alpha)} [n_\beta^2 + 4d_\alpha^2(q|2) + t^2(q) + 4n_\beta d_\alpha(q|2) + 4t(q)d_\alpha(q|2) + 2n_\beta t(q)]\\ &\quad \sum_{q\in V(\beta)} [n_\beta^2 + n_\alpha^2 + 2n_\beta n_\alpha + d_\beta^2(q) + 1 + 2d_\beta(q) - 2n_\beta d_\beta(q) - 2n_\beta\\ &\quad - 2n_\alpha d_\beta(q) - 2n_\alpha] + \sum_{q_{xy}\in|N(\alpha)|} [|N(\alpha)|^2 + (d_\alpha^2(x) + d_\alpha^2(y))\end{aligned}$$

$$
\begin{aligned}
&+2\,|N(\alpha)|\,(d_\alpha(x)+d_\alpha(y))+2d_\alpha(x)d_\alpha(y)+9+|N_\alpha(x)\cap N_\alpha(y)|^2\\
&+6|N_\alpha(x)\cap N_\alpha(y)|-6\,|N(\alpha)|-2\,|N(\alpha)|\,|N_\alpha(x)\cap N_\alpha(y)|\\
&-6(d_\alpha(x)+d_\alpha(y))-2|N_\alpha(x)\cap N_\alpha(y)|(d_\alpha(x)+d_\alpha(y))]\\
=\;& n_\beta^2 n_\alpha+4\sum_{q\in V(\alpha)} d_\alpha^2(q|2)+\sum_{q\in V(\alpha)} t^2(q)+4n_\beta\sum_{q\in V(\alpha)} d_\alpha(q|2)+4\sum_{q\in V(\alpha)} t(q)d_\alpha(q|2)\\
&+2n_\beta\sum_{q\in V(\alpha)} t(q)+n_\beta^3+n_\alpha^2 n_\beta+2n_\beta^2 n_\alpha+\sum_{q\in V(\beta)} d_\beta^2(q)+n_\beta+2\sum_{q\in V(\beta)} d_\beta(q)\\
&-2n_\beta\sum_{q\in V(\beta)} d_\beta(q)-2n_\beta^2-2n_\alpha\sum_{q\in V(\beta)} d_\beta(q)-2n_\alpha n_\beta+\sum_{xy\in E(\alpha)} |N(\alpha)|^2\\
&+\sum_{xy\in E(\alpha)}[d_\alpha^2(x)+d_\alpha^2(y)]+2\sum_{xy\in E(\alpha)}|N(\alpha)|\,(d_\alpha(x)+d_\alpha(y))\\
&+2\sum_{xy\in E(\alpha)} d_\alpha(x)d_\alpha(y)+9m_\alpha+\sum_{xy\in E(\alpha)}|N_\alpha(x)\cap N_\alpha(y)|^2+6\sum_{xy\in E(\alpha)}|N_\alpha(x)\cap N_\alpha(y)|\\
&-6\sum_{xy\in E(\alpha)}|N(\alpha)|-2\sum_{xy\in E(\alpha)}|N(\alpha)|\,|N_\alpha(x)\cap N_\alpha(y)|-6\sum_{xy\in E(\alpha)}[d_\alpha(x)+d_(y)]\\
&-2\sum_{xy\in E(\alpha)}|N_\alpha(x)\cap N_\alpha(y)|[d_\alpha(x)+d_(y)]\\
=\;& n_\alpha n_\beta^2+4LM_1(\alpha)+t^2+4n_\beta X+4t^e+2n_\beta t+n_\beta^3\\
&+n_\alpha^2 n_\beta+2n_\alpha n_\beta^2+M_1(\beta)+n_\beta+2(2m_\beta)-2n_\beta(2m_\beta)\\
&-2n_\beta^2-2n_\alpha(2m_\beta)-2n_\alpha n_\beta+m_\alpha^3+F(\alpha)+2m_\alpha M_1(\alpha)\\
&+2M_2(\alpha)+9m_\alpha+C^2+6C-6m_\alpha^2-2m_\alpha C-6M_1(\alpha)-2D\\
=\;& n_\alpha n_\beta^2+4LM_1(\alpha)+t^2+4n_\beta X+4t^e+2n_\beta t+n_\beta^3+n_\alpha^2 n_\beta+2n_\alpha n_\beta^2\\
&+M_1(\beta)+n_\beta+4m_\beta-4m_\beta n_\beta-2n_\beta^2-4n_\alpha m_\beta-2n_\alpha n_\beta+m_\alpha^3\\
&+F(\alpha)+2m_\alpha M_1(\alpha)+2M_2(\alpha)+9m_\alpha+C^2+6C-6m_\alpha^2-2m_\alpha C\\
&-6M_1(\alpha)-2D\\
=\;& 3n_\alpha n_\beta^2+4LM_1(\alpha)+t^2+4n_\beta X+4t^e+2n_\beta t+n_\beta^3+n_\alpha^2 n_\beta\\
&+M_1(\beta)+n_\beta+4m_\beta-4m_\beta n_\beta-2n_\beta^2-4m_\beta n_\alpha-2n_\alpha n_\beta\\
&+m_\alpha^3+F(\alpha)+2m_\alpha M_1(\alpha)+2M_2(\alpha)+9m_\alpha+C^2+6C-6m_\alpha^2-2m_\alpha C\\
&-6M_1(\alpha)-2D.
\end{aligned}
$$

□

**Theorem 6.** *The following is $\alpha_e$'s second leap Zagreb index:*

$$
\begin{aligned}
LM_2(\alpha_e)\;=\;& m_\alpha n_\beta^2+2n_\beta LM_3(\alpha)+n_\beta T_1+4LM_2(\alpha)+2D_1+2D_2+T_2+m_\beta n_\alpha^2+m_\beta n_\beta^2\\
&+2m_\beta n_\alpha n_\beta-n_\beta M_1(\beta)-2m_\beta n_\beta-n_\alpha M_1(\beta)+M_2(\beta)+M_1(\beta)-2m_\beta n_\alpha+m_\beta\\
&+m_\alpha^2 n_\alpha n_\beta+m_\alpha^2 n_\beta^2-m_\alpha^2 n_\beta+n_\beta^2 M_1(\alpha)-3m_\alpha n_\alpha n_\beta-3m_\alpha n_\beta^2+3m_\alpha n_\beta-\\
&(n_\alpha n_\beta+n_\beta^2-n_\beta)B_1+n_\alpha n_\beta M_1(\alpha)+6m_\alpha m_\beta-2m_\alpha^2 m_\beta-2m_\beta M_1(\alpha)+2m_\beta B_1\\
&-n_\beta M_1(\alpha)+2m_\alpha^2 n_\beta+2n_\beta M_1(\alpha)-6m_\alpha n_\beta-2n_\beta B_1+(2m_\alpha^2-6m_\alpha)D_3\\
&+2M_1(\alpha)D_3-2D_4+(m_\alpha^2-3m_\alpha)C_1+M_1(\alpha)C_1-C_2,
\end{aligned}
$$

*where,*
$T_1=\sum_{sq\in E(\alpha)}(t_\alpha(s)+t_\alpha(q))$,
$T_2=\sum_{sq\in E(\alpha)}t_\alpha(s)t_\alpha(q)$,
$D_1=\sum_{sq\in E(\alpha)}t_\alpha(q)d_\alpha(s|2)$,

$D_2 = \sum\limits_{sq\in E(\alpha)} t_\alpha(s)d_\alpha(q|2)$,
$B_1 = \sum\limits_{s\in\{x,y\}} (d_\alpha(s|2)\,|N_\alpha(x)\cap N_\alpha(y)|$,
$D_3 = \sum\limits_{s\in\{x,y\}} (d_\alpha(s|2)$,
$D_4 = \sum\limits_{xy\in E(\alpha)} \sum\limits_{s\in\{x,y\}} (d_\alpha(s|2)\,|N_\alpha(x)\cap N_\alpha(y)|$,
$C_1 = \sum\limits_{s\in\{x,y\}} t_\alpha(s)$ *and*
$C_2 = \sum\limits_{xy\in E(\alpha)} \sum\limits_{s\in\{x,y\}} t_\alpha(s)\,|N_\alpha(x)\cap N_\alpha(y)|$

**Proof.** Using (2) and edge set of $\alpha_e$, we obtain

$$\begin{aligned}
LM_2(\alpha_e) &= \sum_{sq\in E(\alpha_e)} d_{\alpha_e}(s|2)d_{\alpha_e}(q|2)\\
&= \sum_{sq\in E(\alpha)} [n_\beta + 2d_\alpha(s|2) + t_\alpha(s)][n_\beta + 2d_\alpha(q|2) + t_\alpha(q)]\\
&\quad + \sum_{sq\in E(\beta)} [n_\beta + n_\alpha - d_\beta(s) - 1][n_\beta + n_\alpha - d_\beta(q) - 1] + \sum_{uq_{xy}\in k_e} [n_\beta + n_\alpha - d_\beta(s) - 1]\\
&\quad [|N(\alpha)| + d_\alpha(x) + d_\alpha(y) - 3 - |N_\alpha(x)\cap N_\alpha(y)|] + \sum_{uq_{xy}\in J(\alpha)} [n_\beta + 2d_\alpha(s|2) + t_\alpha(s)]\\
&\quad [|N(\alpha)| + d_\alpha(x) + d_\alpha(y) - 3 - |N_\alpha(x)\cap N_\alpha(y)|]\\
&= \sum_{sq\in E(\alpha)} [n_\beta^2 + 2n_\beta d_\alpha(q|2) + n_\beta t_\alpha(q) + 2n_\beta d_\alpha(s|2) + 4d_\alpha(s|2)d_\alpha(q|2) + 2d_\alpha(s|2)t_\alpha(q)\\
&\quad + n_\beta t_\alpha(s) + 2d_\alpha(q|2)t_\alpha(s) + t_\alpha(s)t_\alpha(q)] + \sum_{sq\in E(\beta)} [n_\beta^2 + n_\alpha n_\beta - n_\beta d_\beta(q) - n_\beta + n_\alpha n_\beta\\
&\quad + n_\alpha^2 - n_\alpha d_\beta(q) - n_\alpha - n_\beta d_\beta(s) - n_\alpha d_\beta(s) + d_\beta(s)d_\beta(q) + d_\beta(s) - n_\beta - n_\alpha + d_\beta(q) + 1]\\
&\quad + \sum_{s\in V(\beta)} \sum_{q_{xy}\in|N(\alpha)|} [n_\beta\,|N(\alpha)| + n_\beta d_\alpha(x) + n_\beta d_\alpha(y) - 3n_\beta - n_\beta\,|N_\alpha(x)\cap N_\alpha(y)|\\
&\quad + n_\alpha\,|N(\alpha)| + n_\alpha d_\alpha(x) + n_\alpha d_\alpha(y) - 3n_\alpha - n_\alpha\,|N_\alpha(x)\cap N_\alpha(y)| - |N(\alpha)|\,d_\beta(s) - d_\beta(s)d_\alpha(x)\\
&\quad - d_\beta(s)d_\alpha(y) + 3d_\beta(s) + d_\beta(s)\,|N_\alpha(x)\cap N_\alpha(y)| - |N(\alpha)| - d_\alpha(x) - d_\alpha(y) + 3 + |N_\alpha(x)\cap N_\alpha(y)|]\\
&\quad + \sum_{q_{xy}\in|N(\alpha)|} \sum_{s\in\{x,y\}} [n_\beta\,|N(\alpha)| + n_\beta d_\alpha(x) + n_\beta d_\alpha(y) - 3n_\beta - n_\beta\,|N_\alpha(x)\cap N_\alpha(y)|\\
&\quad + 2\,|N(\alpha)|\,d_\alpha(s|2) + 2d_\alpha(s|2)d_\alpha(x) + 2d_\alpha(s|2)d_\alpha(y) - 6d_\alpha(s|2) - 2d_\alpha(s|2)\,|N_\alpha(x)\cap N_\alpha(y)|\\
&\quad + |N(\alpha)|\,t_\alpha(s) + d_\alpha(x)t_\alpha(s) + d_\alpha(y)t_\alpha(s) - 3t_\alpha(s) - t_\alpha(s)\,|N_\alpha(x)\cap N_\alpha(y)|]\\
&= \sum_{sq\in E(\alpha)} [n_\beta^2 + 2n_\beta(d_\alpha(s|2) + d_\alpha(q|2)) + n_\beta(t_\alpha(s) + t_\alpha(q)) + 4d_\alpha(s|2)d_\alpha(q|2) + 2t_\alpha(q)d_\alpha(s|2)\\
&\quad + 2t_\alpha(s)d_\alpha(q|2) + t_\alpha(s)t_\alpha(q)] + \sum_{sq\in E(\beta)} [n_\alpha^2 + n_\beta^2 + 2n_\alpha n_\beta - n_\beta(d_\beta(s) + d_\beta(q)) - 2n_\beta\\
&\quad - n_\alpha(d_\beta(s) + d_\beta(q)) + d_\beta(s)d_\beta(q) + (d_\beta(s) + d_\beta(q)) - 2n_\alpha + 1] + \sum_{s\in V(\beta)} \sum_{xy\in E(\alpha)} [(n_\alpha + n_\beta\\
&\quad - 1)\,|N(\alpha)| + n_\beta(d_\alpha(x) + d_\alpha(y)) - 3(n_\alpha + n_\beta - 1) - (n_\alpha + n_\beta - 1)\,|N_\alpha(x)\cap N_\alpha(y)|\\
&\quad + n_\alpha(d_\alpha(x) + d_\alpha(y)) + (3 - |N(\alpha)|)d_\beta(s) - d_\beta(s)(d_\alpha(x) + d_\alpha(y)) + d_\beta(s)\,|N_\alpha(x)\cap N_\alpha(y)|\\
&\quad - (d_\alpha(x) + d_\alpha(y))] + \sum_{xy\in E(\alpha)} \sum_{s\in\{x,y\}} [n_\beta\,|N(\alpha)| + n_\beta(d_\alpha(x) + d_\alpha(y)) - 3n_\beta - n_\beta\,|N_\alpha(x)\cap N_\alpha(y)|\\
&\quad + (2\,|N(\alpha)| - 6)d_\alpha(s|2) + 2d_\alpha(s|2)(d_\alpha(x) + d_\alpha(y)) - 2d_\alpha(s|2)\,|N_\alpha(x)\cap N_\alpha(y)| + (|N(\alpha)| - 3)t_\alpha(s)\\
&\quad + t_\alpha(s)(d_\alpha(x) + d_\alpha(y)) - t_\alpha(s)\,|N_\alpha(x)\cap N_\alpha(y)|]\\
&= n_\beta^2 m_\alpha + 2n_\beta \sum_{sq\in E(\alpha)} (d_\alpha(s|2) + d_\alpha(q|2)) + n_\beta \sum_{sq\in E(\alpha)} (t_\alpha(s) + t_\alpha(q)) + 4\sum_{sq\in E(\alpha)} d_\alpha(s|2)d_\alpha(q|2)\\
&\quad + 2\sum_{sq\in E(\alpha)} t_\alpha(q)d_\alpha(s|2) + 2\sum_{sq\in E(\alpha)} t_\alpha(s)d_\alpha(q|2) + \sum_{sq\in E(\alpha)} t_\alpha(s)t_\alpha(q) + m_\beta n_\alpha^2 + m_\beta n_\beta^2
\end{aligned}$$

$$
\begin{aligned}
&+2m_\beta n_\alpha n_\beta - n_\beta \sum_{sq\in E(\beta)} (d_\beta(s)+d_\beta(q)) - 2m_\beta n_\beta - n_\alpha \sum_{sq\in E(\beta)} (d_\beta(s)+d_\beta(q)) \\
&+\sum_{sq\in E(\beta)} d_\beta(s)d_\beta(q) + \sum_{sq\in E(\beta)} (d_\beta(s)+d_\beta(q)) - 2m_\beta n_\alpha + m_\beta + (n_\alpha+n_\beta-1)m_\alpha^2 n_\beta \\
&+n_\beta^2 \sum_{xy\in E(\alpha)} (d_\alpha(x)+d_\alpha(y)) - 3m_\alpha n_\beta(n_\alpha+n_\beta-1) - n_\beta(n_\alpha+n_\beta-1) \sum_{xy\in E(\alpha)} |N_\alpha(x)\cap N_\alpha(y)| \\
&+n_\alpha n_\beta \sum_{xy\in E(\alpha)} (d_\alpha(x)+d_\alpha(y)) + 6m_\alpha m_\beta - 2m_\alpha^2 m_\beta - 2m_\beta \sum_{xy\in E(\alpha)} (d_\alpha(x)+d_\alpha(y)) \\
&+2m_\beta \sum_{xy\in E(\alpha)} |N_\alpha(x)\cap N_\alpha(y)| - n_\beta \sum_{xy\in E(\alpha)} (d_\alpha(x)+d_\alpha(y)) + 2m_\alpha^2 n_\beta + 2n_\beta \sum_{xy\in E(\alpha)} (d_\alpha(x)+ \\
&d_\alpha(y)) - 6m_\alpha n_\beta - 2n_\beta \sum_{xy\in E(\alpha)} |N_\alpha(x)\cap N_\alpha(y)| + (2m_\alpha^2 - 6m_\alpha) \sum_{s\in\{x,y\}} (d_\alpha(s|2) + \\
&2\sum_{xy\in E(\alpha)} \sum_{s\in\{x,y\}} (d_\alpha(x)+d_\alpha(y))(d_\alpha(s|2) - 2\sum_{xy\in E(\alpha)} \sum_{s\in\{x,y\}} (d_\alpha(s|2)\,|N_\alpha(x)\cap N_\alpha(y)| + (m_\alpha^2 - 3m_\alpha) \\
&\sum_{s\in\{x,y\}} t_\alpha(s) + \sum_{xy\in E(\alpha)} \sum_{s\in\{x,y\}} (d_\alpha(x)+d_\alpha(y))t_\alpha(s) - \sum_{xy\in E(\alpha)} \sum_{s\in\{x,y\}} t_\alpha(s)\,|N_\alpha(x)\cap N_\alpha(y)| \\
=\;& m_\alpha n_\beta^2 + 2n_\beta LM_3(\alpha) + n_\beta T_1 + 4LM_2(\alpha) + 2D_1 + 2D_2 + T_2 + m_\beta n_\alpha^2 + m_\beta n_\beta^2 + 2m_\beta n_\alpha n_\beta \\
&-n_\beta M_1(\beta) - 2m_\beta n_\beta - n_\alpha M_1(\beta) + M_2(\beta) + M_1(\beta) - 2m_\beta n_\alpha + m_\beta + m_\alpha^2 n_\alpha n_\beta + m_\alpha^2 n_\beta^2 \\
&-m_\alpha^2 n_\beta + n_\beta^2 M_1(\alpha) - 3m_\alpha n_\alpha n_\beta - 3m_\alpha n_\beta^2 + 3m_\alpha n_\beta - (n_\alpha n_\beta + n_\beta^2 - n_\beta)B_1 + n_\alpha n_\beta M_1(\alpha) \\
&+6m_\alpha m_\beta - 2m_\alpha^2 m_\beta - 2m_\beta M_1(\alpha) + 2m_\beta B_1 - n_\beta M_1(\alpha) + 2m_\alpha^2 n_\beta + 2n_\beta M_1(\alpha) - 6m_\alpha n_\beta \\
&-2n_\beta B_1 + (2m_\alpha^2 - 6m_\alpha)D_3 + 2M_1(\alpha)D_3 - 2D_4 + (m_\alpha^2 - 3m_\alpha)C_1 + M_1(\alpha)C_1 - C_2.
\end{aligned}
$$

□

**Theorem 7.** *The third leap Zagreb index of $\alpha_e$ is as follows*

$$
\begin{aligned}
LM_3(\alpha_e) \;=\;& 4LM_3(\alpha) + 2t^c + 2m_\beta n_\beta + 2m_\beta n_\alpha - M_1(\beta) - 2m_\beta + m_\alpha n_\beta^2 + m_\alpha n_\alpha n_\beta - 2m_\alpha m_\beta + (n_\beta+2)M_1(\alpha) \\
&+2m_\alpha^2 - (n_\beta+2)C - 6m_\alpha + m_\alpha^2 n_\beta,
\end{aligned}
$$

*where* $t^c = \sum_{q\in V(\alpha)} t(q)d_\alpha(q)$ *and* $C = \sum_{xy\in E(\alpha)} |N_\alpha(x)\cap N_\alpha(y)|$.

**Proof.** Using (3) and vertex set of $\alpha_e$, we obtain

$$
\begin{aligned}
LM_3(\alpha_e) \;=\;& \sum_{q\in V(\alpha_e)} d(q)d_2(q) \\
=\;& \sum_{q\in V(\alpha)} [2d_\alpha(q)(n_\beta + 2d_\alpha(q|2) + t(q))] + \sum_{q\in V(\beta)} [(d_\beta(q)+m_\alpha)(n_\beta+n_\alpha-d_\beta(q)-1)] \\
&\sum_{q_{xy}\in|N(\alpha)|} [(2+n_\beta)(d_\alpha(x)+d_\alpha(y)+|N(\alpha)| - |N_\alpha(x)\cap N_\alpha(y)| - 3)] \\
=\;& \sum_{q\in V(\alpha)} [2n_\beta d_\alpha(q) + 4d_\alpha(q)d_\alpha(q|2) + 2t(q)d_\alpha(q)] \\
&\sum_{q\in V(\beta)} [n_\beta d_\beta(q) + n_\alpha d_\beta(q) - d_\beta^2(q) - d_\beta(q) + m_\alpha n_\beta + m_\alpha n_\alpha - m_\alpha d_\beta(q) - m_\alpha] \\
&\sum_{q_{xy}\in|N(\alpha)|} [2d_\alpha(x) + 2d_\alpha(y) + 2\,|N(\alpha)| - 2|N_\alpha(x)\cap N_\alpha(y)| - 6 + n_\beta d_\alpha(x) \\
&+n_\beta d_d(y) + n_\beta\,|N(\alpha)| - n_\beta|N_\alpha(x)\cap N_\alpha(y)| - 3n_\beta] \\
=\;& 2n_\beta \sum_{q\in V(\alpha)} d_\alpha(q) + 4\sum_{q\in V(\alpha)} d_\alpha(q)d_\alpha(q|2) + 2\sum_{q\in V(\alpha)} t(q)d_\alpha(q) \\
&n_\beta \sum_{q\in V(\beta)} d_\beta(q) + n_\alpha \sum_{q\in V(\beta)} d_\beta(q) - \sum_{q\in V(\beta)} d_\beta^2(q)
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{q\in V(\beta)} d_\beta(q) + m_\alpha n_\beta^2 + m_\alpha n_\alpha n_\beta - m_\alpha \sum_{q\in V(\beta)} d_\beta(q) - m_\alpha n_\beta \\
& +2\sum_{xy\in E(\alpha)} [d_\alpha(x)+d_\alpha(y)] + 2\sum_{q_{xy}\in|N(\alpha)|} |N(\alpha)| - 2\sum_{xy\in E(\alpha)} |N_\alpha(x)\cap N_\alpha(y)| \\
& -6m_\alpha + n_\beta \sum_{xy\in E(\alpha)} [d_\alpha(x)+d_\alpha(y)] + n_\beta \sum_{q_{xy}\in|N(\alpha)|} |N(\alpha)| \\
& -n_\beta \sum_{xy\in E(\alpha)} |N_\alpha(x)\cap N_\alpha(y)| - 3n_\beta m_\alpha \\
= {} & 2n_\beta(2m_\alpha) + 4LM_3(\alpha) + 2t^c + n_\beta(2m_\beta) + n_\alpha(2m_\beta) \\
& -M_1(\beta) - 2m_\beta + m_\alpha n_\beta^2 + m_\alpha n_\alpha n_\beta - m_\alpha(2m_\beta) - m_\alpha n_\beta + 2M_1(\alpha) \\
& +2m_\alpha^2 - 2C - 6m_\alpha + n_\beta M_1(\alpha) + n_\beta m_\alpha^2 - n_\beta C - 3n_\beta m_\alpha \\
= {} & 4m_\alpha n_\beta + 4LM_3(\alpha) + 2t^c + 2m_\beta n_\beta + 2m_\beta n_\alpha - M_1(\beta) - 2m_\beta + m_\alpha n_\beta^2 \\
& +m_\alpha n_\alpha n_\beta - 2m_\alpha m_\beta - m_\alpha n_\beta + 2M_1(\alpha) + 2m_\alpha^2 - 2C - 6m_\alpha + n_\beta M_1(\alpha) + m_\alpha^2 n_\beta - n_\beta C - 3n_\beta m_\alpha \\
= {} & 4LM_3(\alpha) + 2t^c + 2m_\beta n_\beta + 2m_\beta n_\alpha - M_1(\beta) \\
& -2m_\beta + m_\alpha n_\beta^2 + m_\alpha n_\alpha n_\beta - 2m_\alpha m_\beta + (n_\beta+2)M_1(\alpha) + 2m_\alpha^2 - (n_\beta+2)C - 6m_\alpha + m_\alpha^2 n_\beta.
\end{aligned}
$$

□

**Corollary 8.** *Let $G = H = S_n$, $n \geq 3$ be a star graph with $n$ vertices. Then*

$$LM_3(S_n\langle e\rangle S_n) = 4n^3 + 4n^2 - 24n + 16.$$

The comparison between Corollary 4 and Corollary 8 is shown in Figure 2.

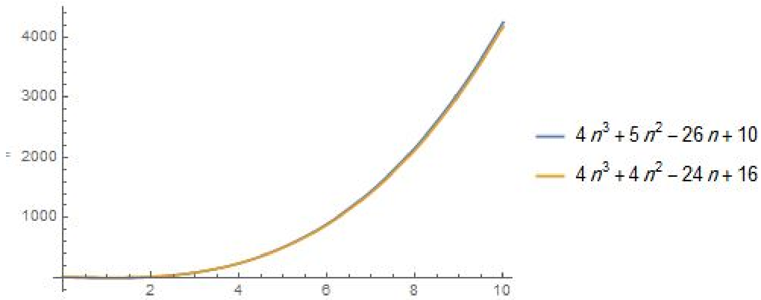


**Figure 2.** Comparison between corollary 4 and 8

## 5. Wiener polarity index of $R$-vertex and edge join of graphs

In this section, we will discuss some results based on third distances indices of $R$-vertex and edge join of graphs.

**Theorem 9.** *The Wiener polarity index of $\alpha_q$ is as follows:*

$$W_p(\alpha_q) = \frac{1}{2}m_\alpha n_\alpha - m_\alpha - \frac{1}{2}p - \frac{1}{2}t + \frac{1}{2}L - \frac{1}{2}M,$$

*where*
$p = \sum_{q\in V(\alpha)} p_\alpha(s|2)$,
$t = \sum_{q\in V(\alpha)} t(s)$,
$M = \sum_{xy\in E(\alpha)} |N_\alpha(x)\cap N_\alpha(y|2) \cup N_\alpha(x|2)\cap N_\alpha(y)|$ *and*
$L = \sum_{xy\in E(\alpha)} \left|N_{\alpha_q}(x|2)\cap N_{\alpha_q}(y|2)\right|$.

**Proof.** By using Eq. (4) and vertex set of $\alpha_q$, we obtain

$$\begin{aligned}
W_p(\alpha_q) &= \frac{1}{2}\sum_{s\in V(\alpha_q)} d_3(s) \\
&= \frac{1}{2}\sum_{s\in V(\alpha)} [|N(\alpha)| - d_\alpha(s) - p_\alpha(s|2) - t(s)] \\
&\quad + \frac{1}{2}\sum_{s_{xy}\in V(|N(\alpha)|)} [|N_{\alpha_q}(x|2)\cup N_{\alpha_q}(y|2)| \\
&\quad -|(N_\alpha(x)\cap N_\alpha(y|2))\cup(|N_\alpha(x|2)\cap N_\alpha(y))|] \\
&= \frac{1}{2}\sum_{s\in V(\alpha)} |N(\alpha)| - \frac{1}{2}\sum_{s\in V(\alpha)} d_\alpha(s) - \frac{1}{2}\sum_{s\in V(\alpha)} p_\alpha(s|2) - \frac{1}{2}\sum_{s\in V(\alpha)} t(s) \\
&\quad + \frac{1}{2}\sum_{xy\in E(\alpha)} |N_{\alpha_q}(x|2)\cup N_{\alpha_q}(y|2)| \\
&\quad - \frac{1}{2}\sum_{xy\in E(\alpha)} |(N_\alpha(x)\cap N_\alpha(y|2))\cup(|N_\alpha(x|2)\cap N_\alpha(y))| \\
&= \frac{1}{2}m_\alpha n_\alpha - \frac{1}{2}(2m_\alpha) - \frac{1}{2}p - \frac{1}{2}t + \frac{1}{2}L - \frac{1}{2}M \\
&= \frac{1}{2}m_\alpha n_\alpha - m_\alpha - \frac{1}{2}p - \frac{1}{2}t + \frac{1}{2}L - \frac{1}{2}M.
\end{aligned}$$

□

**Theorem 10.** *The Wiener polarity index of $\alpha_e$ is as follows:*

$$W_p(\alpha_e) = W_3(\alpha) + m_\alpha n_\alpha - \frac{3}{2}m_\alpha - \frac{1}{2}p - \frac{1}{2}t - \frac{1}{2}m_\alpha^2 - \frac{1}{2}Q - \frac{1}{2}N,$$

*where*
$p = \sum_{q\in V(\alpha)} p_\alpha(s|2)$,
$t = \sum_{q\in V(\alpha)} t(s)$,
$Q = \sum_{s_{x,y}\in|N(\alpha)|} |N_\alpha(x)\cup N_\alpha(y)|$ *and*
$N = \sum_{s_{x,y}\in|N(\alpha)|} |x_{x'y'}|x' or y' \in \{x,y\}|$.

**Proof.** By using Eq. (4) and edge set of $\alpha_e$, we obtain

$$\begin{aligned}
W_p(\alpha_e) &= \frac{1}{2}\sum_{s\in V(\alpha_e)} d(|3u) \\
&= \frac{1}{2}\sum_{s\in V(\alpha)} [d_\alpha(s|3) + |N(\alpha)| - d_\alpha(s) - p_\alpha(s|2) - t(s)] \\
&\quad + \frac{1}{2}\sum_{s_{x,y}\in|N(\alpha)|} [n_\alpha + m_\alpha - |N_\alpha(x)\cup N_\alpha(y)| - |s_{x'y'}|x' \text{ or } y' \in \{x,y\}| - 1] \\
&= \frac{1}{2}\sum_{s\in V(\alpha)} d_\alpha(s|3) + \frac{1}{2}\sum_{s\in V(\alpha)} |N(\alpha)| - \frac{1}{2}\sum_{s\in V(\alpha)} d_\alpha(s) - \frac{1}{2}\sum_{s\in V(\alpha)} p_\alpha(s|2) \\
&\quad - \frac{1}{2}\sum_{s\in V(\alpha)} t(s) + \frac{1}{2}\sum_{s_{x,y}\in|N(\alpha)|} n_\alpha - \frac{1}{2}\sum_{s_{x,y}\in|N(\alpha)|} m_\alpha \\
&\quad - \frac{1}{2}\sum_{s_{x,y}\in|N(\alpha)|} |N_\alpha(x)\cup N_\alpha(y)| - \sum_{s_{x,y}\in|N(\alpha)|} |s_{x'y'}|x' \text{ or } y' \in \{x,y\}| - \frac{1}{2}m_\alpha
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{s\in V(\alpha)} d_\alpha(s|3) + \frac{1}{2}\sum_{s\in V(\alpha)} |N(\alpha)| - \frac{1}{2}\sum_{s\in V(\alpha)} d_\alpha(s) - \frac{1}{2}\sum_{s\in V(\alpha)} p_\alpha(s|2) \\
&\quad -\frac{1}{2}\sum_{s\in V(\alpha)} t(s) + \frac{1}{2}\sum_{xy\in E(\alpha)} n_\alpha - \frac{1}{2}\sum_{xy\in E(\alpha)} m_\alpha \\
&\quad -\frac{1}{2}\sum_{s_{x,y}\in|N(\alpha)|} |N_\alpha(x)\cup N_\alpha(y)| - \sum_{s_{x,y}\in|N(\alpha)|} |s_{x'y'}|x' \text{ or } y' \in \{x,y\}| - \frac{1}{2}m_\alpha \\
&= W_3(\alpha) + \frac{1}{2}m_\alpha n_\alpha - m_\alpha - \frac{1}{2}p - \frac{1}{2}t + \frac{1}{2}m_\alpha n_\alpha - \frac{1}{2}m_\alpha^2 - \frac{1}{2}Q - \frac{1}{2}N - \frac{1}{2}m_\alpha \\
&= W_3(\alpha) + m_\alpha n_\alpha - \frac{3}{2}m_\alpha - \frac{1}{2}p - \frac{1}{2}t - \frac{1}{2}m_\alpha^2 - \frac{1}{2}Q - \frac{1}{2}N.
\end{aligned}
$$

□

## 6. Conclusion

In this paper, we studied the leap Zagreb indices and Wiener polarity index of $R$-vertex join and $R$-edge join of graphs. The authors are invited to investigate the topological properties and descriptors of $R$-vertex join and $R$-edge join of graphs based on distance, which appears to be an interesting and difficult issue.

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