

Article

# Castling tree of tight Dyck nests with applications to odd and middle-levels graphs

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**Abstract:** A subfamily of Dyck words called tight Dyck words is seen to correspond, via a “castling” procedure, to the vertex set of an ordered tree  $T$ . From  $T$ , a “blowing” operation recreates the whole family of Dyck words. The vertices of  $T$  can be elementarily updated all along  $T$ . This simplifies an edge-supplementary arc-factorization view of Hamilton cycles of odd and middle-levels graphs found by T. Mütze et al. This takes into account that the Dyck words represent: **(a)** the cyclic and dihedral vertex classes of odd and middle-levels graphs, respectively, and **(b)** the cycles of their 2-factors, as found by T. Mütze et al.

**Keywords:** Dyck words, ordered trees, Hamilton cycles

**MSC:** 05C45, 05A19.

## 1. Introduction

**F**or  $0 < k < n \in \mathbb{Z}$ , the Kneser graph  $K_{n,k}$  [1,2] has as its vertices all  $k$ -subsets of the set  $[0, n - 1] = \{0, 1, \dots, n - 1\}$ , with any two such vertices adjacent if and only if their intersection as  $k$ -subsets of  $[0, n - 1]$  is empty. For  $n = 2k + 1$ ,  $K(n, k)$  is the *odd graph*  $O_k$ . We represent the vertices of  $O_k$  by the *characteristic vectors* of those  $k$ -subsets of  $[0, n - 1]$ , so that each such  $k$ -subset is the *support* of its characteristic vector.

Those vectors represent the members of the  $k$ -level  $L_k$  of the Boolean lattice  $B_{2k+1}$  on  $[0, 2k]$ . The complements of the reversed strings of those vectors are taken to represent the  $(k + 1)$ -level  $L_{k+1}$  of  $B_{2k+1}$ . The union of these two levels,  $L_k \cup L_{k+1}$ , is the vertex set of the *middle-levels graph*  $M_k$  [3], with adjacency given by the inclusion of  $L_k$  in  $L_{k+1}$ . This yields a 2-covering graph map  $\Psi_k : M_k \rightarrow O_k$  expressible via the *reversed complement* bijection  $\aleph : V(M_k) \rightarrow V(M_k)$  given by  $\aleph(v_0 v_1 \cdots v_{2k-1} v_{2k}) = \bar{v}_{2k} \bar{v}_{2k-1} \cdots \bar{v}_1 \bar{v}_0$ , where  $\bar{0} = 1$  and  $\bar{1} = 0$ , so that  $\aleph(L_k) = L_{k+1}$  and  $\aleph(L_{k+1}) = L_k$ , and extending to a graph automorphism of  $M_k$ , again denoted w.l.o.g. by  $\aleph$ . In fact,  $\Psi : M_k \rightarrow O_k$  is characterized by its restriction to the identity map over  $L_k$  and by its restriction to  $\aleph$  over  $L_{k+1}$ .

Odd graphs were shown to be Hamiltonian in [2], as an initial step in proving that sparse Kneser graphs were Hamiltonian. And this result was extended in [1] to all Kneser graphs.

More specifically, an investigation of relations of Dyck words [2] controlled by the infinite ordered tree  $T$  of *restricted-growth strings* ([4, p. 325]) that are *tight* (see definitions in §2), to the graphs  $O_k$  and  $M_k$  is undertaken. A definition of  $T$  is developed in §3, while *Dyck nests* and *tight Dyck nests* are introduced in §4 associated to corresponding Dyck words and *tight Dyck words* via their *Dyck path heights*. As said, a *castling* procedure assigning tight Dyck nests to the vertices of  $T$  is introduced in §5.

Thus, the main objective of the present work is to apply the tree  $T$  (§3) of tight restricted-growth strings, or TRGSs, (§2) lexicographically ordered by the nonnegative integers  $n$  to the Dyck words and nests of §4, so that to each TRGS  $b(n)$  corresponds a specific Dyck nest  $f(n)$ , and to each child of  $T$  corresponds the castling (§5) of its parent node.

§6-§7 introduce how to *blow* and *anchor* these words and nests, adapting them as vertex representatives of the graphs  $O_k$  and  $M_k$ . In §8, an edge-supplementary arc-factorization [5] of each odd graph based on those representatives is given.

Based on the work of Mütze et al. [2,6], a permutation is assigned in §9 to each Dyck nest  $F$ , leading §10 to establish uniform 2-factors both in the  $O_k$  and the  $M_k$ , as in [6]. The said uniform 2-factors yield a partition

of  $V(O_k)$  (§11), and thus a partition of  $V(M_k)$ , too. However, another partition can be used, instead, based on the action of the cyclic and the dihedral group on  $V(O_k)$  and  $V(M_k)$ , respectively.

In §12, we assign a *clone* to each Dyck nest  $F$ . This clone is equivalent to  $F$  (Theorem 2). Moreover, the clone is universally updated along  $T$  (Theorem 1). Thus, it generates an infinite sequence of updates; see also [7].

A convenient set of strings is established in §13, allowing to associate the elements of the graphs  $O_k$  and  $M_k$ , as arising from the tree  $T$  (Theorems 6-7 and Corollary 3) based on either of the two mentioned partitions. This allows the cited arc-factorization approach to take to the Hamilton cycles of odd graphs and middle-levels graphs of [2] (§14, Theorem 8).

In sum, our approach differs from that of [2] by reversed complementation, so it constitutes an alternative reinterpretation of [2]. That far can be said as for a comparison of this work with that of [2], as we just have provided a universal treatment for all odd and middle-levels graph that [2] did not. As our treatment differs from that of [2] just by reversed complementation, adapting [2] to our approach did not offer any additional improvement in time complexity. About the time complexity of castling, see after Example 8.

### 2. Tight restricted-growth strings

**Definition 1.** Given two nonnegative-integer strings  $A$  and  $B$ , of lengths  $n > 0$  and  $\ell \geq n$ , respectively, with  $B \neq A$  and  $B = d_{\ell-1}d_{\ell-2} \cdots d_2d_1$ , where  $\ell \geq n$ , we say that  $A$  *precedes*  $B$  if the concatenated string  $0^{\ell-n}|A = c_{\ell-1}c_{\ell-2} \cdots c_2c_1$  has  $c_i \leq d_i$ , for  $0 < i < \ell$ .

**Definition 2.** A set of nonnegative-integer strings is said to be *lexicographically ordered* if the precedence criterion of Definition 1 holds for all its pairs of elements.

**Definition 3.** A *tight restricted-growth string*, or TRGS is a nonnegative-integer string of the form

$$a_{k-1}a_{k-2} \cdots a_2a_1, \tag{1}$$

where either

$$k > 2, a_{k-1} = 1 \text{ and } a_{j+1} \geq a_j + 1, \text{ for } k - 1 > j > 0,$$

or

$$k = 2 \text{ and } a_1 \in \{0, 1\}.$$

Let  $\mathcal{S}$  be the lexicographically-ordered sequence of TRGS's. Then,  $\mathcal{S}$  is the infinite extension of the sequence [8, A239903] (finite, as it only uses decimal digit entries  $a_j$ , while in  $\mathcal{S}$  the digit entries  $a_j$  grow unbounded).

**Example 1.**  $\mathcal{S}$  starts as  $\mathcal{S} = (b(0), b(1), b(2), \dots, b(17), \dots) =$

$$(0, 1, 10, 11, 12, 100, 101, 110, 111, 112, 120, 121, 122, 123, 1000, 1001, 1010, 1011, \dots). \tag{2}$$

The following definition yields the original notion of restricted-growth string in [4, p. 325].

**Definition 4.** A *restricted-growth string*, or RGS, is defined as either a TRGS  $b(n)$  or obtained from a TRGS  $b(n)$  by prefixing to  $b(n)$  a finite string of zeros, e.g.  $0^\ell|b(n)$  is an RGS if  $0 \leq \ell \in \mathbb{Z}$  and  $b(n) \in \mathcal{S}$ .

Note that no TRGS starts with 0 if it contains more than one entry. On the other hand, 01 is an RGS, not an TRGS.

### 3. Tree of tight restricted-growth strings

**Definition 5.** Let  $\mathbb{N}$  be the set of nonnegative integers. Let  $n \in \mathbb{N}$  and let  $\gamma(n) \in [0, k - 1]$  be the right-to-left position of the rightmost nonzero entry of  $b(n)$  (that is: counting positions from right to left, like the subindices in (1)).

**Example 2.** Definition 5 yields a sequence  $\Gamma = (\gamma(n); 0 < n \in \mathbb{Z})$  that starts, accompanying  $\mathcal{S} \setminus b(0)$ , as follows:

$\Gamma = (\gamma(1), \dots, \gamma(17), \dots) = (1, 2, 1, 1, 3, 1, 2, 1, 1, 2, 1, 1, 1, 4, 1, 2, \dots)$ , or longer, up to  $\gamma(43)$ :

$$\Gamma = (1, 2, 1, 1, 3, 1, 2, 1, 1, 2, 1, 1, 1, 4, 1, 2, 1, 1, 3, 1, 2, 1, 1, 2, 1, 1, 1, 3, 1, 2, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 5, \dots). \tag{3}$$

**Definition 6.** For each  $0 < n \in \mathbb{N}$ , let us define a string  $c(n) \in \mathcal{S}$  as follows. Express

$$\begin{aligned} b(n) &= a_{k-1}a_{k-2} \cdots a_j a_{j-1} \cdots \cdots a_2 a_1, & \text{as in (1), with } j = \gamma(n) \text{ as in (3), and let} \\ c(n) &= a_{k-1}a_{k-2} \cdots (a_j - 1)a_{j-1} \cdots a_2 a_1, & \text{unless } j = k - 1, \text{ in which case} \\ c(n) &= b(0) = a_1 = 0 \in \mathcal{S} \text{ is the TRGS for the RGS } a_{k1} \cdots a_1 = 0^{k-1}. \end{aligned} \tag{4}$$

Let  $T$  be the tree obtained by taking  $c(n)$  to be the parent of  $b(n)$  in  $T$ , so  $T$  is given by

$$T = \{\mathcal{S}, E(T)\}, \text{ with } (c(n), b(n)) \in E(T), \forall n > 0. \tag{5}$$

**Definition 7.** For each  $0 < n \in \mathbb{N}$ , we have that  $c(n) = b(m)$ , for some  $m = \rho(n) \in \mathbb{N}$ , ( $m < n$ ), where  $\rho$  is the parent function  $\rho : \mathbb{N} \rightarrow \mathbb{N}$  of  $T$ . This yields the parent sequence  $\rho(1) = 0, \rho(2) = 0, \rho(3) = 2, \rho(4) = 3, \rho(5) = 0, \rho(6) = 5, \rho(7) = 5$ , etc., for the edges  $(0, 1), (0, 10), (10, 11), (11, 12), (0, 100), (100, 101), (100, 110)$ , etc.

**Example 3.** Accompanying display (3),  $\rho(\mathbb{N} \setminus \{0\}) = \rho(\{1, 2, \dots, 43, \dots\})$  is:

$$(0, 0, 2, 3, 0, 5, 5, 7, 8, 7, 10, 11, 12, 0, 14, 16, 17, 14, 19, 19, 20, 21, 19, 24, 25, 26, 14, 28, 28, 30, 31, 30, 33, 34, 35, 33, 37, 38, 39, 40, \dots). \tag{6}$$

Figure 2 further exemplifies Definitions 6–7 in the context of §5.

### 4. Dyck words, Dyck nests and blowing

We just have seen that to each  $n \in \mathcal{N}$  corresponds a TRGS that constitutes a vertex of the tree  $T$ , and that  $V(T)$  is formed by such TRGSs. In §5, we will see that  $V(T)$  is the domain of a *castling* correspondence whose image is formed by *Dyck nests*, that we introduce in this section.

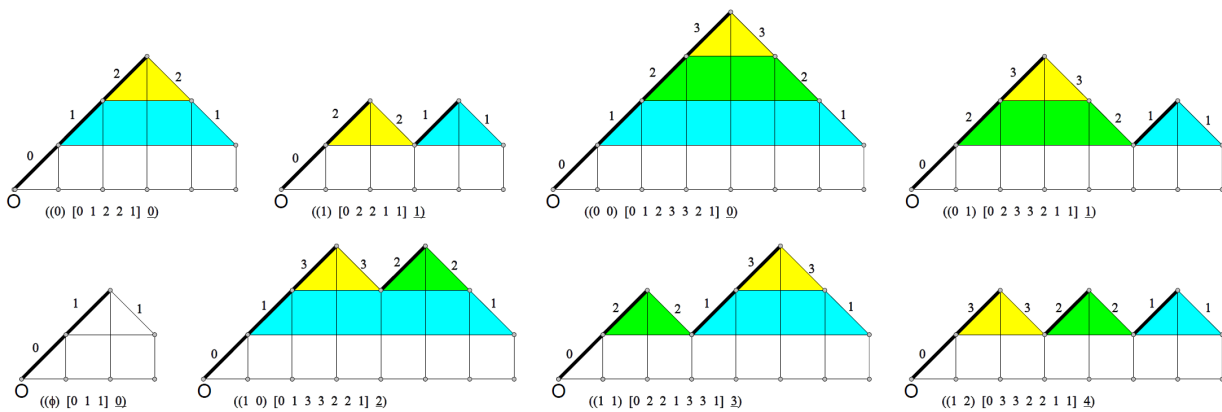
**Definition 8.** By a *Dyck word* we will understand any binary  $2k$ -string  $f = f_1 f_2 \cdots f_{2k}$  of weight  $k$  with the number of 0-bits at least equal to the number of 1-bits in each prefix of  $f$ , (which differs from [2,6,9,10] just by binary complementation). The concept of *empty Dyck word*  $\epsilon$  also makes sense in this context and is used in §14.

**Definition 9.** Each Dyck word  $f \neq \epsilon$  as in Definition 8 determines a *Dyck path*, given as a continuous piecewise-linear curve  $g_f$  in the Cartesian plane  $\mathbb{R}^2$  such that  $g_f(0) = g_f(2k) = 0$  with  $g_f(x) > 0$ , for  $0 < x < 2k$ , and formed by replacing successively from left to right each 0-bit of  $f$  by an *up-step* and each 1-bit of  $f$  by a *down-step*, where *up-steps* and *down-steps* are segments of the forms  $(x, y)(x + 1, y + 1)$  and  $(x, y)(x + 1, y - 1)$ , respectively.

**Definition 10.** We assign the integers in the interval  $[1, k]$  successively to the up-steps, respectively, down-steps, of  $g_f$  in the horizontal unit layers  $[y, y + 1] \subset \mathbb{R}^2$ , for  $y = 0, 1, 2, \dots$ , as needed, and *from right to left* at each layer. The resulting  $2k$ -string is said to be the *Dyck nest*  $F = F_1 F_2 \cdots F_{2k}$  associated to  $f = f_1 f_2 \cdots f_{2k}$ .

**Definition 11.** A Dyck nest  $F = \cdots F_{j-1} F_j F_{j+1} F_{j+2} \cdots = \cdots i(i + 1)(i + 1)i \cdots$ , arising from a Dyck word  $f = \cdots f_{j-1} f_j f_{j+1} f_{j+2} \cdots = \cdots 0011 \cdots$ , is said to be obtained by *blowing* the shorter Dyck nest  $F' = \cdots F'_{j-1} F'_j \cdots = \cdots ii \cdots$ , arising from a corresponding Dyck word  $f = \cdots f'_{j-1} f'_j \cdots = \cdots 01 \cdots$ , where  $F_{j-1} = F_{j+2} = F'_{j-1} = F'_j = i, F_j = F_{j+1} = i + 1, f_{j-1} = f_j = f'_{j-1} = 0$  and  $f_{j+1} = f_{j+2} = f'_j = 1$ . We also say that  $F'$  is obtain by *reduction* from  $F$ , or that the corresponding  $f'$  (with  $F' = F(f')$ ) is obtained by *reduction* from  $f$ . In particular, each Dyck word and corresponding Dyck nest are equal or can be reduced to a TRGS and corresponding irreducible Dyck nest.

**Example 4.** Figure 1 illustrates Definitions 9–10–11. In fact, for each one of the eight cases in the figure, a piecewise-linear curve  $g_f$  is constructed iteratively that starts at the shown origin  $O$  in the Cartesian plane  $\mathbb{R}^2$  by replacing successively the 0-bits and 1-bits of  $f$  by *up-steps* and *down-steps*, namely diagonal segments  $(x, y)(x + 1, y + 1)$  and  $(x, y)(x + 1, y - 1)$ , respectively. We assign the integers in the interval  $[0, k]$  in decreasing order (from  $k$  to  $0$ ) to the up-steps, respectively down-steps, of  $g_f$ , from the top unit layer of  $g_f$  in  $\mathbb{R}^2$  to the bottom one and from left to right at each pertaining unit layer between contiguous lines  $y, y + 1 \in \mathbb{Z}$ . Then, by reading and successively writing the number entries assigned to the steps of  $g_f$ , the  $n$ -tuple  $F(f)$  is obtained. The figure is provided, underneath each instance, with the corresponding  $f$  followed by  $F(f)$  and its (underlined> order of presentation via the castling procedure of §5, below. We assume that all elements of  $V(O_k)$  are represented by means of such piecewise-linear curves, for each fixed integer  $k > 0$ .



**Figure 1.** Recovering  $F(f)$  from  $f$ :  $PLC(f)$  for triples  $((f) [F(f)], \underline{ord}(f)), k = 1, 2, 3$

**Definition 12.** The inverse operation to blowing, applied initially to a Dyck nest  $F$  and iterated successively through  $F', (F')', \dots$  etc., leads to a *tight* Dyck nest, meaning it cannot be further reduced to a shorter Dyck nest. Dyck words  $f$  yielding tight Dyck nests  $F$  will be said to be *tight*.

Each Dyck word  $f$  as in Definition 8 determines a Dyck path  $g_f$  as in Definition 9 which in turn determines a Dyck nest  $F(f)$  as in Definition 10. If  $f$  is tight, then  $F(f)$  is irreducible, as in Definition 11, or tight as in Definition 12. If not, then reduction takes  $f$  into a tight  $f'$ , and  $F = F(f)$  into an irreducible or tight Dyck nest  $F' = F(f')$ .

**Example 5.** To the Dyck words  $f = 01, f = 0011, f = 000111, \dots, f = 000 \dots 111$ , etc., correspond the respective nests  $F = 11, F = 1221, F = 123321, \dots, F = 123 \dots 321$ , etc. The first Dyck nest here,  $F = 11$ , is the tight Dyck nest of each of its subsequent ones, and each of these are obtained by *blowing*  $F = 11$ .

**Example 6.** Tight Dyck nest  $F = 2211$  (for tight Dyck word  $f = 0101$ ) is the tight Dyck nest of  $F = 233211$  (for  $f = 001101$ ), which is therefore obtained by blowing  $F = 2211$ .

**Example 7.** When lexicographically ordered, the tight Dyck nests start as follows, according to (2)-(3) and (6):  $\mathcal{N} = (F(0), F(1), \dots, F(13), \dots)$ , where  $F(0), F(1), \dots, F(13), \dots$  are

$$11, 2211, 133221, 221331, 332211, 12443321, 24433211, 13324421, 24421331, 33244211, 14433221, 22144331, 33221441, 44332211, \dots \tag{7}$$

respectively. The corresponding originating Dyck words here start the sequence of associated *tight* Dyck words:  $\mathcal{W} = (f(0), f(1), \dots, f(13), \dots)$ , where  $f(0), f(1), \dots, f(13), \dots$  are

$$01, 0101, 001011, 010011, 010101, 00010111, 00101101, 00100111, 00110011, 01001101, 00101011, 01001011, 01010011, 01010101, \dots \tag{8}$$

respectively.

### 5. Castling procedure

Modifying Theorem 3.2 [11] or Theorem 2 [12] for our present purposes, we assign a Dyck word  $f(n) = f_1 f_2 \cdots f_{2k}$  to each TRGS  $b(n) = a_{k-1} \cdots a_1 \in \mathcal{S}$  as in (1)-(2) in such a way that all such Dyck words become represented (once each) via uniquely corresponding  $F(n)$ 's generated via the procedure contained in the subsequent items (a)-(f), each  $F(n)$  yielding its  $f(n)$  by replacing each first appearance  $j_1$  of an integer  $j$  by a 0-bit and its second appearance  $j_2$  of  $j$  by a 1-bit, and using the function  $\lambda : \mathbb{N} \rightarrow \mathbb{Z}$  given by  $\lambda(n) = k$ , ( $n \in \mathbb{N}$ ).

- (a) set  $n = 1$  and set  $F(0) = 11$ , with  $\lambda(0) = k = 2$ .
- (b) Let  $\gamma(n) \in \Gamma$  as in (3), let  $(c(n), b(n)) \in T$  as in (5), let  $0^\ell c(n) = c'(n) = 0^\ell b(m)$  as in (4)-(6) ( $0 \leq m < n$ ) and let  $j$  be such that  $\lambda(q^j(F(m))) = \lambda(F(n))$ .
- (c) Set  $q^j(F(m)) = W|X|Y|Z$ , with  $W$  and  $Z$  of lengths  $\gamma(n) - 1$  and  $\gamma(n)$ , respectively, and with  $Y$  starting at entry  $x + 1$ , where  $x$  is the leftmost entry of  $X$ .
- (d) Set  $k = \lambda(n)$  and write  $F^k(n) = F(n)$ .
- (e) Determine  $F^k(n)$  via *castling* of  $X$  and  $Y$ , i.e., transposing:  $X|Y \rightarrow Y|X$ . This yields  $F^k(n) = W|Y|X|Z$ .
- (f) Let  $n := n + 1$  and repeat the sequence of items (b)-(f).

The procedure in items (a)-(f) yields the lexicographically-ordered tight Dyck nests  $F(n) = F^k(n) \in \mathcal{N}$  for all  $n \geq 0$ , and the corresponding Dyck words  $f(n) = f^k(n) \in \mathcal{W}$ .

**Example 8.** While working with a Dyck word  $f$  or a Dyck nest  $F$ , the order  $n \in \mathbb{N}$  of any working tool relating to  $f$  or  $F$  will be indicated by  $ord(\cdot)$ . The RGSs of length  $k - 1$  conform a tree  $T_k$  controlled by the tree  $T$  of TRGSs by blowing each such TRGS to an RGS of length  $k - 1$ . Then,  $T_k$  can be considered subtree of  $T$  corresponding to the prefix  $[0, C_k - 1]$  of  $\mathbb{N}$ , where  $C_k = \frac{(2k)!}{k!(k+1)!}$  is the  $k$ -th Catalan number [8, A000108], [13]. Figure 2 contains representations of the effects of the castling operation on the trees  $T_k$  ( $k = 1, 2, 3, 4$ , with  $C_k = 1, 2, 5, 14$ ), each such tree with its root  $0^{k-1}$  represented in a box containing the order  $ord(0^{k-1}) = 0 \in \mathbb{N}$ , the root  $0^{k-1}$  and  $F(0^{k-1})$ . Each other node  $f$  of  $T_k$  is represented by a box of two levels: the top level contains the order  $ord(\rho(f)) \in \mathbb{N}$ , the parent  $\rho(f)$  and  $F(\rho(f))$ ; the lower level contains the order  $ord(f) \in \mathbb{N}$ ,  $f$  and  $F(f)$ . In these two levels, the entry at which parent  $c$  and child  $b$  differ in (4), namely at position  $\gamma(n)$ , is in red in contrast with the remaining entries, in black. In all boxes,  $F(\rho(f)) = "W|X|Y|Z"$  and  $F(f) = "W|Y|X|Z"$  have  $X$  and  $Y$  colored blue and red, respectively, while  $W$  and  $Z$  are left black. In addition, the edge leading from  $\rho(f)$  to  $f$  is labeled with its subindex  $i$ .

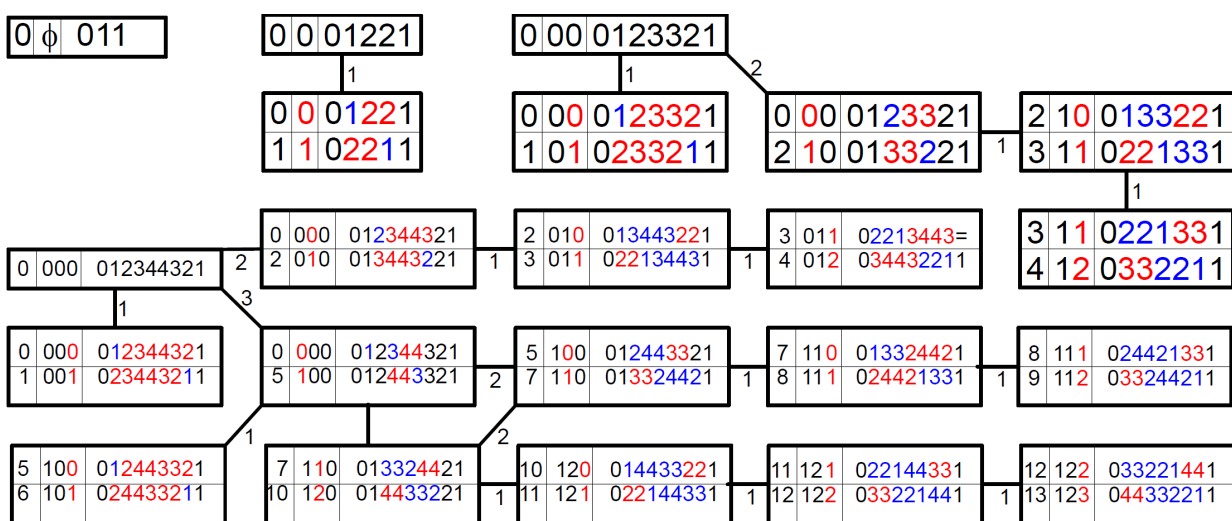


Figure 2. A section of  $T$  accompanying Definitions 6-7 and §5

The computational complexity of the castling operation is determined by the value  $k = \lambda(n)$  in item (d) in time  $O(1)$  and includes: the determination of  $\gamma(n)$  and  $j$  in item (b), both of time  $O(k)$ ;  $W, X, Y, Z$  in item (c), again  $O(k)$ ; the string transposition  $XY \rightarrow YX$  of item (e), again  $O(k)$ ; so in sum the time of the castling procedure is  $O(k)$ , for any particular  $n \in \mathbb{N}$ , where  $k = \lambda(n)$ . Considered over all of  $T_k$ , the complexity

amounts to  $O(kC_k)$ , where the time complexity of  $C_k$  can be obtained by Dynamic Programming via recursive  $C_0 = 1$  and  $C_{k+1} = \sum_{j=0}^k C_j C_{k-j}$ , yielding time  $O(k^2)$ , a total time  $O(k^3)$ . Space complexity is  $O(k)$ .

### 6. Dyck nests blown to a specific length

**Lemma 1.** Every Dyck nest  $F = F^k = F(n) = F^k(n)$  of length  $2k$  contains a substring  $kk = k_1k_2$ .

**Proof.** Clearly, the integer  $k$  in the statement appears as the substring  $kk = k_1k_2$  in  $F$ .  $\square$

**Lemma 2.** Given a Dyck nest  $F^k$  of length  $2k$ , let  $F^{k+1} = q(F^k)$  be the string of length  $2(k + 1)$  obtained from  $F^k$  by inserting the substring  $(k + 1)(k + 1) = (k + 1)_1(k + 1)_2$  between the two entries  $k_1$  and  $k_2$  that equal  $k$  in  $F^k$ , so that  $q(F^k)$  contains the substring  $k(k + 1)(k + 1)k = k_1(k + 1)_1(k + 1)_2k_2$ . Then,  $F^{k+1} = q(F^k)$  is a Dyck nest of length  $2k + 2$  and its corresponding  $f^{k+1} = q(f^k)$  is a Dyck word of length  $2k + 2$ .

**Proof.** Clearly, the substring  $k(k + 1)(k + 1)k = k_1(k + 1)_1(k + 1)_2k_2$  in the larger Dyck nest  $F^{k+1} = q(F^k)$ , based in the shorter Dyck nest  $F^k$  that contains just the substring  $kk = k_1k_2$  and is of length  $2k$ , makes it a Dyck nest of length  $2k + 2$ .  $\square$

**Definition 13.** We say that  $F^{k+1} = q(F^k)$  and  $f^{k+1} = q(f^k)$  are blown from  $F^k$  and  $f^k$ , respectively, with  $q(\cdot)$  as in Lemma 2. By repeating the  $q(\cdot)$ -operation, we blow  $F^k$  (resp.,  $f^k$ ) successively to Dyck nests  $q^2(F^k)$ ,  $q^3(F^k)$ , etc. (resp., Dyck words  $q^2(f^k)$ ,  $q^3(f^k)$ , etc.) We also say that  $F^k(n)$  and  $f^k(n)$  are  $k$ -blown, where necessarily  $0 \leq n < C_k = \frac{(2k)!}{k!(k+1)!}$ , with upper bound  $C_k$ .

**Definition 14.** In a likewise fashion to that of Definition 13, given an RGS  $b(n) \in \mathcal{S}$ , we define  $q(b(n)) = 0|b(n) = 0b(n)$ , assignment that can be iterated as:

$$q^2(b(n)) = 00b(n), \dots, q^i(b(n)) = 0^i b(n), \text{ etc.}$$

**Observation 1.**  $\lambda(n) = k, \forall n \in [C_{k-1}, C_k - 1]$ , where  $\lambda$  is defined in §5.

**Lemma 3.** The castling procedure of §5 permutes with the  $q(\cdot)$ -operation, so the compositions  $b(n) \rightarrow F(n) \rightarrow q^i(F(n))$  (resp.,  $b(n) \rightarrow f(n) \rightarrow q^i(f(n))$ ) and  $b(n) \rightarrow q^i(b(n)) \rightarrow q^i(F(n))$  (resp.,  $b(n) \rightarrow q^i(b(n)) \rightarrow q^i(f(n))$ ) conform a commutative diagram.

**Proof.** Direct verifications allow to establish the commutativity of the two diagrams:

$$\begin{array}{ccc} b(n) & \rightarrow & F(n) \\ \downarrow & & \downarrow \\ q^i(b(n)) & \rightarrow & q^i(F(n)) \end{array} \quad \begin{array}{ccc} b(n) & \rightarrow & f(n) \\ \downarrow & & \downarrow \\ q^i(b(n)) & \rightarrow & q^i(f(n)) \end{array}$$

$\square$

**Observation 2. (A)** Recalling Definition 7,  $\rho^{-1}(0) = \{C_k; 0 < k \in \mathbb{Z}\}$ , where  $C_k$  is the smallest  $n$  such that  $b(n)$  is of length  $k$ . **(B)** All vertices  $b(n) = a_{k-1} \dots a_1$  of  $T$  with  $n \notin \rho^{-1}(0)$  have either  $\gamma(n)$  or  $\gamma(n) - 1$  children, depending on whether  $a_{\gamma(n)} \leq a_{\gamma(n)+1}$  or not, where  $\gamma(n)$  is given in Definition 5.

**Example 9. (A)**  $b(C_3) = b(5) = 100$ . **(B)** The children of  $b(7) = 110$  are  $b(8) = 111$  and  $b(10) = 120$ , but  $b(8) = 111$  and  $b(10) = 120$  have only respective children  $b(9) = 112$  and  $b(11) = 121$ ; the children of  $b(30) = 1210$  are  $b(31) = 1211$  and  $b(33) = 1220$ , these having respective children sets  $\{b(32)\} = \{1212\}$  and  $\{b(34), b(37)\} = \{1221, 1230\}$ .

### 7. Anchoring Dyck words to odd graphs

**Lemma 4.** Prefixing a 0-bit to every  $f^k = f^k(n)$  yields exactly  $C_k$  corresponding representatives  $0f^k = 0f^k(n)$  of the cyclic classes mod  $2k + 1$  (or  $\mathbb{Z}_{2k+1}$ -classes) of  $V(O_k)$ .

**Definition 15.** The representatives in Lemma 4 are to be called *anchored k-blown Dyck words*. The tight versions of such representatives will be said to be *tight odd-graph representatives*, or *TOGR's*.

**Example 10.** The list of TOGR's corresponding to (8) starts as follows:

$$\{001,00101,0001011,0010011,0010101,000010111,000101101,000100111,000110011,001001101,000101011,001001011,001010011,001010101\}. \tag{9}$$

**Definition 16.** The tight Dyck nests in (7) associated to the Dyck words in (8) act as the *pull-backs* of those Dyck words. That is: each  $j_1$  in a Dyck nest  $F(n)$  of  $\mathcal{N}$  corresponds to a 0-bit in the corresponding Dyck word  $f(n)$  of  $\mathcal{W}$ , so  $j_2$  in  $F(n)$  corresponds to a corresponding 1-bit in  $f(n)$ . This makes  $F(n)$  to be the pull-back of  $f(n)$ .

**Example 11.** Let  $k = 3$ . Let us begin with the prefix of length  $\mathcal{C}_3 = 5$  of  $0\mathcal{W}$  in (9):

$$(001,00101,0001011,0010011,0010101),$$

and transform its TOGR's of length less than  $2k + 1 = 7$ , namely  $0f(0) = 0f^1(0) = 001$  and  $0f(1) = 0f^2(1) = 00101$ , to their pull-back anchored Dyck nests,  $0F(0) = 0F^1(0) = 011$  and  $0F(1) = 0F^2(1) = 02211$ , respectively. These two TOGR's are 3-blown to  $0F^3(0) = 0q^2(011) = 0123321$  and  $0F^3(1) = 0q(02211) = 0233211$ , that together with the pull-backs of the remaining anchored 3-blown Dyck words, ( $0f^3(2) = 0001011$ ,  $0f^3(3) = 0010011$  and  $0f^3(4) = 0010101$ ), namely the anchored 3-blown Dyck nests  $0F^3(2) = 0122331$ ,  $0F^3(3) = 0221331$  and  $0F^3(4) = 0332211$ , form the pull-backs of their corresponding anchored 3-blown Dyck words. Summarizing: 0000111, 0001101, 0001011, 0010011 and 0010101; these represent in  $V(O_3)$  the  $\mathcal{C}_3 (=5)$   $\mathbb{Z}_7$ -classes  $[0f]$ , displayed in the upper half of Table 1 and correspond to the  $\mathbb{Z}_7$ -classes  $[0F]$  of their pull-back anchored 3-blown Dyck nests  $0F$ , displayed in the lower half of the table.

**Table 1.**  $\mathbb{Z}_7$ -classes of  $V(O_3)$  and their pull-back anchored 3-blown Dyck nests

$[0000111]$	$= \{0000111, 0001110, 0011100, 0111000, 1110000, 1100001, 1000011\}$
$[0001101]$	$= \{0001101, 0011010, 0110100, 1101000, 1010001, 0100011, 1000110\}$
$[0010110]$	$= \{0001011, 0010110, 0101100, 1011000, 0110001, 1100010, 1000101\}$
$[0010011]$	$= \{0010011, 0100110, 1001100, 0011001, 0110010, 1100100, 1001001\}$
$[0010101]$	$= \{0010101, 0101010, 1010100, 0101001, 1010010, 0100101, 1001010\}$
$[0123321]$	$= \{0123321, 1233210, 2332101, 3321012, 3210123, 2101233, 1012332\}$
$[0233211]$	$= \{0233211, 2332110, 3321102, 3211023, 2110233, 1102332, 1023321\}$
$[0122331]$	$= \{0122331, 1223310, 2233101, 2331012, 3310122, 3101223, 1012233\}$
$[0221331]$	$= \{0221331, 2213310, 2133102, 1331022, 3310221, 3102213, 1022133\}$
$[0332211]$	$= \{0332211, 3322110, 3221103, 2211033, 2110332, 1103322, 1033221\}$

The characteristic vectors of the vertices of  $O_k$  in the upper half of Table 1 also represent the elements of  $L_k \subset V(M_k)$ . The complements of their reversed strings represent  $L_{k+1}$  so their union  $L_k \cup L_{k+1}$  forms  $V(M_K)$ , where adjacency is given by inclusion of  $L_k$  into  $L_{k+1}$ .

### 8. Arc factorizations of odd graphs

Let  ${}^0F = 0F_1 \cdots F_{2k}$  be an anchored Dyck nest representing a  $\mathbb{Z}_{2k+1}$ -class of  $V(O_k)$  ( $F_i \in [1, k]$ , for  $i \in [1, 2k]$ ). Then, such a class can be expressed (Example 11, Table 1) as

$$[{}^0F] = \{{}^0F = 0F_1 \cdots F_{2k}, \quad {}^1F = F_1 \cdots F_{2k}0, \quad {}^2F = F_2 \cdots F_{2k}0F_1, \dots, \quad {}^{2k}F = F_{2k}0F_1 \cdots F_{2k-1}\}. \tag{10}$$

Each entry of value  $j \in [0, k]$  in a  $(2k + 1)$ -tuple  $\chi = {}^\ell F$  as in (10) ( $\ell \in [0, 2k]$ ) is either the *first*,  $j_1$ , or *second*,  $j_2$ , *appearance* of  $j$  in  $\chi$  counting from the entry of value 0, from left to right and cyclically mod  $2k + 1$ , e.g.,  ${}^2F$  has value 0 in the penultimate entry (see (10)).

Let  $\ell \in [0, 2k]$ . Given the entry of value  $j = 0$  in  ${}^\ell F$  or the first appearance  $j_1$  of an integer  $j \in [1, k]$  in  ${}^\ell F$ , there exists just one anchored  $k$ -blown Dyck nest  ${}_{j+\ell}{}^\ell F \neq {}^\ell F$  (with  $j + \ell$  taken mod  $n$ ) adjacent as a vertex of  $O_k$  to  ${}^\ell F$  via one of the two arcs forming an edge  $e$  of  $O_k$ , namely the arc  $\vec{e} = ({}^\ell F, {}_{j+\ell}{}^\ell F)$ , and having:

1.  $(k - j)_1$  as first appearance of  $k - j$  in the same position as  $j_1$  in  ${}^0F$ , and
2. the other  $j = 0$  or first (resp., second) appearance positions of integers  $j \in [1, k]$  in  ${}^\ell F$  as second (resp.,  $j = 0$  or first) appearance positions of integers  $j \in [1, k]$  in  ${}_{j+\ell}{}^\ell F$ .

**Remark 1.** Assigning color  $j$  to the arc  $\overrightarrow{e}$  implies assigning color  $(k - j)$  to its opposite arc  $\overleftarrow{e} = ({}_{j+\ell}{}^\ell F, {}^\ell F)$ . This insures an arc factorization of  $O_k$  that we call the *edge-supplementary 1-arc-factorization* of  $O_k$ , since by considering the color set  $[0, k]$  as if it were a set of weights with values in  $\mathbb{N}$ , then the weight of each edge seen as the sum of the weights of its two arcs is constantly equal to  $k$  in  $O_k$ . This justifies our supplementary terminology.

We use the following notation for the vertices  $v \in V(O_k)$ :  $v = n_k^j$ , where  $n$  is the order of the TRGS  $b(n) \in S$  yielding the  $\mathbb{Z}_{2k+1}$ -class of  $v$ , and  $j$  is taken by expressing  $v$  as  ${}^jF$  in (10).

Thus, for each  $k \in \mathbb{N}$ , the arcs of  $O_k$  form an *arc factorization*  $\mathcal{F}$  composed by the *arc factors*  $\mathcal{F}_j$  formed by those arcs  $(n_k^j, \ell_k^j)$  with fixed  $j \in [0, k]$  and adequate  $n, \ell$  and  $j'$ .

**Example 12.** Table 2 shows the twenty adjacency pairs that represent the arcs  $(n_k^0, \ell_k^j)$ , for  $k = 3$ , in a sandwich fashion, with each of the two shown layers being " $v_k^j = {}^jF$ ";  ${}^jF$  shown with concatenation bars delimiting either  $j = 0$  or the first appearance  $j_1$  of  $j \in [1, 2k]$  on top (showing  $n_k^0$ ) and the first appearance  $(k - j)_1$  of  $k - j$  below (showing  $\ell_k^j$ ).

**Table 2.** Twenty adjacencies in  $O_3$

$0_3^0 = 0 123321$	$0_3^0 = 0 1 23321$	$0_3^0 = 01 2 3321$	$0_3^0 = 012 3 321$
$0_3^4 = 3 321012$	$2_3^4 = 3 2 21013$	$1_3^4 = 32 1 1023$	$0_3^3 = 321 0 123$
$1_3^0 = 0 233211$	$1_3^0 = 0 2 33211$	$1_3^0 = 02 3 3211$	$1_3^0 = 02332 1 1$
$3_3^3 = 3 310221$	$4_3^3 = 2 1 10332$	$2_3^2 = 21 0 1332$	$0_3^3 = 32101 2 3$
$2_3^0 = 0 133221$	$2_3^0 = 0 1 33221$	$2_3^0 = 01 3 3221$	$2_3^0 = 0133 2 21$
$1_3^0 = 3 321102$	$4_3^5 = 3 2 21103$	$3_3^2 = 21 0 3312$	$0_3^3 = 3210 1 23$
$3_3^0 = 0 221331$	$3_3^0 = 0 2 21331$	$3_3^0 = 022 1 331$	$3_3^0 = 0221 3 31$
$2_3^3 = 3 322101$	$3_3^3 = 2 1 33102$	$3_3^3 = 310 2 213$	$1_3^4 = 3211 0 23$
$4_3^0 = 0 332211$	$4_3^0 = 0 3 32211$	$4_3^0 = 033 2 211$	$4_3^0 = 03322 1 1$
$4_3^6 = 3 322110$	$4_3^1 = 1 0 33221$	$2_3^3 = 210 1 332$	$x_3^0 = 32110 2 3$

Lifting  $\mathcal{F}$  via pull-back from  $O_k$  to  $M_k$  via the 2-covering graph map  $\Psi_k : M_k \rightarrow O_k$  yields the so-called *modular arc factorization* of  $M_k$  [2,6,14], again mentioned in relation to item 2 in §11.

### 9. Permutations associated to Dyck words

Let us associate a  $2k$ -permutation  $\pi$  to each anchored  $k$ -blown Dyck nest  $F$ , or word  $f$ , (and to the  $\mathbb{Z}_{2k+1}$ -class of  $V(O_k)$  they represent), via the following procedure (compare [2,5]):

1. Set parentheses or commas between each two entries of  $f$ , so that the four substrings

"01" , "10" , "00" and "11" are transformed into the substrings  
 "0,1" , "1)(0" , "0(0" and "1)1" , respectively, resulting in a string  $f'$ .

Add a terminal parenthesis to  $f'$ , so that the last "1" in  $f'$  is transformed into "1)". Denote by  $g$  the string resulting from such addition of a closing parenthesis to  $f'$ .

2. By proceeding from left to right, replace the bits of  $g$  by the successive integers from 1 to  $|g|$ , keeping all pre-inserted parentheses and commas of  $g$  in their position. This yields a version  $h_0$  of  $g$ .

3. Set  $h_0$  as a concatenation  $(w_1)|(w_2)|\dots|(w_t)$  of expressions  $(w_i)$ ,  $(1 \leq i \leq t)$ , the terminal “)” of each  $(w_i)$  being the closing “)” nearest to its opening “(”. Let  $w'_i$  be the number string obtained from  $w_i$  by removing parentheses and commas. For  $i = 1, \dots, t$ , perform a recursive step  $\mathcal{R}$  consisting in transforming  $w'_i$  into its reverse substring  $w''_i$  and then resetting  $w''_i$  in place of  $w'_i$  in  $(w_i)$ , with the parentheses and commas of  $(w_i)$  kept in place. Denote the resulting expression by  $\mathcal{R}(w_i)$ . This yields a string  $h_1 = \mathcal{R}(w_1)|\mathcal{R}(w_2)|\dots|\mathcal{R}(w_t)$ .
4. For  $i \in [1, t]$ , let  $\mathcal{R}(w_i) = (a_{i,1}^1 \eta_{i,1}^1 b_{i,1}^1)|(a_{i,2}^1 \eta_{i,2}^1 b_{i,2}^1)|\dots|(a_{i,t_i}^1 \eta_{i,t_i}^1 b_{i,t_i}^1)$ , where  $a_{i,j}, b_{i,j} \in \mathbb{N}$  and  $\eta_{i,j}^1 = (w_{i,j})$  has terminal “)” being the closing “)” nearest to its opening “(”. Apply the treatment of the  $(w_i)$ 's in item 3 to each  $\eta_{i,j} = (w_{i,j}) \neq “(”$ , for  $j \in [1, t_i]$ . Replace the resulting strings  $\mathcal{R}(w_{i,j})$  in place of the corresponding  $(w_{i,j})$  in  $\mathcal{R}(w_i)$ , yielding a modified version  $\mathcal{R}^2(w_i)$  of  $\mathcal{R}(w_i)$ . Let  $h_2 = \mathcal{R}^2(w_1)|\mathcal{R}^2(w_2)|\dots|\mathcal{R}^2(w_t)$ .
5. Each  $\mathcal{R}(w_{i,j})$  is a concatenation of terms of the form  $a_I^2 \eta_I^2 b_I^2$  with  $I = \{i, j_1, j_2\}$ , where  $j_1 := j$ . In each such concatenation, the strings  $\eta_I^2 \neq “(”$  are of the form  $(w_I)$  and must be treated as  $(w_{i,j})$  is in item 4 (or  $(w_i)$  in item 3), producing a modified string  $\mathcal{R}(w_I)$  that forms part of the subsequent string  $h_3$ . Eventually ahead, to pass from  $h_{\ell-1}$  to  $h_\ell$  ( $\ell > 3$ ), each  $\mathcal{R}(w_I)$  in  $h_{\ell-1}$  with  $I = \{i, j_1, \dots, j_{\ell-2}\}$  would be a concatenation of terms of the form  $a_{I'}^{\ell-1} |\eta_{I'}^{\ell-1} | b_{I'}^{\ell-1}$  with  $I' = \{i, j_1, \dots, j_{\ell-1}\}$ . In each such concatenation, those  $\eta_{I'}^{\ell-1} \neq “(”$  would be of the form  $(w_{I'})$ , to be treated again as in items 3-4.
6. A sequence  $(h_0, \dots, h_{s+1})$  is eventually obtained for some  $s \geq 0$  when all innermost expressions  $(w_I) = (a, a \pm 1)$  with  $a, a \pm 1 \in [1, 2k]$  have been processed. Disregarding parentheses and commas in  $h_{s+1}$  yields a  $2k$ -string  $g'$  and an assignment  $i \rightarrow p(i)$ , ( $i \in [1, 2k]$ ), by making correspond the places  $i \in [1, 2k]$  of  $g'$  to the values in the places of  $g'$ . Define  $\pi = p^{-1}$ , the inverse  $2k$ -permutation of  $p = (p(1)p(2) \dots p(2k))$ .

**Example 13.** Table 3 illustrates the determination of the permutation  $\pi$  for the five cases with  $k = 3$  and just four of the 14 cases with  $k = 4$ , (exemplifying that not necessarily  $\pi = p$ ). Each such case is headed by an indication  $b(n) \rightarrow b'(n) = 0^{k-\ell(n)}|b(n)$ , where  $\ell(n)$  is the length of  $b(n)$  and  $k$  is the length of  $b'(n)$ . The second and third lines of Table 3 show  $F'(b'(n))$  and  $f'(b'(n))$  with parentheses and commas as in item 1 of the procedure. The rest of each case follows items 2-6 in order to produce  $p$  expressed without parentheses or commas, followed by the identity permutation  $\iota = 12 \dots (2k)$  to ease visualizing  $\pi$  as the inverse permutation of  $p$ , in the final line of each case of the table.

**Table 3.** Examples of determination of the permutation  $\pi$

$b(0) \rightarrow 00$	$b(1) \rightarrow 01$	$b(2) \rightarrow 10$	$b(3) \rightarrow 11$	$b(4) \rightarrow 12$
(1(2(3,3)2)1)	(2(3,3)2)(1,1)	(1(3,3)(2,2)1)	(3,3)(1(2,2)1)	(3,3)(2,2)(1,1)
(0(0(0,1)1)1)	(0(0,1)1)(0,1)	(0(0,1)(0,1)1)	(0,1)(0(0,1)1)	(0,1)(0,1)(0,1)
(1(2(3,4)5)6)	(1(2,3)4)(5,6)	(1(2,3)(4,5)6)	(1,2)(3(4,5)6)	(1,2)(3,4)(5,6)
(6(5(4,3)2)1)	(4(3,2)1)(6,5)	(6(5,4)(3,2)1)	(2,1)(6(5,4)3)	(2,1)(4,3)(6,5)
(6(2(3,4)5)1)	(4(2,3)1)(6,5)	(6(4,5)(2,3)1)	(2,1)(6(4,5)3)	
(6(2(4,3)5)1)				
$p = 624351$	$p = 423165$	$p = 645231$	$p = 216453$	$p = 214365$
$\iota = 123456$	$\iota = 123456$	$\iota = 123456$	$\iota = 123456$	$\iota = 123456$
$\pi = 624351$	$\pi = 423165$	$\pi = 645231$	$\pi = 216453$	$\pi = 214365$

$b(1) \rightarrow 001$	$b(2) \rightarrow 010$	$b(7) \rightarrow 110$	$b(10) \rightarrow 120$
(2(3(4,4)3)2)(1,1)	(1(3(4,4)3)(2,2)1)	(1(3,3)(2(4,4)2)1)	(1(4,4)(3,3)(2,2)1)
(0(0(0,1)1)1)(0,1)	(0(0(0,1)1)(0,1)1)	(0(0,1)(0(01)1)1)	(0(0,1)(0,1)(0,1)1)
(1(2(3,4)5)6)(7,8)	(1(2(3,4)5)(6,7)8)	(1(2,3)(4(5,6)7)8)	(1(2,3)(4,5)(6,7)8)
(6(5(4,3)2)1)(8,7)	(8(7(6,5)4)(3,2)1)	(8(7,6)(5(4,3)2)1)	(8(7,6)(5,4)(3,2)1)
(6(2(3,4)5)1)(8,7)	(8(4(5,6)7)(2,3)1)	(8(6,7)(2(3,4)5)1)	(8(6,7)(4,5)(2,3)1)
(6(2(4,3)5)1)(8,7)	(8(4(6,5)7)(2,3)1)	(8(6,7)(2(4,3)5)1)	
$p = 62435187$	$p = 84657231$	$p = 86724351$	$p = 86745231$
$\iota = 12345678$	$\iota = 12345678$	$\iota = 12345678$	$\iota = 12345678$
$\pi = 62435187$	$\pi = 86724351$	$\pi = 84657231$	$\pi = 86745231$

### 10. Uniform 2-factors of the odd graphs

Departing from each anchored  $k$ -blown Dyck word taken as a vertex  $v$  of  $O_k$ , an oriented path  $\overrightarrow{P_v^{2k}}$  of length  $2k$  is grown in  $2k$  stages  $v = v_0, \overrightarrow{P_v^1} = v_0v_1, \overrightarrow{P_v^2} = v_0v_1v_2, \dots, \overrightarrow{P_v^{2k}} = v_0v_1 \dots v_{2k}$  by traversing successively from  $v_i$  ( $i \in [0, 2k - 1]$ ) the edge arc whose assigned color, according to Remark 1, is the corresponding entry of the reversed permutation  $rev(\pi)$  of  $\pi$ . Now, the terminal vertex  $v_{2k}$  of  $\overrightarrow{P_v^{2k}}$  is at distance 1 from  $v$  by means of an arc  $\overrightarrow{e_v}$  in the arc factor  $\mathcal{F}_0$ . An oriented  $(2k + 1)$ -cycle  $\overrightarrow{C_v^k}$  in  $O_k$  is created by adding  $\overrightarrow{e_v}$  to  $\overrightarrow{P_v^{2k}}$ . Lifting such  $\overrightarrow{C_v^k}$  to  $M_k$  via  $\Psi_k^{-1}$  yields an oriented  $2(2k + 1)$ -cycle  $\overrightarrow{M C_v^k}$  in  $M_k$  containing all pairs of opposite vertices  $(w, \aleph(w))$  (i.e, at distance  $k$  from each other along  $\overrightarrow{M C_v^k}$  via two internally disjoint paths, one oriented, the other one anti-oriented). This provides  $O_k$  (resp.,  $M_k$ ) with a 2-factor of  $\mathcal{C}_k$  components, all as oriented cycles of uniform length  $2k + 1$  (resp.,  $2(2k + 1)$ ) [2,5,6].

**Example 14.** Table 4 is headed by  $k = 1, 2, 3$  and  $n \in [0, \mathcal{C}_k - 1]$  followed by the corresponding anchored Dyck words  $0f^k(n)$ ; for  $k = 3$  it contains information clarified in §14. Below its second horizontal line, the table presents each  $\overrightarrow{C_v^k} + = \overrightarrow{P_v^k} + \overrightarrow{e_v^k}$  in vertical fashion, with each pair of contiguous downward lines, say  $\chi, \chi'$ , representing an arc whose  $j$ -th entry is underlined, where  $j$ , shown as a subindex to the right, is the position containing the pair of  $k$ -supplementary entries in  $\chi, \chi'$ . The vertical column of such subindices  $j$  conform the reversed permutation  $rev(\pi_v)$  of  $\pi_v$  associated to the anchored  $k$ -blown Dyck word  $v$ , for each such word  $v$  in  $V(O_k)$ . For each value of  $k$ , the columns of  $(2k + 1)$ -tuples  $\chi$  will be denoted  $L(n)$  ( $0 \leq n < \mathcal{C}_k$ ). For  $k = 3$ , the subindex  $j$  of each  $(2k + 1)$ -tuple  $\chi$  in a vertical list  $L(n)$  is further extended, first with the value  $m$  of the TRGS  $b(m)$  such that  $0F^k(m)$  is the Dyck nest representing the corresponding  $\mathbb{Z}_{2k+1}$ -class  $[0F^k(m)]$  of  $\chi$  in  $V(O_k)$ , and second with the index  $j'$  such that  $\chi = {}^{j'}F^k(m)$ , in the notation of (10). Additional underlined entries and superindices with middle up-or-down vertical arrows are explained in §14.

**Table 4.** Uniform 2-factors provided by permutations  $\pi$  for  $k = 1, 2, 3$ .

$k=1$ $n=0$	$k=2$ $n=0$	$k=2$ $n=1$	$k=3$ $n=0$	$k=3$ $n=1$	$k=3$ $n=2$	$k=3$ $n=3$	$k=3$ $n=4$
001	00011	00101	<u>0000111</u> $ \zeta_1^3$ <u>0000111</u> $ \zeta_1^1$	<u>0001101</u> $ \zeta_3^2$	<u>0001011</u> $ \zeta_1^3$	<u>0010011</u> $ \zeta_3^2$	<u>0010101</u> $ \zeta_4^3$
<u>011</u> $ \zeta_1$	<u>01221</u> $ \zeta_1$	<u>02211</u> $ \zeta_3$	<u>0123321</u> $ \zeta_{100}$	<u>0233211</u> $ \zeta_{510}$	<u>0133221</u> $ \zeta_{120}$	<u>0221331</u> $ \zeta_{330}$	<u>0332211</u> $ \zeta_{540}$
<u>101</u> $ \zeta_2$	<u>21101</u> $ \zeta_3$	<u>21012</u> $ \zeta_4$	<u>3221013</u> $ \zeta_{524}$	<u>3210123</u> $ \zeta_{603}$	<u>3221103</u> $ \zeta_{345}$	<u>3102213</u> $ \zeta_{532}$	<u>3211023</u> $ \zeta_{614}$
<u>110</u> $ \zeta_0$	<u>11022</u> $ \zeta_2$	<u>12210</u> $ \zeta_1$	<u>1103322</u> $ \zeta_{342}^{0,2}$	<u>1233210</u> $ \zeta_{106}$	<u>1102332</u> $ \zeta_{212}$	<u>1331022</u> $ \zeta_{434}^{1,0}$	<u>1332210</u> $ \zeta_{326}$
	<u>10221</u> $ \zeta_4$	<u>10122</u> $ \zeta_2$	<u>2110332</u> $ \zeta_{443}$	<u>2110233</u> $ \zeta_{313}$	<u>2133102</u> $ \zeta_{535}$	<u>2101332</u> $ \zeta_{622}$	<u>2101233</u> $ \zeta_{402}$
	<u>21102</u> $ \zeta_0$	<u>22110</u> $ \zeta_0$	<u>2211033</u> $ \zeta_{244}$	<u>2213310</u> $ \zeta_{236}$	<u>2210133</u> $ \zeta_{423}$	<u>2332101</u> $ \zeta_{105}$	<u>2332110</u> $ \zeta_{116}$
			<u>1023321</u> $ \zeta_{601}$	<u>1022133</u> $ \zeta_{431}^{0,1}$	<u>1033221</u> $ \zeta_{641}$	<u>1012332</u> $ \zeta_{201}$	<u>1013322</u> $ \zeta_{221}^{1,3}$
			<u>3321012</u> $ \zeta_{004}^{1,4}$	<u>3310221</u> $ \zeta_{033}$	<u>3321102</u> $ \zeta_{015}^{0,1}$	<u>3322101</u> $ \zeta_{025}$	<u>3322110</u> $ \zeta_{006}$

### 11. Partitions of odd-graph vertex sets

So far, we have two different partitions of  $V(O_k)$  into  $(2k + 1)$ -subsets. In terms of anchored  $k$ -blown Dyck nests, these partitions are:

1. the  $\mathbb{Z}_{2k+1}$ -classes  $[0F^k(n)]$  of  $V(O_k)$ , for  $0 \leq n < \mathcal{C}_k$  in Sections 7-9;
2. the vertex sets  $V(\overrightarrow{C_v^k})$ , where  $v = 0F^k(n)$ , for  $0 \leq n < \mathcal{C}_k$  in §10, from [6].

Item 1 refers to the algebraic partition of  $V(O_k)$  that leads to the determination of Hamilton cycles in  $M_k$  via the lexical 1-factorization of  $M_k$  [9-12]. Here, the  $\mathbb{Z}_{2k+1}$ -classes of  $O_k$  correspond via the inverse image  $\Psi_k^{-1}$  to the dihedral classes, or  $\mathbb{D}_{2k+1}$ -classes, of  $M_k$ , where  $\mathbb{D}_{2k+1}$  is the dihedral group with  $2(2k + 1)$  elements in which the cyclic group  $\mathbb{Z}_{2k+1}$  appears as a subgroup of index 2.

In fact, each anchored  $k$ -blown Dyck word  $0f^k(n)$  is a binary  $(2k + 1)$ -string of weight  $k$  whose support is a vertex of  $O_k$  as well as an element of  $L_k$ , while  $\aleph(0f(n))$  is an element of  $L_{k+1}$ . Note that both the pair  $\{0f(n), \aleph(0f(n))\}$  and the  $\mathbb{Z}_{2k+1}$ -class of  $0f(n)$  in  $L_k$  ( $= V(O_k)$ ) generate together the  $\mathbb{D}_{2k+1}$ -class of  $0f(n)$  of  $V(M_k)$ . Thus,  $0f(n)$  represents both a  $\mathbb{Z}_{2k+1}$ -class of  $V(O_k)$  and a  $\mathbb{D}_{2k+1}$ -class of  $V(M_k)$ .

Item 2 refers to the *graph-theoretical* partition of  $V(O_k)$  that leads to the determination of Hamilton cycles both in  $O_k$  and  $M_k$  ( $k > 3$ ) [2] via the *modular arc factorization* of  $M_k$  mentioned at the end of §8. Note that the Petersen graph  $M_3$  is hypo-hamiltonian, a constraint for §14 and its Theorem 8 that reformulate those determinations.

Let  $(P^0, P^1, P^2, \dots)$  be a partition of  $V(T)$  into *threads*  $P^i$ , each thread inducing a path  $T[P^i]$  with an initial vertex  $b(n_0^i)$  such that  $\gamma(n_0^i) > 1$  and its remaining vertices  $b(n_j^i) = b(n_0^i + j)$  such that  $\gamma(n_j^i) = \gamma(n_0^i + j) = 1$ , for  $0 < j \leq s^i$ , where the length  $s^i$  of  $P^i$  is maximal. Here, the indices  $n_0^i$  form an integer sequence  $(n_0^0, n_0^1, n_0^2, \dots)$ , as shown vertically on the second, sixth and tenth columns on the triptych left of Table 5.

**Table 5.** Table of initial values of  $n_0$ ,  $s$  and  $\gamma(n_0)$ . Final data of Table 6

$i$	$n_0^i$	$s^i$	$\gamma^i$	$i$	$n_0^i$	$s^i$	$\gamma^i$	$i$	$n_0^i$	$s^i$	$\gamma^i$
0	0	1	*	14	42	1	5	28	84	1	4
1	2	2	2	15	44	2	2	29	86	2	2
2	5	1	3	16	47	1	3	30	89	1	3
3	7	2	2	17	49	2	2	31	91	2	2
4	10	3	2	18	52	3	3	32	94	3	2
5	14	1	4	19	56	1	4	33	98	1	3
6	16	2	2	20	58	2	2	34	100	2	2
7	19	1	3	21	61	1	3	35	103	3	2
8	21	2	2	22	63	2	2	36	107	4	2
9	24	3	2	23	66	3	2	37	112	1	3
10	28	1	3	24	70	1	3	38	114	2	2
11	30	2	2	25	72	2	2	39	117	3	2
12	33	3	2	26	75	3	2	40	121	4	2
13	37	4	2	27	79	4	2	41	126	5	2

3	2	2	2	2	2
0	-3	-4	-5	-6	0
0	1	2	3	4	-2
	0	1	2	3	-3
		0	1	2	-4
			0	1	-5
				0	-6
					0

The induced paths  $T[P^i]$  have respective lengths  $s^i$  and values  $\gamma^i = \gamma(n_0^i)$  shown subsequently in the triptych.

Table 7 shows the first 42 TRGS  $b(n) = b(n_j^i)$ , with the columns (divided again as in a triptych) headed  $n_j^i, b(n_j^i), \rho(n_j^i), \gamma(n_j^i)$  and  $h(n_j^i)$ , where  $h(\cdot)$  is introduced in Observation 3.

We reunite the threads  $P^i$  into *braids*, which are subsets  $Q^\ell$  ( $0 \leq \ell \in \mathbb{N}$ ) of  $V(T)$ , each inducing a maximal connected ordered subtree with initial vertex  $b(m_0^\ell)$  such that  $\gamma(m_0^\ell) > 2$  and its remaining vertices  $b(m_j^\ell)$  such that  $\gamma(m_j^\ell) = \gamma(m_0^\ell + j) \in \{1, 2\}$ , for  $0 < j \leq \sum_{P^i \subseteq Q^\ell} s^i$ . This yields a partition  $\{Q^\ell\}$  of  $V(T)$  coarser than  $\{P^i\}$ .

### 12. Clones of Dyck nests

Each anchored  $k$ -blown Dyck nest  $0F(n)$  (arising from its anchored  $k$ -blown Dyck word  $0f(n)$ ) is encoded by its *clone* or *signature*, defined as the vector  $\sigma(n) = (\sigma_{k-1}(n), \dots, \sigma_2(n), \sigma_1(n))$  of halfway-distance floors  $\sigma_j(n)$  between the first,  $j_1$ , and second,  $j_2$ , appearances of each integer  $j$  assigned to the respective up- and down-steps of the path  $g_f$ , where  $k > j > 0$ .

For example, if  $j_1 k_1 k_2 j_2$ , (or  $j_1(k-1)_1 k_1 k_2(k-1)_2 j_2$ ), is a substring of  $0F(n)$ , (resp.,  $0F(n')$ ), then the halfway-distance floor of  $j$  is  $\lfloor d(j_1, j_2) \rfloor = \lfloor 3/2 \rfloor = 1$ , (resp.  $\lfloor d(j_1, j_2) \rfloor = \lfloor 5/2 \rfloor = 2$ ), engaged as the  $j$ -th entry of  $\sigma(n)$ , (resp.,  $\sigma(n')$ ), where  $0 \leq n < C_k$ , (resp.,  $0 \leq n' < C_k$ ). We will write  $\sigma(n) = \sigma_{k-1}(n) \cdots \sigma_2(n) \sigma_1(n)$ .

The growth of the tree  $T$  of Dyck nests  $F(n)$  will be simplified further from that given in the procedure of §5 by recursively updating just one entry of the parent clone  $\sigma(\rho(n))$  to obtain the clone  $\sigma(n)$ . This uses an equivalence of the set of anchored  $k$ -blown Dyck nests  $F(n)$  and that of their clones  $\sigma(n)$ , provided in Theorem 2, below.

**Observation 3.** In the transformation from  $F(\rho(n))$  to  $F(n)$  in §5,  $k_1 k_2$  is present either in  $X$  or in  $Y$ ; if  $k_1 k_2$  is in  $X$ , then  $\sigma_{\gamma(n)}(n)$  depends on  $k$  because of blowing, and so  $\sigma_{\gamma(n)}(n) = k + h(n)$ , for some value  $h(n) < 0$ ; if on the contrary  $k_1 k_2$  is in  $Y$ , then  $\sigma_{\gamma(n)}(n)$  does not depend on  $k$ , and so  $\sigma_{\gamma(n)}(n) = h(n)$ , for some  $h(n) \geq 0$ . In both cases, the remaining entries of  $\sigma(n)$  other than  $\sigma_{\gamma(n)}(n)$  are kept the same in  $F(n)$  as in  $F(\rho(n))$ .



**Theorem 1.** For each  $n \neq 0$ ,  $\sigma(\rho(n))$  and  $\sigma(n)$  differ solely at the  $\gamma(n)$ -th entry.

**Proof.** There is a sole difference between the parent  $\sigma(\rho(n))$  of  $\sigma(n)$  and  $\sigma(n)$  itself, occurring at the  $\gamma(n)$ -th (right-to-left) position; its entry is just increased in one unit from  $\sigma(\rho(n))$  to  $\sigma(n)$ , namely  $\sigma_{\gamma(n)}(n) = \sigma_{\gamma(n)}(\rho(n)) + 1$ . The effect of this via the permutation (transposition, or castling) of the inner strings  $X$  and  $Y$  from  $F(\rho(n)) = Z|X|Y|W$  to  $F(n) = Z|Y|X|W$  (§5) modifies just one of the halfway-distance floors  $\sigma_j(n) = \lfloor d(j_1, j_2)/2 \rfloor$  between the first appearance,  $j_1$ , of the corresponding  $j \in [0, k - 1]$  in  $F(n)$  and its second appearance,  $j_2$ , namely  $\sigma_{\gamma(n)}(n) = \lfloor d(\gamma(n)_1, \gamma(n)_2)/2 \rfloor$ .  $\square$

Table 8 exemplifies how the procedure in §5 is solely determined by the integer entries  $\sigma_{\gamma(n)}(n)$  of the corresponding permutation  $\sigma(n)$  in Theorem 1, supplying in the

1. first column: the value of  $k$  that depends on the value of  $n$  in the third column;
2. second and third columns: the values of the parent index  $\rho(n)$  and  $n$  itself;
3. fourth and fifth columns: the TRGS  $b(n)$  and the value of  $\gamma(n)$ , respectively;
4. sixth column:  $0F(\rho(n)) = 0|Z|\underline{X}|Y|W$ ; with  $X$  underlined and  $Y$  in Italics;
5. seventh column:  $0F(n) = 0|Z|Y|X|W$ , with  $X, Y$  as in the sixth column;
6. last three columns: the values of  $\sigma(\rho(n))$ ,  $\sigma(n)$  and  $\sigma_{\gamma(n)}(n)$ , this last value given in terms of  $k$  via Observation 3.

**Table 8.** Development of clone updates  $\sigma_{\gamma(n)}(n)$  expressed in terms of  $k$

$k$	$\rho(n)$	$n$	$b(\rho(n))$	$b(n)$	$\gamma(n)$	$0F(\rho(n))$	$0F(n)$	$\sigma(\rho(n))$	$\sigma(n)$	$\sigma_{\gamma(n)}(n)$
1	*	0	*	0	*	0	011	*	1	
2	0	1	0	1	1	01221	02211	1	0	0
3	0	2	00	10	2	0123321	0133221	12	02	0
3	2	3	10	11	1	0133221	0221331	02	01	$k - 2$
3	3	4	11	12	1	0221331	0332211	01	00	0
4	0	5	000	100	3	012344321	012443321	123	023	0
4	5	6	100	101	1	012443321	024433211	023	020	0
4	5	7	100	110	2	012443321	013324421	023	013	$k - 3$
4	7	8	110	111	1	013324421	024421331	013	011	1
4	8	9	111	112	1	024421331	033244211	011	010	0
4	7	10	110	120	2	013324421	014433221	013	003	0
4	10	11	120	121	1	014433221	022144331	003	002	$k - 2$
4	11	12	121	122	1	022144331	033221441	002	001	$k - 3$
4	12	13	122	123	1	033221441	044332211	001	000	0

**Theorem 2.** The correspondence that assigns each  $n$ -nest to its clone is a bijection.

**Proof.** Let  $b(n) = a_{k-1} \dots a_2 a_1$  be a TRGS. The anchored  $k$ -blown (or tight) Dyck nest  ${}^0F(n) = 0F(n) = 0F_1 \dots F_{2k}$  has rightmost entry  $F_{2k} = 1_2$ , so  $\sigma_1(n)$  determines the position of  $1_1$ . For example, if  $\sigma_1(n) = 0$ , then  $F_{2k-1} = 1_1$ , so  $a_1$  is a local maximum. To obtain  $0F(n)$  from  $\sigma(n)$ , we initialize a candidate for  $F(n)$  as the  $2k$ -string  $F(n, 0) = 00 \dots 0$ . Setting successively  $1_2, 1_1, 2_2, 2_1, \dots, (k-1)_2, (k-1)_1$  instead of the zeros of  $F(n, 0)$  from right to left according to the indications  $\sigma_i(n)$ , for  $i = 1, 2, \dots, k-1$ , is done in stages  $F(n, i-1) \rightarrow F(n, i)$  by setting each pair  $(i_1, i_2)$  as an outermost pair, only constrained by the presence of already replaced positions; after setting a value  $i_1$  in the initial position, we restart if necessary on the right again with the replacement of the remaining zeros by the remaining pairs  $(i_1, i_2)$  in ascending order from right to left. This allows to recover  $F(n)$  from the  $\sigma(n)$ 's by finally replacing the only resulting substring  $00$  in  $F(n, k-1)$  by  $k_1 k_2$ .  $\square$

We introduce a family  $\{\Phi_j | j > 0\}$  of subsequences of  $\mathbb{N}$  defined by the following properties:

1.  $\gamma(\Phi_1)$  is the subsequence of  $\gamma(\mathbb{N})$  formed by all indices  $\gamma(n)$  larger than 1.
2. The first term of  $\Phi_j$  is 1 if  $j=1$ ; and  $n$  such that  $b(n)$  is the smallest TRGS having suffix  $(j-1)(j-1)$  for  $j > 1$ .



The lower box of Table 6 shows an upper level composed by threads  $P^0, \dots, P^{13}$ , a middle level composed by threads  $P^{14}, \dots, P^{25}$  and a lower level composed by threads  $P^{26}, \dots, P^{41}$ , disposed vertically as to facilitate verifying our assertions.

Similarly with Table 9, where the threads  $P^0, \dots, P^{126}$  are disposed in four levels with their underlined headings  $\gamma(\cdot)$  as above and the vertically disposed values of  $h(\cdot)$  replaced by rings "o" if  $h(\cdot) \notin \Phi_1$  and bullets "•" if  $h(\cdot) \in \Phi_1$ . The final five threads here are shown in the box on the right hand side of Table 5 with the data disposed as in Table 6.

**Theorem 5.** Let  $1 < j \leq k \in \mathbb{N}$  and let  $n \in \mathbb{N}$  be such that  $b(n) = 1 \cdots (j-1)(j-1)a_{k-j-1} \cdots a_1$ . Then,  $\exists r > n$  such that  $b(r) = 1 \cdots (j-1)ja_{k-j-1} \cdots a_1$  is a vertex of  $T$  and

1. if  $h(n) \in \Phi_j$ , then  $h(r) \in \Phi_j$  and  $h(r) = k - h(n)$ ;
2. if  $h(n) \notin \Phi_j$ , then  $h(r) \notin \Phi_j$  and  $h(r) = h(n)$ .

**Proof.** The argument in the proof of Theorem 4 applies here, too.  $\square$

**Corollary 1.** Let  $n \in \mathbb{N}$  be such that  $b(n) = 11a_{k-3} \cdots a_1$ . Then,  $\exists m > n$  such that  $b(m) = 12a_{k-3} \cdots a_1$  and

1. if  $h(n) \in \Phi_2$ , then  $h(m) \in \Phi_2$  and  $h(m) = k - h(n)$ ;
2. if  $h(n) \notin \Phi_2$ , then  $h(m) \notin \Phi_2$  and  $h(m) = h(n)$ .

**Example 16.** Applying Corollary 1 to  $n = 3, 7, 8, 9$ , so that  $b(n) = 11, 110, 111, 112$ , and  $h(n) = -2, -3, 1, 0 \in \Phi_2$  yields  $r = 4, 10, 11, 12$ , with  $b(r) = 12, 120, 121, 122$  and  $h(r) = 0, 0, -2, -3$ .

**Corollary 2.** Let  $n \in \mathbb{N}$  be such that  $b(n) = 122a_{k-4} \cdots a_1$ . Then,  $\exists r > n$  such that  $b(r) = 123a_{k-4} \cdots a_1$  and

1. if  $h(n) \in \Phi_3$ , then  $h(r) \in \Phi_3$  and  $h(r) = k - h(n)$ ;
2. if  $h(n) \notin \Phi_3$ , then  $h(r) \notin \Phi_3$  and  $h(r) = h(n)$ .

**Example 17.** Applying Corollary 2 to  $n = 12, 33, 34, 35, 36$ , so  $b(n) = 122, 1220, 1221, 1222, 1223$  and  $h(\alpha_3) = -3, -4, 2, 1, 0 \in \Phi_3$  yields  $r = 13, 37, 38, 39, 40$  with  $b(r) = 123, 1230, 1231, 1232, 1233$  and  $h(\alpha'_3) = 0, 0, -2, -3, -4$ .

### 13. Controlling odd and middle-levels graphs via T

**Table 10.** Introduction of strings  $\zeta_\gamma^b$ , for all pairs  $(\gamma, b) \in \mathbb{N}^2$  such that  $1 < \gamma \leq b$ .

$\zeta_2^2 = 2_1 1_1 1_2;$
$\zeta_2^3 = 2_2 1_1 1_2 1_3;$
$\zeta_2^4 = 2_3 1_1 1_2 1_3 1_4;$
$\zeta_2^5 = 2_4 1_1 1_2 1_3 1_4 1_5;$
$\dots;$
$\zeta_3^3 = 3_1 1_1 \zeta_2^2 \zeta_2^3 = 3_1 1_1 2_1 1_1 2_2 1_1 2_2 1_3;$
$\zeta_3^4 = 3_2 1_1 \zeta_2^2 \zeta_2^3 \zeta_2^4 = 3_2 1_1 2_1 1_1 2_2 2_1 1_2 1_3 2_3 1_1 2_2 1_3 1_4;$
$\zeta_3^5 = 3_3 1_1 \zeta_2^2 \zeta_2^3 \zeta_2^4 \zeta_2^5 = 3_3 1_1 2_1 1_1 2_2 2_1 1_2 1_3 2_3 1_1 2_2 1_3 1_4 2_4 1_1 2_2 1_3 1_4 1_5;$
$\dots;$
$\zeta_4^4 = 4_1 1_1 \zeta_2^2 \zeta_3^3 \zeta_3^4 = 4_1 1_1 2_1 1_1 2_2 3_1 1_2 1_1 2_2 1_1 2_2 1_3 3_2 1_1 2_1 1_2 2_2 1_1 2_2 1_3 2_3 1_1 2_2 1_3 1_4;$
$\zeta_4^5 = 4_2 1_1 \zeta_2^2 \zeta_3^3 \zeta_3^4 \zeta_3^5;$
$\dots$
$\zeta_5^5 = 5_1 1_1 \zeta_2^2 \zeta_3^3 \zeta_4^4 \zeta_4^5;$
$\zeta_5^6 = 5_2 1_1 \zeta_2^2 \zeta_3^3 \zeta_4^4 \zeta_4^5 \zeta_4^6;$
$\dots$
$\zeta_{a+b}^{a+b} = a_{a+b} 1_1 \zeta_2^2 \dots \zeta_{a-1}^{a-1} \zeta_{a-1}^a \dots \zeta_{a-1}^{a+b}, \forall 0 < a \in \mathbb{N}, \forall 0 < b \in \mathbb{N}.$

We introduce strings  $\zeta_\gamma^b$ , for all pairs  $(\gamma, b) \in \mathbb{N}^2$  with  $1 < \gamma \leq b$ . The entries of each  $\zeta_\gamma^b$  are integer pairs  $(\alpha, \beta)$ , denoted  $\alpha_\beta$ , starting with  $\alpha_\beta = 1_1$ , initial case of the more general notation  $1_\beta$ , for  $\beta \geq 1$ . The strings

$\zeta_\gamma^b$  are determined following Table 10. The components  $\alpha$  in the entries  $\alpha_\beta$  represent the indices  $\gamma = \gamma(b(n))$  (see §3) in their order of appearance in  $\mathcal{S}$ , and  $\beta$  is an indicator to distinguish different entries  $\alpha_\beta$  with  $\alpha$  locally constant.

Next, consider the infinite string  $J$  of integer pairs  $\alpha_\beta$  formed as the concatenation

$$J = \zeta_1^1 | \zeta_2^2 | \cdots | \zeta_\gamma^\gamma | \cdots = *|1_1 | \zeta_2^2 | \cdots | \zeta_\gamma^\gamma | \cdots, \tag{11}$$

with  $\zeta_1^1 = *|1_1$  standing for the first two lines of Tables 7 and 8, where  $*$ , representing the root of  $T$ , stands for the first such line, and  $\zeta_1^1$  for the second line.

A partition of a string  $A$  is a sequence of substrings  $\sigma_1, \sigma_2, \dots, \sigma_n$  whose concatenation  $\sigma_1 | \sigma_2 | \cdots | \sigma_n$  is equal to  $A$ .

We recur to Catalan’s reversed triangle  $\Delta'$ , whose lines are obtained from Catalan’s triangle  $\Delta$  (see [11]) by reversing its lines, so that they may be written as in Table 11 that shows the first eight lines  $\Delta'_k$  of  $\Delta'$ , for  $k \in [0, 7]$ . Each  $\zeta_\gamma^b$  in the statement of Theorem 6 is presented in the Tables 7-8 in  $\gamma(n)$ -columnwise disposition.

**Table 11.** An initial detailed portion of Catalan’s reversed triangle  $\Delta'$ .

				$\tau_4^4 = 14$	$\tau_3^3 = 5$	$\tau_2^2 = 2$	$\tau_1^1 = 1$	$\tau_0^0 = 1$
			$\tau_5^5 = 42$	$\tau_4^4 = 14$	$\tau_3^3 = 5$	$\tau_2^2 = 2$	$\tau_1^1 = 1$	$\tau_0^0 = 1$
		$\tau_6^6 = 132$	$\tau_5^5 = 42$	$\tau_4^4 = 14$	$\tau_3^3 = 5$	$\tau_2^2 = 2$	$\tau_1^1 = 1$	$\tau_0^0 = 1$
		$\tau_6^6 = 429$	$\tau_5^5 = 132$	$\tau_4^4 = 42$	$\tau_3^3 = 14$	$\tau_2^2 = 5$	$\tau_1^1 = 3$	$\tau_0^0 = 1$
			$\tau_5^5 = 297$	$\tau_4^4 = 90$	$\tau_3^3 = 28$	$\tau_2^2 = 14$	$\tau_1^1 = 4$	$\tau_0^0 = 1$
				$\tau_5^5 = 90$	$\tau_4^4 = 48$	$\tau_3^3 = 20$	$\tau_2^2 = 5$	$\tau_1^1 = 1$
				$\tau_5^5 = 165$	$\tau_4^4 = 75$	$\tau_3^3 = 27$	$\tau_2^2 = 6$	$\tau_1^1 = 1$
.....	.....	.....	.....	.....	.....	.....	.....	.....

**Theorem 6.** Let  $k > 0$ . The vertex subset of  $T$  representing the  $\mathbb{Z}_{2k+1}$ -classes of  $V(O_k)$  and the  $\mathbb{D}_{2k+1}$ -classes of  $V(M_k)$  is given in  $J = *|\gamma(\mathcal{S} \setminus \{b(0)\}) = *|\Gamma = *|\gamma(\mathbb{N})$  by the prefix

$$\zeta_k^k = \zeta_1^1 | \zeta_2^2 | \cdots | \zeta_{k-1}^{k-1} | \zeta_k^k, \text{ with partition } \{\zeta_1^1, \zeta_2^2, \dots, \zeta_{k-1}^{k-1}, \zeta_k^k\},$$

further refined by splitting the last column  $\zeta_{k-2}^k$  of  $\zeta_{k-1}^k$  into the sets  $\zeta_{k-2}^k$  of its first  $k - 1$  entries and  $X_{k-2}^k$  of its last entry,  $a_{k-1} a_{k-2} \cdots a_1 = 12 \cdots (k - 1)$ . The sizes  $|\zeta_1^1|, |\zeta_2^2|, \dots, |\zeta_{k-1}^{k-1}|, |\zeta_{k-2}^k|, |X_{k-2}^k|$  form the line  $\Delta'_{k-1}$  of  $\Delta'$ .

**Proof.** The statement represents the set of vertices of the induced truncated tree  $T[\mathcal{S} \cap b([0, C_k - 1])]$ , ( $1 \leq k \in \mathbb{N}$ ) via the prefix  $\zeta_k^k$  of  $J$  and the line  $\Delta'_{k-1}$  of  $\Delta'$ .  $\square$

**Theorem 7.** The sequence  $h(\mathcal{S} \setminus b(0))$  can be recreated by stepwise generation of the induced truncated trees  $T[\mathcal{S} \cap b([0, C_k - 1])]$ , ( $1 \leq k \in \mathbb{N}$ ). In the  $k$ -th step, the determinations specified in Theorems 3-4 are performed in the natural order of the TRGS’s. The  $k$ -step completes those determinations, namely  $(n, h(n)) \rightarrow (r, h(r))$ , for the lines of  $\Delta'$  corresponding to the sets  $\zeta_j^j$  ( $j = 1, \dots, k - 1$ ), and ends up with the determinations  $(n, h(n)) \rightarrow (r, h(r))$  for  $j = k$  in the line corresponding to  $\zeta_{k-2}^k$  and  $(n, h(n)) \rightarrow (r, h(r))$  in the final line for  $j = k + 1$ , corresponding to  $X_{k-2}^k$ .

**Proof.** Theorem 7 is used to express the stepwise nature of the generation of the sequence  $h(\mathcal{S} \setminus b(0))$ . The methodology in the statement is obtained by integrating steps applying Theorem 5 in the way prescribed, that yields the correspondence with the lines of  $\Delta'$ .  $\square$

**Corollary 3.** Let  $v \in V(O_n)$  (resp.,  $v \in V(M_k)$ ). The sequence of quadruples

$$(b, \gamma, \rho, h)(\mathbb{N}) = (b(\mathbb{N}), \gamma(\mathbb{N}), \rho(\mathbb{N}), h(\mathbb{N})) \tag{12}$$

allows to retrieve  $v$  by locating either its oriented  $(2k + 1)$ - (resp.,  $(4k + 2)$ -) cycle in the uniform 2-factor (§10) or in a specific  $\mathbb{Z}_{2k+1}$ - (resp.,  $\mathbb{D}_{2k+1}$ -) class (Sections 7-9) and then locating  $v$  in such cycle or class by departing from its only

anchored Dyck word. The sequence (12) allows to enlist all vertices  $v$  by ordering their cycles or classes, including all vertices in each such cycle or class, starting with its anchored Dyck word.

**Proof.** Let  $n \in \mathbb{N}$ . Then,  $\gamma(n)$  yields the required update location in the TRGS  $b(n) \in \mathcal{S}$  with respect to the parent TRGS  $b(\rho(n)) \in \mathcal{S}$ , while  $h(n)$  yields the specific update, as determined in Theorems 6-7. This produces the corresponding clone. Then, Theorem 1 allows to recover the original Dyck word from that clone, and thus the corresponding vertex of  $O_k$  (resp.,  $M_k$ ) by local translation in its containing cycle in the cycle factor of §10, or cyclic (resp., dihedral) class (as pointed out in §11).  $\square$

### 14. Hamilton cycles in odd and middle-levels graphs

A *flippable tuple* [2] in a vertical list  $L(n)$  is a pair  $FT(n, j)$  of contiguous lines in  $L(n)$  having its  $k$ -supplementary entry pair at the  $j$ -th position counted from the right ( $j \in [0, 2k]$ ). Let  $2 < \kappa \in \mathbb{Z}$ . A *flipping  $\kappa$ -cycle* [2] is a finite sequence of pairwise different vertical lists  $L(n_j)$ , ( $n = 1, \dots, \kappa$ ) determining a  $2\kappa$ -cycle in  $O_k$  containing successive pairwise disjoint edges whose endvertex pairs  $\{\chi_0^j, \chi_1^j\}$  are flippable tuples in their corresponding vertical lists  $L(n_j)$  ( $n = 1, \dots, \kappa$ ), with the vertical pairs of number  $k$ -supplementary entries happening at pairwise different coordinate positions.

**Example 18.** As mentioned for  $k = 3$ , the five right columns in Table 4 contain the lists  $L(n)$  ( $n \in [0, 4]$ ). These lists contain in those five final columns additional information that allows to assemble the flipping triples  $\tau_0 = (L(0), L(1), L(2))$  and  $\tau_1 = (L(0), L(3), L(4))$ , with the initial line  $0F^3(n)$  of each such  $L(n)$  having a sole underlined entry per flippable tuple corresponding to the underlined entries of its two constituent contiguous  $(2k + 1)$ -tuples, say  $\chi_0, \chi_1$ . Here,  $\chi_1$  is provided with a superindex containing: (i) an index  $z \in \{0, 1\}$  relating to a sole associated triple  $\tau_z$  ( $z \in \{0, 1\}$ ); (ii) a vertical arrow indicating a definite orientation of the edge  $\chi_0\chi_1$  which determines an arc  $\chi'\chi''$  with  $\{\chi_0, \chi_1\} = \{\chi', \chi''\}$ ; (iii) an index  $n'$  such that  $L(n')$  is in  $\tau_z$  and contains a flippable tuple determining an arc  $\chi'''\chi''''$  so that the arc  $\chi''\chi'''$  is in  $\tau_z$ . By considering the three flippable tuples obtained in this way and the additional neighbor adjacencies, an oriented 6-cycle is obtained. The triangles  $\tau_0, \tau_1$  form the two hyperedges of a connected acyclic hypergraph on the vertex set  $\{L(n) | i \in [0, 4]\}$  that yields the simplest case of the construction of a Hamilton cycle in the odd graphs, in this case in  $O_3$ . Each of  $\tau_0$  and  $\tau_1$  yields a 21-cycle in  $O_3$  by means of symmetric differences. The presence of  $L(0)$  in both  $\tau_0$  and  $\tau_1$  then allows to transform both 21-cycles into the claimed Hamilton cycle of  $O_3$  by means of corresponding flippable tuples in  $L(0)$ , one for  $\tau_0$  and the other one for  $\tau_1$ .

Consider the following Dyck-word collections (triples, quadruples, etc.):

$$\begin{cases} S_1(w) &= \{\zeta_{1w}^1 = 0w0011\bar{1}, \quad \bar{\zeta}_{1w}^2 = 0w0\bar{1}101, \quad \bar{\zeta}_{1w}^3 = 0w0101\bar{1}\}, \\ S_2 &= \{\bar{\zeta}_2^1 = 00\bar{1}10011, \quad \bar{\zeta}_2^2 = 001001\bar{1}, \quad \bar{\zeta}_2^3 = 000101\bar{1}\}, \\ S_3 &= \{\bar{\zeta}_3^1 = 00011\bar{1}, \quad \bar{\zeta}_3^2 = 0100\bar{1}, \quad \bar{\zeta}_3^3 = 0\bar{1}0101\}, \\ S_4 &= \{\bar{\zeta}_4^1 = 00011\bar{1}, \quad \bar{\zeta}_4^2 = 0010\bar{1}, \quad \bar{\zeta}_4^3 = 0100\bar{1}, \quad \bar{\zeta}_4^4 = 0\bar{1}0101\}, \end{cases} \tag{13}$$

(based on [2, display (4.2)]) where  $w$  is any (possibly empty) Dyck word. Consider also the sets  $\underline{S}_1(w), \underline{S}_2, \underline{S}_3, \underline{S}_4$  obtained respectively from  $S_1(w), S_2, S_3, S_4$  by having their component Dyck paths  $\bar{\zeta}_{1w}^j, \bar{\zeta}_2^j, \bar{\zeta}_3^j, \bar{\zeta}_4^j$  defined as the complements of the reversed strings of the corresponding Dyck paths  $\zeta_{1w}^j, \zeta_2^j, \zeta_3^j, \zeta_4^j$ . Note that each Dyck word in the subsets of display (13) has just one underlined entry. By denoting

$$\bar{\zeta}_{1w}^j = x_s x_{s-1} \cdots x_2 x_1 x_0 \quad \text{and} \quad \bar{\zeta}_i^j = x_s x_{s-1} \cdots x_2 x_1 x_0, \quad \text{for } i = 2, 3, 4, \tag{14}$$

where  $j = 1, 2, 3$  for  $i = 1, 2, 3$  and  $j = 1, 2, 3, 4$  for  $i = 4$  and adequate  $s$  in each case, the barred positions in (13) are the targets of the following correspondence  $\Phi$ :

$$\begin{cases} \Phi(\bar{\zeta}_{1w}^1) = 1, & \Phi(\bar{\zeta}_{1w}^2) = 4, & \Phi(\bar{\zeta}_{1w}^3) = 0, \\ \Phi(\bar{\zeta}_2^1) = 6, & \Phi(\bar{\zeta}_2^2) = 0, & \Phi(\bar{\zeta}_2^3) = 2, \\ \Phi(\bar{\zeta}_3^1) = 0, & \Phi(\bar{\zeta}_3^2) = 1, & \Phi(\bar{\zeta}_3^3) = 5, \\ \Phi(\bar{\zeta}_4^1) = 0, & \Phi(\bar{\zeta}_4^2) = 1, & \Phi(\bar{\zeta}_4^3) = 3, & \Phi(\bar{\zeta}_4^4) = 5. \end{cases} \tag{15}$$

The correspondence  $\Phi$  is extended over the Dyck words  $\underline{\zeta}_{1^w}^j, \underline{\zeta}_2^j, \underline{\zeta}_3^j, \underline{\zeta}_4^j$  with their barred positions taken reversed with respect to the corresponding barred positions in  $\bar{\zeta}_{1^w}^j, \bar{\zeta}_2^j, \bar{\zeta}_3^j, \bar{\zeta}_4^j$ , respectively.

Adapting from [2], we define an hypergraph  $H_k$  with  $V(H_k)$  formed by the lists  $L(i)$  with  $b(i) \in V(T) \cap b([0, C_k - 1])$  and as hyperedges the subsets

$$\{L(i_j) | j \in \{1, 2, 3\}\} \subset V(H_k) \text{ and } \{L(i_j) | j \in \{1, 2, 3, 4\}\} \subset V(H_k)$$

whose member lists  $L(i_j)$  have the Dyck words  $f(i_j)$  corresponding to their initial rows, the Dyck nests  $F(i_j)$ , for  $j = 1, 2, 3$  or  $j = 1, 2, 3, 4$ , containing respective Dyck subwords as members in the collection (indexed by  $j$ )

$$\{\bar{\zeta}_{1^w}^j, \bar{\zeta}_2^j, \bar{\zeta}_3^j, \bar{\zeta}_4^j, \underline{\zeta}_{1^w}^j, \underline{\zeta}_2^j, \underline{\zeta}_3^j, \underline{\zeta}_4^j\}$$

in the same 6 or 8 fixed positions  $x_i$  (for specific indices  $i \in \{0, 1, \dots, s\}$  in (14)) and forming respective subsets

$$\{\bar{\zeta}_{1^w}^j | j = 1, 2, 3\}, \{\underline{\zeta}_{1^w}^j | j = 1, 2, 3\}, \{\bar{\zeta}_4^j | j = 1, 2, 3, 4\}, \{\underline{\zeta}_4^j | j = 1, 2, 3, 4\},$$

$$\{\bar{\zeta}_i^j | j = 1, 2, 3\} \text{ and } \{\underline{\zeta}_i^j | j = 1, 2, 3\}, \text{ for both } i = 2 \text{ and } 3.$$

**Example 19.** Two hyperedges  $h_0, h_1$  of  $H_3$  are obtained from the six indices  $z \in \{0, 1\}$  in items (i)-(iii) (see Example 18) of corresponding superindices in Table 4. These two hyperedges are obtained from the triples  $\tau_0, \tau_1$  in Example 18, with substrings  $\bar{\zeta}_{1^e}^j$  or  $\bar{\zeta}_i^j$  ( $j = 3, 4$ ) realized via the last six entries of the initial lines  $F(i)$  of the lists  $L(i)$  ( $i \in [0, 4]$ ), where the involved flippable tuples have those six entries overlined in the lower one of its two component lines. For  $h_0$  (resp.,  $h_1$ ),  $f(0), f(1), f(2)$  (resp.,  $f(0), f(3), f(4)$ ) are Dyck words  $\bar{\zeta}_{1^e}^1, \bar{\zeta}_3^2, \bar{\zeta}_{1^e}^3$  (resp.,  $\bar{\zeta}_{1^e}^3, \bar{\zeta}_3^2, \bar{\zeta}_4^3$ ), with their entries in positions  $\Phi(\bar{\zeta}_{1^e}^1) = 1, \Phi(\bar{\zeta}_4^2) = 4, \Phi(\bar{\zeta}_{1^e}^3) = 0$  (resp.,  $\Phi(\bar{\zeta}_{1^e}^3) = 0, \Phi(\bar{\zeta}_3^2) = 1, \Phi(\bar{\zeta}_4^3) = 5$ ) in thick trace. Then  $H_3$  contains the connected subhypergraph  $H'_3$  formed by  $h_0, h_1$ , that are both incident to the vertex  $L(0)$  of  $H_3$ . This is used to construct the Hamilton cycle mentioned in Example 18;  $H'_3$  is schematically represented over the lower right in Table 12.

**Example 20.** For  $k = 4$ , in a likewise manner to that of Example 19, Table 12 shows the 14 columns  $L(i)$ , for  $i \in [0, 12]$ , where we use hexadecimal notation with  $a = 10, b = 11, c = 12, d = 13$ . The information contained in the table leads to a subhypergraph  $H'_4$  of  $H_4$  represented in its lower-right corner with the hyperedges

$$h_0 = (0, 2, a), h_1 = (8, 7, 5), h_2 = (7, 6, a), h_3 = (1, 4, 6), h_4 = (1, 9, d), h_5 = (3, b, c, d).$$

The respective triples of Dyck words  $\bar{\zeta}_{1^w}^j$  or  $\bar{\zeta}_i^j$  or  $\bar{\zeta}_i^j$  or  $\bar{\zeta}_i^j$  ( $j = 2, 3, 4$ ) may be expressed as follows by replacing the Greek letters  $\zeta$  by the values of the correspondence  $\Phi$ :

$$(\underline{5}_3^1, \underline{4}_3^2, \underline{0}_3^3), (6_2^1, 0_2^2, 2_2^3), (1_{101}^1, 4_{101}^2, 0_{101}^3), (1_{1^e}^1, 4_{1^e}^2, 0_{1^e}^3), (0_3^1, 1_3^2, 5_3^3), (0_4^1, 1_4^2, 3_4^3, 5_4^4),$$

where we can also write  $(\underline{5}_3^1, \underline{4}_3^2, \underline{0}_3^3) = (\underline{0}_3^1, \underline{1}_3^2, \underline{5}_3^3)$ . The flippable tuples  $FT(i, j)$  allow to compose five flipping 6-cycles and one flipping 8-cycle, allowing to integrate by symmetric differences a Hamilton cycle in  $O_4$ .

We represent  $H_k$  as a simple graph  $\psi(H_k)$  with  $V(\psi(H_k)) = V(H_k)$  by replacing each hyperedge  $e$  of  $H_k$  by the clique  $K(e) = K(V(e))$  so that  $\psi(H_k[e]) = K(e)$ , being such replacements the only source of cliques of  $\psi(H_k)$ . A tree  $T$  of  $H_k$  is a subhypergraph of  $H_k$  such that: **(a)**  $\psi(T)$  is a connected union of cliques  $K(V(e))$ ; **(b)** for each cycle  $C$  of  $\psi(H_k)$ , there exist a unique clique  $K(V(e))$  such that  $C$  is a subgraph of  $K(e)$ . A spanning tree  $T$  of  $H_k$  is a tree of  $H_k$  with  $V(T) = V(H_k)$ . Clearly, the subhypergraphs  $H'_k$  of  $H_k$  for  $k = 3$  and 4 are corresponding spanning trees.

A subset  $G$  of hyperedges of  $H_k$  is said to be *conflict-free* [2] if: **(a)** any two hyperedges of  $G$  have at most one vertex in common; **(b)** for any two hyperedges  $g, g'$  of  $G$  with a vertex in common, the corresponding images by  $\Phi$  (as in display (15)) in  $g$  and  $g'$  are distinct. A proof of the following final result is included, as our viewpoint and notation differs from that of its proof in [2].

**Table 12.** Uniform 2-factors provided by permutations  $\pi$  for  $k = 4$ .

$k=4$ $i=0$ 000001111  $\zeta_4^1$	$k=4$ $i=1$ 000011101  $\zeta_4^1$ 000011101  $\zeta_4^1$	$k=4$ $i=2$ 000011011  $\zeta_4^1$	$k=4$ $i=3$ 001000111  $\zeta_4^1$	$k=4$ $i=4$ 000110101  $\zeta_4^2$
012344321 1000 433210124 7550 110344322 3420 322110443 5d50 221104433 4d40 211044332 6d30 332210144 2a50 102344321 0 <sup>1a</sup> <sub>8110</sub> 443210123 0050	023443211 7101 432101234 8041 123443210 1081 322101344 5241 221443310 3 <sup>16</sup> <sub>3b81</sub> 211033244 4931 332214410 2c81 102442133 6811 442101332 4 <sup>1d</sup> <sub>0741</sub>	013443221 1202 433102214 5b42 110332442 3922 322144103 0 <sup>10</sup> <sub>4c72</sub> 221103443 2442 213443102 7372 332101244 6542 103443221 8412 443102213 0342	022134431 3303 421331024 7863 144103322 5c73 310221443 6c23 244211033 4963 210134432 8223 344321012 5 <sup>1b</sup> <sub>1063</sub> 101234432 2013 443321012 0563	034432211 7404 431022134 8334 134432210 5284 321012344 6034 234432110 1184 211024433 3634 332442110 2984 102214433 3 <sup>11</sup> <sub>4b14</sub> 441033221 0c34
$k=4$ $i=5$ 000010111  $\zeta_2^3$	$k=4$ $i=6$ 000101101  $\zeta_3^3$ 000101101  $\zeta_3^3$	$k=4$ $i=7$ 000100111  $\zeta_2^2$ 000100111  $\zeta_2^2$	$k=4$ $i=8$ 000110011  $\zeta_2^1$	$k=4$ $i=9$ 001001101  $\zeta_3^2$
012443321 1505 433211024 7665 110443322 5d25 322101443 6a45 221441033 1 <sup>18</sup> <sub>3c65</sub> 211034432 4435 332211044 2d65 102443321 8615 443211023 0155	024433211 7606 432110234 8156 124433210 1586 322110344 3456 221344310 2386 213310244 5856 331024421 4846 103322144 2 <sup>18</sup> <sub>6c16</sub> 442110332 3 <sup>14</sup> <sub>0956</sub>	013324421 1707 433221014 3a67 110234432 2127 324421013 7767 221014433 2 <sup>1a</sup> <sub>5a37</sub> 214410332 6c47 331022144 4b37 103324421 8c17 443221013 1 <sup>15</sup> <sub>0267</sub>	024421331 5808 421013324 7738 134431022 6368 321012443 8538 234432101 1078 211023443 3138 332442101 2778 102213443 1 <sup>17</sup> <sub>4318</sub> 442133102 0878	033244211 7909 432210134 8259 133244210 3789 310221344 5329 244213310 4 <sup>11</sup> <sub>4889</sub> 210133244 6729 344310221 1359 101344322 2219 443310221 0b59
$k=4$ $i=a$ 000101011  $\zeta_4^3$ 000101011  $\zeta_4^4$	$k=4$ $i=b$ 001001011  $\zeta_4^2$	$k=4$ $i=c$ 001010011  $\zeta_4^3$	$k=4$ $i=d$ 001010101  $\zeta_4^4$ 001010101  $\zeta_4^4$	$H'_3 :$ 1 > 0 < 3 2 > 4
014433221 1a0a 433221104 3d7a 120244331 262a 324421103 597a 221013443 423a 214433102 7b7a 332110244 0 <sup>12</sup> <sub>665a</sub> 104433221 8d1a 443221103 2 <sup>16</sup> <sub>047a</sub>	022144331 3b0b 410332214 5c2b 133102442 484b 310244213 785b 244210133 5 <sup>1c</sup> <sub>675b</sub> 210144332 8a2b 344321102 117b 101244332 251b 443321102 067b	033221441 5c0c 421103324 5 <sup>1d</sup> <sub>794c</sub> 144331022 6b6c 321102443 864c 244332101 357c 210123443 402c 344322101 127c 101332442 271c 443322101 0a7c	044332211 7d0d 432211034 846d 144332210 5a8d 321102344 614d 244332110 368d 210124433 5 <sup>1c</sup> <sub>452d</sub> 344322110 148d 101443322 4 <sup>19</sup> <sub>2a1d</sub> 443322110 0d8d	$H'_4 :$ 5 > 8 7 > a < 0 6 > 2 1 > 4 1 > 9 d - 3   -   c - b

**Theorem 8.** A conflict-free spanning tree of  $H_k$  yields a Hamilton cycle of  $O_k$ , for every  $k \geq 3$ . Moreover, distinct conflict-free spanning trees of  $H_k$  yield distinct Hamilton cycles of  $H_k$ , for every  $k \geq 6$ .

**Proof.** Let  $D_k$  be the set of all Dyck words of length  $2k$  and, recalling display (13), let

$$\begin{cases} E_2 = \{0101\}, E_3 = S_4, & E_k = 01D_{k-1}, \forall k > 3, \\ F_2 = \{0011\}, F_3 = D_3 \setminus E_3 = \{001101\}, & F_k = D_k \setminus 01D_{k-1}, \forall k > 3. \end{cases} \tag{16}$$

In particular,  $0101(01)^{k-2} \in E_k$  and  $0011(01)^{k-2} \in F_k$ . Now, let

$$\begin{cases} \mathcal{E}_2 = \emptyset, \quad \mathcal{E}_3 = \{S_4\}, \quad \mathcal{T}_3 = \{S_1(\epsilon), S_3\}, & \mathcal{E}_k = 01\mathcal{T}_{k-1}, \forall k > 3, \\ \mathcal{F}_2 = \emptyset, \quad \mathcal{F}_3 = \emptyset, \quad \mathcal{F}_4 = \{S_1(01), S_2, 0S_31, S_1(\epsilon)01\}. \end{cases} \tag{17}$$

Let us set  $\mathcal{F}_k$  as a function of  $\mathcal{E}_2, \dots, \mathcal{E}_{k-1}, \mathcal{F}_2, \dots, \mathcal{F}_{k-1}, \mathcal{T}_{k-2}$ , as follows: For  $1 < j \leq k$ , let  $F_k^j = \cup_{i=2}^j \{0\underline{u}1v \mid u \in D_{i-1}, v \in D_{k-1}\}$ . Since  $F_k = F_k^k$ , then the following implies the existence of a spanning tree of  $H_k[F_k]$ .

**Lemma 5.** For every  $1 < j \leq k$ , there exists a spanning tree  $\mathcal{F}_k^j$  of  $H_k[F_k^j]$ .

**Proof.** Lemma 7 [2] asserts that if  $\tau$  is a flippable tuple and  $u, v$  are Dyck words, then: (i)  $u\tau v$  is a flippable tuple if  $|u|$  is even; (ii)  $u\tau v$  is a flippable tuple if  $|u|$  is odd. Lemma 8 [2] insures that the collections in (13) are flippable tuples. Using those two lemmas of [2], we define  $\Psi$  as the set of all the flippable tuples  $u\tau v$  and  $u\tau v$  arising from (13). Moreover, we define  $\Psi_2 = \emptyset$  and  $\Psi_k = \Psi \cap D_k$ , for  $k > 2$ .

Since  $F_k^2 = 0011D_{k-2}$ , we let  $\mathcal{F}_k^2 = 0011\mathcal{T}_{k-2}$ . Assuming  $2 < j \leq k$ , since  $D_{j-2} = E_{j-1} \cup F_{j-1}$  is a disjoint union, then we have the following partition:

$$F_k^j = F_k^{j-1} \cup_{v \in D_{k-j}} (0\underline{D}_{j-1}1v) = F_k^{j-1} \cup_{v \in D_{k-j}} ((0\underline{E}_{j-1}1v) \cup (0\underline{F}_{j-1}1v)). \tag{18}$$

For every  $v \in D_{k-j}$ , the elements of  $\tau(v) = S_1((01)^{j-3})v \in \Psi_k$ , are:

$$0(01)^{j-3}0011\underline{1}v \in 0\underline{E}_{j-1}1v, \quad 0(01)^{j-3}0101\underline{1}v \in 0\underline{E}_{j-1}1v, \quad 0(01)^{j-3}0\underline{1}101v \in F_k^{j-1}. \tag{19}$$

Now, we let

$$\mathcal{F}_k^j = \mathcal{F}_k^{j-1} \cup (\cup_{v \in D_{k-j}} (\{\tau(v)\} \cup (0\underline{\mathcal{E}}_{j-1}1v) \cup (0\underline{\mathcal{F}}_{j-1}1v))), \tag{20}$$

which defines a spanning tree of  $H_k[F_k^j]$ .  $\square$

Now, the elements of  $\tau = S_3(01)^{k-3} \in \Psi_k$  are:

$$00011\underline{1}(01)^{k-3} \in F_k, \quad (k > 3), \quad 01001\underline{1}(01)^{k-3} \in 01E_{k-1}, \quad 010101(01)^{k-3} \in 01F_{k-1}. \tag{21}$$

The sets  $F_k, 01E_{k-1}$  and  $01F_{k-1}$  form a partition of  $D_k$ . We take the spanning trees of the subhypergraphs induced by these three sets and connect them into a single spanning tree of  $H_k$  by means of the triple  $\tau$ , that is:

$$H'_k = F_k \cup \{\tau\} \cup 01E_{k-1} \cup 01F_{k-1}. \tag{22}$$

$\square$

**Example 21.** Example 19 uses  $H'_3$  in display (17), with  $S_1(\epsilon) = 012$  and  $S_3 = 034$  yielding the hypergraph  $H'_3$  depicted over the lower-right enclosure of Tabledia26. Example 20 uses  $H'_k$  in display (22) for  $k = 4, \mathcal{F}_4$  and  $\mathcal{E}_3$  in display (17) and  $\tau$  in display (21), with  $S_1(01) = 67a, S_2 = 875, 0\underline{S}_31 = 02a, S_1(\epsilon) = 146$ , being these four triples the elements in  $\mathcal{F}_4; 01S_4 = 3bcd$ , this one as the only element of  $01E_3$ , (while  $\mathcal{F}_3 = \emptyset$ ); and  $\tau = 02a$ , yielding the hypergraph  $H'_4$  depicted in the lower-right enclosure of Table 4.

**Corollary 4.** To each Hamilton cycle in  $O_k$  produced by Theorem 8 corresponds a Hamilton cycle in  $M_k$ .

**Proof.** For each vertical list  $L(i)$ , let  $L^M(i)$  be a corresponding vertical list in  $M_k$  which is obtained from  $L(i)$ . Then, Theorem 8 can be adapted to producing Hamilton cycles in the  $M_k$  by repeating the argument in its proof in replacing the lists  $L(\alpha)$  by lists  $L^M(\alpha)$ , since they have locally similar behaviors, being the cycles provided by the lists  $L^M(\alpha)$  twice as long as the corresponding lists  $L(\alpha)$ , so the said local behavior happens twice around opposite (rather short) subpaths. Combining Dyck-word triples and quadruples as in display (13) into adequate pullback liftings (of the covering graph map  $M_k \rightarrow O_k$  in the lists  $L^M(\alpha)$  of those parts of the lists  $L(\alpha)$  in which the necessary symmetric differences take place to produce the Hamilton cycles in  $O_k$  will produce corresponding Hamilton cycles in  $M_k$ .  $\square$

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