Open Journal of
Discrete Applied Mathematics

PSR Press

Article

The general energy of a graph

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Received: 27 October 2025; Accepted: 04 February 2026; Published: 07 March 2026

Abstract: For any real number α, the general energy of a graph is defined as the sum of the α-th powers of the nonzero singular values of its adjacency matrix. This definition unifies several classical spectral invariants, such as the graph energy and spectral moments. In this paper, we establish bounds on the general energy of graphs. These bounds, in turn, yield new estimates for the ordinary energy and spectral moments, and lead to a more general relationship between these quantities.

Keywords: the general energy, singular value, bound

MSC: 05C09, 05C50, 05C92.

1. Introduction

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For a vertex $v_i \in V(G)$, the set of its neighbors in G is denoted as $N(v_i) = \{v_j \in V(G) : v_iv_j \in E(G)\}$. The degree of v_i is the number of its neighbors, that is, $d(v_i) = |N(v_i)|$. The maximum degree $\Delta = \Delta(G)$ is defined as $\Delta = \max_{v_i \in V(G)} d(v_i)$. The first Zagreb index $Z_1(G)$ of G is equal to the sum of squares of the degrees of the vertices of G, that is, $Z_1(G) = \sum\limits_{i=1}^{n} d^2(v_i)$. The first Zagreb index is a highly regarded descriptor of graphs and is widely used in chemistry and mathematics, see [1,2].

The adjacency matrix of G with n vertices, denoted by $A(G) = (a_{ij})$, is the $n \times n$ matrix whose rows and columns are indexed by the vertex set of G and the (i,j)-th entry $a_{ij} = 1$, if the vertices v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be the eigenvalues of its adjacency matrix $A(G)$. The energy $\mathcal{E}(G)$ of G was introduced by Gutman in connection with the π-molecular energy, see [3]. It is defined by

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

It has an important role in the calculation of electron and molecule energy. Thus, its mathematical chemistry applications gave rise to the research study of the energy of graphs [4,5].

Since $A(G)$ is a real symmetric matrix, Nikiforov [6] recognized that the energy of the graph G is equal to the sum of the singular values of its adjacency matrix $A(G)$, i.e.,

$$\mathcal{E}(G) = \sum_{i=1}^{n} \sigma_i(G),$$

where $\sigma_1(G) \geq \sigma_2(G) \geq \cdots \geq \sigma_n(G)$. This observation is crucial for graph energy theory as it establishes a connection with classical singular value theory. On this basis, Nikiforov proposed the Schatten p-norms and the Ky Fan k-norms of graphs, and conducted a series of studies that greatly enriched the theory of graph energy, see [7–11]. Recently, Akbari et al. [12] proposed the positive square energy and the negative square energy of graphs for studying the conjecture of Elphick, Farber, Goldberg and Wocjan [13]. Subsequently, Tang, Liu and Wang [14] investigated the positive and negative p-th power energy of graphs and their behavior under edge addition. Moreover, they [15] showed that the graph with the smallest p-th power energy among all connected graphs is a star, which resolves Nikiforov's conjecture [10]. Very recently, Akbari et al. [16] obtained the lower bounds for p-energy of graphs. Tang and Elphick [17] gave the lower bound of the chromatic number and quantum chromatic number in terms of the positive and negative p-energies of graphs.

Inspired by the above work, we define the general energy of a graph G as the sum of the α-th power of the non-zero singular values of G, i.e.,

$$\mathcal{E}_\alpha(G) = \sum_{i=1}^{n-\eta} \sigma_i^\alpha = \sum_{i=1}^{n-\eta} |\lambda_i|^\alpha,$$

where α is any real number, and η is the nullity of $A(G)$. Obviously, $\mathcal{E}_1(G) = \mathcal{E}(G)$, $\mathcal{E}_{\alpha=p}(G) = \mathcal{E}_p^+(G) + \mathcal{E}_p^-(G)$, where $\mathcal{E}_p^+(G)$ and $\mathcal{E}_p^-(G)$ are the positive and negative p-th power energy of graphs. Moreover, $\mathcal{E}_\alpha(G)$ is a classic spectral moment when α is a positive integer. Therefore, the general energy of the graph is a generalized expression of many classical parameters. In this paper, we establish several bounds on the general energy of a graph. By establishing these results, we further derive new bounds for the conventional energy and the spectral moment, and obtain a more general relationship between the two quantities.

2. Preliminaries

A graph G is strongly regular with parameters (n, k, a, b) whenever G is regular of degree k, every pair of adjacent vertices has a common neighbors, and every pair of distinct nonadjacent vertices has b common neighbors. A strongly regular graph with parameters (n, k, a, a) is called a design graph. As usual, denote by K_n and $K_{s,n-s}$ the the complete graph and the complete bipartite graph with n vertices, respectively.

For two non-increasing sequences $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$, Y is majorized by X, denoted by $Y \preceq X$, if

$$\sum_{i=1}^{j} y_i \leq \sum_{i=1}^{j} x_i \quad \text{for } j = 1, 2, \ldots, n-1, \text{ and } \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i.$$

Lemma 1. [19,20] *Let G be a graph with n vertices. Then*

$$\sqrt{\frac{Z_1(G)}{n}} \leq \lambda_1(G) \leq \Delta(G).$$

The equality in the left hand side holds if and only if G is regular or semiregular. If G is a connected graph, then the equality in the right hand side holds if and only if G is regular.

Lemma 2. [19,21] *Let G be a connected graph with n vertices and m edges, then*

$$\frac{2m}{n} \leq \lambda_1(G) \leq \sqrt{2m-n+1}.$$

The equality in the left hand side holds if and only if G is a regular graph, and the equality in the right hand side holds if and only if $G \cong K_{1,n-1}$ or $G \cong K_n$.

Lemma 3. [22] *Let n and k be integers, and M be a $(0,1)$-matrix of order n.*

(i) *If $n \geq k \geq 1$, then*

$$\sigma_1 + \sigma_2 + \cdots + \sigma_k \leq \frac{1}{2}(1+\sqrt{k})n.$$

(ii) *Given $\varepsilon > 0$, for all sufficiently large k and n,*

$$\sigma_1 + \sigma_2 + \cdots + \sigma_k \geq \frac{1}{2}\left(\frac{1}{2} + \sqrt{k} - \varepsilon k^{-2/5}\right) n.$$

Lemma 4. [23] *Suppose $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ are non-increasing sequences of real numbers. If $Y \preceq X$, then for any convex function f,*

$$\sum_{i=1}^{n} f(y_i) \leq \sum_{i=1}^{n} f(x_i).$$

3. Main results

Theorem 1. *Let G be a graph with n vertices and m edges.*

(i) *If* $\alpha < 0$ *or* $\alpha \geq 1$, *then*

$$\mathcal{E}_\alpha(G) \geq \sigma_1^\alpha + \frac{1}{(n-\eta-1)^{\alpha-1}}(\mathcal{E}-\sigma_1)^\alpha. \tag{1}$$

The equality holds if and only if $\sigma_2 = \sigma_3 = \cdots = \sigma_{n-\eta}$.

(ii) *If* $0 < \alpha < 1$, *then*

$$\mathcal{E}_\alpha(G) \leq \sigma_1^\alpha + \frac{1}{(n-\eta-1)^{\alpha-1}}(\mathcal{E}-\sigma_1)^\alpha. \tag{2}$$

The equality holds if and only if $\sigma_2 = \sigma_3 = \cdots = \sigma_{n-\eta}$.

Proof. For $x > 0$, we know that $|x|^\alpha$ is a convex function if and only if $\alpha < 0$ or $\alpha \geq 1$. By Jensen's inequality, we have

$$\sum_{i=2}^{n-\eta} \frac{\sigma_i^\alpha}{n-\eta-1} \geq \left(\sum_{i=2}^{n-\eta} \frac{\sigma_i}{n-\eta-1}\right)^\alpha,$$

with equality if and only if $\sigma_2 = \sigma_3 = \cdots = \sigma_{n-\eta}$. Thus, we have

$$\begin{aligned}
\mathcal{E}_\alpha(G) &= \sum_{i=1}^{n-\eta} \sigma_i^\alpha \\
&= \sigma_1^\alpha + \sum_{i=2}^{n-\eta} \sigma_i^\alpha \\
&\geq \sigma_1^\alpha + \frac{1}{(n-\eta-1)^{\alpha-1}} \left(\sum_{i=2}^{n-\eta} \sigma_i\right)^\alpha \\
&= \sigma_1^\alpha + \frac{1}{(n-\eta-1)^{\alpha-1}}(\mathcal{E}-\sigma_1)^\alpha,
\end{aligned}$$

with equality if and only if $\sigma_2 = \sigma_3 = \cdots = \sigma_{n-\eta}$. Now suppose that $0 < \alpha < 1$. Then

$$\sum_{i=2}^{n-\eta} \frac{\sigma_i^\alpha}{n-\eta-1} \leq \left(\sum_{i=2}^{n-\eta} \frac{\sigma_i}{n-\eta-1}\right)^\alpha,$$

with equality if and only if $\sigma_2 = \sigma_3 = \cdots = \sigma_{n-\eta}$. By similar arguments as above, the second part of the theorem follows. □

Remark 1. One can verify that equality holds in (1) and (2) for complete graphs, complete bipartite graphs, and strongly regular graphs. In addition, for positive integer values of α, inequality (1) establishes a relation linking the energy, spectral moments, and spectral radius.

Theorem 2. *Let* G *be a graph with* n *vertices and* m *edges.*

(i) *If* $\alpha < 0$ *or* $\alpha \geq 1$, *then*

$$\begin{aligned}
\mathcal{E}_\alpha(G) \leq& \Delta^\alpha + \left(\frac{1}{2}(1+\sqrt{2})n - \Delta\right)^\alpha + \frac{1}{2^\alpha}(\sqrt{3}-\sqrt{2})^\alpha n^\alpha \\
&+ \cdots + \frac{1}{2^\alpha}(\sqrt{n-\eta}-\sqrt{n-\eta-1})^\alpha n^\alpha.
\end{aligned}$$

(ii) *If* $0 < \alpha < 1$, *then*

$$\begin{aligned}
\mathcal{E}_\alpha(G) \geq& \Delta^\alpha + \left(\frac{1}{2}(1+\sqrt{2})n - \Delta\right)^\alpha + \frac{1}{2^\alpha}(\sqrt{3}-\sqrt{2})^\alpha n^\alpha \\
&+ \cdots + \frac{1}{2^\alpha}(\sqrt{n-\eta}-\sqrt{n-\eta-1})^\alpha n^\alpha.
\end{aligned}$$

Proof. (i) Observe that $|x|^\alpha$ is a convex function if $\alpha < 0$ and $\alpha \geq 1$. Let $X = \left(\Delta, \frac{1}{2}(1+\sqrt{2})n - \Delta, \frac{1}{2}(\sqrt{3}-\sqrt{2})n, \ldots, \frac{1}{2}(\sqrt{n-\eta}-\sqrt{n-\eta-1})n\right)$ and $Y = (\sigma_1, \sigma_2, \ldots, \sigma_{n-\eta})$. By Lemmas 1 and 3, we have $Y \preceq X$. By Lemma 4, we have

$$\begin{aligned}\mathcal{E}_\alpha(G) &= \sum_{i=1}^{n-\eta} \sigma_i^\alpha \\ &\leq \Delta^\alpha + \left(\frac{1}{2}(1+\sqrt{2})n - \Delta\right)^\alpha + \frac{1}{2^\alpha}(\sqrt{3}-\sqrt{2})^\alpha n^\alpha + \cdots + \frac{1}{2^\alpha}(\sqrt{n-\eta}-\sqrt{n-\eta-1})^\alpha n^\alpha.\end{aligned}$$

(ii) Observe that $-|x|^\alpha$ is a convex function if is a convex function if $0 < \alpha < 1$. Using similar arguments as in the proof of (i), we may prove (ii). □

Remark 2. By Lemma 2 and the proof of Theorem 2, the same substitution of Δ with $\sqrt{2m-n+1}$ extends to Theorem 2. Furthermore, for $\alpha = 1$, we obtain a new upper bound on the graph energy.

Theorem 3. *Let G be a graph with n vertices and m edges.*

(i) *If $\alpha < 0$ or $\alpha \geq 1$, given $\varepsilon > 0$, for all sufficiently large k and n,*

$$\begin{aligned}\mathcal{E}_\alpha(G) \geq & \left(\frac{Z_1}{n}\right)^{\alpha/2} + \frac{1}{2^\alpha}\left(\left(\frac{1}{2}+\sqrt{2}-\varepsilon 2^{-2/5}\right)n - \sqrt{\frac{Z_1}{n}}\right)^\alpha + \\ & \frac{1}{2^\alpha}(\sqrt{3}-\sqrt{2}-\varepsilon(3^{-2/5}+2^{-2/5}))^\alpha n^\alpha + \cdots + \\ & \frac{1}{2^\alpha}(\sqrt{n-\eta}-\sqrt{n-\eta-1}-\varepsilon((n-\eta)^{-2/5}+(n-\eta-1)^{-2/5}))^\alpha n^\alpha.\end{aligned}$$

(ii) *If $0 < \alpha < 1$, given $\varepsilon > 0$, for all sufficiently large k and n,*

$$\begin{aligned}\mathcal{E}_\alpha(G) \leq & \left(\frac{Z_1}{n}\right)^{\alpha/2} + \frac{1}{2^\alpha}\left(\left(\frac{1}{2}+\sqrt{2}-\varepsilon 2^{-2/5}\right)n - \sqrt{\frac{Z_1}{n}}\right)^\alpha + \\ & \frac{1}{2^\alpha}(\sqrt{3}-\sqrt{2}-\varepsilon(3^{-2/5}+2^{-2/5}))^\alpha n^\alpha + \cdots + \\ & \frac{1}{2^\alpha}(\sqrt{n-\eta}-\sqrt{n-\eta-1}-\varepsilon((n-\eta)^{-2/5}+(n-\eta-1)^{-2/5}))^\alpha n^\alpha.\end{aligned}$$

Proof. (i) Observe that $|x|^\alpha$ is a convex function if $\alpha < 0$ and $\alpha \geq 1$. Let $X = (\sigma_1, \sigma_2, \ldots, \sigma_{n-\eta})$ and $Y = (\sqrt{\frac{Z_1}{n}}, \frac{1}{2}(\frac{1}{2}+\sqrt{2}-\varepsilon 2^{-2/5})n - \sqrt{\frac{Z_1}{n}}, \frac{1}{2}(\sqrt{3}-\sqrt{2}-\varepsilon(3^{-2/5}+2^{-2/5}))n, \ldots, \frac{1}{2}(\sqrt{n-\eta}-\sqrt{n-\eta-1}-\varepsilon((n-\eta)^{-2/5}+(n-\eta-1)^{-2/5}))n)$. By Lemmas 1 and3, we have $Y \preceq X$. By Lemma 4, we have

$$\begin{aligned}\mathcal{E}_\alpha(G) &= \sum_{i=1}^{n-\eta} \sigma_i^\alpha \\ &\geq \left(\frac{Z_1}{n}\right)^\alpha + \frac{1}{2^\alpha}\left(\left(\frac{1}{2}+\sqrt{2}-\varepsilon 2^{-2/5}\right)n - \sqrt{\frac{Z_1}{n}}\right)^\alpha \\ &\quad + \frac{1}{2^\alpha}(\sqrt{3}-\sqrt{2}-\varepsilon(3^{-2/5}+2^{-2/5}))^\alpha n^\alpha + \cdots \\ &\quad + \frac{1}{2^\alpha}(\sqrt{n-\eta}-\sqrt{n-\eta-1}-\varepsilon((n-\eta)^{-2/5}+(n-\eta-1)^{-2/5}))^\alpha n^\alpha.\end{aligned}$$

(ii) Observe that $-|x|^\alpha$ is a strictly convex function if is a strictly convex function if $0 < \alpha < 1$. Using similar arguments as in the proof of (i), we may prove (ii). □

Remark 3. By Lemma 2 and the proof of Theorem 3, the same substitution of $\sqrt{\frac{Z_1}{n}}$ with $\frac{2m}{n}$ applies to Theorem 3. Furthermore, for $\alpha = 1$, we obtain a new lower bound on the graph energy.

Theorem 4. *Let G be a graph with n vertices and m edges.*

(i) *If* $0 < \alpha < 1$, *then*

$$\mathcal{E}_\alpha(G) \leq k^{1-\alpha}\left(\frac{k\mathcal{E}}{n-\eta}\right)^\alpha + (n-\eta-k)^{1-\alpha}\left(\mathcal{E} - \frac{k\mathcal{E}}{n-\eta}\right)^\alpha, \tag{3}$$

with equality if and only if $G \cong nK_1$.

(ii) *If* $1 < \alpha$, *then*

$$\mathcal{E}_\alpha(G) \geq k^{1-\alpha}\left(\sum_{i=1}^{k}\sigma_i\right)^\alpha + (n-\eta-k)^{1-\alpha}\left(\mathcal{E} - \sum_{i=1}^{k}\sigma_i\right)^\alpha, \tag{4}$$

with equality if and only if $G \cong nK_1$.

(iii) *If* $\alpha < 0$, *then*

$$\mathcal{E}_\alpha(G) \leq k^{1-\alpha}\left(\frac{1}{2}(1+\sqrt{k})n\right)^\alpha + (n-\eta-k)^{1-\alpha}\left(\mathcal{E} - \frac{1}{2}(1+\sqrt{k})n\right)^\alpha, \tag{5}$$

with equality if and only if $\sigma_1 = \cdots = \sigma_k = \frac{1}{2k}(1+\sqrt{k})n$, $\sigma_{k+1} = \cdots = \sigma_{n-\eta}$.

Proof. (i) Since $0 < \alpha < 1$, by power mean inequality, we have

$$\left(\frac{\sum_{i=1}^{k}\sigma_i^\alpha}{k}\right) \leq \left(\frac{\sum_{i=1}^{k}\sigma_i}{k}\right)^\alpha, \quad \text{i.e.,} \sum_{i=1}^{k}\sigma_i^\alpha \leq k^{1-\alpha}\left(\sum_{i=1}^{k}\sigma_i\right)^\alpha, \tag{6}$$

with equality if and only if $\sigma_1 = \sigma_2 = \cdots = \sigma_k$. Similarly, we have

$$\sum_{i=k+1}^{n-\eta}\sigma_i^\alpha \leq (n-\eta-k)^{1-\alpha}\left(\mathcal{E} - \sum_{i=1}^{k}\sigma_i\right)^\alpha, \tag{7}$$

with equality if and only if $\sigma_{k+1} = \sigma_{k+2} = \cdots = \sigma_{n-\eta}$. Note that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n-\eta}$. Then

$$\frac{\sum_{i=1}^{k}\sigma_i}{k} \geq \frac{\sum_{i=k+1}^{n-\eta}\sigma_i}{n-\eta-k} = \frac{\mathcal{E} - \sum_{i=1}^{k}\sigma_i}{n-\eta-k},$$

that is,

$$\sum_{i=1}^{k}\sigma_i \geq \frac{k\mathcal{E}}{n-\eta}. \tag{8}$$

By (6) and (7), we have

$$\begin{aligned}\mathcal{E}_\alpha(G) = \sum_{i=1}^{n-\eta}\sigma_i^\alpha &= \sum_{i=1}^{k}\sigma_i^\alpha + \sum_{i=k+1}^{n-\eta}\sigma_i^\alpha \\ &\leq k^{1-\alpha}\left(\sum_{i=1}^{k}\sigma_i\right)^\alpha + (n-\eta-k)^{1-\alpha}\left(\mathcal{E} - \sum_{i=1}^{k}\sigma_i\right)^\alpha.\end{aligned}$$

Let $g(x) = k^{1-\alpha}x^\alpha + (n-\eta-k)^{1-\alpha}(\mathcal{E}-x)^\alpha$, $x \geq \frac{k\mathcal{E}}{n-\eta}$. Then we have

$$g'(x) = \alpha\left[\left(\frac{x}{k}\right)^{\alpha-1} - \left(\frac{\mathcal{E}-x}{n-\eta-k}\right)^{\alpha-1}\right] \leq 0,$$

for $0 < \alpha < 1$ and $x \geq \frac{k\mathcal{E}}{n-\eta}$. Thus $g(x)$ is a decreasing function on $\left[\frac{k\mathcal{E}}{n-\eta}, +\infty\right)$, and

$$g(x) \leq g\left(\frac{k\mathcal{E}}{n-\eta}\right) = k^{1-\alpha}\left(\frac{k\mathcal{E}}{n-\eta}\right)^{\alpha} + (n-\eta-k)^{1-\alpha}\left(\mathcal{E} - \frac{k\mathcal{E}}{n-\eta}\right)^{\alpha}.$$

Combining inequalities (6)-(8), we know that the equality holds in (3) if and only if $\sigma_1 = \sigma_2 = \cdots = \sigma_{n-\eta}$, that is, $G \cong nK_1$.

(ii) Since $\alpha > 1$, using power mean inequality, from (i), we have

$$\mathcal{E}_\alpha(G) \geq k^{1-\alpha}\left(\sum_{i=1}^{k}\sigma_i\right)^{\alpha} + (n-\eta-k)^{1-\alpha}\left(\mathcal{E} - \sum_{i=1}^{k}\sigma_i\right)^{\alpha}.$$

Also $g(x)$ is an increasing function on $\left[\frac{k\mathcal{E}}{n-\eta}, +\infty\right)$. Using the same technique as in (i), we get the required result in (4). Moreover, the equality holds in (4) if and only if $G \cong nK_1$.

(iii) By Lemma 3, we have $\sum_{i=1}^{k}\sigma_i \leq \frac{1}{2}(1+\sqrt{k})n$. Since $\alpha < 0$, from (i), we know that $g(x)$ is an increasing function on $\left[\frac{k\mathcal{E}}{n-\eta}, \frac{1}{2}(1+\sqrt{k})n\right]$. Thus we have

$$g(x) \leq g\left(\frac{1}{2}(1+\sqrt{k})n\right) = k^{1-\alpha}\left(\frac{1}{2}(1+\sqrt{k})n\right)^{\alpha} + (n-\eta-k)^{1-\alpha}\left(\mathcal{E} - \frac{1}{2}(1+\sqrt{k})n\right)^{\alpha},$$

with equality if and only if $\sigma_1 = \cdots = \sigma_k = \frac{1}{2k}(1+\sqrt{k})n$, $\sigma_{k+1} = \cdots = \sigma_{n-\eta}$. This completes the proof. □

Remark 4. For positive integer values of α, inequality (4) provides a relation between the energy and the spectral moments.

According to [24], the adjacency matrix of any graph possessing a unique perfect matching is necessarily nonsingular. Therefore, we obtain the following result.

Theorem 5. *Let G be a graph with n vertices and m edges. If $A(G)$ is non-singular, then*

$$\mathcal{E}_\alpha(G) \geq \left(\frac{2m}{n}\right)^{\alpha} + (n-1)\left(\frac{n|\det(A(G))|}{2m}\right)^{\frac{\alpha}{n-1}} \tag{9}$$

with equality if and only if either $G \cong (n/2)K_2$, or $G \cong K_n$, or G is a design graph.

Proof. Since $A(G)$ is non-singular, we have $\eta = 0$. Thus $\prod_{i=1}^{n}\sigma_i = |\det(A(G))|$. By the arithmetic-geometric mean inequality, we have

$$\begin{aligned}\mathcal{E}_\alpha(G) =& \sigma_1^\alpha + \sum_{i=2}^{n}\sigma_i^\alpha \\ \geq& \sigma_1^\alpha + (n-1)\left(\prod_{i=2}^{n}\sigma_i^\alpha\right)^{\frac{1}{n-1}} \\ =& \sigma_1^\alpha + (n-1)\left(\frac{|\det(A(G))|}{\sigma_1}\right)^{\frac{\alpha}{n-1}},\end{aligned}$$

with equality if and only if $\sigma_2 = \cdots = \sigma_n$. Let $h(x) = x^\alpha + (n-1)\left(\frac{|\det(A(G))|}{x}\right)^{\frac{\alpha}{n-1}}$. Then we have

$$h'(x) = \alpha\left[x^{\alpha-1} - (|\det(A(G))|)^{\frac{\alpha}{n-1}}x^{-\frac{\alpha}{n-1}-1}\right] \geq 0,$$

for $x \geq (|\det(A(G))|)^{\frac{1}{n}}$ whether $\alpha \neq 0$. Thus $h(x)$ is a decreasing function on $\left[(|\det(A(G))|)^{\frac{1}{n}}, +\infty\right)$. By Lemma 2,

$$\sigma_1(G) = \lambda_1(G) \geq \frac{2m}{n} \geq \sqrt{\frac{2m}{n}} \geq \frac{\mathcal{E}(G)}{n} \geq ((|\det(A(G))|)^{\frac{1}{n}},$$

and then

$$\mathcal{E}_\alpha(G) \geq h\left(\frac{2m}{n}\right) = \left(\frac{2m}{n}\right)^\alpha + (n-1)\left(\frac{n|\det(A(G))|}{2m}\right)^{\frac{\alpha}{n-1}},$$

with equality if and only if $\sigma_2 = \cdots = \sigma_n$, and G is regular. From the proof of Theorem 1 in [25], the equality holds in (9) if and only if either $G \cong (n/2)K_2$, or $G \cong K_n$, or G is a design graph. This completes the proof. □

Remark 5. For positive integer values of α, inequality (9) provides a lower bound for the spectral moments.

4. Concluding remarks

It is well known that the sum of the inverses of all nonzero eigenvalues of the Laplacian matrix defines the Kirchhoff index [26], while the sum of their squared inverses corresponds to the biharmonic index [27]. Both indices hold significant application value in fields such as mathematical chemistry and network science [28–31]. Inspired by this, we naturally consider: does the sum of negative powers of all nonzero singular values of the adjacency matrix also possess corresponding mathematical or practical significance? In what areas might its potential applications lie?

Acknowledgments: We would like to thank the anonymous referees for reading the document very carefully and for many helpful comments and suggestions. This work is supported by National Natural Science Foundation of China (No. 12261074).

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