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First degcity index of some graph operations

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Abstract: The first degcity index $\mathrm{DC}_1(G)$ of a connected graph G is the edge sum

$$\mathrm{DC}_1(G) = \sum_{uv \in E(G)} \left[e_G(u) + e_G(v)\right] \left[d_G(u) + d_G(v)\right],$$

where $d_G(u)$ and $e_G(u)$ denote the degree and eccentricity of a vertex u, respectively. The index combines local valency and global distance information in a single degree–eccentricity descriptor. This paper determines closed expressions for the first degcity index under six standard graph operations: disjoint union, join, Cartesian product, composition, symmetric difference and disjunction. The formulas separate the contributions of edges inherited from the factor graphs from the contributions created by the operation. The statements use the eccentricity behaviour in joins and the edge and degree relations in product-type operations, giving formulas that are consistent with the usual definitions of these graph operations.

Keywords: graph operation, topological index, first Zagreb index, eccentric connectivity index, degcity index

MSC: 05C07, 05C09, 05C10, 05C12, 05C76.

1. Introduction

Let $G = (V, E)$ be a finite, simple and connected graph with n vertices and m edges. The degree of a vertex $u \in V(G)$ is denoted by $d_G(u)$, and its eccentricity is

$$e_G(u) = \max\{\mathrm{dist}_G(u, v) : v \in V(G)\},$$

where $\mathrm{dist}_G(u, v)$ is the length of a shortest u–v path in G. Standard graph-theoretic terminology follows Harary [1].

Degree-based and distance-based topological indices are useful because they compress structural information into algebraic quantities that can be compared across graph families. The first Zagreb index, introduced by Gutman and Trinajstić [2], is

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2. \tag{1}$$

Došlić et al. [3] observed the equivalent edge representation

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 = \sum_{uv \in E(G)} \left[d_G(u) + d_G(v)\right]. \tag{2}$$

The eccentric connectivity index of Sharma et al. [4] is

$$\xi^c(G) = \sum_{u \in V(G)} d_G(u) e_G(u) = \sum_{uv \in E(G)} \left[e_G(u) + e_G(v)\right], \tag{3}$$

and the total eccentricity index is

$$\zeta(G) = \sum_{u \in V(G)} e_G(u). \tag{4}$$

The first degcity index [5] is defined by

$$\mathrm{DC}_1(G) = \sum_{uv \in E(G)} \left[e_G(u) + e_G(v)\right]\left[d_G(u) + d_G(v)\right]. \tag{5}$$

Unlike an index based only on degrees or only on distances, $\mathrm{DC}_1(G)$ assigns a larger contribution to an edge when its end vertices are simultaneously high-degree and distant from some part of the graph. Closed formulas for graph operations therefore clarify how this combined local–global descriptor changes when graphs are assembled into larger graphs.

The main contribution of this paper is a collection of operation formulas for DC_1 for disjoint union, join, Cartesian product, composition, symmetric difference and disjunction. The formulas are expressed in terms of M_1, ξ^c, ζ, DC_1, and the orders and sizes of the factor graphs. This gives a direct way to compute DC_1 for compound graphs without recomputing all vertex degrees and eccentricities in the resulting graph.

Throughout the paper, G_1 and G_2 have disjoint vertex sets $V_1 = V(G_1)$ and $V_2 = V(G_2)$ and edge sets $E_1 = E(G_1)$ and $E_2 = E(G_2)$, respectively. Let $|V_1| = n_1$, $|V_2| = n_2$, $|E_1| = m_1$ and $|E_2| = m_2$. Unless otherwise specified, each graph under consideration is connected.

Definition 1 (Union)**.** The union of G_1 and G_2 is denoted by $G_1 \cup G_2$, with

$$V(G_1 \cup G_2) = V_1 \cup V_2, \qquad E(G_1 \cup G_2) = E_1 \cup E_2.$$

Thus $|V(G_1 \cup G_2)| = n_1 + n_2$ and $|E(G_1 \cup G_2)| = m_1 + m_2$. Since the disjoint union is not connected when both factors are nonempty, the degcity index of $G_1 \cup G_2$ is understood componentwise. Hence

$$d_{G_1 \cup G_2}(u) = \begin{cases} d_{G_1}(u), & u \in V_1, \\ d_{G_2}(u), & u \in V_2, \end{cases} \qquad e_{G_1 \cup G_2}(u) = \begin{cases} e_{G_1}(u), & u \in V_1, \\ e_{G_2}(u), & u \in V_2. \end{cases}$$

Definition 2 (Join)**.** The join of G_1 and G_2, denoted by $G_1 + G_2$, is obtained from $G_1 \cup G_2$ by adding all edges between V_1 and V_2. Thus

$$V(G_1 + G_2) = V_1 \cup V_2,$$
$$E(G_1 + G_2) = E_1 \cup E_2 \cup \{ab : a \in V_1,\ b \in V_2\},$$
$$|V(G_1 + G_2)| = n_1 + n_2, \qquad |E(G_1 + G_2)| = m_1 + m_2 + n_1 n_2,$$

and

$$d_{G_1 + G_2}(u) = \begin{cases} d_{G_1}(u) + n_2, & u \in V_1, \\ d_{G_2}(u) + n_1, & u \in V_2. \end{cases}$$

The eccentricity in a join is not inherited unchanged from the factor graph. For $u \in V_i$,

$$e_{G_1 + G_2}(u) = \begin{cases} 1, & d_{G_i}(u) = n_i - 1, \\ 2, & d_{G_i}(u) < n_i - 1. \end{cases}$$

This distinction is needed because the added cross edges reduce all distances in the join to at most two.

Definition 3 (Cartesian product)**.** The Cartesian product $G_1 \times G_2$ has vertex set $V_1 \times V_2$. For vertices (a, c) and (b, d) in this set,

$$E(G_1 \times G_2) = \{(a, c)(b, d) : ab \in E_1,\ c = d \text{ or } cd \in E_2,\ a = b\}.$$

For connected factors, as in [3,6],

$$d_{G_1 \times G_2}(a, c) = d_{G_1}(a) + d_{G_2}(c), \qquad e_{G_1 \times G_2}(a, c) = e_{G_1}(a) + e_{G_2}(c),$$

and

$$|E(G_1 \times G_2)| = n_1 m_2 + n_2 m_1, \qquad |V(G_1 \times G_2)| = n_1 n_2.$$

A vertex $v \in V(G)$ is called well connected [6] if $d_G(v) = |V(G)| - 1$; equivalently, in a nontrivial connected graph, $e_G(v) = 1$.

Definition 4 (Composition)**.** Let G_1 and G_2 be graphs such that G_1 has no well connected vertices. The composition, or lexicographic product, of G_1 and G_2 is denoted by $G_1 \circ G_2$ or $G_1[G_2]$. It has vertex set $V_1 \times V_2$ and edge set

$$E(G_1 \circ G_2) = \{(a,c)(b,d) : ab \in E_1,\ c,d \in V_2\} \cup \{(a,c)(a,d) : cd \in E_2,\ a \in V_1\}.$$

Furthermore,

$$d_{G_1 \circ G_2}(u,v) = n_2 d_{G_1}(u) + d_{G_2}(v), \qquad e_{G_1 \circ G_2}(u,v) = e_{G_1}(u),$$

and

$$|E(G_1 \circ G_2)| = n_2^2 m_1 + n_1 m_2, \qquad |V(G_1 \circ G_2)| = n_1 n_2.$$

Definition 5 (Symmetric difference)**.** Let G_1 and G_2 be graphs without well connected vertices. The symmetric difference of G_1 and G_2, denoted by $G_1 \oplus G_2$, has vertex set $V_1 \times V_2$ and edge set

$$E(G_1 \oplus G_2) = \{(a,c)(b,d) : \text{exactly one of } ab \in E_1 \text{ and } cd \in E_2 \text{ holds}\}.$$

For $(u,v) \in V_1 \times V_2$,

$$d_{G_1 \oplus G_2}(u,v) = n_2 d_{G_1}(u) + n_1 d_{G_2}(v) - 2 d_{G_1}(u) d_{G_2}(v), \qquad e_{G_1 \oplus G_2}(u,v) = 2,$$

and

$$|E(G_1 \oplus G_2)| = n_1^2 m_2 + n_2^2 m_1 - 4 m_1 m_2, \qquad |V(G_1 \oplus G_2)| = n_1 n_2.$$

Definition 6 (Disjunction)**.** Let G_1 and G_2 be graphs without well connected vertices. The disjunction of G_1 and G_2, denoted by $G_1 \vee G_2$, has vertex set $V_1 \times V_2$ and edge set

$$E(G_1 \vee G_2) = \{(a,c)(b,d) : ab \in E_1 \text{ or } cd \in E_2\}.$$

For $(u,v) \in V_1 \times V_2$,

$$d_{G_1 \vee G_2}(u,v) = n_2 d_{G_1}(u) + n_1 d_{G_2}(v) - d_{G_1}(u) d_{G_2}(v), \qquad e_{G_1 \vee G_2}(u,v) = 2,$$

and

$$|E(G_1 \vee G_2)| = n_1^2 m_2 + n_2^2 m_1 - 2 m_1 m_2, \qquad |V(G_1 \vee G_2)| = n_1 n_2.$$

2. Main Results

Theorem 1. *For connected graphs G_1 and G_2, the componentwise degcity index of their disjoint union satisfies*

$$\mathrm{DC}_1(G_1 \cup G_2) = \mathrm{DC}_1(G_1) + \mathrm{DC}_1(G_2).$$

Proof. Since the edge set of $G_1 \cup G_2$ is the disjoint union of E_1 and E_2, and since degrees and eccentricities are evaluated within the corresponding component,

$$\begin{aligned} \mathrm{DC}_1(G_1 \cup G_2) &= \sum_{ab \in E(G_1 \cup G_2)} [d(a) + d(b)]\,[e(a) + e(b)] \\ &= \sum_{ab \in E_1} [d_{G_1}(a) + d_{G_1}(b)]\,[e_{G_1}(a) + e_{G_1}(b)] + \sum_{ab \in E_2} [d_{G_2}(a) + d_{G_2}(b)]\,[e_{G_2}(a) + e_{G_2}(b)] \\ &= \mathrm{DC}_1(G_1) + \mathrm{DC}_1(G_2). \end{aligned}$$

□

The formula shows that no interaction term is created by a disjoint union. The index is additive because every edge remains inside its original component.

Theorem 2. *Let G_1 and G_2 be connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. For $u \in V_i$, put*

$$\varepsilon_i(u) = e_{G_1+G_2}(u) = \begin{cases} 1, & d_{G_i}(u) = n_i - 1, \\ 2, & d_{G_i}(u) < n_i - 1, \end{cases}$$

and define

$$\widehat{\zeta}_i = \sum_{u \in V_i} \varepsilon_i(u), \qquad \widehat{\xi}_i = \sum_{u \in V_i} d_{G_i}(u)\varepsilon_i(u),$$

$$\widehat{D}_i = \sum_{uv \in E_i} \left[\varepsilon_i(u) + \varepsilon_i(v)\right]\left[d_{G_i}(u) + d_{G_i}(v)\right].$$

Then

$$\mathrm{DC}_1(G_1 + G_2) = \widehat{D}_1 + \widehat{D}_2 + 3n_2\widehat{\xi}_1 + 3n_1\widehat{\xi}_2 + \left(n_1n_2 + n_2^2 + 2m_2\right)\widehat{\zeta}_1 + \left(n_1^2 + n_1n_2 + 2m_1\right)\widehat{\zeta}_2.$$

Proof. The edge set of $G_1 + G_2$ consists of the edges of G_1, the edges of G_2, and the n_1n_2 edges between V_1 and V_2. For $ab \in E_1$, the degree sum in the join is $d_{G_1}(a) + d_{G_1}(b) + 2n_2$, while the eccentricity sum is $\varepsilon_1(a) + \varepsilon_1(b)$. Hence these edges contribute

$$\widehat{D}_1 + 2n_2\widehat{\xi}_1.$$

Similarly, the edges of G_2 contribute

$$\widehat{D}_2 + 2n_1\widehat{\xi}_2.$$

For the cross edges ab with $a \in V_1$ and $b \in V_2$, the contribution is

$$\begin{aligned} S &= \sum_{a \in V_1} \sum_{b \in V_2} \left[d_{G_1}(a) + n_2 + d_{G_2}(b) + n_1\right]\left[\varepsilon_1(a) + \varepsilon_2(b)\right] \\ &= n_2\widehat{\xi}_1 + n_1\widehat{\xi}_2 + 2m_2\widehat{\zeta}_1 + 2m_1\widehat{\zeta}_2 + (n_1 + n_2)n_2\widehat{\zeta}_1 + (n_1 + n_2)n_1\widehat{\zeta}_2. \end{aligned}$$

Adding the three contributions gives the stated identity. □

The join formula records the effect of the added complete bipartite part. Vertices that are already universal in a factor have eccentricity one in the join, while all other vertices have eccentricity two; this is why the quantities with hats are required.

Theorem 3. *For connected graphs G_1 and G_2 with n_1, n_2 vertices and m_1, m_2 edges, respectively,*

$$\mathrm{DC}_1(G_1 \times G_2) = n_2\,\mathrm{DC}_1(G_1) + n_1\,\mathrm{DC}_1(G_2) + 2M_1(G_1)\zeta(G_2) + 2M_1(G_2)\zeta(G_1) + 8m_2\xi^c(G_1) + 8m_1\xi^c(G_2).$$

Proof. Edges of $G_1 \times G_2$ are of two types. If $cd \in E_2$ and $a \in V_1$, then the edge $(a,c)(a,d)$ contributes

$$\left[2d_{G_1}(a) + d_{G_2}(c) + d_{G_2}(d)\right]\left[2e_{G_1}(a) + e_{G_2}(c) + e_{G_2}(d)\right].$$

Summing over all $a \in V_1$ and $cd \in E_2$ gives

$$4m_2\xi^c(G_1) + 4m_1\xi^c(G_2) + 2M_1(G_2)\zeta(G_1) + n_1\,\mathrm{DC}_1(G_2).$$

If $ab \in E_1$ and $c \in V_2$, then the edge $(a,c)(b,c)$ contributes

$$\left[d_{G_1}(a) + d_{G_1}(b) + 2d_{G_2}(c)\right]\left[e_{G_1}(a) + e_{G_1}(b) + 2e_{G_2}(c)\right].$$

Summing over all $ab \in E_1$ and $c \in V_2$ gives

$$n_2\,\mathrm{DC}_1(G_1) + 2M_1(G_1)\zeta(G_2) + 4m_2\xi^c(G_1) + 4m_1\xi^c(G_2).$$

The sum of these two edge-type contributions is the asserted expression. □

The Cartesian product keeps both factor directions visible. The terms $n_2\,\mathrm{DC}_1(G_1)$ and $n_1\,\mathrm{DC}_1(G_2)$ arise from copies of the factor edges, whereas the remaining four terms measure the coupling between degrees from one factor and eccentricities from the other.

Theorem 4. *Let G_1 and G_2 be graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. If G_1 has no well connected vertices, then*

$$\mathrm{DC}_1(G_1 \circ G_2) = n_2^3\,\mathrm{DC}_1(G_1) + 8n_2m_2\xi^c(G_1) + 2M_1(G_2)\zeta(G_1).$$

Proof. In the composition $G_1 \circ G_2$, an edge either joins two different G_2-fibres over an edge of G_1 or lies inside one copy of G_2. For $ab \in E_1$ and $c, d \in V_2$, the edge $(a,c)(b,d)$ contributes

$$\left[n_2 d_{G_1}(a) + d_{G_2}(c) + n_2 d_{G_1}(b) + d_{G_2}(d)\right]\left[e_{G_1}(a) + e_{G_1}(b)\right].$$

The sum over all such edges is

$$n_2^3\,\mathrm{DC}_1(G_1) + 4n_2m_2\xi^c(G_1).$$

For $a \in V_1$ and $cd \in E_2$, the edge $(a,c)(a,d)$ contributes

$$\left[2n_2 d_{G_1}(a) + d_{G_2}(c) + d_{G_2}(d)\right] 2e_{G_1}(a).$$

The sum over these edges is

$$4n_2m_2\xi^c(G_1) + 2M_1(G_2)\zeta(G_1).$$

Combining the two sums proves the result. □

The composition formula is asymmetric, as expected from the definition of the lexicographic product. The eccentricity is governed by G_1, while the internal edges of the G_2-fibres enter through m_2 and $M_1(G_2)$.

Theorem 5. *Let G_1 and G_2 be connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. If neither graph has a well connected vertex, then*

$$\mathrm{DC}_1(G_1 \oplus G_2) = 4M_1(G_1)\left(n_2^3 - 8n_2m_2\right) + 4M_1(G_2)\left(n_1^3 - 8n_1m_1\right) + 16M_1(G_1)M_1(G_2) + 32n_1n_2m_1m_2.$$

Proof. For $H = G_1 \oplus G_2$, every vertex has eccentricity 2 under the stated hypothesis. Therefore

$$\mathrm{DC}_1(H) = 4M_1(H) = 4\sum_{(u,v)\in V_1\times V_2} d_H(u,v)^2.$$

Using

$$d_H(u,v) = n_2 d_{G_1}(u) + n_1 d_{G_2}(v) - 2d_{G_1}(u)d_{G_2}(v),$$

we obtain

$$M_1(H) = n_2^3M_1(G_1) + n_1^3M_1(G_2) + 4M_1(G_1)M_1(G_2) + 8n_1n_2m_1m_2 - 8n_2m_2M_1(G_1) - 8n_1m_1M_1(G_2).$$

Multiplication by 4 yields the stated expression. □

The symmetric-difference expression depends only on the first Zagreb indices and the orders and sizes of the factors. The constant eccentricity equal to two collapses the degcity computation to a Zagreb-index computation for the product graph.

Theorem 6. *Let G_1 and G_2 be connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. If neither graph has a well connected vertex, then*

$$\mathrm{DC}_1(G_1 \vee G_2) = 4M_1(G_1)\big(n_2^3 - 4n_2m_2\big) + 4M_1(G_2)\big(n_1^3 - 4n_1m_1\big) + 4M_1(G_1)M_1(G_2) + 32n_1n_2m_1m_2.$$

Proof. Let $H = G_1 \vee G_2$. As in the previous theorem, all eccentricities in H are equal to 2, and hence $\mathrm{DC}_1(H) = 4M_1(H)$. The vertex degree in the disjunction is

$$d_H(u,v) = n_2 d_{G_1}(u) + n_1 d_{G_2}(v) - d_{G_1}(u) d_{G_2}(v).$$

Expanding $\sum_{(u,v)\in V_1 \times V_2} d_H(u,v)^2$ gives

$$M_1(H) = n_2^3 M_1(G_1) + n_1^3 M_1(G_2) + M_1(G_1)M_1(G_2) + 8n_1n_2m_1m_2 - 4n_2m_2M_1(G_1) - 4n_1m_1M_1(G_2).$$

Multiplying by 4 completes the proof. □

The disjunction differs from the symmetric difference only in the treatment of pairs for which both factor adjacencies occur. This changes the coefficient of the mixed Zagreb term and halves the coefficient attached to m_1 and m_2.

3. Conclusion

This paper has determined closed formulas for the first degcity index under six graph operations. The disjoint union is additive, while the join requires the eccentricities induced by the joined graph rather than the original eccentricities of the factors. For the Cartesian product, the formula separates inherited edge contributions from degree–eccentricity interactions between the two factors. For the composition, the first factor controls eccentricity and the second factor contributes through its size and first Zagreb index. For the symmetric difference and disjunction, the absence of well connected vertices forces all eccentricities in the resulting graph to be two, reducing the computation of DC_1 to the first Zagreb index of the corresponding product graph.

The resulting identities provide direct computation rules for compound graphs and clarify how local degree information and global eccentricity information combine under standard graph operations. They also identify the conditions under which simplified expressions are valid, preventing the use of factor eccentricities in situations where the operation changes all relevant distances.

Conflicts of Interest: "The authors declare no conflict of interest."

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