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Multiple sums involving the terms of a general second order sequence of numbers

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Abstract: Closed forms are derived for nested finite sums of the form

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}x^{a_0},$$

where a_n and c are integers and x is real or complex. This elementary identity is then used to evaluate multiple sums whose summands contain terms of the Horadam sequence $\big(W_j(a,b;p,q)\big)$. The sequence is defined by

$$W_0=a,\qquad W_1=b,\qquad W_j=pW_{j-1}-qW_{j-2}\quad (j\geq 2),$$

where $a,b,p,q\in\mathbb{C}$ with $p\neq 0$ and $q\neq 0$. The resulting identities include weighted sums involving Lucas sequences of the first and second kinds, Fibonacci and Lucas numbers, gibonacci sequences, and products of two and three shifted terms. The formulas show how the depth of summation is absorbed into binomial coefficients and shifted sequence indices, yielding compact expressions suitable for direct use in recurrence and summation problems.

Keywords: Horadam sequence, Fibonacci number, Lucas number, Lucas sequence, summation identity, nested sum, multiple sum

MSC: Primary 11B39; Secondary 11B37.

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1. Introduction

Let F_j denote the jth Fibonacci number. Ivie [1] proved the identities

$$\sum_{s=1}^{m}\sum_{r=1}^{s}F_r=F_{m+4}-F_4-m,$$
$$\sum_{m=1}^{n}\sum_{s=1}^{m}\sum_{r=1}^{s}F_r=F_{n+6}-F_6-nF_4-\frac{n(n+1)}{2},$$

and, more generally,

$$\sum_{a_{n-1}=1}^{a_n}\sum_{a_{n-2}=1}^{a_{n-1}}\cdots\sum_{a_0=1}^{a_1}F_{a_0}=F_{a_n+2n}-\sum_{j=0}^{n-1}F_{2(n-j)}\binom{a_n+j-1}{j}. \tag{1}$$

The purpose of this work is to place identities of this type in a general second-order recurrence setting and to derive closed forms in which the lower summation limit may be any integer c. The central observation is that a nested geometric sum can be evaluated before specializing the base to quantities generated by a recurrence. This separates the combinatorial part of the problem from the recurrence-theoretic part: the former contributes binomial coefficients, while the latter contributes shifted Horadam, Lucas, Fibonacci, or gibonacci terms.

A representative identity obtained below is

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{W_{ra_0+s}}{V_r^{a_0}}=(-1)^n\frac{W_{r(a_n+2n)+s}}{q^{rn}V_r^{a_n}}-\frac{1}{V_r^{c-1}}\sum_{j=0}^{n-1}(-1)^{n-j}\frac{W_{r(2n-2j+c-1)+s}}{q^{r(n-j)}}\binom{a_n+j-c}{j}. \tag{2}$$

Formula (1) follows as a particular case. Thus the method gives a unified derivation of identities for a broad class of second-order sequences rather than isolated Fibonacci sums.

Throughout the paper, $\big(W_j(a,b;p,q)\big)$ is the Horadam sequence [2], defined for non-negative integers j by

$$W_0=a,\qquad W_1=b,\qquad W_j=pW_{j-1}-qW_{j-2}\quad (j\geq 2), \tag{3}$$

where $a,b,p,q\in\mathbb{C}$, $p\neq 0$, and $q\neq 0$. Two important special cases are the Lucas sequences of the first and second kinds,

$$\big(U_j(p,q)\big)=\big(W_j(0,1;p,q)\big),\qquad \big(V_j(p,q)\big)=\big(W_j(2,p;p,q)\big),$$

so that

$$U_0=0,\quad U_1=1,\quad U_j=pU_{j-1}-qU_{j-2}\quad (j\geq 2),$$

and

$$V_0=2,\quad V_1=p,\quad V_j=pV_{j-1}-qV_{j-2}\quad (j\geq 2).$$

The Fibonacci and Lucas numbers are recovered as $(F_j)=\big(U_j(1,-1)\big)$ and $(L_j)=\big(V_j(1,-1)\big)$.

The case $p=1$ gives the restricted Horadam sequence

$$\big(w_j^*(a,b;q)\big)=\big(W_j(a,b;1,q)\big),$$

with corresponding Lucas sequences $(u_j^*(q))=\big(U_j(1,q)\big)$ and $(v_j^*(q))=\big(V_j(1,q)\big)$. The particular choice $q=-1$ gives the gibonacci sequence

$$(G_j(a,b))=\big(w_j^*(a,b;-1)\big),\qquad G_0=a,\quad G_1=b,\quad G_j=G_{j-1}+G_{j-2}\quad (j\geq 2).$$

Another useful specialization is $q=-1$, for which

$$(w_j(a,b;p))=\big(W_j(a,b;p,-1)\big),$$

with associated Lucas sequences $(u_j(p))=\big(U_j(p,-1)\big)$ and $(v_j(p))=\big(V_j(p,-1)\big)$. In particular, $(G_j(a,b))=(w_j(a,b;1))$.

Let

$$\delta=\sqrt{p^2-4q},\qquad \tau=\frac{p+\delta}{2},\qquad \sigma=\frac{p-\delta}{2}.$$

In the non-degenerate case $\delta\neq 0$, the Binet formulas are

$$U_j=\frac{\tau^j-\sigma^j}{\tau-\sigma}=\frac{\tau^j-\sigma^j}{\delta},\qquad V_j=\tau^j+\sigma^j,\qquad W_j=A\tau^j+B\sigma^j, \tag{4}$$

where

$$A=\frac{b-a\sigma}{\tau-\sigma},\qquad B=\frac{a\tau-b}{\tau-\sigma}.$$

Here τ and σ are the roots of x^2-px+q, so that $\tau\sigma=q$ and $\tau+\sigma=p$. For the Fibonacci and Lucas numbers,

$$F_j=\frac{\alpha^j-\beta^j}{\alpha-\beta}=\frac{\alpha^j-\beta^j}{\sqrt{5}},\qquad L_j=\alpha^j+\beta^j, \tag{5}$$

where $\alpha=(1+\sqrt{5})/2$ and $\beta=(1-\sqrt{5})/2=-1/\alpha$. Negative subscripts are interpreted through the recurrence, equivalently by writing

$$W_{-n}=\frac{pW_{-n+1}-W_{-n+2}}{q}.$$

This convention keeps the identities valid when shifted indices become negative.

2. Preliminary results

Let x be a real or complex variable, let a_n be an integer, and let n be a positive integer. The fundamental object is

$$\sum_{a_{n-1}=1}^{a_n}\sum_{a_{n-2}=1}^{a_{n-1}}\cdots\sum_{a_0=1}^{a_1}x^{a_0},$$

which will later be specialized by choosing x as a function of the roots of a second-order recurrence.

The geometric sum

$$\sum_{k=1}^{m}x^k=\frac{x^{m+1}-x}{x-1},$$

may be written as

$$\frac{x-1}{x}\sum_{j=1}^{m}x^j=x^m-1. \tag{6}$$

For $m=a_1$ this gives

$$\frac{x-1}{x}\sum_{a_0=1}^{a_1}x^{a_0}=x^{a_1}-1. \tag{7}$$

Iterating this identity through the successive summation levels produces the binomial coefficients that appear in all later formulas. The following lemma records the required counting identities.

Lemma 1. *Let k, m and b_s be non-negative integers and let s be a positive integer. Then*

$$\sum_{j=1}^{m}\binom{j+k-1}{k}=\binom{k+m}{k+1}, \tag{8}$$

and

$$\sum_{b_{s-1}=1}^{b_s}\sum_{b_{s-2}=1}^{b_{s-1}}\cdots\sum_{b_0=1}^{b_1}1=\binom{b_s+s-1}{s}. \tag{9}$$

Proof. Identity (8) follows by induction on m with k fixed. The induction step is Pascal's identity,

$$\binom{k+r}{k+1}+\binom{k+r}{k}=\binom{k+r+1}{k+1}.$$

For (9), the case $s=1$ is immediate because $\sum_{b_0=1}^{b_1}1=b_1=\binom{b_1}{1}$. If the result holds for $s=k$, then

$$\sum_{b_k=1}^{b_{k+1}}\sum_{b_{k-1}=1}^{b_k}\cdots\sum_{b_0=1}^{b_1}1=\sum_{b_k=1}^{b_{k+1}}\binom{b_k+k-1}{k}=\binom{b_{k+1}+k}{k+1},$$

by (8). This proves the assertion for every positive s. □

Using Lemma 1, the iteration of (7) yields

$$\left(\frac{x-1}{x}\right)^n\sum_{a_{n-1}=1}^{a_n}\sum_{a_{n-2}=1}^{a_{n-1}}\cdots\sum_{a_0=1}^{a_1}x^{a_0}=x^{a_n}-\sum_{j=0}^{n-1}\left(\frac{x-1}{x}\right)^j\binom{a_n+j-1}{j}. \tag{10}$$

The next lemma gives a formal proof and establishes the form used later.

Lemma 2. *Let n be a positive integer and let x be real or complex. Then identity (10) holds.*

Proof. The case $n = 1$ is exactly (7). Assume that the identity is true for $n = k$:

$$\left(\frac{x-1}{x}\right)^k \sum_{a_{k-1}=1}^{a_k} \sum_{a_{k-2}=1}^{a_{k-1}} \cdots \sum_{a_0=1}^{a_1} x^{a_0} = x^{a_k} - \sum_{j=0}^{k-1} \left(\frac{x-1}{x}\right)^j \binom{a_k+j-1}{j}.$$

Multiplying by $(x-1)/x$ and summing over a_k gives

$$\left(\frac{x-1}{x}\right)^{k+1} \sum_{a_k=1}^{a_{k+1}} \sum_{a_{k-1}=1}^{a_k} \cdots \sum_{a_0=1}^{a_1} x^{a_0} = \frac{x-1}{x} \sum_{a_k=1}^{a_{k+1}} x^{a_k} - \sum_{j=0}^{k-1} \left(\frac{x-1}{x}\right)^{j+1} \sum_{a_k=1}^{a_{k+1}} \binom{a_k+j-1}{j}. \tag{11}$$

The first sum is

$$\frac{x-1}{x} \sum_{a_k=1}^{a_{k+1}} x^{a_k} = x^{a_{k+1}} - 1, \tag{12}$$

and Lemma 1 gives

$$\sum_{j=0}^{k-1} \left(\frac{x-1}{x}\right)^{j+1} \sum_{a_k=1}^{a_{k+1}} \binom{a_k+j-1}{j} = \sum_{j=0}^{k} \left(\frac{x-1}{x}\right)^j \binom{a_{k+1}+j-1}{j} - 1. \tag{13}$$

Substituting (12) and (13) into (11) proves the result for $k+1$. □

The lower limit in all nested sums can be shifted from 1 to an arbitrary integer c. Replacing each a_i with $a_i - c + 1$ gives

$$\left(\frac{x-1}{x}\right)^n \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} x^{a_0} = x^{a_n} - x^{c-1} \sum_{j=0}^{n-1} \left(\frac{x-1}{x}\right)^j \binom{a_n+j-c}{j}. \tag{14}$$

The corresponding forms of Lemma 1 are

$$\sum_{j=c}^{m} \binom{j-c+k}{k} = \binom{m-c+k+1}{k+1}, \tag{15}$$

and

$$\sum_{b_{s-1}=c}^{b_s} \sum_{b_{s-2}=c}^{b_{s-1}} \cdots \sum_{b_0=c}^{b_1} 1 = \binom{b_s+s-c}{s}. \tag{16}$$

Multiplying (14) by $(x/(x-1))^n$ and substituting x/y and $-x/y$, respectively, gives two useful identities. Define

$$\begin{aligned} f(x,y;a_n,n,c) &= \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} \left(\frac{x}{y}\right)^{a_0} \\ &= \left(\frac{x}{x-y}\right)^n \left(\frac{x}{y}\right)^{a_n} - \sum_{j=0}^{n-1} \left(\frac{x}{x-y}\right)^{n-j} \left(\frac{x}{y}\right)^{c-1} \binom{a_n+j-c}{j}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} g(x,y;a_n,n,c) &= \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} (-1)^{a_0} \left(\frac{x}{y}\right)^{a_0} \\ &= (-1)^{a_n} \left(\frac{x}{x+y}\right)^n \left(\frac{x}{y}\right)^{a_n} + (-1)^c \sum_{j=0}^{n-1} \left(\frac{x}{x+y}\right)^{n-j} \left(\frac{x}{y}\right)^{c-1} \binom{a_n+j-c}{j}. \end{aligned} \tag{18}$$

These two identities are the main computational tool. They show that a nested sum is reduced to a finite single sum whose coefficients depend only on the summation depth and the lower limit.

3. Main results

The identities in this section follow by substituting appropriate functions of τ and σ into (17) and (18), and then simplifying the result with the Binet formulas.

Theorem 1. *Let r, s, c and a_n be integers and let n be a positive integer. Then*

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{W_{ra_0+s}}{V_r^{a_0}}=(-1)^n\frac{W_{r(a_n+2n)+s}}{q^{rn}V_r^{a_n}}-\frac{1}{V_r^{c-1}}\sum_{j=0}^{n-1}(-1)^{n-j}\frac{W_{r(2n-2j+c-1)+s}}{q^{r(n-j)}}\binom{a_n+j-c}{j}. \tag{19}$$

Proof. Use (17) to simplify

$$A\tau^s f(\tau^r, V_r; a_n, n, c)+B\sigma^s f(\sigma^r, V_r; a_n, n, c),$$

and then apply (4), together with $\tau\sigma = q$ and $\tau^r+\sigma^r = V_r$. $\square$

The restricted Horadam cases obtained from (19) are

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{w^*_{ra_0+s}}{(v_r^*)^{a_0}}=(-1)^n\frac{w^*_{r(a_n+2n)+s}}{q^{rn}(v_r^*)^{a_n}}-\frac{1}{(v_r^*)^{c-1}}\sum_{j=0}^{n-1}(-1)^{n-j}\frac{w^*_{r(2n-2j+c-1)+s}}{q^{r(n-j)}}\binom{a_n+j-c}{j},$$

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{w_{ra_0+s}}{v_r^{a_0}}=(-1)^{n(r-1)}\frac{w_{r(a_n+2n)+s}}{v_r^{a_n}}-\frac{1}{v_r^{c-1}}\sum_{j=0}^{n-1}(-1)^{(n-j)(r-1)}w_{r(2n-2j+c-1)+s}\binom{a_n+j-c}{j}.$$

In particular,

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}w^*_{a_0+s}=(-1)^n\frac{w^*_{a_n+2n+s}}{q^n}-\sum_{j=0}^{n-1}(-1)^{n-j}\frac{w^*_{2n-2j+c-1+s}}{q^{n-j}}\binom{a_n+j-c}{j}.$$

For the gibonacci sequence,

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{G_{ra_0+s}}{L_r^{a_0}}=(-1)^{n(r-1)}\frac{G_{r(a_n+2n)+s}}{L_r^{a_n}}-\frac{1}{L_r^{c-1}}\sum_{j=0}^{n-1}(-1)^{(n-j)(r-1)}G_{r(2n-2j+c-1)+s}\binom{a_n+j-c}{j},$$

and the special case $r=1, s=0$ gives

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}G_{a_0}=G_{a_n+2n}-\sum_{j=0}^{n-1}G_{2(n-j)}\binom{a_n+j-c}{j}.$$

Identity (1) is recovered by choosing $c=1$ and $G_j = F_j$. This shows that the Fibonacci identity is not an isolated formula but the first member of a Horadam family.

Theorem 2. *Let r, s, c and a_n be integers and let n be a positive integer. Then*

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{a_0}\frac{W_{2ra_0+s}}{q^{ra_0}}=(-1)^{a_n}\frac{W_{r(2a_n+n)+s}}{q^{ra_n}V_r^n}+\frac{(-1)^c}{q^{r(c-1)}}\sum_{j=0}^{n-1}\frac{W_{r(n-j+2c-2)+s}}{V_r^{n-j}}\binom{a_n+j-c}{j}. \tag{20}$$

Proof. Apply (18) to

$$A\tau^s g(\tau^r, \sigma^r; a_n, n, c)+B\sigma^s g(\sigma^r, \tau^r; a_n, n, c),$$

and use (4). $\square$

For $q=-1$ this becomes

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{(r-1)a_0}w_{2ra_0+s}=(-1)^{(r-1)a_n}\frac{w_{r(2a_n+n)+s}}{v_r^n}+(-1)^{r(c-1)+c}\sum_{j=0}^{n-1}\frac{w_{r(n-j+2c-2)+s}}{v_r^{n-j}}\binom{a_n+j-c}{j}.$$

The gibonacci version is

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{(r-1)a_0}G_{2ra_0+s}=(-1)^{(r-1)a_n}\frac{G_{r(2a_n+n)+s}}{L_r^n}+(-1)^{r(c-1)+c}\sum_{j=0}^{n-1}\frac{G_{r(n-j+2c-2)+s}}{L_r^{n-j}}\binom{a_n+j-c}{j}.$$

In particular,

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}G_{2a_0+s}=G_{2a_n+n+s}-\sum_{j=0}^{n-1}G_{n-j+2c-2+s}\binom{a_n+j-c}{j}.$$

The alternating sign in (20) converts the argument ra_0 into the doubled argument $2ra_0$ and changes the denominator from $V_r^{a_0}$ to q^{ra_0}. This is the algebraic mechanism behind the shorter special forms above.

For the next two theorems we use four standard relations among Lucas sequences.

Lemma 3. *For integers r and d,*

$$U_{r+d}-\tau^rU_d=\sigma^dU_r, \tag{21}$$
$$U_{r+d}-\sigma^rU_d=\tau^dU_r, \tag{22}$$
$$V_{r+d}-\tau^rV_d=-\sigma^dU_r\delta, \tag{23}$$
$$V_{r+d}-\sigma^rV_d=\tau^dU_r\delta. \tag{24}$$

Proof. Each identity follows by direct substitution from (4). For instance,

$$U_{r+d}-\tau^rU_d=\frac{\tau^{r+d}-\sigma^{r+d}}{\tau-\sigma}-\tau^r\frac{\tau^d-\sigma^d}{\tau-\sigma}=\sigma^d\frac{\tau^r-\sigma^r}{\tau-\sigma}=\sigma^dU_r.$$

The other identities are obtained in the same way. □

Theorem 3. *Let r, s, c, d and a_n be integers with $r \neq 0$ and $r + d \neq 0$, and let n be a positive integer. Then*

$$\begin{aligned}\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{U_d}{U_{r+d}}\right)^{a_0}W_{ra_0+s}&=(-1)^n\frac{U_d^{n+a_n}}{q^{dn}U_r^nU_{r+d}^{a_n}}W_{(r+d)n+ra_n+s}\\&-\left(\frac{U_d}{U_{r+d}}\right)^{c-1}\sum_{j=0}^{n-1}(-1)^{n-j}q^{d(n-j)}\left(\frac{U_d}{U_r}\right)^{n-j}W_{r(n-j+c-1)+d(n-j)+s}\binom{a_n+j-c}{j}.\end{aligned} \tag{25}$$

Proof. Use (17) to simplify

$$A\tau^sf(\tau^rU_d,U_{r+d};a_n,n,c)+B\sigma^sf(\sigma^rU_d,U_{r+d};a_n,n,c),$$

and then apply (4), (21), and (22). □

For $q=-1$ the identity becomes

$$\begin{aligned}\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{u_d}{u_{r+d}}\right)^{a_0}w_{ra_0+s}&=(-1)^{n(d+1)}\frac{u_d^{n+a_n}}{u_r^nu_{r+d}^{a_n}}w_{(r+d)n+ra_n+s}\\&-\left(\frac{u_d}{u_{r+d}}\right)^{c-1}\sum_{j=0}^{n-1}(-1)^{(n-j)(d+1)}\left(\frac{u_d}{u_r}\right)^{n-j}w_{r(n-j+c-1)+d(n-j)+s}\binom{a_n+j-c}{j}.\end{aligned}$$

Since $u_{-1}=1$, for $r\neq 0$ and $r\neq 1$,

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{w_{ra_0+s}}{u_{r-1}^{a_0}}=\frac{w_{(r-1)n+ra_n+s}}{u_r^nu_{r-1}^{a_n}}-\frac{1}{u_{r-1}^{c-1}}\sum_{j=0}^{n-1}\frac{w_{r(n-j+c-1)-n+j+s}}{u_r^{n-j}}\binom{a_n+j-c}{j}.$$

A useful special case is

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1} w_{2a_0+s}=\frac{w_{n+2a_n+s}}{p^n}-\sum_{j=0}^{n-1}\frac{w_{n-j+2c-2+s}}{p^{n-j}}\binom{a_n+j-c}{j}.$$

For the gibonacci sequence,

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{F_d}{F_{r+d}}\right)^{a_0} G_{ra_0+s}=(-1)^{n(d+1)}\frac{F_d^{n+a_n}}{F_r^n F_{r+d}^{a_n}}G_{(r+d)n+ra_n+s}$$
$$-\left(\frac{F_d}{F_{r+d}}\right)^{c-1}\sum_{j=0}^{n-1}(-1)^{(n-j)(d+1)}\left(\frac{F_d}{F_r}\right)^{n-j}G_{r(n-j+c-1)+d(n-j)+s}\binom{a_n+j-c}{j}. \tag{26}$$

Choosing $d=-2, r=1$ gives

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{a_0}G_{a_0+s}=(-1)^{a_n}G_{-n+a_n+s}+(-1)^c\sum_{j=0}^{n-1}G_{-n+j+c-1+s}\binom{a_n+j-c}{j}, \tag{27}$$

while $d=-2, r=3$ gives

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{a_0}G_{3a_0+s}=\frac{(-1)^{a_n}}{2^n}G_{n+3a_n+s}+(-1)^c\sum_{j=0}^{n-1}\frac{G_{n-j+3(c-1)+s}}{2^{n-j}}\binom{a_n+j-c}{j}. \tag{28}$$

Finally, $d=-1, r=3$ gives

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}G_{3a_0+s}=\frac{G_{2n+3a_n+s}}{2^n}-\sum_{j=0}^{n-1}\frac{G_{2(n-j)+3(c-1)+s}}{2^{n-j}}\binom{a_n+j-c}{j}. \tag{29}$$

These examples display the practical value of (25): suitable choices of d and r collapse a weighted Horadam identity into concise Fibonacci and gibonacci formulas.

Lemma 4 ([3, Lemma 1])**.** *For every integer j,*

$$A\tau^j-B\sigma^j=\frac{W_{j+1}-qW_{j-1}}{\delta}. \tag{30}$$

Theorem 4. *Let r,s,c,d and a_n be integers with $r\neq 0$. If n is a positive even integer, then*

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{V_d}{V_{r+d}}\right)^{a_0}W_{ra_0+s}=\frac{1}{q^{dn}\delta^n}\left(\frac{V_d}{U_r}\right)^n\left(\frac{V_d}{V_{r+d}}\right)^{a_n}W_{r(n+a_n)+dn+s}$$
$$-\left(\frac{V_d}{V_{r+d}}\right)^{c-1}\frac{1}{\delta^n}\sum_{j=0}^{(n-2)/2}\frac{\delta^{2j}}{q^{d(n-2j)}}\left(\frac{V_d}{U_r}\right)^{n-2j}W_{(r+d)(n-2j)+r(c-1)+s}\binom{a_n+2j-c}{2j}$$
$$-\left(\frac{V_d}{V_{r+d}}\right)^{c-1}\frac{1}{\delta^{n+2}}\sum_{j=1}^{n/2}\frac{\delta^{2j}}{q^{d(n-2j+1)}}\left(\frac{V_d}{U_r}\right)^{n-2j+1}\big(W_{(r+d)(n-2j+1)+r(c-1)+s+1}$$
$$-qW_{(r+d)(n-2j+1)+r(c-1)+s-1}\big)\binom{a_n+2j-1-c}{2j-1}. \tag{31}$$

If n is a positive odd integer, then

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{V_d}{V_{r+d}}\right)^{a_0}W_{ra_0+s}=\frac{1}{q^{dn}\delta^{n+1}}\left(\frac{V_d}{U_r}\right)^n\left(\frac{V_d}{V_{r+d}}\right)^{a_n}\big(W_{r(n+a_n)+dn+s+1}-qW_{r(n+a_n)+dn+s-1}\big)$$
$$-\left(\frac{V_d}{V_{r+d}}\right)^{c-1}\frac{1}{\delta^{n+1}}\sum_{j=0}^{(n-1)/2}\frac{\delta^{2j}}{q^{d(n-2j)}}\left(\frac{V_d}{U_r}\right)^{n-2j}\big(W_{(r+d)(n-2j)+r(c-1)+s+1}-qW_{(r+d)(n-2j)+r(c-1)+s-1}\big)$$

$$\binom{a_n+2j-c}{2j} - \left(\frac{V_d}{V_{r+d}}\right)^{c-1} \frac{1}{\delta^{n+1}} \sum_{j=1}^{(n-1)/2} \frac{\delta^{2j}}{q^{d(n-2j+1)}} \left(\frac{V_d}{U_r}\right)^{n-2j+1} W_{(r+d)(n-2j+1)+r(c-1)+s} \binom{a_n+2j-1-c}{2j-1}. \tag{32}$$

Proof. Split the finite sum in (17) into its even and odd parts:

$$\sum_{j=0}^{m} f_j = \sum_{j=0}^{\lfloor m/2 \rfloor} f_{2j} + \sum_{j=1}^{\lceil m/2 \rceil} f_{2j-1}.$$

Thus

$$\begin{aligned} f(x,y;a_n,n,c) &= \left(\frac{x}{x-y}\right)^n \left(\frac{x}{y}\right)^{a_n} - \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \left(\frac{x}{x-y}\right)^{n-2j} \left(\frac{x}{y}\right)^{c-1} \binom{a_n+2j-c}{2j} \\ &\quad - \sum_{j=1}^{\lceil (n-1)/2 \rceil} \left(\frac{x}{x-y}\right)^{n-2j+1} \left(\frac{x}{y}\right)^{c-1} \binom{a_n+2j-1-c}{2j-1}. \end{aligned} \tag{33}$$

Apply (33) to

$$A\tau^s f(\tau^r V_d, V_{r+d}; a_n, n, c) + B\sigma^s f(\sigma^r V_d, V_{r+d}; a_n, n, c),$$

then use (4), (23), (24), and Lemma 4. The parity of n determines whether the terms involving $A\tau^m + B\sigma^m$ or $A\tau^m - B\sigma^m$ occur, giving (31) and (32). □

For the gibonacci sequence, the even case is

$$\begin{aligned} \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} \left(\frac{L_d}{L_{r+d}}\right)^{a_0} G_{ra_0+s} &= \frac{1}{5^{n/2}} \left(\frac{L_d}{F_r}\right)^n \left(\frac{L_d}{L_{r+d}}\right)^{a_n} G_{r(n+a_n)+dn+s} \\ &- \left(\frac{L_d}{L_{r+d}}\right)^{c-1} \frac{1}{5^{n/2}} \sum_{j=0}^{(n-2)/2} 5^j \left(\frac{L_d}{F_r}\right)^{n-2j} G_{(r+d)(n-2j)+r(c-1)+s} \binom{a_n+2j-c}{2j} \\ &- \left(\frac{L_d}{L_{r+d}}\right)^{c-1} \frac{(-1)^d}{5^{(n+2)/2}} \sum_{j=1}^{n/2} 5^j \left(\frac{L_d}{F_r}\right)^{n-2j+1} (G_{(r+d)(n-2j+1)+r(c-1)+s+1} \\ &+ G_{(r+d)(n-2j+1)+r(c-1)+s-1}) \binom{a_n+2j-1-c}{2j-1}, \end{aligned}$$

and the odd case is

$$\begin{aligned} \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} \left(\frac{L_d}{L_{r+d}}\right)^{a_0} G_{ra_0+s} &= \frac{(-1)^d}{5^{(n+1)/2}} \left(\frac{L_d}{F_r}\right)^n \left(\frac{L_d}{L_{r+d}}\right)^{a_n} (G_{r(n+a_n)+dn+s+1} + G_{r(n+a_n)+dn+s-1}) \\ - \left(\frac{L_d}{L_{r+d}}\right)^{c-1} \frac{(-1)^d}{5^{(n+1)/2}} \sum_{j=0}^{(n-1)/2} 5^j \left(\frac{L_d}{F_r}\right)^{n-2j} & (G_{(r+d)(n-2j)+r(c-1)+s+1} + G_{(r+d)(n-2j)+r(c-1)+s-1}) \binom{a_n+2j-c}{2j} \\ - \left(\frac{L_d}{L_{r+d}}\right)^{c-1} \frac{1}{5^{(n+1)/2}} \sum_{j=1}^{(n-1)/2} 5^j \left(\frac{L_d}{F_r}\right)^{n-2j+1} & G_{(r+d)(n-2j+1)+r(c-1)+s} \binom{a_n+2j-1-c}{2j-1}. \end{aligned}$$

These parity-dependent forms are longer than the previous theorems, but they provide identities in which the weight is governed by the second-kind Lucas sequence rather than the first-kind sequence.

Theorem 5. *Let r, s, d, a_n and c be integers with $r+1 \neq d$. If $W_{r+s} \neq 0$ and $W_{s+d} \neq 0$, then*

$$\begin{aligned} &\sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} q^{a_0} \left(\frac{U_{r-d}}{U_{r-d+1}}\right)^{a_0} \left(\frac{W_{s+d-1}}{W_{s+d}}\right)^{a_0} \\ &= (-1)^n q^{n+a_n} U_{r-d}^n \left(\frac{U_{r-d}}{U_{r-d+1}}\right)^{a_n} \left(\frac{W_{s+d-1}}{W_{s+d}}\right)^{a_n} \left(\frac{W_{s+d-1}}{W_{r+s}}\right)^n \end{aligned}$$

$$- q^{c-1}\left(\frac{U_{r-d}}{U_{r-d+1}}\right)^{c-1}\left(\frac{W_{s+d-1}}{W_{s+d}}\right)^{c-1}\sum_{j=0}^{n-1}(-1)^{n-j}q^{n-j}U_{r-d}^{n-j}\left(\frac{W_{s+d-1}}{W_{r+s}}\right)^{n-j}\binom{a_n+j-c}{j}. \tag{34}$$

Proof. Apply (17) to

$$f(qU_{r-d}W_{s+d-1}, U_{r-d+1}W_{s+d}; a_n, n, c)$$

and use Horadam's identity [2, Identity 3.15]

$$W_{r+s} = U_{r-d+1}W_{s+d} - qU_{r-d}W_{s+d-1}.$$

□

The gibonacci version is

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{a_0}\left(\frac{F_{r-d}}{F_{r-d+1}}\right)^{a_0}\left(\frac{G_{s+d-1}}{G_{s+d}}\right)^{a_0}\\
&= (-1)^{a_n}F_{r-d}^n\left(\frac{F_{r-d}}{F_{r-d+1}}\right)^{a_n}\left(\frac{G_{s+d-1}}{G_{s+d}}\right)^{a_n}\left(\frac{G_{s+d-1}}{G_{r+s}}\right)^n\\
&\quad + (-1)^c\left(\frac{F_{r-d}}{F_{r-d+1}}\right)^{c-1}\left(\frac{G_{s+d-1}}{G_{s+d}}\right)^{c-1}\sum_{j=0}^{n-1}F_{r-d}^{n-j}\left(\frac{G_{s+d-1}}{G_{r+s}}\right)^{n-j}\binom{a_n+j-c}{j}.
\end{aligned}$$

Taking $r = 1$ and $d = 0$ gives

$$\begin{aligned}
\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{a_0}\left(\frac{G_{s-1}}{G_s}\right)^{a_0} &= (-1)^{a_n}\left(\frac{G_{s-1}}{G_s}\right)^{a_n}\left(\frac{G_{s-1}}{G_{s+1}}\right)^n\\
&\quad + (-1)^c\left(\frac{G_{s-1}}{G_s}\right)^{c-1}\sum_{j=0}^{n-1}\left(\frac{G_{s-1}}{G_{s+1}}\right)^{n-j}\binom{a_n+j-c}{j}.
\end{aligned}$$

This result differs from the preceding theorems because the summand contains no explicit Horadam term after simplification; all recurrence information is encoded in the ratios of shifted terms.

4. Multiple sums involving products of terms

The preceding identities also evaluate nested sums whose summands are products of recurrence terms. The product formulas below convert such products into linear combinations of single recurrence terms, after which the identities in Section 3 apply.

Lemma 5. *For a positive integer s and integers b_s and c,*

$$\sum_{b_{s-1}=c}^{b_s}\sum_{b_{s-2}=c}^{b_{s-1}}\cdots\sum_{b_0=c}^{b_1}(-1)^{b_0} = \frac{(-1)^{b_s}}{2^s} + \frac{(-1)^c}{2^s}\sum_{j=0}^{s-1}2^j\binom{b_s+j-c}{j}. \tag{35}$$

Proof. Set $x = 1$ and $y = -1$ in (17). □

Lemma 6. *For integers k, m and s,*

$$U_{k+m}U_{k+s}\delta^2 = V_{2k+m+s} - q^{k+s}V_{m-s}, \tag{36}$$

$$V_{k+m}U_{k+s} = U_{2k+m+s} - q^{k+s}U_{m-s}, \tag{37}$$

$$V_{k+m}V_{k+s} = V_{2k+m+s} + q^{k+s}V_{m-s}. \tag{38}$$

Proof. These are the general Lucas sequence analogues of standard identities for Fibonacci and Lucas numbers [4]. The case $k = 0$ follows directly from (4), and the stated form follows by shifting the index by k. □

Theorem 6. *Let r, s, c, m and a_n be integers and let n be a positive integer. Then*

$$\delta^2 \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} (-1)^{a_0} \frac{U_{ra_0+m}U_{ra_0+s}}{q^{ra_0}} = (-1)^{a_n} \frac{V_{r(2a_n+n)+s+m}}{q^{ra_n} V_r^n} + \frac{(-1)^c}{q^{r(c-1)}} \sum_{j=0}^{n-1} \frac{V_{r(n-j+2c-2)+s+m}}{V_r^{n-j}} \binom{a_n+j-c}{j} - \frac{(-1)^{a_n}}{2^n} q^s V_{m-s} - \frac{(-1)^c}{2^n} q^s V_{m-s} \sum_{j=0}^{n-1} 2^j \binom{a_n+j-c}{j}, \tag{39}$$

$$\sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} (-1)^{a_0} \frac{V_{ra_0+m}U_{ra_0+s}}{q^{ra_0}} = (-1)^{a_n} \frac{U_{r(2a_n+n)+s+m}}{q^{ra_n} V_r^n} + \frac{(-1)^c}{q^{r(c-1)}} \sum_{j=0}^{n-1} \frac{U_{r(n-j+2c-2)+s+m}}{V_r^{n-j}} \binom{a_n+j-c}{j} - \frac{(-1)^{a_n}}{2^n} q^s U_{m-s} - \frac{(-1)^c}{2^n} q^s U_{m-s} \sum_{j=0}^{n-1} 2^j \binom{a_n+j-c}{j}, \tag{40}$$

$$\sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} (-1)^{a_0} \frac{V_{ra_0+m}V_{ra_0+s}}{q^{ra_0}} = (-1)^{a_n} \frac{V_{r(2a_n+n)+s+m}}{q^{ra_n} V_r^n} + \frac{(-1)^c}{q^{r(c-1)}} \sum_{j=0}^{n-1} \frac{V_{r(n-j+2c-2)+s+m}}{V_r^{n-j}} \binom{a_n+j-c}{j} + \frac{(-1)^{a_n}}{2^n} q^s V_{m-s} + \frac{(-1)^c}{2^n} q^s V_{m-s} \sum_{j=0}^{n-1} 2^j \binom{a_n+j-c}{j}. \tag{41}$$

Proof. We prove (39); the remaining identities follow similarly from (37) and (38). By (36),

$$\delta^2 \sum_{a_0=c}^{a_1} (-1)^{a_0} \frac{U_{ra_0+m}U_{ra_0+s}}{q^{ra_0}} = \sum_{a_0=c}^{a_1} (-1)^{a_0} \frac{V_{2ra_0+m+s}}{q^{ra_0}} - q^s V_{m-s} \sum_{a_0=c}^{a_1} (-1)^{a_0}.$$

Repeated summation, followed by (20) and (35), gives (39). □

Lemma 7. *For integers k and s,*

$$U_{k+s+1}^2 - qU_{k+s}^2 = U_{2k+2s+1}, \tag{42}$$

$$V_{k+s+1}^2 - qV_{k+s}^2 = U_{2k+2s+1}\delta^2. \tag{43}$$

Theorem 7. *If s, a_n and c are integers and n is a positive integer, then*

$$\sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} (-1)^{a_0} \frac{U_{a_0+s}^2}{q^{a_0}} = \frac{(-1)^{a_n+1}}{2} \frac{U_{2a_n+n+s}}{q^{a_n+1}p^n} - \frac{(-1)^c}{2q^c p^n} \sum_{j=0}^{n-1} p^j U_{n-j+2c-2+s} \binom{a_n+j-c}{j} + \frac{(-1)^{n+c}}{q^{c-1}} \sum_{k=1}^{n-1} \frac{U_{s+c+k-1}^2}{2^k q^k} \binom{a_n+n-k-c}{n-k} - \frac{(-1)^{a_n+n}}{2q^{a_n+1}p^n} \sum_{k=1}^{n-1} \frac{p^k}{2^k q^k} U_{2a_n+n+k+2s+1} - \frac{(-1)^{n+c}}{2q^c p^n} \sum_{k=1}^{n-1} \frac{1}{2^k q^k} \sum_{j=0}^{n-k-1} p^{k+j} U_{n+k-j+2c-1+2s} \binom{a_n+j-c}{j} + \frac{1}{2^n} \frac{(-1)^c}{q^{n+c-1}} U_{c+s+n-1}^2 + \frac{1}{2^n} \frac{(-1)^{a_n}}{q^{n+a_n}} U_{a_n+s+n}^2, \tag{44}$$

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{a_0}\frac{V_{a_0+s}^2}{q^{a_0}}=\delta^2\frac{(-1)^{a_n+1}}{2}\frac{U_{2a_n+n+s}}{q^{a_n+1}p^n}-\delta^2\frac{(-1)^c}{2q^cp^n}\sum_{j=0}^{n-1}p^jU_{n-j+2c-2+s}\binom{a_n+j-c}{j}$$
$$+\frac{(-1)^{n+c}}{q^{c-1}}\sum_{k=1}^{n-1}\frac{V_{s+c+k-1}^2}{2^kq^k}\binom{a_n+n-k-c}{n-k}-\frac{(-1)^{a_n+n}}{2q^{a_n+1}p^n}\delta^2\sum_{k=1}^{n-1}\frac{p^k}{2^kq^k}U_{2a_n+n+k+2s+1}$$
$$-\delta^2\frac{(-1)^{n+c}}{2q^cp^n}\sum_{k=1}^{n-1}\frac{1}{2^kq^k}\sum_{j=0}^{n-k-1}p^{k+j}U_{n+k-j+2c-1+2s}\binom{a_n+j-c}{j}+\frac{1}{2^n}\frac{(-1)^c}{q^{n+c-1}}V_{c+s+n-1}^2+\frac{1}{2^n}\frac{(-1)^{a_n}}{q^{n+a_n}}V_{a_n+s+n}^2. \tag{45}$$

Proof. We derive (44); identity (45) follows by replacing (42) with (43). Shifting the summation index gives

$$\sum_{k=0}^{n}(-1)^k\frac{U_{k+s+1}^2}{q^k}=-q\sum_{k=0}^{n}(-1)^k\frac{U_{k+s}^2}{q^k}+qU_s^2+(-1)^n\frac{U_{n+s+1}^2}{q^n}.$$

Multiplying (42) by $(-1)^k/q^k$ and summing over k yields

$$\sum_{k=0}^{n}(-1)^k\frac{U_{k+s}^2}{q^k}=-\frac{1}{2q}\sum_{k=0}^{n}(-1)^k\frac{U_{2k+2s+1}}{q^k}+\frac{U_s^2}{2}+\frac{(-1)^n}{2q}\frac{U_{n+s+1}^2}{q^n}.$$

Starting the sum at $k=c$ and iterating this relation through the nested summation levels reduces the first part to (20) and the remaining constant levels to (16). This gives (44). □

Lemma 8. *For integers k, m, s and t,*

$$U_{k+m}U_{k+s}U_{k+t}\delta^2=U_{3k+m+s+t}-q^{k+m}U_{k+s+t-m}-q^{k+t}V_{s-t}U_{k+m}, \tag{46}$$
$$V_{k+m}U_{k+s}U_{k+t}\delta^2=V_{3k+m+s+t}+q^{k+m}V_{k+s+t-m}-q^{k+t}V_{s-t}V_{k+m}, \tag{47}$$
$$V_{k+m}V_{k+s}U_{k+t}=U_{3k+m+s+t}+q^{k+m}U_{k+s+t-m}-q^{k+t}U_{s-t}V_{k+m}, \tag{48}$$
$$V_{k+m}V_{k+s}V_{k+t}=V_{3k+m+s+t}+q^{k+m}V_{k+s+t-m}+q^{k+t}V_{s-t}V_{k+m}. \tag{49}$$

Proof. The four identities follow from Lemma 6 by multiplying the two-factor identities by the remaining shifted term and simplifying the resulting Lucas expressions. □

Theorem 8. *If m, s, t, a_n and c are integers and n is a positive integer, then*

$$5\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}F_{a_0+m}F_{a_0+s}F_{a_0+t}$$
$$=\frac{F_{2n+3a_n+m+s+t}}{2^n}-\sum_{j=0}^{n-1}\frac{F_{2(n-j)+3(c-1)+m+s+t}}{2^{n-j}}\binom{a_n+j-c}{j}$$
$$-(-1)^{a_n+m}F_{-n+a_n+s+t-m}-(-1)^{c+m}\sum_{j=0}^{n-1}F_{-n+j+c-1+s+t-m}\binom{a_n+j-c}{j}$$
$$-(-1)^{a_n+t}L_{s-t}F_{-n+a_n+m}-(-1)^{c+t}L_{s-t}\sum_{j=0}^{n-1}F_{-n+j+c-1+m}\binom{a_n+j-c}{j}, \tag{50}$$

$$5\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}L_{a_0+m}F_{a_0+s}F_{a_0+t}$$
$$=\frac{L_{2n+3a_n+m+s+t}}{2^n}-\sum_{j=0}^{n-1}\frac{L_{2(n-j)+3(c-1)+m+s+t}}{2^{n-j}}\binom{a_n+j-c}{j}$$
$$+(-1)^{a_n+m}L_{-n+a_n+s+t-m}+(-1)^{c+m}\sum_{j=0}^{n-1}L_{-n+j+c-1+s+t-m}\binom{a_n+j-c}{j}$$

$$- (-1)^{a_n+t} L_{s-t} L_{-n+a_n+m} - (-1)^{c+t} L_{s-t} \sum_{j=0}^{n-1} L_{-n+j+c-1+m} \binom{a_n+j-c}{j}, \tag{51}$$

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} L_{a_0+m} L_{a_0+s} F_{a_0+t} \\
&= \frac{F_{2n+3a_n+m+s+t}}{2^n} - \sum_{j=0}^{n-1} \frac{F_{2(n-j)+3(c-1)+m+s+t}}{2^{n-j}} \binom{a_n+j-c}{j} \\
&\quad + (-1)^{a_n+m} F_{-n+a_n+s+t-m} + (-1)^{c+m} \sum_{j=0}^{n-1} F_{-n+j+c-1+s+t-m} \binom{a_n+j-c}{j} \\
&\quad - (-1)^{a_n+t} F_{s-t} L_{-n+a_n+m} - (-1)^{c+t} F_{s-t} \sum_{j=0}^{n-1} L_{-n+j+c-1+m} \binom{a_n+j-c}{j},
\end{aligned} \tag{52}$$

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} L_{a_0+m} L_{a_0+s} L_{a_0+t} \\
&= \frac{L_{2n+3a_n+m+s+t}}{2^n} - \sum_{j=0}^{n-1} \frac{L_{2(n-j)+3(c-1)+m+s+t}}{2^{n-j}} \binom{a_n+j-c}{j} \\
&\quad + (-1)^{a_n+m} L_{-n+a_n+s+t-m} + (-1)^{c+m} \sum_{j=0}^{n-1} L_{-n+j+c-1+s+t-m} \binom{a_n+j-c}{j} \\
&\quad + (-1)^{a_n+t} L_{s-t} L_{-n+a_n+m} + (-1)^{c+t} L_{s-t} \sum_{j=0}^{n-1} L_{-n+j+c-1+m} \binom{a_n+j-c}{j}.
\end{aligned} \tag{53}$$

Proof. We prove (50). The Fibonacci case of (46) is

$$5F_{a_0+m} F_{a_0+s} F_{a_0+t} = F_{3a_0+m+s+t} - (-1)^{a_0+m} F_{a_0+s+t-m} - (-1)^{a_0+t} L_{s-t} F_{a_0+m}.$$

Summing this identity n times gives

$$\begin{aligned}
&5 \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} F_{a_0+m} F_{a_0+s} F_{a_0+t} \\
&= \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} F_{3a_0+m+s+t} - (-1)^m \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} (-1)^{a_0} F_{a_0+s+t-m} \\
&\quad - (-1)^t L_{s-t} \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} (-1)^{a_0} F_{a_0+m}.
\end{aligned}$$

Substitution from (27) and (29) yields (50). The remaining identities are obtained in the same manner from (47), (48), and (49). □

The product identities demonstrate that nested sums of quadratic and cubic products do not require separate summation techniques. Once the products are decomposed into shifted Lucas or Fibonacci terms, the closed forms from Section 3 complete the calculation.

5. Concluding comments

The paper has derived closed forms for nested sums involving terms of the Horadam sequence and its principal specializations. The results cover

$$\sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} \frac{W_{ra_0+s}}{V_r^{a_0}},$$

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{a_0}\frac{W_{2ra_0+s}}{q^{ra_0}},$$
$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{U_d}{U_{r+d}}\right)^{a_0}W_{ra_0+s},$$
$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{V_d}{V_{r+d}}\right)^{a_0}W_{ra_0+s},$$
$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}q^{a_0}\left(\frac{U_{r-d}}{U_{r-d+1}}\right)^{a_0}\left(\frac{W_{s+d-1}}{W_{s+d}}\right)^{a_0},$$

as well as nested sums containing products of two and three shifted terms. The essential contribution is the reduction of a multiple summation problem to a one-dimensional binomial sum through the geometric identity (10). When this reduction is combined with Binet formulas and product identities for Lucas sequences, it produces compact Horadam, Fibonacci, Lucas, and gibonacci identities with arbitrary lower limit c.

The formulas clarify how the depth n affects both the shifted indices and the binomial coefficients. In particular, the term $\binom{a_n+j-c}{j}$ records the number of admissible paths through the nested summation limits, while the recurrence parameters determine the shifted sequence terms that remain after simplification. This separation explains why the same binomial structure occurs throughout the paper, even when the summand changes from a single Horadam term to weighted products.

The present treatment uses the common lower limit $a_i = c$ for $i = 0, 1, \dots, n-1$. A natural next problem is to allow different lower limits in successive levels. One would then seek a formula for

$$\sum_{a_{n-1}=c_{n-1}}^{a_n}\sum_{a_{n-2}=c_{n-2}}^{a_{n-1}}\cdots\sum_{a_1=c_1}^{a_2}\sum_{a_0=c_0}^{a_1}x^{a_0}, \tag{54}$$

where the integers c_i may differ. The corresponding iteration gives

$$\left(\frac{x-1}{x}\right)^n\sum_{a_{n-1}=c_{n-1}}^{a_n}\sum_{a_{n-2}=c_{n-2}}^{a_{n-1}}\cdots\sum_{a_0=c_0}^{a_1}x^{a_0}$$
$$= x^{a_n} - x^{c_{n-1}-1} - \sum_{j=1}^{n-1}\left\{\left(\frac{x-1}{x}\right)^j x^{c_{n-j-1}-1}\sum_{a_{n-1}=c_{n-1}}^{a_n}\sum_{a_{n-2}=c_{n-2}}^{a_{n-1}}\cdots\sum_{a_{n-j}=c_{n-j}}^{a_{n-j+1}}1\right\}.$$

Thus the key remaining ingredient is a closed form for

$$\sum_{b_{s-1}=c_{s-1}}^{b_s}\sum_{b_{s-2}=c_{s-2}}^{b_{s-1}}\cdots\sum_{b_1=c_1}^{b_2}\sum_{b_0=c_0}^{b_1}1.$$

Such a formula would permit the same recurrence-based substitutions used here and would broaden the class of Horadam multiple sums available in closed form.

References

[1] Ivie, J. (1969). Multiple Fibonacci sums. *The Fibonacci Quarterly*, *7*(3), 303-309.

[2] Horadam, A. F. (1965). Basic properties of a certain generalized sequence of numbers. *The Fibonacci Quarterly*, *3*(3), 161-176.

[3] Adegoke, K., Frontczak, R., & Goy, T. Special formulas involving polygonal numbers and Horadam numbers. *Carpathian Mathematical Publications*, *13*(1), 207-216.

[4] Vajda, S. (2008). *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*. Courier Corporation.