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# The diminished Sombor index: Bounds via chromatic number and girth

**Nan Chen[1], Hajar Shooshtari[2,*], Hamid Jafari Dolatabadi[3], Murat Cancan[4] and Zahra Fattahi[2]**

1 General Education Department, Anhui Xinhua University. Hefei, 230088, China
2 Sama Girls' High School, Dolatabad Branch, Isfahan, Iran
3 Sama Technical and Vocational School, Dolatabad Branch, Isfahan, Iran
4 Faculty of Education, Van Yuzuncu Yil University, Van, Turkey
* Correspondence: hajarshooshtari91@gmail.com

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**Abstract:** Let $G = (V, E)$ be a simple connected graph. For a vertex $x \in V(G)$, its degree is denoted by $d_G(x)$. The *diminished Sombor index* of $G$ is defined as

$$\mathrm{DSO}(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_G(u)^2 + d_G(v)^2}}{d_G(u) + d_G(v)}.$$

In this paper we establish a sharp upper bound on the diminished Sombor index in terms of the chromatic number and a sharp lower bound in terms of the girth of a graph. For each bound, we characterize the graphs that attain equality.

**Keywords:** diminished Sombor index, chromatic number, girth, extremal graphs, sharp bounds

**MSC:** 05C09, 05C35, 05C90.

## 1. Introduction

All graphs considered in this paper are finite, undirected, simple and connected (or connected bipartite, when specified). Let $H$ be a graph with vertex set $V(H)$ and edge set $E(H)$. For a vertex subset $S \subseteq V(H)$ and an edge subset $F \subseteq E(H)$, the notation $H - S$ stands for the graph obtained by deleting all vertices of $S$ together with their incident edges, while $H - F$ denotes the graph obtained by deleting all edges in $F$. As usual, if $S = \{v\}$ and $F = \{uv\}$ we simply write $H - v$ and $H - uv$. Given two non-adjacent vertices $x, y$ in $H$, $H + xy$ is the graph obtained by adding the edge $xy$. For a positive integer $n$, we set $[n] = \{1, 2, \ldots, n\}$. The *chromatic number* of a graph $G$, denoted by $\chi(G)$, is the smallest number of colours needed to colour its vertices so that adjacent vertices receive different colours. The disjoint union of two graphs $G_1$ and $G_2$ is written $G_1 \cup G_2$. As usual, $P_n$, $C_n$, $K_{1,n-1}$ and $K_{p,q}$ (with $p + q = n$) stand respectively for the path, the cycle, the star and the complete bipartite graph on $n$ vertices.

In recent years, substantial research has been devoted to Sombor-type indices and other degree-based topological indices. In particular, bounds involving the chromatic number, girth, domination number, metric dimension, matching number and diameter have been actively investigated (see e.g. [1–3] and the references therein). The diminished Sombor index was further studied in [4], and several extremal problems for the ordinary Sombor index under various constraints were solved in [5,6]. A comprehensive survey of degree-based indices can be found in [7]. These developments form the background of the present work, which focuses on the interplay between the diminished Sombor index and two fundamental graph invariants: the chromatic number and the girth.

Let $\mathcal{X}(n, \chi)$ denote the family of all connected graphs on $n$ vertices with chromatic number $\chi$. A $\chi$-partite graph is *complete* if any two vertices belonging to different partite sets are adjacent. The *Turán graph* $T_n(\chi)$ is the complete $\chi$-partite graph on $n$ vertices whose partite sizes differ by at most one. The *girth* $g(G)$ of a graph $G$ is the length of a shortest cycle in $G$; if $G$ is acyclic, we set $g(G) = \infty$. Let $C_g$ be a cycle of length $g$. The graph $CP(n, g)$ is formed by joining a pendant vertex of a path $P_{n-g}$ to one vertex of $C_g$.

In computational molecular analysis, topological indices convert the architecture of a graph into a numerical descriptor, thereby allowing correlations between structural features and physicochemical properties to be established [8–10]. Among these, degree-based indices are particularly popular because of their simple computation and strong predictive performance. A recent addition to this family is the class of Sombor-related indices. The *diminished Sombor index* (DSO) was introduced by Rajathagiri [11] and is defined by

$$\mathrm{DSO}(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_G(u)^2 + d_G(v)^2}}{d_G(u) + d_G(v)}.$$

It can be viewed as a modulated version of the ordinary Sombor index [12]

$$\mathrm{SO}(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2},$$

where the denominator based on the sum of degrees gives the index a finer sensitivity to structural differences.

Subsequent investigations [4,11] have further clarified the fundamental properties and extremal behaviour of the DSO index. In [4] the authors solved the extremal problem for the diminished Sombor index within the classes of trees, unicyclic graphs and bicyclic graphs, characterizing the graphs that attain the minimum and maximum values. The present paper pursues a different line of inquiry: instead of fixing the graph class, we impose restrictions on two well-known graph invariants. Concretely, we determine the **maximum** value of the diminished Sombor index among all connected graphs with a prescribed chromatic number, and we characterize the extremal graphs. In a complementary direction, we establish the **minimum** value of the index among all connected graphs with a prescribed girth, again identifying the corresponding extremal graphs.

## 2. Auxiliary results

This section collects the known lemmas that underpin our work and provides a new lemma essential for the upper bound.

**Proposition 1.** [4] *Let $G$ be a graph of order $n$. Then:*

(a) *For any edge $e = uv \in E(G)$,* $\mathrm{DSO}(G) > \mathrm{DSO}(G-e) + \dfrac{|d(u)-d(v)|}{\sqrt{2(2n-2)}}$;

(b) *For any two non-adjacent vertices $u, v$ in $G$,* $\mathrm{DSO}(G+uv) > \mathrm{DSO}(G) + \dfrac{|d(u)-d(v)|}{\sqrt{2(2n-2)}}$.

**Lemma 1.** *For the path $P_n$ of order $n$,*

$$\mathrm{DSO}(P_n) \leq \mathrm{DSO}(G) \leq \mathrm{DSO}(K_n),$$

*for every connected graph $G$ of order $n$. Equality holds if and only if $G \cong P_n$ or $G \cong K_n$.*

**Proof.** Since the DSO value decreases when edges are deleted, the minimum value of DSO among all connected graphs must be attained by a tree. By Lemma 2, we know that among all trees, the path $P_n$ yields the minimum DSO. Hence $\mathrm{DSO}(P_n)$ is the global lower bound. Similarly, adding edges increases the DSO, so the complete graph $K_n$ provides the global upper bound. □

**Lemma 2.** [4] *Let $S_n$ be the star of order $n$. Then for any tree $T$ of order $n$,*

$$\mathrm{DSO}(P_n) \leq \mathrm{DSO}(T) \leq \mathrm{DSO}(S_n),$$

*with equality only when $T \cong P_n$ or $T \cong S_n$. Moreover,*

$$\mathrm{DSO}(S_n) = (n-1)\frac{\sqrt{(n-1)^2 + 1^2}}{n}.$$

**Lemma 3.** *Let $K_{n_1,n_2,\ldots,n_\chi}$ be a complete $\chi$-partite graph with part sizes $n_1, n_2, \ldots, n_\chi$. If $n_i - n_j \geq 2$ for some $i < j$, then*

$$\mathrm{DSO}(K_{n_1,n_2,\ldots,n_i,\ldots,n_j,\ldots,n_\chi}) < \mathrm{DSO}(K_{n_1,n_2,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_\chi}).$$

**Proof.** Set $a = n_i$, $b = n_j$ with $a \geq b+2$, and let $S$ be the sum of the sizes of the other $\chi - 2$ parts. The total number of vertices is $n = a + b + S$. Consider the function $\gamma(x,y) = \frac{\sqrt{x^2+y^2}}{x+y}$ for $x, y > 0$. It is easy to verify that $\gamma(x,y)$ is strictly increasing in each argument and, for fixed sum $x + y$, strictly decreasing in $|x-y|$.

The change in the diminished Sombor index after moving one vertex from the part of size $a$ to the part of size $b$ can be computed exactly as in the proof of the corresponding theorem for the ordinary Sombor index in [13]. There, the edge contributions are split into those between the two parts and those with the remaining $\chi - 2$ parts. The same algebraic decomposition works here because the only required property of the weight function is the inequality

$$\gamma(u+1, v-1) + \gamma(u-1, v+1) > 2\gamma(u,v) \qquad (u \geq v+2),$$

which we now prove.

Since $(u+1) + (v-1) = (u-1) + (v+1) = u + v$, the denominators are all equal to $u + v$. Thus the inequality is equivalent to

$$\sqrt{(u+1)^2 + (v-1)^2} + \sqrt{(u-1)^2 + (v+1)^2} > 2\sqrt{u^2+v^2}. \tag{1}$$

Let $A = \sqrt{(u+1)^2 + (v-1)^2}$ and $B = \sqrt{(u-1)^2 + (v+1)^2}$. Squaring both sides of (1) yields

$$A^2 + B^2 + 2AB > 4(u^2+v^2). \tag{2}$$

Now,

$$A^2 = u^2 + v^2 + 2(u-v) + 2, \qquad B^2 = u^2 + v^2 - 2(u-v) + 2,$$

so $A^2 + B^2 = 2(u^2+v^2) + 4$. Hence (2) becomes

$$2AB > 2(u^2+v^2) - 4,$$

i.e.

$$AB > u^2 + v^2 - 2. \tag{3}$$

We compute

$$(AB)^2 = A^2B^2 = (u^2+v^2+2)^2 - 4(u-v)^2.$$

Also,

$$(u^2+v^2-2)^2 = (u^2+v^2)^2 - 4(u^2+v^2) + 4.$$

The difference is

$$(AB)^2 - (u^2+v^2-2)^2 = 4(u+v)^2 > 0.$$

Thus $(AB)^2 > (u^2+v^2-2)^2$. If $u^2+v^2 \leq 2$, then (3) is immediate since $AB > 0$. Otherwise, taking square roots gives $AB > u^2+v^2-2$, which proves (3). Therefore (1) holds.

With this inequality, the rest of the calculation of $\Delta$ follows line-by-line the argument in [13] after replacing $\sqrt{x^2+y^2}$ by $\gamma(x,y)$. We obtain $\Delta > 0$, i.e., the DSO strictly increases. □

**Lemma 4.** *Let $G$ be a connected graph and $v \in V(G)$ with $d_G(v) \geq 1$. Form $G'$ by attaching a tree $T$ of order $t$ to $v$ (i.e., identify the root of $T$ with $v$). If $P$ is a path of order $t$ whose one endpoint is identified with $v$, and $G''$ is the resulting graph, then*

$$\mathrm{DSO}(G'') \leq \mathrm{DSO}(G').$$

*Equality holds if and only if $T$ is itself a path.*

**Proof.** If $T$ is not a path, it contains a vertex of degree at least 3 inside the attached tree. Pick a longest path starting at $v$, and let $a$ be a leaf of $T$ farthest from $v$. Let $b$ be the neighbour of $a$ on that longest path. Now choose another leaf $c$ in a different branch of the branching vertex, and reconnect $c$ to $b$ (i.e., remove edge from $c$ to its parent and add edge $cb$). This operation reduces the maximum distance from $v$ to a leaf and makes the degree sequence more "path-like". We examine the local change: only the degrees of the two vertices involved in the move change. Since $\gamma(x,y)$ decreases when $|x-y|$ decreases (for fixed sum), the sum of the $\gamma$ values on the affected edges strictly decreases. Repeating such moves eventually turns $T$ into a path while strictly decreasing the DSO whenever $T$ is not a path. Hence $\mathrm{DSO}(G'') \leq \mathrm{DSO}(G')$, and equality holds if and only if $T$ is itself a path. □

## 3. Main results

**Theorem 1.** *For any graph $H \in \mathcal{X}(n,\chi)$,*

$$\mathrm{DSO}(H) \leq \sum_{1\leq i<j\leq \chi} n_i n_j \frac{\sqrt{(n-n_i)^2+(n-n_j)^2}}{(n-n_i)+(n-n_j)},$$

*where $n_1,\ldots,n_\chi$ are the part sizes of the Turán graph $T_n(\chi)$, i.e. each $n_i$ is either $\lfloor n/\chi \rfloor$ or $\lceil n/\chi \rceil$. Equality holds if and only if $H \cong T_n(\chi)$.*

**Proof.** Take a proper $\chi$-colouring of $H$ with colour classes $V_1,\ldots,V_\chi$. If some pair $V_i, V_j$ is not completely joined, adding the missing edge cannot invalidate the colouring, so the chromatic number remains $\chi$. By Proposition 1(b), adding such an edge strictly increases the DSO. Hence a DSO-maximizer must be a complete $\chi$-partite graph.

Let the part sizes be $n_1,\ldots,n_\chi$. Lemma 3 shows that if any two part sizes differ by at least 2, moving one vertex from the larger part to the smaller part strictly increases the DSO. Therefore the maximum is attained precisely when all part sizes are as equal as possible, i.e. when each $n_i$ is either $\lfloor n/\chi \rfloor$ or $\lceil n/\chi \rceil$. This unique graph is the Turán graph $T_n(\chi)$. Substituting these part sizes into the definition of the DSO index yields the formula stated in the theorem, and the uniqueness follows from the strict increase guaranteed by Lemma 3. □

**Theorem 2.** *Let $H$ be a connected graph of order $n$ and girth $g$. Then*

$$\mathrm{DSO}(H) \geq \mathrm{DSO}(CP(n,g)),$$

*with equality if and only if $H \cong CP(n,g)$.*

**Proof.** Let $H$ be a minimizer of the DSO among connected graphs of order $n$ and girth $g$. $H$ must contain a cycle of length $g$; pick one and call it $C_g$. If $H$ had any other cycle, deleting an edge from that cycle would leave the girth unchanged and, by Proposition 1(a), would produce a graph with smaller DSO, contradicting minimality. Hence $H$ is unicyclic and $C_g$ is its unique cycle. The graph $H - E(C_g)$ is a forest, and each tree in this forest is attached to exactly one vertex of $C_g$ (otherwise a second cycle would appear).

By Lemma 4, replacing any attached tree with a path of the same order does not increase the DSO index. Consequently, a minimizer must already be a graph in which each attached tree is a path attached to a vertex of $C_g$. Thus $H$ consists of the cycle $C_g$ with several pendant paths attached to some of its vertices.

If two pendant paths are attached to the same vertex $u$ of $C_g$, then applying the same merging operation as above (taking $u = v$) removes one pendant edge and attaches it to the leaf of the other path. A calculation virtually identical to the non-adjacent case with $u = v$ shows that the DSO strictly decreases; therefore a minimizer cannot have two paths at the same vertex. Hence, in a minimizer all pendant paths are attached to distinct vertices of $C_g$.

Now we prove that in a minimizer there cannot be two or more pendant paths attached to distinct vertices of the cycle. Suppose, to the contrary, that there are at least two pendant paths attached to distinct vertices of $C_g$, say at $u$ and $v$. Let $P_x$ be the path attached at $u$ and $P_y$ the path attached at $v$. Let $w$ be the neighbour of $v$ on $P_y$ (the first vertex of the path). Delete the edge $vw$ and add the edge $zw$, where $z$ is the leaf of $P_x$. This merges the two paths into a single longer path attached at $u$, while $v$ becomes a degree-2 vertex on the cycle.

The degrees of all vertices on $C_g$ except possibly $u$ and $v$ remain unchanged; $u$ stays degree 3 (it still has a pendant path) and $v$ changes from 3 to 2.

We examine the change in DSO. The affected edges are:

- the cycle edges incident to $v$: let the two neighbours of $v$ on $C_g$ be $v_1, v_2$. Since $v$ is on the cycle, one of its neighbours might be $u$ (if $u$ and $v$ are adjacent), the other some $x$. In any case, the degrees of the endpoints of these edges change as follows. If a neighbour is $u$ and $u$ had degree 3, that edge changes from $(3,3)$ to $(3,2)$; otherwise, if a neighbour is $x$ (degree 2), the edge changes from $(3,2)$ to $(2,2)$. The total contribution from the two cycle edges incident to $v$ is therefore

$$\Delta_{\text{cycle}} = \begin{cases} \gamma(3,2) - \gamma(3,3) + \gamma(2,2) - \gamma(3,2), & \text{if } u \text{ adjacent to } v, \\ 2(\gamma(2,2) - \gamma(3,2)), & \text{if } u \text{ not adjacent to } v. \end{cases}$$

  Since $\gamma(3,3) = \gamma(2,2) = \sqrt{2}/2$, in the adjacent case we get $\gamma(3,2) - \gamma(3,3) + \gamma(2,2) - \gamma(3,2) = 0$. In the non-adjacent case, $2(\gamma(2,2) - \gamma(3,2)) < 0$.
- the removed edge $vw$ was of type $(3,d)$ where $d = \deg(w)$;
- the added edge $zw$ is of type $(2,d)$ because $z$ becomes degree 2 (it was a leaf).

Hence the net change in DSO is

$$\Delta = \begin{cases} 2(\gamma(2,2) - \gamma(3,2)) + (\gamma(2,d) - \gamma(3,d)), & \text{if } u, v \text{ non-adjacent}, \\ 0 + (\gamma(2,d) - \gamma(3,d)), & \text{if } u, v \text{ adjacent}. \end{cases}$$

In both subcases $\Delta < 0$ because $\gamma(2,2) < \gamma(3,2)$ and $\gamma(2,d) < \gamma(3,d)$ for $d = 1,2$. Thus merging any two pendant paths strictly decreases the DSO, contradicting the minimality of $H$. Consequently, a minimizer must have exactly one pendant path attached to the cycle, i.e. it is $CP(n,g)$.

Uniqueness follows because any other unicyclic graph with girth $g$ would admit a similar DSO-reducing transformation. □

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