Open J. Math. Anal., Vol. **2***(2018), No. 1, pp. 08 - 18*
Website: https://pisrt.org/psr-press/journals/oma/
ISSN: 2616-8111 (online) 2616-8103 (Print)
http://dx.doi.org/10.30538/psrp-oma2018.0008

NON-CONVEX HYBRID METHOD CORRESPONDING TO KARAKAYA ITERATIVE PROCESS

SAMINA KAUSAR, MUHAMMAD ASIF, MOBEEN MUNIR[1]

Abstract. In this article we present non-convex hybrid iteration algorithm corollaryresponding to Karakaya iterative scheme [1] as done by Guan *et al.* in [2] corollaryresponding to Mann iterative scheme [3]. We also prove some strong convergence results about common fixed points for a uniformly closed asymptotic family of countable quasi-Lipschitz mappings in Hilbert spaces.

AMS Mathematics Subject Classification: 47H05; 47H09; 47H10.
Key words and phrases: hybrid algorithm; nonexpansive mapping; Karakaya iteration process; closed quasi-nonexpansive.

1. Introduction

Fixed points of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings has become a field of interest on its on and has a variety of applications in related fields like image recovery, signal processing and geometry of objects [4]. Almost in all branches of mathematics we see some versions of theorems relating to fixed points of functions of special nature. As a result we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. A fixed-point iteration scheme has been applied in IMRT optimization to pre-compute dose-deposition coefficient (DDC) matrix, see [5]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present now.
Constructive fixed point theorems (e.g. Banach fixed point theorem) which not only claim the existence of a fixed point but yield an algorithm, too (in the

Received 09-01-2018. Revised 16-04-2018.
[1] Corresponding Author

Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some map f that is contracting with respect to some (complete) metric on X will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and is invariably used in most of the occasions see[3]. But it only ensures weak convergence, see [6] but more often then not, we require strong convergence in many real world problems relating to Hilbert spaces, see [7]. So mathematician are in search for the modifications of the Mann's process to control and ensure the strong convergence, ([2, 6, 8, 9, 10, 11, 12, 13] and references therein).

Most probably the first noticeable modification of Mann's Iteration process was propositionosed by Nakajo *et al.* in [9] in 2003. They introduced this modification for only one nonexpansive mapping, where as Kim *et al.* introduced a variant for asymptotically nonexpansive mapping in the context of Hilbert spaces in the year 2006, see [10]. In the same year Martinez *et al.* in [11] introduced a variation of the Ishikawa Iteration process for a nonexpansive mapping for a Hilbert space. They also gave a variant of Halpern method. Su *et al.* in [12] gave a hybrid iteration process for nonexpansive mapping which is monotone. Liu *et al.* in [13] gave a novel method for quasi-asymptotically finite family of pseudo-contractive mapping. Let H be the reserved symbol for Hilbert space and C be nonempty, closed and convex subset of it. First we recall some basic definitions that will accompany us throughout this paper. Let $P_c(.)$ be the metric projection onto C. A mapping $T : C \to C$ is said to be non-expensive if $\|Tx - Ty\| \leq \|x - y\| \; \forall \; x, y \in C$. And $T : C \to C$ is said to be quasi-Lipschitz if

1) $FixT \neq \phi$

2) For all $p \in FixT$, $\|Tx - p\| \leq L\|x - p\|$ where L is a constant $1 \leq L < \infty$.

If $L = 1$ then T is known as quasi-nonexpansive. It is well-known that T is said to be closed if for $n \to \infty$, $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$. T is said to be weak closed if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$. as $n \to \infty$. It is trivial fact that a mapping which is weak closed should be closed but converse is no longer true.

Let $\{T_n\}$ be a sequence of mappings having non-empty fixed points sets. Then $\{T_n\}$ is called uniformly closed if for all convergent sequences $\{Z_n\} \subset C$ with conditions $\|Zx_n - Z_n\| \to 0$,$n \to \infty$ implies the limit of $\{Z_n\}$ belongs to $FixT_i$.

In 1953 [3], we have Mann iterative sheme:

$$x_{n+1} = (1 - a_n)x_n n + a_n T(x_n); n = 0, 1, 2, \ldots.$$

In [2] Guan *et al.* established non-convex hybrid iteration algorithm corollaryresponding to Mann iterative scheme:

$$\begin{cases} x_0 \in C = Q_0, & \text{arbitrarily,} \\ y_n = (1 - a_n)x_n + a_n T_n x_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq (1 + (L_n - 1)a_n)\|x_n - z\| \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, & \end{cases}$$

and proved some strong convergence results about common fixed points relating to a family of countable uniformly closed asymptotic quasi-Lipschitz mappings in H. They applied their results for the finite case to obtain fixed points.
The Karakaya iterative scheme [1] was defined in 2013 as

$$\begin{cases} x_{n+1} = (1-\alpha_n-\beta_n)y_n + \alpha_n T(y_n) + \beta_n T(z_n); \\ y_n = (1-a_n-b_n)z_n + a_n T(z_n) + b_n T(x_n); \\ z_n = (1-\gamma_n)x_n + \gamma_n T(x_n); \qquad\qquad n = 0,1,2,\ldots. \end{cases}$$

where $\alpha_n, \beta_n, \gamma_n, a_n, b_n \in [0,1], \alpha_n + \beta_n \in [0,1], a_n + b_n \in [0,1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n+\beta_n) = \infty$.
In this article, we establish a non-convex hybrid algorithms corollaryresponding to Karakaya iteration scheme. Then we also establish strong convergence theorems with proofs about common fixed points related to a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the realm of Hilbert spaces. An application of this algorithm is also given. We fix $\overline{co}C_n$ for closed convex closure of C_n for all $n \geq 1$, $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$, T_n for countable quasi-L_n-Lipschitz mappings from C into itself, and T be closed quasi-nonexpansive mapping from C into itself to avoid redundancy.

2. Main results

In this section we give our main results.

Definition 2.1. $\{T_n\}$ is said to be asymptotic, if $\lim_{n\to\infty} L_n = 1$

Proposition 2.2. *For $x \in H$ and $z \in C$, $z = P_C x$ iff we have*

$$\langle x - z, z - y\rangle \geq 0$$

for all $y \in C$.

Proposition 2.3. *The common fixed point set F of above said T_n is closed and convex.*

Proposition 2.4. *For any given $x_0 \in H$, we have*

$$p = P_C x_0 \iff \langle p - z, x_0 - p\rangle \geq 0,\ \forall z \in C.$$

Theorem 2.5. *Assume that α_n, β_n, γ_n, a_n and $b_n \in [0,1]$, $\alpha_n+\beta_n \in [0,1]$ and $a_n + b_n \in [0,1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n+\beta_n) = \infty$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, & \textit{arbitrarily,} \\ y_n = (1-\alpha_n-\beta_n)z_n + \alpha_n T_n z_n + \beta_n T_n t_n, & n \geq 0, \\ z_n = (1-a_n-b_n)t_n + a_n T_n t_n + b_n T_n x_n, & n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T_n x_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n(1-b_n-2\gamma_n-2a_n+3a_n\gamma_n & \\ \qquad +2b_n\gamma_n) + L_n^2(-3a_n\gamma_n+\gamma_n-b_n\gamma_n+a_n+b_n) + a_n\gamma_n L_n^3 & \\ \qquad +a_n+b_n-a_n\gamma_n-b_n\gamma_n-1)\alpha_n + (L_n(1-a_n-b_n-2\gamma_n & \\ \qquad +2a_n\gamma_n-b_n\gamma_n) + L_n^2(-a_n\gamma_n+\gamma_n) - b_n\gamma_n - a_n\gamma_n - b_n & \\ \qquad +a_n+\gamma_n-1)\beta_n + (L_n(1-2a_n-2b_n) + a_nL_n^2 + b_n - a_n & \\ \qquad -1)\gamma_n - a_n - b_n + L_n(a_n+b_n)]\|x_n - z\|\} \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, & \end{cases}$$

converges strongly to $P_F x_0$*.*

Proof. We partition our proof in following seven steps.
Step 1. We know that $\overline{co}C_n$ and Q_n are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$\begin{aligned}
&\|y_n - p\| \\
&= \|(1-\alpha_n-\beta_n)z_n + \alpha_n T_n z_n + \beta_n T_n t_n - p\| \\
&= \|(1-\alpha_n-\beta_n)[(1-a_n-b_n)t_n + a_n T_n t_n + b_n T_n x_n] \\
&+ \alpha_n T_n[(1-a_n-b_n)t_n + a_n T_n t_n + b_n T_n x_n] + \beta_n T_n t_n - p\| \\
&= \|(1-\alpha_n-\beta_n)[(1-a_n-b_n)((1-\gamma_n)x_n + \gamma_n T_n x_n) \\
&+ a_n T_n((1-\gamma_n)x_n + \gamma_n T_n x_n) + b_n T_n x_n] \\
&+ \alpha_n T_n[(1-a_n-b_n)((1-\gamma_n)x_n + \gamma_n T_n x_n) \\
&+ a_n T_n((1-\gamma_n)x_n + \gamma_n T_n x_n) + b_n T_n x_n] + \beta_n T_n[(1-\gamma_n)x_n + \gamma_n T_n x_n] - p\| \\
&= \|(1-\gamma_n-a_n-b_n-\alpha_n-\beta_n-a_n\gamma_n+b_n\gamma_n+\alpha_n\gamma_n+a_n\alpha_n \\
&+ b_n\alpha_n+\beta_n\gamma_n+a_n\beta_n-b_n\beta_n-a_n\alpha_n\gamma_n-b_n\alpha_n\gamma_n-a_n\beta_n\gamma_n-b_n\beta_n\gamma_n) \\
&\times (x_n - p) + (\gamma_n+a_n+b_n+\beta_n+\alpha_n-b_n\alpha_n-a_n\beta_n-b_n\beta_n-2a_n\gamma_n \\
&- 2b_n\gamma_n-2\alpha_n\gamma_n-2a_n\alpha_n-2\beta_n\gamma_n+3a_n\alpha_n\gamma_n+2b_n\alpha_n\gamma_n+2a_n\beta_n\gamma_n \\
&- b_n\beta_n\gamma_n)(T_n x_n - p) + (a_n\gamma_n-3a_n\alpha_n\gamma_n-a_n\beta_n\gamma_n+\alpha_n\gamma_n-b_n\alpha_n\gamma_n \\
&+ a_n\alpha_n+b_n\alpha_n+\beta_n\gamma_n)(T_n^2 x_n - p) + (a_n\alpha_n\gamma_n)(T_n^3 x_n - p)\| \\
&\leq (1-\gamma_n-a_n-b_n-\alpha_n-\beta_n-a_n\gamma_n+b_n\gamma_n+\alpha_n\gamma_n+a_n\alpha_n \\
&+ b_n\alpha_n+\beta_n\gamma_n+a_n\beta_n-b_n\beta_n-a_n\alpha_n\gamma_n-b_n\alpha_n\gamma_n-a_n\beta_n\gamma_n \\
&- b_n\beta_n\gamma_n)\|x_n - p\| + (\gamma_n+a_n+b_n+\beta_n+\alpha_n-b_n\alpha_n-a_n\beta_n-b_n\beta_n \\
&- 2a_n\gamma_n-2b_n\gamma_n-2\alpha_n\gamma_n-2a_n\alpha_n-2\beta_n\gamma_n+3a_n\alpha_n\gamma_n+2b_n\alpha_n\gamma_n \\
&+ 2a_n\beta_n\gamma_n-b_n\beta_n\gamma_n)L_n\|x_n - p\| + (a_n\gamma_n-3a_n\alpha_n\gamma_n-a_n\beta_n\gamma_n \\
&+ \alpha_n\gamma_n-b_n\alpha_n\gamma_n+a_n\alpha_n+b_n\alpha_n+\beta_n\gamma_n)L_n^2\|x_n - p\| + (a_n\alpha_n\gamma_n)L_n^3\|x_n - p\|
\end{aligned}$$

$$= [1 + (L_n(1 - b_n - 2\gamma_n - 2a_n + 3a_n\gamma_n + 2b_n\gamma_n) + L_n^2(-3a_n\gamma_n + \gamma_n - b_n\gamma_n + a_n + b_n) + a_n\gamma_n L_n^3 + a_n + b_n - a_n\gamma_n - b_n\gamma_n - 1)\alpha_n + (L_n(1 - a_n - b_n - 2\gamma_n + 2a_n\gamma_n - b_n\gamma_n) + L_n^2(-a_n\gamma_n + \gamma_n) - b_n\gamma_n - a_n\gamma_n - b_n + a_n + \gamma_n - 1)\beta_n + (L_n(1 - 2a_n - 2b_n) + a_nL_n^2 + b_n - a_n - 1)\gamma_n - a_n - b_n + L_n(a_n + b_n)]\|x_n - p\|$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. Therefore, $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$.

Step 2. We show that $F \cap A \subset \overline{co}C_n \cap Q_n$ for all $n \geq 0$. It suffices to show that $F \cap A \subset Q_n$, for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$ we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $\overline{co}C_n \cap Q_n$, from proposition 2.2, we have
$\langle x_{n+1} - z, x_{n+1} - x_0\rangle \leq 0, \forall z \in \overline{co}C_n \cap Q_n$
as $F \cap A \subset \overline{co}C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of Q_{n+1} implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{co}C_n \cap Q_n$ holds for all $n \geq 0$.

Step 3. We prove $\{x_n\}$ is bounded. Since F is a nonepmty, closed, and convex subset of C, there exists a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0$, we have
$\|x_{n+1} - x_0\| \leq \|z - x_0\|$
for every $z \in \overline{co}C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{co}C_n \cap Q_n$, we get
$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$
for each $n \geq 0$. This implies that $\{x_n\}$ is bounded.

Step 4. We show that $\{x_n\}$ converges strongly to a point of C by showing that $\{x_n\}$ is a cauchy sequence. As $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Proposition 2.4), we have
$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$
for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exsists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0\rangle \leq 0$ and hence

$$\begin{aligned} \|x_{n+m} - x_n\|^2 &= \|(x_{n+m} - x_0) - (x_n - x_0)\|^2 \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0,\ n \to \infty \end{aligned}$$

for any $m \geq 1$. Therefore $\{x_n\}$ is a cauchy sequence in C, then there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

Step 5. We show that $y_n \to q$, as $n \to \infty$. Let
$D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4)\}$.
From the definition of D_n, we have

$$D_n = \{z \in C : \langle y_n - z, y_n - z\rangle \leq \langle x_n - z, x_n - z\rangle + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4)\}$$

$$= \{z \in C : \|y_n\|^2 - 2\langle y_n, z\rangle + \|z\|^2 \leq \|x_n\|^2 - 2\langle x_n, z\rangle$$
$$+ \|z\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4)\}$$
$$= \{z \in C : 2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4)\}$$

This shows that D_n is convex and closed, $n \in \mathbb{Z}^+ \cup \{0\}$. Next, we want to prove that
$C_n \subset D_n$, $n \geq 0$. In fact, for any $z \in C_n$, we have

$$\begin{aligned}
\|y_n - z\|^2 &\leq [1 + (L_n(1 - b_n - 2\gamma_n - 2a_n + 3a_n\gamma_n + 2b_n\gamma_n) + L_n^2(-3a_n\gamma_n + \gamma_n \\
&\quad - b_n\gamma_n + a_n + b_n) + a_n\gamma_n L_n^3 + a_n + b_n - a_n\gamma_n - b_n\gamma_n - 1)\alpha_n \\
&\quad + (L_n(1 - a_n - b_n - 2\gamma_n + 2a_n\gamma_n - b_n\gamma_n) + L_n^2(-a_n\gamma_n + \gamma_n) \\
&\quad - b_n\gamma_n - a_n\gamma_n - b_n + a_n + \gamma_n - 1)\beta_n + (L_n(1 - 2a_n - 2b_n) \\
&\quad + a_nL_n^2 + b_n - a_n - 1)\gamma_n - a_n - b_n + L_n(a_n + b_n)]^2\|x_n - z\|^2 \\
&= \|x_n - z\|^2 + 2[(L_n(1 - b_n - 2\gamma_n - 2a_n + 3a_n\gamma_n + 2b_n\gamma_n) \\
&\quad + L_n^2(-3a_n\gamma_n + \gamma_n - b_n\gamma_n + a_n + b_n) + a_n\gamma_n L_n^3 + a_n + b_n - a_n\gamma_n \\
&\quad - b_n\gamma_n - 1)\alpha_n + (L_n(1 - a_n - b_n - 2\gamma_n + 2a_n\gamma_n - b_n\gamma_n) \\
&\quad + L_n^2(-a_n\gamma_n + \gamma_n) - b_n\gamma_n - a_n\gamma_n - b_n + a_n + \gamma_n - 1)\beta_n \\
&\quad + (L_n(1 - 2a_n - 2b_n) + a_nL_n^2 + b_n - a_n - 1)\gamma_n - a_n - b_n + L_n(a_n + b_n)] \\
&\quad + [(L_n(1 - b_n - 2\gamma_n - 2a_n + 3a_n\gamma_n + 2b_n\gamma_n) + L_n^2(-3a_n\gamma_n + \gamma_n \\
&\quad - b_n\gamma_n + a_n + b_n) + a_n\gamma_n L_n^3 + a_n + b_n - a_n\gamma_n - b_n\gamma_n - 1)\alpha_n \\
&\quad + (L_n(1 - a_n - b_n - 2\gamma_n + 2a_n\gamma_n - b_n\gamma_n) + L_n^2(-a_n\gamma_n + \gamma_n) \\
&\quad - b_n\gamma_n - a_n\gamma_n - b_n + a_n + \gamma_n - 1)\beta_n + (L_n(1 - 2a_n - 2b_n) \\
&\quad + a_nL_n^2 + b_n - a_n - 1)\gamma_n - a_n - b_n + L_n(a_n + b_n)]^2\alpha_n^2]\|x_n - z\|^2 \\
&\leq \|x_n - z\|^2 + [2(L_n^3 - 2L_n - 6) + (L_n^3 - 2L_n - 6)^2]\|x_n - z\|^2 \\
&= \|x_n - z\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4)\|x_n - z\|^2.
\end{aligned}$$

From $C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n(1 - b_n - 2\gamma_n - 2a_n + 3a_n\gamma_n + 2b_n\gamma_n) + L_n^2(-3a_n\gamma_n + \gamma_n - b_n\gamma_n + a_n + b_n) + a_n\gamma_n L_n^3 + a_n + b_n - a_n\gamma_n - b_n\gamma_n - 1)\alpha_n + (L_n(1 - a_n - b_n - 2\gamma_n + 2a_n\gamma_n - b_n\gamma_n) + L_n^2(-a_n\gamma_n + \gamma_n) - b_n\gamma_n - a_n\gamma_n - b_n + a_n + \gamma_n - 1)\beta_n + (L_n(1 - 2a_n - 2b_n) + a_nL_n^2 + b_n - a_n - 1)\gamma_n - a_n - b_n + L_n(a_n + b_n)]\|x_n - z\|\} \cap A$, $n \geq 0$, we have $C_n \subset A$, $n \geq 0$. Since A is convex, we also have $\overline{co}C_n \subset A$, $n \geq 0$. Consider $x_n \in \overline{co}C_{n-1}$, we know that

$$\begin{aligned}
\|y_n - z\| &\leq \|x_n - z\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4)\|x_n - z\|^2 \\
&\leq \|x_n - z\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4).
\end{aligned}$$

This implies that $z \in D_n$ and hence $C_n \subset D_n$, $n \geq 0$. Sinnce D_n is convex, we have $\overline{co}(C_n) \subset D_n$, $n \geq 0$. Therefore $\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4) \to 0$, as $n \to \infty$. That is, $y_n \to q$ as $n \to \infty$.

Step 6. To prove that $q \in F$, we use definition of y_n. So we have $(\alpha_n + \beta_n + \gamma_n + a_n + b_n - a_n\gamma_n - b_n\gamma_n - \alpha_n\gamma_n - a_n\alpha_n + a_n\alpha_n\gamma_n - b_n\alpha_n + b_n\alpha_n\gamma_n - \beta_n\gamma_n - a_n\beta_n + a_n\beta_n\gamma_n - b_n\beta_n + b_n\beta_n\gamma_n - a_n\beta_n\gamma_n T_n - a_n\alpha_n\gamma_n T_n + \alpha_n\gamma_n T_n + a_n\alpha_n T_n - a_n\alpha_n\gamma_n T_n + b_n\alpha_n T_n - b_n\alpha_n\gamma_n T_n + \beta_n\gamma_n T_n + a_n\alpha_n\gamma_n T_n^2 + a_n\gamma_n T_n)\|T_n x_n - x_n\| = \|y_n - x_n\| \to 0$, as $n \to \infty$. Since $\alpha_n \in (a, 1] \subset [0, 1]$, from the above limit we have $\lim_n \to \infty \|T_n x_n - x_n\| = 0$.
Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.
Step 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exist a positive integer N, if $n > N$ then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to $\|z_0 - x_n\|^2 = \|z_0 - x_n + x_n - x_0\|^2 = \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle$. It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$ which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F$, this is a contradiction. This completes the proof. $\square$

Now, we present an example of C_n which does not involve a convex subset.

Example 2.6. Take $H = R^2$, and a sequence of mappings $T_n : R^2 \to R^2$ given by $T_n : (t_1, t_2) \mapsto (\frac{1}{8}t_1, t_2)$, $\forall (t_1, t_2) \in R^2$, $\forall n \geq 0$.
It is clear that $\{T_n\}$ satisfies the desired definition of with $F = \{(t_1, 0) : t_1 \in (-\infty, +\infty)\}$ common fixed point set. Take $x_0 = (4, 0)$, $a_0 = \frac{6}{7}$, we have
$y_0 = \frac{1}{7}x_0 + \frac{6}{7}T_0 x_0 = (4 \times \frac{1}{7} + \frac{4}{8} \times \frac{6}{7}, 0) = (1, 0)$.
Take $1 + (L_0 - 1)a_0 = \sqrt{\frac{5}{2}}$, we have
$C_0 = \{z \in R^2 : \|y_0 - z\| \leq \sqrt{\frac{5}{2}}\|x_0 - z\|\}$.
It is easy to show that $z_1 = (1, 3)$, $z_2 = (-1, 3) \in C_0$. But
$z' = \frac{1}{2}z_1 + \frac{1}{2}z_2 = (0, 3) \overline{\in} C_0$,
since $\|y_0 - z\| = 2$, $\|x_0 - z\| = 1$. Therefore C_0 is not convex.

Corollary 2.7. *Assume that α_n, β_n, γ_n, a_n and $b_n \in [0, 1]$, $\alpha_n + \beta_n \in [0, 1]$ and $a_n + b_n \in [0, 1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, & \textit{arbitrarily,} \\ y_n = (1 - \alpha_n - \beta_n)z_n + \alpha_n T z_n + \beta_n T t_n, & n \geq 0, \\ z_n = (1 - a_n - b_n)t_n + a_n T t_n + b_n T x_n, & n \geq 0, \\ t_n = (1 - \gamma_n)x_n + \gamma_n T x_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 - (\gamma_n - b_n)\alpha_n & \\ \qquad -(2a_n + b_n)\gamma_n - 2b_n\beta_n(1 + \gamma_n)]\|x_n - z\|\} \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, & \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

Proof. Take $T_n \equiv T$, $L_n \equiv 1$ in Theorem 2.5, in this case, C_n is convex and closed and , for all $n \geq 0$, by using Theorem 2.5, we obtain Corollary 2.7. Take $T_n \equiv T$, $L_n \equiv 1$ in Theorem 2.5, in this case, C_n is closed and convex, for all $n \geq 0$, by using Theorem 2.5, we obtain Corollary 2.7. $\square$

Corollary 2.8. *Assume that* α_n, β_n, γ_n, a_n *and* $b_n \in [0,1]$, $\alpha_n+\beta_n \in [0,1]$ *and* $a_n + b_n \in [0,1]$ *for all* $n \in N$ *and* $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. *Then* $\{x_n\}$ *generated by*

$$\begin{cases} x_0 \in C = Q_0, & \textit{arbitrarily,} \\ y_n = (1-\alpha_n-\beta_n)z_n + \alpha_n T z_n + \beta_n T t_n, & n \geq 0, \\ z_n = (1-a_n-b_n)t_n + a_n T t_n + b_n T x_n, & n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T x_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1-(\gamma_n - b_n)\alpha_n & \\ \qquad -(2a_n+b_n)\gamma_n - 2b_n\beta_n(1+\gamma_n)]\|x_n - z\|\} \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, & \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

3. Applications

Here, we give an application of our result for the following case of finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let

$$\|T_i^j x - p\| \leq k_{i,j}\|x - p\|, \forall x \in C, \, p \in F,$$

where F is common fixed point set of $\{T_n\}_{n=0}^{N-1}$,$\lim_j \to \infty k_{i,j} = 1$ for all $0 \leq i \leq N-1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is uniformly $L-Lipschitz$, if

$$\|T_i^j x - T_i^j y\| \leq L_{i,j}\|x - y\|, \forall x, y \in C$$

for all $i \in \{0,1,2,...,N-1\}$, $j \geq 1$, where $L \geq 1$.

Theorem 3.1. *Let* $\{T_n\}_{n=0}^{N-1} : C \to C$ *be a uniformly L-Lipschitz finit family of asymptotically quasi-nonexpansive mappings with nonempty common fixed point set* F. *Assume that* α_n, β_n, γ_n, a_n *and* $b_n \in [0,1]$, $\alpha_n+\beta_n \in [0,1]$ *and* $a_n+b_n \in$

$[0,1]$ *for all* $n \in N$ *and* $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. *Then* $\{x_n\}$ *generated by*

$$\begin{cases} x_0 \in C = Q_0, & \textit{arbitrarily}, \\ y_n = (1-\alpha_n-\beta_n)z_n + \alpha_n T_{i(n)}^{j(n)} z_n + \beta_n T_{i(n)}^{j(n)} t_n, & n \geq 0, \\ z_n = (1-a_n-b_n)t_n + a_n T_{i(n)}^{j(n)} t_n + b_n T_{i(n)}^{j(n)} x_n, & n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T_{i(n)}^{j(n)} x_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (k_{i(n),j(n)}(1-b_n-2\gamma_n-2a_n & \\ \qquad +3a_n\gamma_n+2b_n\gamma_n) + k_{i(n),j(n)}^2(-3a_n\gamma_n+\gamma_n-b_n\gamma_n+a_n & \\ \qquad +b_n) + a_n\gamma_n k_{i(n),j(n)}^3 + a_n + b_n - a_n\gamma_n - b_n\gamma_n - 1)\alpha_n & \\ \qquad +(k_{i(n),j(n)}(1-a_n-b_n-2\gamma_n+2a_n\gamma_n-b_n\gamma_n) & \\ \qquad +k_{i(n),j(n)}^2(-a_n\gamma_n+\gamma_n) - b_n\gamma_n - a_n\gamma_n - b_n + a_n & \\ \qquad +\gamma_n - 1)\beta_n + (k_{i(n),j(n)}(1-2a_n-2b_n) + a_n k_{i(n),j(n)}^2 & \\ \qquad +b_n - a_n - 1)\gamma_n - a_n - b_n + k_{i(n),j(n)}(a_n+b_n)] & \\ \qquad \|x_n - z\|\} \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, & \end{cases}$$

converges strongly to $P_F x_0$, *where* $n = (j(n)-1)N + i(n)$ *for all* $n \geq 0$.

Proof. We can drive the prove from the following two conclusions.

conclusion1. $\{T_{n=0}^{N-1}\}_{n=0}^{\infty}$ *is a uniformly closed asymptotically family of countable quasi-*L_n*-Lipschitz mappings from* C *into itself.*

conclusion2. $F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, *where* $F(T)$ *denotes the fixed point set of the mappings* T.

□

Corollary 3.2. *Let* $T : C \to C$ *be a L-Lipschitz asymptotically quasi-nonexpansive mappings with nonempty common fixed point set* F. *Assume that* α_n, β_n, γ_n, a_n *and* $b_n \in [0,1]$, $\alpha_n + \beta_n \in [0,1]$ *and* $a_n + b_n \in [0,1]$ *for all* $n \in N$ *and* $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. *Then* $\{x_n\}$ *generated by*

$$\begin{cases} x_0 \in C = Q_0, & \textit{arbitrarily}, \\ y_n = (1-\alpha_n-\beta_n)z_n + \alpha_n T^n z_n + \beta_n T^n t_n, & n \geq 0, \\ z_n = (1-a_n-b_n)t_n + a_n T^n t_n + b_n T^n x_n, & n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (k_n(1-b_n-2\gamma_n-2a_n+3a_n\gamma_n & \\ \qquad +2b_n\gamma_n) + k_n^2(-3a_n\gamma_n+\gamma_n-b_n\gamma_n+a_n+b_n) + a_n\gamma_n k_n^3 & \\ \qquad +a_n+b_n-a_n\gamma_n-b_n\gamma_n-1)\alpha_n + (k_n(1-a_n-b_n-2\gamma_n & \\ \qquad +2a_n\gamma_n-b_n\gamma_n) + k_n^2(-a_n\gamma_n+\gamma_n) - b_n\gamma_n - a_n\gamma_n - b_n & \\ \qquad +a_n+\gamma_n-1)\beta_n + (k_n(1-2a_n-2b_n) + a_n k_n^2 + b_n - a_n & \\ \qquad -1)\gamma_n - a_n - b_n + k_n(a_n+b_n)]\|x_n - z\|\} \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, & \end{cases}$$

converges strongly to $P_F x_0$*, where* $\overline{co}C_n$ *denotes the closed convex closure of* C_n *for all* $n \geq 1$*,* $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$ in theorem 3.1, we proved. □

Competing interests

The author(s) declare that they have no competing interests.

References

1. Karakaya, V., Dogan, K., Grsoy, F., & Ertürk, M. (2013). Fixed point of a new three-step iteration algorithm under contractive-like operators over normed spaces. *In Abstract and Applied Analysis* (Vol. 2013). Hindawi. http://dx.doi.org/10.1155/2013/560258
2. Guan, J., Tang, Y., Ma, P., Xu, Y., & Su, Y. (2015). Non-convex hybrid algorithm for a family of countable quasi-Lipschitz mappings and application. *Fixed Point Theory and Applications*, 2015(1), 214. https://doi.org/10.1186/s13663-015-0457-4
3. Mann, W. R. (1953). Mean value methods in iteration. *Proceedings of the American Mathematical Society*, 4(3), 506-510.
4. Youla, D. C. (1987). Mathematical theory of image restoration by the method of convex projections. *Image Recovery: Theory and Application,* 29-77.
5. Tian, Z., Zarepisheh, M., Jia, X., & Jiang, S. B. (2013). The fixed-point iteration method for IMRT optimization with truncated dose deposition coefficient matrix. *arXiv preprint arXiv:1303.3504*.
6. Genel, A., & Lindenstrauss, J. (1975). An example concerning fixed points. *Israel Journal of Mathematics*, 22(1), 81-86.
7. Bauschke, H. H., & Combettes, P. L. (2001). A weak-to-strong convergence principle for Fejr-monotone methods in Hilbert spaces. *Mathematics of operations research*, 26(2), 248-264. https://doi.org/10.1287/moor.26.2.248.10558
8. Matsushita, S. Y., & Takahashi, W. (2005). A strong convergence theorem for relatively nonexpansive mappings in a Banach space. *Journal of Approximation Theory,* 134(2), 257-266.https://doi.org/10.1016/j.jat.2005.02.007
9. Nakajo, K., & Takahashi, W. (2003). Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. *Journal of Mathematical Analysis and Applications*, 279(2), 372-379. https://doi.org/10.1016/S0022-247X(02)00458-4
10. Kim, T. H., & Xu, H. K. (2006). Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups. *Nonlinear Analysis: Theory, Methods & Applications*, 64(5), 1140-1152. https://doi.org/10.1016/j.na.2005.05.059
11. Martinez-Yanes, C., & Xu, H. K. (2006). Strong convergence of the CQ method for fixed point iteration processes. *Nonlinear Analysis: Theory, Methods & Applications*, 64(11), 2400-2411. https://doi.org/10.1016/j.na.2005.08.018
12. Su, Y., & Qin, X. (2008). Monotone CQ iteration processes for nonexpansive semigroups and maximal monotone operators. *Nonlinear Analysis: Theory, Methods & Applications*, 68(12), 3657-3664. https://doi.org/10.1016/j.na.2007.04.008
13. Liu, Y., Zheng, L., Wang, P., & Zhou, H. (2015). Three kinds of new hybrid projection methods for a finite family of quasi-asymptotically pseudocontractive mappings in Hilbert spaces. *Fixed Point Theory and Applications,* 2015(1), 118. https://doi.org/10.1186/s13663-015-0371-9
14. Moudafi, A. (2000). Viscosity approximation methods for fixed-points problems. *Journal of Mathematical Analysis and Applications,* 241(1), 46-55..
15. Xu, H. K. (2004). Viscosity approximation methods for nonexpansive mappings. *Journal of Mathematical Analysis and Applications,* 298(1), 279-291.

16. Podilchuk, C. I., & Mammone, R. J. (1990). Image recovery by convex projections using a least-squares constraint. JOSA A, 7(3), 517-521.

Samina Kausar
Division of Science and Technology, University of Education, Lahore, 54000, Pakistan.
e-mail: sminasaddique@gmail.com

Muhammad Asif

Department of Mathematics, Govt. Post Graduate College, Chistian, Pakistan.
e-mail: asifnazeer.n@gmail.com

Mubeen Munir
Division of Science and Technology, University of Education, Lahore, 54000, Pakistan.
e-mail: mmunir@ue.edu.pk