NON-CONVEX HYBRID METHOD CORRESPONDING TO KARAKAYA ITERATIVE PROCESS

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ABSTRACT. In this article we present non-convex hybrid iteration algorithm corresponding to Karakaya iterative scheme [1] as done by Guan et al. in [2] corresponding to Mann iterative scheme [3]. We also prove some strong convergence results about common fixed points for a uniformly closed asymptotic family of countable quasi-Lipschitz mappings in Hilbert spaces.

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1. Introduction

Fixed points of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings has become a field of interest on its own and has a variety of applications in related fields like image recovery, signal processing and geometry of objects [4]. Almost in all branches of mathematics we see some versions of theorems relating to fixed points of functions of special nature. As a result we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. A fixed-point iteration scheme has been applied in IMRT optimization to pre-compute dose-deposition coefficient (DDC) matrix, see [5]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present now. Constructive fixed point theorems (e.g. Banach fixed point theorem) which not only claim the existence of a fixed point but yield an algorithm, too (in the
Hilbert space. They also gave a variant of Halpern method. Su et al. a variation of the Ishikawa Iteration process for a nonexpansive mapping in the context of Hilbert spaces, see [10]. In the same year Martinez et al. introduced a variation of the Ishikawa Iteration process for a nonexpansive mapping which is monotone. Liu et al. in [13] gave a novel method for quasi-asymptotically finite family of pseudo-contractive mapping. Let H be the reserved symbol for Hilbert space and C be nonempty, closed and convex subset of it. First we recall some basic definitions that will accompany us throughout this paper. Let \( P_c(.) \) be the metric projection onto C. A mapping \( T : C \rightarrow C \) is said to be non-expansive if \( \|Tx - Ty\| \leq \|x - y\| \forall x,y \in C \). And \( T : C \rightarrow C \) is said to be quasi-Lipschitz if

1) \( \text{Fix}T \neq \phi \) 
2) For all \( p \in \text{Fix}T, \|Tx - p\| \leq L\|x - p\| \) where L is a constant \( 1 \leq L < \infty \).

If \( L = 1 \) then \( T \) is known as quasi-nonexpansive. It is well-known that \( T \) is said to be closed if for \( n \rightarrow \infty, x_n \rightarrow x \) and \( \|Tx_n - x_n\| \rightarrow 0 \) implies \( Tx = x \). \( T \) is said to be weak closed if \( x_n \rightarrow x \) and \( \|Tx_n - x_n\| \rightarrow 0 \) implies \( Tx = x \). as \( n \rightarrow \infty \). It is trivial fact that a mapping which is weak closed should be closed but converse is no longer true.

Let \( \{T_n\} \) be a sequence of mappings having non-empty fixed points sets. Then \( \{T_n\} \) is called uniformly closed if for all convergent sequences \( \{Z_n\} \subset C \) with conditions \( \|Z_n - Z\| \rightarrow 0, n \rightarrow \infty \) implies the limit of \( \{Z_n\} \) belongs to \( \text{Fix}T \).

In 1953 [9], we have Mann iterative scheme:

\[
x_{n+1} = (1 - a_n)x_n + a_nT(x_n); \quad n = 0, 1, 2, \ldots.
\]

In [2] Guan et al. established non-convex hybrid iteration algorithm corollary corresponding to Mann iterative scheme:

\[
\begin{align*}
x_0 & \in C = Q_0, \\
y_n & = (1 - a_n)x_n + a_nT_nx_n, \\
C_n & = \{z \in C : \|y_n - z\| \leq (1 + (L_n - 1)a_n)\|x_n - z\| \cap A, \quad n \geq 0, \\
Q_n & = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} & = P_{C_n \cap Q_n}x_0,
\end{align*}
\]

arbitrarily, 
\( n \geq 0, \)
\( n \geq 0, \)
\( n \geq 1, \)
and proved some strong convergence results about common fixed points relating to a family of countable uniformly closed asymptotic quasi-Lipschitz mappings in $H$. They applied their results for the finite case to obtain fixed points.

The Karakaya iterative scheme \cite{1} was defined in 2013 as

\[
\begin{align*}
    x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_n T(y_n) + \beta_n T(z_n); \\
    y_n &= (1 - a_n - b_n)z_n + a_n T(z_n) + b_n T(x_n); \\
    z_n &= (1 - \gamma_n)x_n + \gamma_n T(x_n); \quad n = 0, 1, 2, \ldots,
\end{align*}
\]

where $\alpha_n, \beta_n, \gamma_n, a_n, b_n \in [0, 1], \alpha_n + \beta_n \in [0, 1], a_n + b_n \in [0, 1]$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} (\alpha_n + \beta_n) = \infty$.

In this article, we establish a non-convex hybrid algorithms corollary corresponding to Karakaya iteration scheme. Then we also establish strong convergence theorems with proofs about common fixed points related to a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the realm of Hilbert spaces. An application of this algorithm is also given. We fix $\overline{co}C_n$ for closed convex closure of $C_n$ for all $n \geq 1$, $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$, $T_n$ for countable quasi-$L_n$-Lipschitz mappings from $C$ into itself, and $T$ be closed quasi-nonexpansive mapping from $C$ into itself to avoid redundancy.

### 2. Main results

In this section we give our main results.

**Definition 2.1.** \{$T_n\$} is said to be asymptotic, if $\lim_{n \to \infty} L_n = 1$

**Proposition 2.2.** For $x \in H$ and $z \in C$, $z = P_C x$ iff we have

$\langle x - z, z - y \rangle \geq 0$

for all $y \in C$.

**Proposition 2.3.** The common fixed point set $F$ of above said $T_n$ is closed and convex.

**Proposition 2.4.** For any given $x_0 \in H$, we have

$p = P_C x_0 \iff \langle p - z, x_0 - p \rangle \geq 0, \forall z \in C$.

**Theorem 2.5.** Assume that $\alpha_n, \beta_n, \gamma_n, a_n$ and $b_n \in [0, 1], \alpha_n + \beta_n \in [0, 1]$ and $a_n + b_n \in [0, 1]$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} (\alpha_n + \beta_n) = \infty$. Then \{$x_n\$} generated by

...
\[
\begin{align*}
x_0 & \in C = Q_0, \\
y_n &= (1 - \alpha_n - \beta_n) z_n + \alpha_n T_n z_n + \beta_n T_n t_n, \quad n \geq 0, \\
z_n &= (1 - \alpha_n - \beta_n) t_n + \alpha_n T_n t_n + b_n T_n x_n, \\
t_n &= (1 - \gamma_n) x_n + \gamma_n T_n x_n, \\
C_n &= \{ z \in C : \| y_n - z \| \leq [1 + (L_n(1 - b_n - 2\gamma_n - 2a_n + 3\alpha_n \gamma_n \\
+ 2b_n \gamma_n) + L_n^2(3a_n \gamma_n + \gamma_n - b_n \gamma_n + a_n + b_n) + a_n \gamma_n L_n^3 \\
+ a_n + b_n - a_n \gamma_n - b_n \gamma_n - 1) \alpha_n + (L_n(1 - a_n - b_n - 2\gamma_n \\
+ 2a_n \gamma_n - b_n \gamma_n) + L_n^2(-a_n \gamma_n + \gamma_n) - b_n \gamma_n - a_n \gamma_n - b_n \\
+ a_n + \gamma_n - 1) \beta_n + (L_n(1 - 2a_n - 2b_n) + a_n L_n^2 + b_n - a_n \\
- 1) \gamma_n - a_n - b_n + L_n(a_n + b_n) \| x_n - z \|} \cap A, \\
Q_n &= \{ z \in Q_n - 1 : (x_n - z, x_0 - x_n) \geq 0 \}, \\
x_{n+1} &= P_{\Sigma C_n \cap Q_n} x_0,
\end{align*}
\]

converges strongly to \( P_F x_0 \).

**Proof.** We partition our proof in following seven steps.

**Step 1.** We know that \( \overline{\omega} C_n \) and \( Q_n \) are closed and convex for all \( n \geq 0 \). Next, we show that \( F \cap A \subset \overline{\omega} C_n \) for all \( n \geq 0 \). Indeed, for each \( p \in F \cap A \), we have

\[
\begin{align*}
\| y_n - p \| &= \| (1 - \alpha_n - \beta_n) z_n + \alpha_n T_n z_n + \beta_n T_n t_n - p \|
\leq & \| (1 - \alpha_n - \beta_n) \| (1 - a_n - b_n) t_n + a_n T_n t_n + b_n T_n x_n \\
+ & \alpha_n T_n(1 - a_n - b_n) t_n + a_n T_n t_n + b_n T_n x_n + \beta_n T_n t_n - p \|
\leq & \| (1 - \alpha_n - \beta_n) \| (1 - a_n - b_n) (1 - \gamma_n) x_n + \gamma_n T_n x_n \\
+ & \alpha_n T_n(1 - a_n - b_n) (1 - \gamma_n) x_n + \gamma_n T_n x_n \\
+ & \alpha_n T_n(1 - \gamma_n) x_n + \gamma_n T_n x_n + b_n T_n x_n \\
+ & \beta_n T_n (1 - \gamma_n) x_n + \gamma_n T_n x_n - p \|
\leq & \| (1 - \gamma_n - a_n - b_n - 2a_n \gamma_n - 2b_n \gamma_n + 3a_n \alpha_n \gamma_n + 2b_n \alpha_n \gamma_n + a_n \alpha_n \\
+ & a_n \alpha_n + \beta_n \alpha_n + a_n \beta_n - b_n \alpha_n \gamma_n - a_n \alpha_n \gamma_n - a_n \beta_n \gamma_n - b_n \beta_n \gamma_n \\
\times & (x_n - p) + (\gamma_n + a_n + b_n + \beta_n + a_n - b_n \alpha_n - a_n \beta_n - 2a_n \gamma_n \\
- & 2b_n \gamma_n - 2a_n \gamma_n - 2a_n \alpha_n - 2b_n \gamma_n + 3a_n \alpha_n \gamma_n + 2b_n \alpha_n \gamma_n + 2a_n \beta_n \gamma_n \\
- & b_n \beta_n \gamma_n) (T_n x_n - p) + (a_n \gamma_n - 3a_n \alpha_n \gamma_n - a_n \beta_n \gamma_n + a_n \gamma_n - b_n \alpha_n \gamma_n \\
+ & a_n \alpha_n + b_n \alpha_n + \beta_n \gamma_n) (T_n^3 x_n - p) + (a_n \alpha_n \gamma_n (T_n^3 x_n - p) \| \\
\leq & (1 - \gamma_n - a_n - b_n - 2a_n \gamma_n + 2b_n \gamma_n + a_n \alpha_n + a_n \alpha_n \\
+ & b_n \alpha_n + \beta_n \alpha_n + a_n \beta_n - b_n \beta_n \gamma_n - a_n \beta_n \gamma_n - b_n \alpha_n \gamma_n - a_n \beta_n \gamma_n \\
- & b_n \beta_n \gamma_n) \| x_n - p \| + (\gamma_n + a_n + b_n + \beta_n + a_n - b_n \alpha_n - a_n \beta_n - b_n \beta_n \\
- & 2a_n \gamma_n - 2b_n \gamma_n - 2a_n \alpha_n - 2b_n \alpha_n - 2a_n \gamma_n + 3a_n \alpha_n \gamma_n + 2b_n \alpha_n \gamma_n \\
+ & 2a_n \beta_n \gamma_n - b_n \beta_n \gamma_n + a_n \alpha_n \gamma_n - a_n \beta_n \gamma_n \\
+ & a_n \alpha_n - b_n \alpha_n \gamma_n + a_n \alpha_n + b_n \alpha_n + b_n \beta_n \gamma_n) L_n \| x_n - p \| + (a_n \alpha_n \gamma_n - a_n \beta_n \gamma_n \\
+ & a_n \alpha_n - b_n \alpha_n \gamma_n + a_n \alpha_n + b_n \alpha_n + b_n \beta_n \gamma_n) L_n \| x_n - p \| + (a_n \alpha_n \gamma_n) L_n^3 \| x_n - p \|.
\]
\[
= [1 + (L_n(1 - b_n - 2\gamma_n - 2a_n + 3a_n\gamma_n + 2b_n\gamma_n) + L_n^2(-3a_n\gamma_n \\
+ \gamma_n - b_n\gamma_n + a_n + b_n) + a_n\gamma_nL_n + a_n + b_n - a_n\gamma_n - b_n\gamma_n - 1)a_n \\
+ (L_n(1 - a_n - b_n - 2\gamma_n + 2a_n\gamma_n - b_n\gamma_n) + L_n^2(-a_n\gamma_n + \gamma_n) - b_n\gamma_n \\
- a_n\gamma_n - b_n + a_n + \gamma_n - 1)\beta_n + (L_n(1 - 2a_n - 2b_n) + a_nL^2 + b_n \\
- a_n - 1)\gamma_n - a_n - b_n + L_n(a_n + b_n)\|x_n - p\|
\]
and \( p \in A \), so \( p \in C_n \) which implies that \( F \cap A \subset C_n \) for all \( n \geq 0 \). Therefore, \( F \cap A \subset C_n \) for all \( n \geq 0 \).

**Step 2.** We show that \( F \cap A \subset \overline{\varepsilon C_n} \cap Q_n \) for all \( n \geq 0 \). It suffices to show that \( F \cap A \subset Q_n \) for all \( n \geq 0 \). We prove this by mathematical induction. For \( n = 0 \) we have \( F \cap A \subset C = Q_0 \). Assume that \( F \cap A \subset Q_n \). Since \( x_{n+1} \) is the projection of \( x_0 \) onto \( \overline{\varepsilon C_n} \cap Q_n \), from proposition 2.2 we have

\[
\langle x_{n+1} - z, x_{n+1} - x_0 \rangle \leq 0, \forall z \in \overline{\varepsilon C_n} \cap Q_n
\]

as \( F \cap A \subset \overline{\varepsilon C_n} \cap Q_n \), the last inequality holds, in particular, for all \( z \in F \cap A \). This together with the definition of \( Q_{n+1} \) implies that \( F \cap A \subset Q_{n+1} \). Hence the \( F \cap A \subset \overline{\varepsilon C_n} \cap Q_n \) holds for all \( n \geq 0 \).

**Step 3.** We prove \( \{x_n\} \) is bounded. Since \( F \) is a nonempty, closed, and convex subset of \( C \), there exists a unique element \( z_0 \in F \) such that \( z_0 = P_F x_0 \). From \( x_{n+1} = P_{\overline{\varepsilon C_n}}\cap Q_n x_0 \), we have

\[
\|x_{n+1} - x_0\| \leq \|z - x_0\|
\]

for every \( z \in \overline{\varepsilon C_n} \cap Q_n \). As \( z_0 \in F \cap A \subset \overline{\varepsilon C_n} \cap Q_n \), we get

\[
\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|
\]

for each \( n \geq 0 \). This implies that \( \{x_n\} \) is bounded.

**Step 4.** We show that \( \{x_n\} \) converges strongly to a point of \( C \) by showing that \( \{x_n\} \) is a cauchy sequence. As \( x_{n+1} = P_{\overline{\varepsilon C_n}}\cap Q_n x_0 \subset Q_n \) and \( x_n = P_{Q_n} x_0 \) (Proposition 2.4), we have

\[
\|x_{n+1} - x_0\| \geq \|x_n - x_0\|
\]

for every \( n \geq 0 \), which together with the boundedness of \( \|x_n - x_0\| \) implies that there exists the limit of \( \|x_n - x_0\| \). On the other hand, from \( x_{n+m} \in Q_n \), we have \( \langle x_n - x_{n+m}, x_n - x_0 \rangle \leq 0 \) and hence

\[
\|x_{n+m} - x_n\|^2 = \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0 \rangle \\
\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0 \rangle \\
\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \rightarrow 0, \; n \rightarrow \infty
\]

for any \( m \geq 1 \). Therefore \( \{x_n\} \) is a cauchy sequence in \( C \), then there exists a point \( q \in C \) such that \( \lim_{n \rightarrow \infty} x_n = q \).

**Step 5.** We show that \( y_n \rightarrow q \), as \( n \rightarrow \infty \). Let

\[
D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4)\}
\]

From the definition of \( D_n \), we have

\[
D_n = \{z \in C : \langle y_n - z, y_n - z \rangle \leq \langle x_n - z, x_n - z \rangle \\
+ (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4)\}
\]
This implies that \( D_2 \subset C \), From \( \mathbb{R}^n \) + \{z \} \leq ||x_n||^2 - \|y_n\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4) \}

This shows that \( D_n \) is convex and closed, \( n \in \mathbb{Z}^+ \cup \{0\} \). Next, we want to prove that \( C_n \subset D_n, n \geq 0 \). In fact, for any \( z \in C_n \), we have

\[
\|y_n - z\|^2 \leq [1 + (L_n(1-b_n-2\gamma_n-2a_n+3a_n\gamma_n+2b_n\gamma_n))] + L_n^2(-3a_n\gamma_n + \gamma_n - b_n\gamma_n + a_n + b_n - a_n\gamma_n - b_n\gamma_n - 1)\alpha_n \\
+ (L_n(1-b_n-2\gamma_n-2a_n\gamma_n-b_n\gamma_n)) + L_n^2(-a_n\gamma_n + \gamma_n) \\
- b_n\gamma_n - a_n\gamma_n - b_n + a_n + \gamma_n - 1)\beta_n + (L_n(1-2a_n-2b_n)) \\
+ a_nL_n^2 + b_n - a_n - 1)\gamma_n - a_n - b_n + L_n(a_n + b_n))^2\|x_n - z\|^2 \\
= \|x_n - z\|^2 - 2[(L_n(1-b_n-2\gamma_n-2a_n + 3a_n\gamma_n + 2b_n\gamma_n))] + L_n^2(-3a_n\gamma_n + \gamma_n - b_n\gamma_n + a_n + b_n - a_n\gamma_n - b_n\gamma_n - 1)\alpha_n \\
+ (L_n(1-b_n-2\gamma_n-2a_n\gamma_n-b_n\gamma_n)) + L_n^2(-a_n\gamma_n + \gamma_n) \\
- b_n\gamma_n - a_n\gamma_n - b_n + a_n + \gamma_n - 1)\beta_n + (L_n(1-2a_n-2b_n)) \\
+ a_nL_n^2 + b_n - a_n - 1)\gamma_n - a_n - b_n + L_n(a_n + b_n))^2\|x_n - z\|^2 \\
\leq \|x_n - z\|^2 + 2(L_n^3 - 2L_n - 6) + (L_n^3 - 2L_n - 6)^2\|x_n - z\|^2 \\
= \|x_n - z\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4)\|x_n - z\|^2 .
\]

From \( C_n = \{ z \in C : \|y_n - z\| \leq [1 + (L_n(1-b_n-2\gamma_n-2a_n+3a_n\gamma_n+2b_n\gamma_n))] + L_n^2(-3a_n\gamma_n + \gamma_n - b_n\gamma_n + a_n + b_n - a_n\gamma_n - b_n\gamma_n - 1)\alpha_n \\
+ (L_n(1-b_n-2\gamma_n-2a_n\gamma_n-b_n\gamma_n)) + L_n^2(-a_n\gamma_n + \gamma_n) \\
- b_n\gamma_n - a_n\gamma_n - b_n + a_n + \gamma_n - 1)\beta_n + (L_n(1-2a_n-2b_n)) \\
+ a_nL_n^2 + b_n - a_n - 1)\gamma_n - a_n - b_n + L_n(a_n + b_n))^2\|x_n - z\|^2 \} \cap A, n \geq 0, \) we have \( C_n \subset A, n \geq 0 \). Since \( A \) is convex, we also have \( \overline{\text{conv}}C_n \subset A, n \geq 0 \).

Consider \( x_n \in \overline{\text{conv}}C_{n-1} \), we know that

\[
\|y_n - z\| \leq \|x_n - z\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4)\|x_n - z\|^2 \\
\leq \|x_n - z\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4) .
\]

This implies that \( z \in D_n \) and hence \( C_n \subset D_n, n \geq 0 \). Since \( D_n \) is convex, we have \( \overline{\text{conv}}C_n \subset D_n, n \geq 0 \). Therefore \( \|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (L_n^3 - 2L_n - 6)(L_n^3 - 2L_n - 4) \to 0, n \to \infty \). That is, \( y_n \to q \) as \( n \to \infty \).
Step 6. To prove that \( q \in F \), we use definition of \( y_n \). So we have 
\[
\begin{align*}
&\alpha_n + \beta_n + \gamma_n + a_n + b_n - a_n\alpha_n - b_n\gamma_n - a_n\alpha_n + a_n\alpha_n\gamma_n - b_n\alpha_n + b_n\alpha_n - a_n\beta_n + a_n\beta_n\gamma_n - b_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n - a_n\beta_n\gamma_n \end{align*}
\]

which leads to 
\[
\begin{align*}
\|x_n - y_n\| = \|y_n - x_n\| \to 0, \text{ as } n \to \infty. 
\end{align*}
\]
Since \( \alpha_n \in (a, 1) \subseteq [0, 1] \), from the above limit we have \( \lim_n \to \infty \|T_n x_n - x_n\| = 0 \). 

Since \( \{T_n\} \) is uniformly closed and \( x_n \to q \), we have \( q \in F \). 

Step 7. We claim that \( q = z_0 = P_F x_0 \), if not, we have that \( \|x_0 - p\| > \|x_0 - z_0\| \).

There must exist a positive integer \( N \), if \( n > N \) then \( \|x_0 - x_n\| > \|x_0 - z_0\| \), which leads to 
\[
\begin{align*}
\|z_0 - x_n\| = \|x_0 - x_n\| & > \|x_0 - z_0\| + 2\|z_0 - x_n\| + 2\|x_n - x_0\| \\
& > \|x_0 - z_0\| + 2\|z_0 - x_n\| + 2\|x_n - x_0\|.
\end{align*}
\]

It follows that \( \|z_0 - x_n, x_n - x_0\| < 0 \) which implies that \( z_0 \in \text{Q}_n \), so that \( z_0 \in F \), this is a contradiction. This completes the proof. \( \square \)

Now, we present an example of \( C_n \) which does not involve a convex subset.

Example 2.6. Take \( H = R^2 \), and a sequence of mappings \( T_n : R^2 \to R^2 \) given by \( T_n : (t_1, t_2) \mapsto \left( \frac{t_1}{2}, t_2 \right) \), \( \forall (t_1, t_2) \in R^2 \), \( \forall n \geq 0 \).

It is clear that \( \{T_n\} \) satisfies the desired definition of with \( \mathcal{F} = \{(t_1, 0) : t_1 \in (-\infty, +\infty)\} \) common fixed point set. Take \( x_0 = (4, 0), a_0 = \frac{1}{2}, \) we have 
\[
\begin{align*}
y_0 &= \frac{1}{2} x_0 + \frac{1}{2} T_0 x_0 = (1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}, 0) = (1, 0). 
\end{align*}
\]

Take 1 + \( (L_0 - 1)a_0 = \sqrt{2} \), we have 
\[
\begin{align*}
C_0 &= \{z \in R^2 : \|y_0 - z\| \leq \sqrt{2} \|x_0 - z\| \}.
\end{align*}
\]
It is easy to show that \( z_1 = (1, 3), z_2 = (-1, 3) \in C_0 \). But 
\[
\begin{align*}
z' &= \frac{1}{2} z_1 + \frac{1}{2} z_2 = (0, 3) \in C_0, 
\end{align*}
\]

since \( \|y_0 - z\| = 2, \|x_0 - z\| = 1 \). Therefore \( C_0 \) is not convex.

Corollary 2.7. Assume that \( \alpha_n, \beta_n, \gamma_n, a_n \) and \( b_n \in [0, 1] \), \( \alpha_n + \beta_n \in [0, 1] \) and \( \alpha_n + b_n \in [0, 1] \) for all \( n \in N \) and \( \sum_{n=0}^{\infty} (\alpha_n + \beta_n) = \infty \). Then \( \{x_n\} \) generated by 
\[
\begin{align*}
x_0 &\in C = Q_0, \\
y_n &= (1 - \alpha_n - \beta_n)z_n + \alpha_n Tz_n + \beta_n Tt_n, \\
z_n &= (1 - \alpha_n - b_n)z_n + \alpha_n Tt_n + b_n Tz_n, \\
t_n &= (1 - \gamma_n)z_n + \gamma_n Tt_n, \\
C_n &= \{z \in C : \|y_n - z\| \leq \|1 - (\gamma_n - b_n)\alpha_n - (2\alpha_n + b_n)\gamma_n - 2b_n\gamma_n (1 + \gamma_n)\|x_n - z\| \} \cap A, \\
Q_n &= \{\{x_{n-1} : \langle x_n - z, x_{n-1} - x_n \rangle \geq 0 \} \}
\end{align*}
\]

converges strongly to \( P_{F(T) x_0} \).

Proof. Take \( T_n \equiv T, L_n \equiv 1 \) in Theorem 2.5, in this case, \( C_n \) is convex and closed and , for all \( n \geq 0 \), by using Theorem 2.5 we obtain Corollary 2.7. Take \( T_n \equiv T, L_n \equiv 1 \) in Theorem 2.5, in this case, \( C_n \) is closed and convex, for all \( n \geq 0 \), by using Theorem 2.5 we obtain Corollary 2.7. \( \square \)
Corollary 2.8. Assume that \( \alpha_n, \beta_n, \gamma_n, a_n \) and \( b_n \in [0, 1] \), \( \alpha_n + \beta_n \in [0, 1] \) and \( a_n + b_n \in [0, 1] \) for all \( n \in N \) and \( \sum_{n=0}^{\infty} (\alpha_n + \beta_n) = \infty \). Then \( \{x_n\} \) generated by

\[
\begin{cases}
x_0 \in C = Q_0, \\
y_n = (1 - \alpha_n - \beta_n) z_n + \alpha_n T z_n + \beta_n T t_n, \quad n \geq 0, \\
z_n = (1 - a_n - b_n) t_n + a_n T t_n + b_n T x_n, \quad n \geq 0, \\
t_n = (1 - \gamma_n) x_n + \gamma_n T x_n, \quad n \geq 0, \\
C_n = \{z \in C : \|y_n - z\| \leq [1 - (\gamma_n - b_n)\alpha_n \\
- (2a_n + b_n)\gamma_n - 2b_n\beta_n(1 + \gamma_n)]\|x_n - z\|} \cap A, \quad n \geq 0, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\
x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{cases}
\]

converges strongly to \( P_{F(T)} x_0 \).

3. Applications

Here, we give an application of our result for the following case of finite family of asymptotically quasi-nonexpansive mappings \( \{T_n\}_{n=0}^{N-1} \). Let

\[
\|T^j_i x - p\| \leq k_{i,j} \|x - p\|, \forall x \in C, p \in F,
\]

where \( F \) is common fixed point set of \( \{T_n\}_{n=0}^{N-1}, \lim_{j \to \infty} \rightarrow k_{i,j} = 1 \) for all \( 0 \leq i \leq N - 1 \). The finite family of asymptotically quasi-nonexpansive mappings \( \{T_n\}_{n=0}^{N-1} \) is uniformly \( L \)-Lipschitz, if

\[
\|T^j_i x - T^j_i y\| \leq L_{i,j} \|x - y\|, \forall x, y \in C
\]

for all \( i \in \{0, 1, 2, ..., N - 1\} \), \( j \geq 1 \), where \( L \geq 1 \).

Theorem 3.1. Let \( \{T_n\}_{n=0}^{N-1} : C \to C \) be a uniformly \( L \)-Lipschitz finit family of asymptotically quasi-nonexpansive mappings with nonempty common fixed point set \( F \). Assume that \( \alpha_n, \beta_n, \gamma_n, a_n \) and \( b_n \in [0, 1] \), \( \alpha_n + \beta_n \in [0, 1] \) and \( a_n + b_n \in \)
converges strongly to $P_Fx_0$, where $n = (j(n) - 1)N + i(n)$ for all $n \geq 0$.

Proof. We can drive the prove from the following two conclusions.

**Conclusion 1.** $\{T_n^{N-1}\}_{n=0}^{\infty}$ is a uniformly closed asymptotically family of countable quasi-Lipschitz mappings from $C$ into itself.

**Conclusion 2.** $F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_i(n))$, where $F(T)$ denotes the fixed point set of the mappings $T$.

□

**Corollary 3.2.** Let $T : C \to C$ be a $L$-Lipschitz asymptotically quasi-nonexpansive mappings with nonempty common fixed point set $F$. Assume that $\alpha_n$, $\beta_n$, $\gamma_n$, $a_n$ and $b_n \in [0, 1]$, $\alpha_n + \beta_n \in [0, 1]$ and $a_n + b_n \in [0, 1]$ for all $n \in N$ and $\sum_{n=0}^{\infty} (\alpha_n + \beta_n) = \infty$. Then $\{x_n\}$ generated by

\[
\begin{aligned}
x_0 &\in C = Q_0, \\
y_n &= (1 - \alpha_n - \beta_n)z_n + \alpha_nT_n^{n}z_n + \beta_nT_n^{n}t_n, \quad n \geq 0, \\
z_n &= (1 - a_n - b_n)t_n + a_nT_n^{n}t_n + b_nT_n^{n}x_n, \quad n \geq 0, \\
t_n &= (1 - \gamma_n)x_n + \gamma_nT_n^{n}x_n, \quad n \geq 0, \\
C_n &= \{z \in C : \|y_n - z\| \leq 1 + (k_n(1 - b_n - 2\gamma_n - 2a_n + 3a_n\gamma_n + 2b_\gamma n + 2a_n\gamma_n + a_n + b_n) + a_n\gamma_nk_\gamma^2 + a_n + b_n - a_n\gamma_n - b_n\gamma_n - 1)\alpha_n + (k_n(1 - a_n - b_n - 2\gamma_n + 2a_n\gamma_n + a_n + b_n) + a_n\gamma_nk_\gamma^2 + a_n + b_n - a_n\gamma_n - b_n\gamma_n - 1)\alpha_n + (k_n(1 - b_n - 2\gamma_n - 2a_n + 3a_n\gamma_n), \\
Q_n &= \{z \in Q_n - 1 : (x_n - z, x_0 - x_n) \geq 0\}, \quad n \geq 1, \\
x_{n+1} &= P_{\cap C_n \cap Q_n}x_0,
\end{aligned}
\]

converges strongly to $P_Fx_0$, where $n = (j(n) - 1)N + i(n)$ for all $n \geq 0$. 

□
converges strongly to \( P_F x_0 \), where \( \overline{co}C_n \) denotes the closed convex closure of \( C_n \) for all \( n \geq 1 \). \( A = \{ z \in H : \| z - P_F x_0 \| \leq 1 \} \).

**Proof.** Take \( T_n \equiv T \) in theorem 3.1, we proved. \( \square \)

**Competing interests**

The author(s) declare that they have no competing interests.

**References**


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