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L^p- BOUNDEDNESS FOR INTEGRAL TRANSFORMS ASSOCIATED WITH SINGULAR PARTIAL DIFFERENTIAL OPERATORS

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ABSTRACT. We define fractional transforms $\mathscr{R}_\mu$ and $\mathscr{H}_\mu$, $\mu > 0$ on the space $\mathbb{R} \times \mathbb{R}^n$. First, we study these transforms on regular function spaces and we establish that these operators are topological isomorphisms and we give the inverse operators as integro differential operators. Next, we study the L^p-boundedness of these operators. Namely, we give necessary and sufficient condition on the parameter μ for which the transforms $\mathscr{R}_\mu$ and $\mathscr{H}_\mu$ are bounded on the weighted spaces $L^p([0, +\infty[\times\mathbb{R}^n, r^{2a}dr \otimes dx)$ and we give their norms.

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1. Introduction

Let $D_j,\ 1 \leq j \leq n$, and $\Xi_\mu,\ \mu > 0$, be the singular partial differential operators defined by

$$\begin{cases} D_j = \dfrac{\partial}{\partial x_j} \\ \Xi_\mu = (\dfrac{\partial}{\partial r})^2 + \dfrac{2\mu}{r}\dfrac{\partial}{\partial r} + \displaystyle\sum_{j=1}^{n}(\dfrac{\partial}{\partial x_j})^2; (r, x) \in]0, +\infty[\times\mathbb{R}^n, \mu > 0. \end{cases}$$

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Ξ_μ is a Bessel-Laplace operator.
When $\mu = \frac{n-1}{2}$; $n \in \mathbb{N}^*$, $\Xi_{\frac{n-1}{2}}$ is the Laplacien operator on $\mathbb{R}^n \times \mathbb{R}^n$ when acting on the functions $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$, that are radial with respect to the first variable.

For every $(\lambda_0, \lambda) \in \mathbb{C} \times \mathbb{C}^n$, the system

$$\begin{cases} D_j u(r,x) = -i\lambda_j u(r,x), 1 \leqslant j \leqslant n \\ \Xi_\mu u(r,x) = -(\lambda_0^2 + \lambda^2) u(r,x) \\ u(0,0) = 1, \dfrac{\partial}{\partial r} u(0,x) = 0, \forall x \in \mathbb{R}^n \end{cases}$$

admits a unique solution given by

$$\psi_{\lambda_0,\lambda}(r,x) = j_{\mu-\frac{1}{2}}(r\lambda_0) e^{-i\langle\lambda|x\rangle}, \tag{1}$$

where
$\lambda^2 = \lambda_1^2 + \lambda_2^2 + ... + \lambda_n^2,\ \lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$
$\langle\lambda|x\rangle = \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n$
$j_{\mu-\frac{1}{2}}$ is the modified Bessel function given by

$$\begin{aligned} j_{\mu-\frac{1}{2}}(s) &= 2^{\mu-\frac{1}{2}} \Gamma(\mu + \frac{1}{2}) \frac{J_{\mu-\frac{1}{2}}(s)}{s^{\mu-\frac{1}{2}}} \\ &= \Gamma(\mu + \frac{1}{2}) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma(\mu + k + \frac{1}{2})} (\frac{s}{2})^{2k} \\ &= \frac{2\ \Gamma(\mu + \frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)} \int_0^1 (1 - t^2)^{\mu-1} \cos(st) dt, \end{aligned}$$

and $J_{\mu-\frac{1}{2}}$ is the Bessel function of first kind and index $\mu - \frac{1}{2}$ ([1, 2, 3, 4]).
The eigenfunction $\psi_{\lambda_0,\lambda}$ allows us to define the Fourier transform $\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}$ connected with the operators $D_j,\ 1 \leqslant j \leqslant n$ and Ξ_μ by

$$\begin{aligned} \widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}(f)(\lambda_0 \lambda) &= \int_0^\infty \int_{\mathbb{R}^n} f(r,x) \psi_{\lambda_0,\lambda}(r,x) d\nu_\mu(r,x) \\ &= \int_0^\infty \int_{\mathbb{R}^n} f(r,x) j_{\mu-\frac{1}{2}}(r\lambda_0) e^{-i\langle\lambda|x\rangle} d\nu_\mu(r,x), \end{aligned} \tag{2}$$

where f is any integrable function on $[0, +\infty[\times \mathbb{R}^n$ with respect to the measure

$$d\nu_\mu(r,x) = \frac{r^{2\mu} dr}{2^{\mu-\frac{1}{2}}\ \Gamma(\mu + \frac{1}{2})} \otimes \frac{dx}{(2\pi)^{\frac{n}{2}}}. \tag{3}$$

Many harmonic analysis results related to the Fourier transform $\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}$ are established ([5, 6, 7, 8, 9, 10]).
Also, many uncertainty principles have been cheked for this transform ([11, 12, 13, 14]).

On the other hand, the eigenfunction $\psi_{\lambda_0,\lambda}$ admits the Poisson integral representation

$$\begin{aligned}\psi_{\lambda_0,\lambda}(r,x) &= \frac{2\ \Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)} r^{1-2\mu}\int_0^r (r^2-t^2)^{\mu-1}\cos(\lambda_0 t)e^{-i\langle\lambda|x\rangle}dt \\ &= \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)}\int_0^1 (1-t^2)^{\mu-1}\cos(\lambda_0 rt)e^{-i\langle\lambda|x\rangle}dt. \end{aligned} \quad (4)$$

Using the relation (4), we define the fractional transform $\mathscr{R}_\mu$ on $\mathscr{C}_e(\mathbb{R}\times\mathbb{R}^n)$ (the space of continuous functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable) by

$$\begin{aligned}\mathscr{R}_\mu(f)(r,x) &= \frac{2\ \Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)} r^{1-2\mu}\int_0^r (r^2-t^2)^{\mu-1} f(t,x)dt;\,(r,x)\in]0,+\infty[\times\mathbb{R}^n \\ &= \frac{2\ \Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)}\int_0^1 (1-t^2)^{\mu-1} f(tr,x)dt;\ (r,x)\in\mathbb{R}\times\mathbb{R}^n. \end{aligned} \quad (5)$$

This involves in particular, that

$$\psi_{\lambda_0,\lambda}(r,x) = \mathscr{R}_\mu\big(\cos(\lambda_0 \cdot)e^{-i\langle\lambda|\cdot\rangle}\big)(r,x), \quad (6)$$

which gives the mutual connecion between the functions $\psi_{\lambda_0,\lambda}$ and $\cos(\lambda_0\cdot)e^{-i\langle\lambda|\cdot\rangle}$. On the other hand, we shall prove in the next section that for every integrable function f on $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure $d\nu_\mu(r,x)$ and for every bounded function g on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, we have the duality relation

$$\int_0^\infty\int_{\mathbb{R}^n} f(r,x)\mathscr{R}_\mu(g)(r,x)d\nu_\mu(r,x) = \int_0^\infty\int_{\mathbb{R}^n} g(r,x)\mathscr{H}_\mu(f)(r,x)dm(r,x) \quad (7)$$

where

dm is the Lebesgue measure on $]0,+\infty[\times\mathbb{R}^n$,

$$dm(r,x) = \sqrt{\frac{2}{\pi}}dr\otimes\frac{dx}{(2\pi)^{\frac{n}{2}}}. \quad (8)$$

$\mathscr{H}_\mu$ is the fractional transform defined by

$$\mathscr{H}_\mu(f)(r,x) = \frac{1}{2^\mu\ \Gamma(\mu)}\int_r^\infty (t^2-r^2)^{\mu-1} f(t,x)2tdt.$$

The relations (2), (6) and (7) show that for all integrable functions f,g on $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure $d\nu_\mu(r,x)$, we have

$$\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}(f) = \Lambda o\mathscr{H}_\mu(f) \quad (9)$$

and

$$\mathscr{H}_\mu(f*g) = \mathscr{H}_\mu(f)*_o\mathscr{H}_\mu(g), \quad (10)$$

where
Λ is the usual Fourier transform defined by

$$\Lambda(f)(\lambda_0,\lambda) = \int_0^{\infty}\int_{\mathbb{R}^n} f(r,x)\cos(\lambda_0 r)e^{-i\langle\lambda|x\rangle}dm(r,x),$$

$*$ is the convolution product associated with the Fourier transform $\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}$,
$*_o$ is the usual convolution product defined by

$$f *_o g(r,x) = \int_0^{\infty}\int_{\mathbb{R}^n} f(s,y)\sigma_{r,x}(g)(s,-y)dm(s,y)$$

and $\sigma_{r,x}$ is the usual translation operator given by

$$\sigma_{r,x}(f)(s,y) = \frac{1}{2}\,\big(f(r+s,x+y)+f(|r-s|,x+y)\big). \tag{11}$$

Our purpose in this work is to study the fractional transforms $\mathscr{R}_\mu$ and $\mathscr{H}_\mu$ in two ways.
In the second section, we will prove that the operator $\mathscr{R}_\mu$ is a topological isomorphism from $\mathscr{E}_e\big(\mathbb{R}\times\mathbb{R}^n\big)$ (the space of infinitely differentiable functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable) onto itself and we give the inverse operator $\mathscr{R}_\mu^{-1}$ as integro-differential operator .
Next, we show that the fractional transform $\mathscr{H}_\mu$ can be extended to $\mu\in\ \mathbb{R}$ and that for every $\mu\in\ \mathbb{R}$, $\mathscr{H}_\mu$ is a topological isomorphism from the Schwartz's space $\mathscr{S}_e\big(\mathbb{R}\times\mathbb{R}^n\big)$ (the subspace of $\mathscr{E}_e\big(\mathbb{R}\times\mathbb{R}^n\big)$ consisting of rapidly decreasing functions together with all their derivatives) onto itself whose inverse operator is $\mathscr{H}_\mu^{-1} = \mathscr{H}_{-\mu}$.
The precedent results imply in particular that $\mathscr{R}_\mu$ and $\mathscr{H}_\mu$ are transmutation operators of $D_j,\ 1\le j\le n$, and Ξ_μ to $D_j,\ 1\le j\le n$ and Δ, where

$$\Delta = (\frac{\partial}{\partial r})^2+\sum_{j=1}^{n}(\frac{\partial}{\partial x_j})^2.$$

That is, for every $f\in\mathscr{E}_e\big(\mathbb{R}\times\mathbb{R}^n\big)$

$$\begin{aligned} D_j\mathscr{R}_\mu(f) &= \mathscr{R}_\mu D_j(f),\ 1\leqslant j\leqslant n\\ \Xi_\mu\mathscr{R}_\mu(f) &= \mathscr{R}_\mu\ \Delta(f),\end{aligned}$$

and for every $f\in\mathscr{S}_e\big(\mathbb{R}\times\mathbb{R}^n\big)$

$$\begin{aligned} D_j\mathscr{H}_\mu(f) &= \mathscr{H}_\mu D_j(f),\ 1\leqslant j\leqslant n\\ \Delta\mathscr{H}_\mu(f) &= \mathscr{H}_\mu\ \Xi_\mu(f).\end{aligned}$$

The third section contains the main results of this paper. In fact, we study the L^p- boundedness of the operators $\mathscr{R}_\mu$ and $\mathscr{H}_\mu$ on the weighted spaces $L^p\big([0,+\infty[\times\mathbb{R}^n, r^{2a}dr\otimes dx\big),\ p\in\ [1,+\infty]$. We recall in this context, that studing the L^p- boundedness of integral transforms connected with differential systems is an interesting subject because knowing the range of parameters $\mu,\ p$ for which an operator is bounded on Lebesgue space gives quantitative information about

the rate of growth of the transformed functions ([15, 16, 17]) .
In this work, we give necessary and sufficient conditions on the parameters $\mu,\ a,\ p$ for which the operator $\mathscr{R}_\mu$ (respectively $\mathscr{H}_\mu$) satisfies

$$||\mathscr{R}_\mu(f)||_{p,a} \quad \leqslant \quad C_{p,a,\mu}\ ||f||_{p,a}, \tag{12}$$

respectively

$$||\mathscr{H}_\mu(f)||_{p,a} \quad \leqslant \quad D_{p,a,\mu}\ ||r^{2\mu}f||_{p,a}. \tag{13}$$

Moreover, we give the best (the smallest) contants $C_{p,a,\mu}$ and $D_{p,a,\mu}$ that satisfy the relations (12) and (13) .

2. Fractional transforms

2.1. The fractional transform $\mathscr{R}_\mu$. The space $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ is equipped with the topology generated by the family of semi-norms

$$P_{m,k}(f) \quad = \quad \sup_{\substack{||(r,x)||\leqslant m\\ |\alpha|\leqslant k}} \big|D^\alpha(f)(r,x)\big|,\ (m,k)\in\mathbb{N}^2.$$

and the distance

$$d(f,g) \quad = \quad \sum_{m,k=0}^{+\infty}(\frac{1}{2})^{m+k}\frac{P_{m,k}(f-g)}{1+P_{m,k}(f-g)}.$$

Lemma 2.1. ***i.*** *For every $\mu>0$, the transform $\mathscr{R}_\mu$ is continuous from $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ into itself.*
ii. *The operator $\frac{\partial}{\partial r^2}=\frac{1}{r}\frac{\partial}{\partial r}$ is continuous from $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ into itself.*

Proof. **i.** For every $f\in\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\mathscr{R}_\mu(f)(r,x) \quad = \quad \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)}\int_0^1(1-t^2)^{\mu-1}f(tr,x)dt,$$

this shows that the function $\mathscr{R}_\mu(f)$ belongs to the space $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$. Moreover, for every $(\alpha_0,\alpha)\in\mathbb{N}\times\mathbb{N}^n$

$$D^{(\alpha_0,\alpha)}(\mathscr{R}_\mu(f))(r,x) \quad = \quad \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)}\int_0^1(1-t^2)^{\mu-1}t^{\alpha_0}D^{(\alpha_0,\alpha)}(f)(tr,x)dt,$$

thus, for every $(m,k)\in\mathbb{N}^2, P_{m,k}(\mathscr{R}_\mu(f))\leqslant P_{m,k}(f)$.
ii. For every $f\in\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$

$$\frac{\partial}{\partial r^2}(f)(r,x) \quad = \quad \int_0^1\frac{\partial^2 f}{\partial t^2}(rt,x)dt.$$

Hence, the function $\frac{\partial}{\partial r^2}(f)$ belongs to the space $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ and for every $(\alpha_0,\alpha)\in\mathbb{N}\times\mathbb{N}^n$

$$D^{(\alpha_0,\alpha)}(\frac{\partial}{\partial r^2}f)(r,x) \quad = \quad \int_0^1 t^{\alpha_0}D^{(\alpha_0+2,\alpha)}(f)(rt,x)dt,$$

so, for every $(m,k)\in\mathbb{N}^2, P_{m,k}\big(\frac{\partial}{\partial r^2}(f)\big)\leqslant P_{m,k+2}(f)$. □

In the following, we shall prove that $\mathscr{R}_\mu$ is a topological isomorphism from $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ onto itself and we give the inverse operator. For this we need following notations:
$r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ is the space defined by $r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)=\big\{f:\mathbb{R}\backslash\{0\}\times\mathbb{R}^n\longrightarrow\mathbb{C}, f$ is even with respect to the first variable and $f(r,x)=r^{2a}g(r,x),\ g\in\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)\big\}$
$r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ is equipped by the family of semi-norms

$$\widetilde{P}_{m,k,a}(f)=P_{m,k}(r^{-2a}f).$$

$\widetilde{\mathscr{R}_\mu}$ is the transform defined on $r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n),\ a>-\frac{1}{2}$, by

$$\widetilde{\mathscr{R}_\mu}(f)(r,x) = \frac{2r}{2^\mu\ \Gamma(\mu)}\int_0^r (r^2-t^2)^{\mu-1}f(t,x)dt,\ r>0.$$

Proposition 2.2. ***i.*** *For every $a>-\frac{1}{2}$, the operator □ defined by*

$$\Box(f)(r,x) = \frac{\partial}{\partial r}\Big(\frac{f(r,x)}{r}\Big)$$

is continuous from $r^{2(a+1)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ into $r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$.
ii. *The transform $\widetilde{\mathscr{R}_\mu}$ is continuous from $r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ into $r^{2(a+\mu)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$.*

Proof. **i.** Let $f\in r^{2(a+1)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n); f(r,x)=r^{2a+2}g(r,x), g\in\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$

$$\Box f(r,x)=r^{2a}\big((2a+1)g(r,x)+r\frac{\partial g}{\partial r}(r,x)\big).$$

Since, the map : $g\longrightarrow(2a+1)g+r\dfrac{\partial g}{\partial r}$ is continuous from $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ into itself, then, the function $\Box(f)$ belongs to $r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$.
Moreover, for every $(m,k)\in\mathbb{N}^2$

$$\begin{aligned}\widetilde{P}_{m,k,a}(\Box(f)) &= P_{m,k}\big((2a+1)g+r\frac{\partial g}{\partial r}\big)\\ &\leqslant CP_{m',k'}(g)=C\widetilde{P}_{m',k',a+1}(f),\end{aligned}$$

where C is a constant.
ii. For every $f\in r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n), f=r^{2a}g,\ g\in\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ and $a>-\frac{1}{2}$, the function

$$\begin{aligned}\widetilde{\mathscr{R}_\mu}(f)(r,x) &= \frac{2r}{2^\mu\ \Gamma(\mu)}\int_0^r(r^2-t^2)^{\mu-1}t^{2a}g(t,x)dt\\ &= \frac{2r^{2a+2\mu}}{2^\mu\ \Gamma(\mu)}\int_0^1(1-t^2)^{\mu-1}t^{2a}g(tr,x)dt\end{aligned}$$

belongs to the space $r^{2(a+\mu)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$, and for every $(m,k)\in\mathbb{N}^2$

$$\widetilde{P}_{m,k,a+\mu}(\widetilde{\mathscr{R}_\mu}(f)) = P_{m,k}\Big(\frac{2}{2^\mu\ \Gamma(\mu)}\int_0^1(1-t^2)^{a-1}t^{2a}g(tr,x)dt\Big)$$

$$\begin{aligned} &\leqslant \frac{\Gamma(a+\frac{1}{2})}{2^{\mu}\ \Gamma(\mu+a+\frac{1}{2})} P_{m,k}(g) \\ &= \frac{\Gamma(a+\frac{1}{2})}{2^{\mu}\ \Gamma(\mu+a+\frac{1}{2})} \widetilde{P}_{m,k,a}(f). \end{aligned}$$

□

Proposition 2.3. *For all $\mu,\nu>0$ and $f\in r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n),\ a>-\frac{1}{2}$, we have*

$$\widetilde{\mathscr{R}_\mu}\circ\widetilde{\mathscr{R}_\nu}(f) \quad=\quad \widetilde{\mathscr{R}}_{\mu+\nu}(f).$$

Proof. For all $\mu\ ,\nu>0$ and $f\in r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n),\ a>-\frac{1}{2}$,

$$\begin{aligned} &\widetilde{\mathscr{R}_\mu}\circ\widetilde{\mathscr{R}_\nu}(f)(r,x) \\ &= \frac{2r}{2^{\mu+\nu}\ \Gamma(\mu)\ \Gamma(\nu)}\int_0^r (r^2-t^2)^{\mu-1}2t\Big(\int_0^t (t^2-s^2)^{\nu-1}f(s,x)ds\Big)dt. \end{aligned}$$

Applying Fubini's theorem we get

$$\begin{aligned} &\widetilde{\mathscr{R}_\mu}\circ\widetilde{\mathscr{R}_\nu}(f)(r,x) \\ &= \frac{2r}{2^{\mu+\nu}\ \Gamma(\mu)\ \Gamma(\nu)}\int_0^r f(s,x)\Big(\int_s^r (r^2-t^2)^{\mu-1}(t^2-s^2)^{\nu-1}2tdt\Big)ds, \end{aligned}$$

however, $\displaystyle\int_s^r (r^2-t^2)^{\mu-1}(t^2-s^2)^{\nu-1}2tdt = \frac{\Gamma(\mu)\ \Gamma(\nu)}{\Gamma(\mu+\nu)}(r^2-s^2)^{\mu+\nu-1}$.
This completes the proof. □

Proposition 2.4. ***i.*** *For every $\mu>1$ and $f\in r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n),\ a>-\frac{1}{2}$, we have*

$$\Box\widetilde{\mathscr{R}}_\mu(f) \quad=\quad \widetilde{\mathscr{R}}_{\mu-1}(f).$$

In particular, for every $\mu>0,\ k\in\mathbb{N}$

$$\Box^k\widetilde{\mathscr{R}}_{\mu+k}(f) \quad=\quad \widetilde{\mathscr{R}_\mu}(f). \tag{14}$$

ii. *For every $f\in r^{2(a+1)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n),\ a>-\frac{1}{2}$ and $\mu>0$*

$$\widetilde{\mathscr{R}_\mu}(\Box f) \quad=\quad \Box\widetilde{\mathscr{R}_\mu}(f). \tag{15}$$

In particular, for every $f\in r^{2(a+k)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n),\ a>-\frac{1}{2},\ k\in\mathbb{N}$

$$\widetilde{\mathscr{R}_\mu}(\Box^k(f)) \quad=\quad \Box^k\widetilde{\mathscr{R}_\mu}(f). \tag{16}$$

Proof. **i.** Let $f\in r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$,

$$\begin{aligned} \Box\widetilde{\mathscr{R}\mu}(f)(r,x) &= \frac{\partial}{\partial r}\Big(\frac{2}{2^{\mu}\ \Gamma(\mu)}\int_0^r (r^2-t^2)^{\mu-1}f(t,x)dt\Big) \\ &= \frac{2.2r(\mu-1)}{2^{\mu}\ \Gamma(\mu)}\int_0^r (r^2-t^2)^{\mu-2}f(t,x)dt \\ &= \widetilde{\mathscr{R}}_{\mu-1}(f)(r,x), \end{aligned}$$

and by induction, we deduce that for all $\mu > 0,\ k \in \mathbb{N}$

$$\Box^k \widetilde{\mathscr{R}}_{\mu+k}(f) \quad = \quad \widetilde{\mathscr{R}}_{\mu}(f).$$

ii. Let $f \in r^{2(a+1)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$, by Proposition 2.2, the function $\Box(f)$ belongs to the space $r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ and we have

$$\widetilde{\mathscr{R}}_{\mu}(f)(r,x) = \frac{r}{2^{\mu}\ \Gamma(\mu+1)}\int_0^r -\frac{\partial}{\partial t}\big((r^2-t^2)^{\mu}\big)\frac{f(t,x)}{t}dt.$$

Integrating by parts, we get

$$\widetilde{\mathscr{R}}_{\mu}(f)(r,x) = \frac{r}{2^{\mu}\ \Gamma(\mu+1)}\int_0^r (r^2-t^2)^{\mu}\Box f(t,x)dt,$$

so,

$$\begin{aligned}\Box\widetilde{\mathscr{R}}_{\mu}(f)(r,x) &= \frac{2r}{2^{\mu}\ \Gamma(\mu)}\int_0^r (r^2-t^2)^{\mu-1}\Box f(t,x)dt \\ &= \widetilde{\mathscr{R}}_{\mu}(\Box f)(r,x).\end{aligned}$$

Now, suppose that for every $f \in r^{2(a+k)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$, $\Box^k\widetilde{\mathscr{R}}_{\mu}(f) = \widetilde{\mathscr{R}}_{\mu}(\Box^k f)$, let $g \in r^{2(a+k+1)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$.
Then, the function $\Box g$ belongs to $r^{2(a+k)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$, and by hypothesis

$$\Box^k\widetilde{\mathscr{R}}_{\mu}(\Box g)(r,x) = \widetilde{\mathscr{R}}_{\mu}(\Box^{k+1}g),$$

on the other hand, by relation(15) and the fact that $\Box g \in r^{2(a+k)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n) \subset r^{2(a+1)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\Box^k\widetilde{\mathscr{R}}_{\mu}(\Box g)(r,x) = \Box^{k+1}\widetilde{\mathscr{R}}_{\mu}(g).$$

The proof is complete by induction. □

Theorem 2.5. *For every $k \in \mathbb{N}\backslash\{0\}$, the operator $\widetilde{\mathscr{R}}_k$ is an isomorphism from $r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ onto $r^{2(a+k)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$; $a > -\frac{1}{2}$. The inverse operator is given by*

$$\widetilde{\mathscr{R}_k}^{-1} = \Box^k.$$

Proof. Let $f \in r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$. From Proposition 2.2, the function $\widetilde{\mathscr{R}}_k(f)$ belongs to $r^{2(a+k)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ and by relation(14), we have

$$\begin{aligned}\Box^k\widetilde{\mathscr{R}}_k(f) &= \Box\Box^{k-1}\widetilde{\mathscr{R}}_{1+(k-1)}(f) \\ &= \Box\widetilde{\mathscr{R}}_1(f) \\ &= f.\end{aligned}$$

Let $g \in r^{2(a+k)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n) \subset r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$, by relation(16)

$$\begin{aligned}\widetilde{\mathscr{R}}_k(\Box^k(g)) &= \Box^k\widetilde{\mathscr{R}}_k(g) \\ &= g.\end{aligned}$$

This achieves the proof. □

Theorem 2.6. *For every $\mu \in]0,1[$, the fractional transform $\widetilde{\mathscr{R}_\mu}$ is an isomorphism from $r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ onto $r^{2(a+\mu)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$, $a > -\frac{1}{2}$. The inverse operator is given by*

$$\widetilde{\mathscr{R}_\mu}^{-1} = \Box\widetilde{\mathscr{R}}_{1-\mu}.$$

Proof. Let $g \in r^{2(a+\mu)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$,

$$g(r,x) = r^{2a+2\mu}h(r,x);\;\; h \in \mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n),$$

$$\begin{aligned}
\Box\widetilde{\mathscr{R}}_{1-\mu}(g)(r,x) &= \frac{\partial}{\partial r}\Big(\frac{2}{2^{1-\mu}\Gamma(1-\mu)}\int_0^r (r^2-t^2)^{-\mu}t^{2a+2\mu}h(t,x)dt\Big)\\
&= \frac{\partial}{\partial r}\Big(\frac{2r^{2a+1}}{2^{1-\mu}\Gamma(1-\mu)}\int_0^1 (1-t^2)^{-\mu}t^{2a+2\mu}h(tr,x)dt\Big)\\
&= 2(2a+1)\frac{r^{2a}}{2^{1-\mu}\;\Gamma(1-\mu)}\int_0^1 (1-t^2)^{-\mu}t^{2a+2\mu}h(tr,x)dt\\
&+ 2\frac{r^{2a+1}}{2^{1-\mu}\Gamma(1-\mu)}\int_0^1 (1-t^2)^{-\mu}t^{2a+2\mu+1}\frac{\partial h}{\partial t}(tr,x)dt\\
&= 2\frac{(2a+1)}{2^{1-\mu}\Gamma(1-\mu)}\frac{1}{r}\int_0^r (r^2-t^2)^{-\mu}t^{2a+2\mu}h(t,x)dt\\
&+ \frac{2}{2^{1-\mu}\;\Gamma(1-\mu)}\frac{1}{r}\int_0^r (r^2-t^2)^{-\mu}t^{2a+2\mu+1}\frac{\partial h}{\partial t}(t,x)dt.
\end{aligned}$$

We deduce that

$$\begin{aligned}
&\widetilde{\mathscr{R}}_\mu\Big(\Box\widetilde{\mathscr{R}}_{1-\mu}(g)\Big)(r,x)\\
=\;& \frac{2(2a+1)2r}{2\Gamma(\mu)\;\Gamma(1-\mu)}\int_0^r (r^2-t^2)^{\mu-1}\frac{1}{t}\Big(\int_0^t (t^2-s^2)^{-\mu}s^{2a+2\mu}h(s,x)ds\Big)dt\;+\\
& \frac{2.2r}{2\Gamma(\mu)\;\Gamma(1-\mu)}\int_0^r (r^2-t^2)^{\mu-1}\frac{1}{t}\Big(\int_0^t (t^2-s^2)^{-\mu}s^{2a+2\mu+1}\frac{\partial h}{\partial s}(s,x)ds\Big)dt\\
=\;& I_{1,\mu}(r,x) + I_{2,\mu}(r,x).
\end{aligned}$$

From Fubini's theorem, we have

$$I_{1,\mu}(r,x) = \frac{(2a+1)r}{\Gamma(\mu)\;\Gamma(1-\mu)}\int_0^r h(s,x)\Big(\int_s^r (r^2-t^2)^{\mu-1}(t^2-s^2)^{-\mu}\frac{2t}{t^2}dt\Big)s^{2a+2\mu}ds.$$

Let

$$J(r,s) = \int_s^r (r^2-t^2)^{\mu-1}(t^2-s^2)^{-\mu}\frac{2t}{t^2}dt.$$

By the change of variables $\omega = \frac{r^2-t^2}{r^2-s^2}$, we get

$$J(r,s) \;=\; \frac{1}{r^2}\int_0^1 \frac{\omega^{\mu-1}(1-\omega)^{-\mu}}{1-\frac{r^2-s^2}{r^2}\omega}d\omega$$

$$\begin{aligned} &= \frac{1}{r^2}\sum_{k=0}^{\infty}(\frac{r^2-s^2}{r^2})^k\int_0^1 \omega^{k+\mu-1}(1-\omega)^{-\mu}d\omega \\ &= \frac{\Gamma(1-\mu)}{r^2}\sum_{k=0}^{\infty}\frac{\Gamma(k+\mu)}{k!}(\frac{r^2-s^2}{r^2})^k \\ &= \Gamma(\mu)\ \Gamma(1-\mu)r^{2\mu-2}s^{-2\mu}. \end{aligned}$$

So,

$$I_{1,\mu}(r,x) = (2a+1)r^{2\mu-1}\int_0^r h(s,x)s^{2a}ds$$

As the same way,

$$\begin{aligned} &I_{2,\mu}(r,x) \\ =\ & \frac{r}{\Gamma(\mu)\ \Gamma(1-\mu)}\int_0^r \frac{\partial h}{\partial s}(s,x)\big(\int_s^r (r^2-t^2)^{\mu-1}(t^2-s^2)^{-\mu}\frac{2t}{t^2}dt\big)s^{2a+2\mu+1}ds \\ =\ & r^{2\mu-1}\int_0^r \frac{\partial h}{\partial s}(s,x)s^{2a+1}ds. \end{aligned}$$

Consequently,

$$\begin{aligned} \widetilde{\mathscr{R}}_\mu\big(\Box\widetilde{\mathscr{R}}_{1-\mu}(g)\big)(r,x) &= r^{2\mu-1}\int_0^r \Big((2a+1)s^{2a}h(s,x)+s^{2a+1}\frac{\partial h}{\partial s}(s,x)\Big)ds \\ &= r^{2\mu-1}\int_0^r \frac{\partial}{\partial s}\big(s^{2a+1}h(s,x)\big)ds \\ &= r^{2a+2\mu}h(r,x),\ \text{because } a>-\frac{1}{2} \\ &= g(r,x). \end{aligned}$$

On the other hand, from Proposition 2.3 and for every $f\in r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$,

$$\begin{aligned} \Box\widetilde{\mathscr{R}}_{1-\mu}\widetilde{\mathscr{R}}_\mu(f) &= \Box\widetilde{\mathscr{R}}_1(f) \\ &= f. \end{aligned}$$

This completes the proof. □

Lemma 2.7. *Let $\mu\in\mathbb{R},\ \mu\geqslant 0$. For every $k_1,\ k_2\in\mathbb{N}\backslash\{0\},\ k_1-\mu>0,\ k_2-\mu>0$ and for every $f\in r^{2(a+\mu)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$, we have*

$$\Box^{k_1}\widetilde{\mathscr{R}}_{k_1-\mu}(f) = \Box^{k_2}\widetilde{\mathscr{R}}_{k_2-\mu}(f).$$

Proof. Let $k_1,\ k_2\in\mathbb{N}\backslash\{0\},\ k_1-\mu>0,\ k_2-\mu>0$, and $k_1<k_2$,

$$\Box^{k_2}\widetilde{\mathscr{R}}_{k_2-\mu}(f) = \Box^{k_1}\Box^{k_2-k_1}\widetilde{\mathscr{R}}_{k_2-k_1+(k_1-\mu)}(f),$$

applying relation (14), we get

$$\Box^{k_2}\widetilde{\mathscr{R}}_{k_2-\mu}(f) = \Box^{k_1}\widetilde{\mathscr{R}}_{k_1-\mu}(f).$$

□

The previous Lemma allows us to define the fractional transform $\widetilde{\mathscr{R}_\mu}$ for every $\mu \in \mathbb{R}$.

Definition 2.8. For every $\mu \in \mathbb{R},\ \mu \geqslant 0$, the fractional transform $\widetilde{\mathscr{R}_{-\mu}}$ is defined on $r^{2(a+\mu)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ by

$$\widetilde{\mathscr{R}_{-\mu}}(f) = \Box^k \widetilde{\mathscr{R}}_{k-\mu}(f),$$

where $k \in \mathbb{N}\backslash\{0\},\ k-\mu > 0$.
In particular, for $f \in r^{2(a+\mu)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$

$$\widetilde{\mathscr{R}_{-\mu}}(f) = \Box^{E(\mu)+1}\widetilde{\mathscr{R}}_{E(\mu)+1-\mu}(f),$$

where $E(\mu)$ is the entire party of μ.

Remark 2.9. According to definition 2.8 and for every $f \in r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n),\ a > -\frac{1}{2}$, we have

$$\widetilde{\mathscr{R}_0}(f) = \Box\widetilde{\mathscr{R}_1}(f) = f,$$

that is

$$\widetilde{\mathscr{R}_0} = Id_{r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)}.$$

Theorem 2.10. *For $\mu > 0$, the fractional transform $\widetilde{\mathscr{R}_\mu}$ is a topological isomorphism from $r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ onto $r^{2(a+\mu)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n),\ a > -\frac{1}{2}$. The inverse operator is given by*

$$\widetilde{\mathscr{R}_\mu}^{-1} = \widetilde{\mathscr{R}_{-\mu}}.$$

Proof. For $\mu \in \mathbb{N}$, the result follows from Theorem 2.5 and Remark 2.9. Let $\mu \in]0,+\infty[\backslash\mathbb{N}$, for every $f \in r^{2a}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ and from Proposition 2.3 and Theorem 2.5, we have

$$\begin{aligned}\widetilde{\mathscr{R}_{-\mu}}\big(\widetilde{\mathscr{R}_\mu}(f)\big) &= \Box^{E(\mu)+1}\widetilde{\mathscr{R}}_{E(\mu)+1-\mu}\big(\widetilde{\mathscr{R}}_\mu(f)\big)\\ &= \Box^{E(\mu)+1}\widetilde{\mathscr{R}}_{E(\mu)+1}(f)\\ &= f.\end{aligned}$$

Conversely, for every $g \in r^{2(a+\mu)}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$,

$$\widetilde{\mathscr{R}_\mu}\circ\widetilde{\mathscr{R}_{-\mu}}(g) = \widetilde{\mathscr{R}_\mu}\Box^{E(\mu)+1}\widetilde{\mathscr{R}}_{E(\mu)+1-\mu}(g),$$

let $\nu = \mu - E(\mu)$, then $\nu \in]0,1[$, and

$$\widetilde{\mathscr{R}_\mu}\circ\widetilde{\mathscr{R}_{-\mu}}(g) = \widetilde{\mathscr{R}_\nu}\widetilde{\mathscr{R}}_{E(\mu)}\Box^{E(\mu)}\Box\widetilde{\mathscr{R}}_{1-\nu}(g).$$

Since, $\Box\widetilde{\mathscr{R}}_{1-\nu}(g)$ belongs to $r^{2(a+E(\mu))}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$, then, Theorem 2.5 involves that

$$\widetilde{\mathscr{R}}_\mu\circ\widetilde{\mathscr{R}}_{-\mu}(g) = \widetilde{\mathscr{R}}_\nu\Box\widetilde{\mathscr{R}}_{1-\nu}(g).$$

The result follows from Theorem 2.6. $\Box$

Now, we have the following important result.

Theorem 2.11. *For every $\mu > 0$, the fractional transform $\mathscr{R}_\mu$ defined by relation (5) is a topological isomorphism from $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ onto itself.*

Proof. For every $f \in \mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$,

$$\mathscr{R}_\mu(r,x) = \frac{2^\mu\ \Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}} r^{-2\mu}\widetilde{\mathscr{R}_\mu}(f)(r,x).$$

From Theorem 2.10, the transform $\widetilde{\mathscr{R}_\mu}$ is a topological isomorphism from $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ onto $r^{2\mu}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$. On the other hand, the map

$$f \longrightarrow r^{-2\mu} f$$

is a topological isomorphism from $r^{2\mu}\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ onto $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$.
Consequently, $\mathscr{R}_\mu$ is a topological isomorphism from $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ onto itself.
Moreover,

$$\begin{aligned}\mathscr{R}_\mu^{-1}(f)(r,x) &= \frac{\sqrt{\pi}}{2^\mu\ \Gamma(\mu+\frac{1}{2})}\widetilde{\mathscr{R}_{-\mu}}\big(r^{2\mu}f\big)(r,x)\\ &= \frac{\sqrt{\pi}}{2^\mu\ \Gamma(\mu+\frac{1}{2})}\square^{E(\mu)+1}\widetilde{\mathscr{R}}_{E(\mu)+1-\mu}\big(r^{2\mu}f\big)(r,x).\end{aligned}$$

□

2.2. The fractional transform $\mathscr{H}_\mu$. We recall that the space $\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ is equipped with the topology generated by the family of norms

$$N_m(f) = \max_{\substack{(r,x)\in\mathbb{R}\times\mathbb{R}^n\\ k+|\alpha|\leqslant m}} (1+r^2+|x|^2)^k|D^\alpha(f)(r,x)|,\ m\in\mathbb{N}.$$

By a standard argument, for every $f \in \mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$, the function $\frac{\partial}{\partial r^2}(f)$ belongs to $\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ and for every $m\in\mathbb{N}$,

$$N_m\big(\frac{\partial}{\partial r^2}(f)\big) \leqslant 2^{m+1}N_{m+3}(f).$$

This shows that the operator $\frac{\partial}{\partial r^2}$ is continuous from $\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ into itself and consequently the operator Ξ_μ is also continuous from $\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ into itself.
On the other hand, for every $f\in\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ and for every $k\in\mathbb{N}$, we have

$$(1+\lambda_0^2+|\lambda|^2)^k\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}(f)(\lambda_0,\lambda) = \widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}\big((I-\Xi_\mu)^k(f)\big)(\lambda_0,\lambda). \quad (17)$$

Where I is the identity operator.
Using the relation (17) and the inversion formula for $\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}$ that is for every $f\in L^1(d\nu_\mu)$ such that $\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}(f)$ belongs to $L^1(d\nu_\mu)$, we have

$$f = \widetilde{\mathscr{F}}_{\mu-\frac{1}{2}} o \widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}(\check{f}) \text{ a.e},$$

we deduce that the transform $\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}$ is a topolgical isomorphism from $\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ onto itself and

$$\widetilde{\mathscr{F}}^{-1}_{\mu-\frac{1}{2}}(f)=\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}(\check{f})$$

where $\check{f}(r,x) \;=\; f(r,-x)$.

Lemma 2.12. *For every $f\in L^1(d\nu_\mu)$ and $\mu>0$, the function*

$$\mathscr{H}_\mu(f)(t,x)=\frac{1}{2^\mu\;\Gamma(\mu)}\int_t^\infty (r^2-t^2)^{\mu-1}f(r,x)2rdr,$$

is defined almost every where, belongs to $L^1(dm)$, where dm is the Lebesgue measure given by relation (8), and we have

$$||\mathscr{H}_\mu(f)||_{1,m}\leqslant ||f||_{1,\nu_\mu}.$$

Proof. By Fubini-Tonnelli Theorem's, we have

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^n}|\mathscr{H}_\mu(f)(t,x)|dm(t,x)\\
\leqslant\;& \sqrt{\frac{2}{\pi}}\frac{1}{2^\mu\;\Gamma(\mu)(2\pi)^{\frac{n}{2}}}\int_0^\infty\int_{\mathbb{R}^n}\Big(\int_t^\infty (r^2-t^2)^{\mu-1}|f(r,x)|2rdr\Big)dtdx\\
=\;& \sqrt{\frac{2}{\pi}}\frac{1}{2^\mu\;\Gamma(\mu)(2\pi)^{\frac{n}{2}}}\int_0^\infty\int_{\mathbb{R}^n}|f(r,x)|\Big(\int_0^r (r^2-t^2)^{\mu-1}dt\Big)2rdrdx\\
=\;& \frac{1}{2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})(2\pi)^{\frac{n}{2}}}\int_0^\infty\int_{\mathbb{R}^n}|f(r,x)|r^{2\mu}drdx\\
=\;& \|f\|_{1,\nu_\mu}.
\end{aligned}$$

□

Proposition 2.13. ***i.*** *For every $f\in L^1(d\nu_\mu)$ and every bounded measurable function g on $[0,+\infty[\times\mathbb{R}^n$, we have the duality relation*

$$\int_0^\infty\int_{\mathbb{R}^n}f(r,x)\mathscr{R}_\mu(g)(r,x)d\nu_\mu(r,x)=\int_0^\infty\int_{\mathbb{R}^n}\mathscr{H}_\mu(f)(r,x)g(r,x)dm(r,x).$$

ii. *For every $f\in L^1(d\nu_\mu)$*

$$\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}(f)\;=\;\Lambda\circ\mathscr{H}_\mu(f), \tag{18}$$

where, Λ is the usual Fourier transform defined on $L^1(dm)$ by

$$\Lambda(f)(\lambda_0,\lambda)=\int_0^\infty\int_{\mathbb{R}^n}f(r,x)\cos(r\lambda_0)e^{-i\langle\lambda|x\rangle}dm(r,x).$$

Proof. **i.** It is clear that for every bounded function g on $[0,+\infty[\times\mathbb{R}^n$, the function $\mathscr{R}_\mu(g)$ is also bounded on $[0,+\infty[\times\mathbb{R}^n$.

Consequently, the integral $\int_0^\infty\int_{\mathbb{R}^n} f(r,x)\mathscr{R}_\mu(g)(r,x)d\nu_\mu(r,x)$ is well defined, and we have

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^n} f(r,x)\mathscr{R}_\mu(g)(r,x)d\nu_\mu(r,x) &= \int_0^\infty\int_{\mathbb{R}^n} f(r,x)\frac{2r}{2^{\mu-\frac{1}{2}}\sqrt{\pi}\,(2\pi)^{\frac{n}{2}}\Gamma(\mu)}\\ &\times \Big(\int_0^r (r^2-t^2)^{\mu-1}g(t,x)dt\Big)drdx.\end{aligned}$$

By Fubini's Theorem,

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^n} f(r,x)\mathscr{R}_\mu(g)(r,x)d\nu_\mu(r,x)\\ =&\int_0^\infty\int_{\mathbb{R}^n} g(t,x)\big(\frac{1}{2^\mu\,\Gamma(\mu)}\int_t^\infty(r^2-t^2)^{\mu-1}f(r,x)2rdr\big)\times\sqrt{\frac{2}{\pi}}dt\frac{dx}{(2\pi)^{\frac{n}{2}}}\\ =&\int_0^\infty\int_{\mathbb{R}^n} g(t,x)\mathscr{H}_\mu(f)(t,x)dm(t,x).\end{aligned}$$

ii. Let $f\in L^1(d\nu_\mu)$, we have

$$\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}(f)(\lambda_0,\lambda)=\int_0^\infty\int_{\mathbb{R}^n} f(r,x)\Psi_{\lambda_0,\lambda}(r,x)d\nu_\mu(r,x)$$

and by the relation (6),

$$\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}(f)(\lambda_0,\lambda)=\int_0^\infty\int_{\mathbb{R}^n} f(r,x)\mathscr{R}_\mu\big(\cos(\lambda_0.)e^{-i\langle\lambda|.\rangle}\big)(r,x)d\nu_\mu(r,x),$$

and by the relation of duality, Proposition 2.13, we obtain

$$\begin{aligned}\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}(f)(\lambda_0,\lambda) &= \int_0^\infty\int_{\mathbb{R}^n}\mathscr{H}_\mu(f)(r,x)\cos(\lambda_0 r)e^{-i\langle\lambda|x\rangle}dm(r,x)\\ &= \Lambda\circ\mathscr{H}_\mu(f)(\lambda_0,\lambda).\end{aligned}$$

□

Corollary 2.14. *For every $\mu>0$, the fractional transform $\mathscr{H}_\mu$ is a topological isomorphism from $\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ onto itself.*

Proof. Since the Fourier transforms Λ and $\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}$ are topological isomorphisms from $\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ onto itself, the result follows from the relation (18). □

Next, we will prove that the fractional transform $\mathscr{H}_\mu$ can be extended to $\mu\in\mathbb{R}$ and we give the inverse operator $\mathscr{H}_\mu^{-1}$.

Proposition 2.15. *For every $\mu,\ \nu>0$ and $f\in\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$,*

$$\mathscr{H}_\mu\circ\mathscr{H}_\nu(f)=\mathscr{H}_{\mu+\nu}(f).$$

Proof. Let $\mu,\ \nu > 0$ and $f \in \mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$

$$\begin{aligned} & \mathscr{H}_\mu \circ \mathscr{H}_\nu(f)(r,x) \\ = \ & \frac{1}{2^{\mu+\nu}\ \Gamma(\mu)\Gamma(\nu)} \int_r^\infty (t^2-r^2)^{\mu-1}\Big(\int_t^{+\infty}(s^2-t^2)^{\nu-1}f(s,x)2sds\Big)2tdt. \end{aligned}$$

Applying Fubini's Theorem we get

$$\begin{aligned} & \mathscr{H}_\mu \circ \mathscr{H}_\nu(f)(r,x) \\ = \ & \frac{1}{2^{\mu+\nu}\ \Gamma(\mu)\Gamma(\nu)} \int_r^\infty f(s,x)\Big(\int_r^{s}(s^2-t^2)^{\nu-1}(t^2-r^2)^{\mu-1}2tdt\Big)2sds, \end{aligned}$$

however,

$$\int_r^s (s^2-t^2)^{\nu-1}(t^2-r^2)^{\mu-1}2tdt = \frac{\Gamma(\mu)\ \Gamma(\nu)}{\Gamma(\mu+\nu)}(s^2-r^2)^{\mu+\nu-1},$$

this completes the proof. □

Proposition 2.16. ***i.*** *For every $f \in \mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ and $\mu > 0$, we have*

$$\frac{\partial}{\partial t^2}\mathscr{H}_\mu(f) = \mathscr{H}_\mu(\frac{\partial}{\partial t^2}f). \tag{19}$$

ii. *For every $f \in \mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ and $\mu > 0$, we have*

$$-\mathscr{H}_{\mu+1}(\frac{\partial}{\partial t^2}f) = \mathscr{H}_\mu(f). \tag{20}$$

Proof. **i.** Integrating by parts, we get for every $f \in \mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$,

$$\mathscr{H}_\mu(f)(t,x) = -\frac{1}{2^\mu\ \Gamma(\mu+1)}\int_t^\infty (r^2-t^2)^\mu \frac{\partial f}{\partial r}(r,x)dr.$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial t^2}\mathscr{H}_\mu(f)(t,x) &= \frac{1}{2^\mu\ \Gamma(\mu)}\int_t^\infty (r^2-t^2)^{\mu-1}\frac{\partial f}{\partial r^2}(r,x)2rdr \\ &= \mathscr{H}_\mu(\frac{\partial}{\partial r^2}f)(t,x). \end{aligned}$$

ii. For every $f \in \mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$, $\mu > 0$, and from relation (19),

$$\frac{\partial}{\partial t^2}\mathscr{H}_{\mu+1}(f) = \mathscr{H}_{\mu+1}(\frac{\partial}{\partial t^2}f).$$

So, for every $(t,x) \in \mathbb{R}\times\mathbb{R}^n$,

$$\begin{aligned} \mathscr{H}_{\mu+1}(\frac{\partial}{\partial t^2}f)(t,x) &= \frac{\partial}{\partial t^2}\Big(\frac{1}{2^{\mu+1}\ \Gamma(\mu+1)}\int_t^\infty (r^2-t^2)^\mu f(r,x)2rdr\Big) \\ &= -\mathscr{H}_\mu(f)(t,x). \end{aligned}$$

□

Corollary 2.17. *Let μ be a real number. For all $k_1,\ k_2 \in \mathbb{N},\ k_1+\mu > 0,\ k_2+\mu > 0$ and for every $f \in \mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$, we have*

$$(-1)^{k_1}\mathscr{H}_{\mu+k_1}\Big((\frac{\partial}{\partial t^2})^{k_1}f\Big) = (-1)^{k_2}\mathscr{H}_{\mu+k_2}\Big((\frac{\partial}{\partial t^2})^{k_2}f\Big).$$

Proof. Let $k_1,\ k_2 \in \mathbb{N},\ k_1 < k_2,\ k_1+\mu > 0$ and $k_2+\mu > 0$. From Proposition 2.16, it follows that for every $f \in \mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$,

$$\begin{aligned} & (-1)^{k_2}\mathscr{H}_{\mu+k_2}((\frac{\partial}{\partial t^2})^{k_2}f) \\ = \ & (-1)^{k_1}(-1)^{k_2-k_1}\mathscr{H}_{\mu+k_1+(k_2-k_1)}\Big((\frac{\partial}{\partial t^2})^{k_2-k_1}(\frac{\partial}{\partial t^2})^{k_1}(f)\Big) \\ = \ & (-1)^{k_1}\mathscr{H}_{\mu+k_1}((\frac{\partial}{\partial t^2})^{k_1}f). \end{aligned}$$

□

Definition 2.18. For every $\mu \in \mathbb{R}$, the fractional transform $\mathscr{H}_\mu$ is defined on $\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ by

$$\mathscr{H}_\mu(f) = (-1)^k\mathscr{H}_{\mu+k}((\frac{\partial}{\partial t^2})^k f) = (-1)^k(\frac{\partial}{\partial t^2})^k\mathscr{H}_{\mu+k}(f),$$

where $k \in \mathbb{N},\ k+\mu > 0$.

From Corollary 2.17, the expression $\mathscr{H}_\mu$ in Definition 2.18 is independent of the choice of $k \in \mathbb{N},\ k+\mu > 0$.
For every $f \in \mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$,

$$\begin{aligned} \mathscr{H}_0(f)(t,x) &= -\frac{\partial}{\partial t^2}\mathscr{H}_1(f)(t,x) \\ &= -\frac{1}{t}\frac{\partial}{\partial t}\Big(\int_t^\infty f(r,x)rdr\Big) = f(t,x). \end{aligned} \tag{21}$$

Proposition 2.19. ***i.*** *For every $\mu,\ \nu \in \mathbb{R}$ and $f \in \mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$.*

$$\mathscr{H}_\mu \circ \mathscr{H}_\nu(f) = \mathscr{H}_{\mu+\nu}(f) \tag{22}$$

ii. *For every $\mu \in \mathbb{R}$, the fractional transform $\mathscr{H}_\mu$ is a topological isomorphism from $\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ onto itself whose inverse isomorphism is*

$$\mathscr{H}_\mu^{-1} = \mathscr{H}_{-\mu}.$$

Proof. **i.** Let $\mu,\ \nu \in \mathbb{R},\ k_1,\ k_2 \in \mathbb{N},\ k_1+\mu > 0,\ k_2+\mu > 0$ and $f \in \mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\begin{aligned} \mathscr{H}_\mu \circ \mathscr{H}_\nu(f) &= \mathscr{H}_\mu\Big((-1)^{k_2}(\frac{\partial}{\partial t^2})^{k_2}\mathscr{H}_{\nu+k_2}(f)\Big) \\ &= (-1)^{k_1+k_2}\mathscr{H}_{\mu+k_1}\Big((\frac{\partial}{\partial t^2})^{k_1}\mathscr{H}_{\nu+k_2}((\frac{\partial}{\partial t^2})^{k_2}(f))\Big) \\ &= (-1)^{k_1+k_2}\mathscr{H}_{\mu+k_1}\circ\mathscr{H}_{\nu+k_2}((\frac{\partial}{\partial t^2})^{k_1+k_2}(f)). \end{aligned}$$

Now, from Proposition 2.15, we deduce that

$$\begin{aligned}\mathscr{H}_\mu \circ \mathscr{H}_\nu(f) &= (-1)^{k_1+k_2}\mathscr{H}_{\mu+\nu+k_2+k_1}\Big((\frac{\partial}{\partial t^2})^{k_1+k_2}(f)\Big)\\ &= \mathscr{H}_{\mu+\nu}(f),\end{aligned}$$

because $\mu+\nu+k_1+k_2>0$.
ii. The result follows from relations (21) and (22). □

3. L^p-boundedness of the fractional transform $\mathscr{R}_\mu$ and $\mathscr{H}_\mu$

This section contains the main results of this work. In fact, we study the boundedness of the operators $\mathscr{R}_\mu$ and $\mathscr{H}_\mu$ on the the weighted Lebesgue spaces $L^p\big([0,+\infty[\times\mathbb{R}^n, r^{2a}drdx\big), p\in[1,+\infty[$ equipped with the norm

$$||f||_{p,a}=\begin{cases}\Big(\displaystyle\int_0^\infty\int_{\mathbb{R}^n}|f(r,x)|^p r^{2a}drdx\Big)^{\frac{1}{p}}, & \text{if } 1\leqslant p\leqslant+\infty\\ \underset{(r,x)\in\ [0,+\infty[\times\mathbb{R}^n}{\text{ess sup}}\ \big|f(r,x)\big|, & \text{if } p=+\infty.\end{cases}$$

For convenience we refer to this space as $L^p(d\gamma_a)$ with $d\gamma_a(r,x)=r^{2a}drdx$.

3.1. L^p-boundedness of the fractional transform $\mathscr{R}_\mu$.

Proposition 3.1. *For every $a\in\mathbb{R}$ and every $\mu>0$, the fractional transform $\mathscr{R}_\mu$ is bounded from $L^\infty(d\gamma_a)$ into itself and*

$$||\mathscr{R}_\mu||_{\infty,\gamma_a}=\sup_{||f||_{\infty,a}\leqslant 1}||\mathscr{R}_\mu(f)||_{\infty,a}=1.$$

Proof. Let f be a bounded measurable function on $[0,+\infty[\times\mathbb{R}^n$. For every $(r,x)\in[0,+\infty[\times\mathbb{R}^n$,

$$\begin{aligned}|\mathscr{R}_\mu(f)(r,x)| &\leqslant \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\mu)}\int_0^1(1-t^2)^{\mu-1}|f(tr,x)|dt\\ &\leqslant ||f||_{\infty,a}\frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\mu)}\int_0^1(1-t^2)^{\mu-1}dt\\ &= ||f||_{\infty,a}.\end{aligned}$$

This shows that the operator $\mathscr{R}_\mu$ is bounded from $L^\infty(d\gamma_a)$ into itself and that

$$||\mathscr{R}_\mu||_{\infty,\gamma_a}\leqslant 1.$$

However, $\mathscr{R}_\mu(1)=1$, this shows that

$$||\mathscr{R}_\mu||_{\infty,\gamma_a}=1.$$

□

Theorem 3.2. *The operator $\mathscr{R}_\mu;\mu>0$ is bounded from $L^1(d\gamma_a)$ into itself if and only if $a<0$ and in this case*

$$||\mathscr{R}_\mu||_{1,\gamma_a}=\frac{\Gamma(\mu+\frac{1}{2})\Gamma(-a)}{\sqrt{\pi}\ \Gamma(\mu-a)}.$$

Proof. Let $a \in \mathbb{R},\ a < 0$. By Fubini-Tonnelli Theorem's and for every $f \in L^1(d\gamma_a)$,

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} |\mathscr{R}_\mu(f)(r,x)| d\gamma_a(r,x) \\
\leqslant\ & \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)} \int_0^\infty \int_{\mathbb{R}^n} \Big(\int_0^1 (1-t^2)^{\mu-1} |f(tr,x)| dt \Big) d\gamma_a(r,x) \\
=\ & \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)} \int_0^1 (1-t^2)^{\mu-1} \Big(\int_0^\infty \int_{\mathbb{R}^n} |f(tr,x)| d\gamma_a(r,x) \Big) dt \\
=\ & ||f||_{1,a} \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)} \int_0^1 (1-t^2)^{\mu-1} t^{-(2a+1)} dt \\
=\ & \frac{\Gamma(\mu+\frac{1}{2})\Gamma(-a)}{\sqrt{\pi}\ \Gamma(\mu-a)} ||f||_{1,a}.
\end{aligned}$$

Consequently for $a < 0$, the transform $\mathscr{R}_\mu$ is a bounded operator from $L^1(d\gamma_a)$ into itself and

$$||\mathscr{R}_\mu||_{1,\gamma_a} \leqslant \frac{\Gamma(\mu+\frac{1}{2})\Gamma(-a)}{\sqrt{\pi}\ \Gamma(\mu-a)}.$$

On the other hand, for every nonnegative $f \in L^1(d\gamma_a)$, we have

$$||\mathscr{R}_\mu(f)||_{1,a} = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(-a)}{\sqrt{\pi}\ \Gamma(\mu-a)} ||f||_{1,a}$$

We conclude that

$$||\mathscr{R}_\mu||_{1,\gamma_a} = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(-a)}{\sqrt{\pi}\ \Gamma(\mu-a)}.$$

For converse, let $a \in \mathbb{R},\ a \geqslant 0$ and let $f \in L^1(d\gamma_a)$ be a nonnegative function such that $||f||_{1,a} = 1$. We have

$$||\mathscr{R}_\mu(f)||_{1,\gamma_a} = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(-a)}{\sqrt{\pi}\ \Gamma(\mu-a)} = +\infty.$$

This completes the proof. □

Theorem 3.3. *Let $p \in]1,+\infty[$. The operator $\mathscr{R}_\mu,\ \mu > 0$, is bounded from $L^p(d\gamma_a)$ into itself if and only if $2a+1 < p$ and in this case*

$$||\mathscr{R}_\mu||_{p,\gamma_a} = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{p-(2a+1)}{2p})}{\sqrt{\pi}\ \Gamma(\mu+\frac{p-(2a+1)}{2p})}.$$

Proof. Let $p \in\ ]1,+\infty[,\ 2a+1 < p$. From Minkowski's inequality [18] and for every $f \in L^p(d\gamma_a)$,

$$\begin{aligned}
||\mathscr{R}_\mu(f)||_{p,a} &\leqslant \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)} \int_0^1 (1-t^2)^{\mu-1} \Big(\int_0^\infty \int_{\mathbb{R}^n} |f(tr,x)|^p d\gamma_a(r,x) \Big)^{\frac{1}{p}} dt \\
&= \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)} \|f\|_{p,a} \int_0^1 (1-t^2)^{\mu-1} t^{-\frac{2a+1}{p}} dt
\end{aligned}$$

$$= \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{p-(2a+1)}{2p})}{\sqrt{\pi}\ \Gamma(\mu+\frac{p-(2a+1)}{2p})}||f||_{p,a}.$$

This proves that for $2a+1<p$, the fractional transform $\mathscr{R}_\mu$ is bounded from $L^p(d\gamma_a)$ into itself and

$$||\mathscr{R}_\mu||_{p,\gamma_a} \leqslant \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{p-(2a+1)}{2p})}{\sqrt{\pi}\ \Gamma(\mu+\frac{p-(2a+1)}{2p})}. \tag{23}$$

Let $\eta>0$ and let

$$f_0(r,x) = r^{\frac{\eta-(2a+1)}{p}}\mathbf{1}_{]0,1[}(r)\Pi_{j=1}^n\mathbf{1}_{]0,1[}(x_j),$$

then f_0 belongs to $L^p(d\gamma_a)$ and

$$||f_0||_{p,a} = (\frac{1}{\eta})^{\frac{1}{p}}.$$

On the other hand,

$$\begin{aligned}
&|\mathscr{R}_\mu(f_0)(r,x)| \\
\geqslant\ & \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)}r^{1-2\mu}\Big(\int_0^r(r^2-t^2)^{\mu-1}t^{\frac{\eta-(2a+1)}{p}}dt\Big)\mathbf{1}_{]0,1[}(r)\Pi_{j=1}^n\mathbf{1}_{]0,1[}(x_j) \\
=\ & \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)}f_0(r,x)\int_0^1(1-t^2)^{\mu-1}t^{\frac{\eta-(2a+1)}{p}}dt \\
=\ & \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{1}{2}+\frac{\eta-(2a+1)}{2p})}{\sqrt{\pi}\ \Gamma(\mu+\frac{1}{2}+\frac{\eta-(2a+1)}{2p})}f_0(r,x).
\end{aligned}$$

Integrating over $]0,+\infty[\times\mathbb{R}^n$ with respect to the measure $d\gamma_a$, we deduce that for every $\eta>0$,

$$||\mathscr{R}_\mu||_{p,\gamma_a} \geq \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{1}{2}+\frac{\eta-(2a+1)}{2p})}{\sqrt{\pi}\ \Gamma(\mu+\frac{1}{2}+\frac{\eta-(2a+1)}{2p})}.$$

This involves that

$$||\mathscr{R}_\mu||_{p,\gamma_a} \geq \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{p-(2a+1)}{2p})}{\sqrt{\pi}\ \Gamma(\mu+\frac{p-(2a+1)}{2p})}. \tag{24}$$

The relations (23) and (24) imply that for every $a,\ 2a+1<p$

$$||\mathscr{R}_\mu||_{p,\gamma_a} = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{p-(2a+1)}{2p})}{\sqrt{\pi}\ \Gamma(\mu+\frac{p-(2a+1)}{2p})}.$$

Now, we prove that, for $2a+1>p$, $\mathscr{R}_\mu$ does not map $L^p(d\gamma_a)$ into itself. To prove this we have following two cases:

Case 1. Suppose that $2a+1=p$ and let

$$g_0(r,x) = \frac{1}{r(1-\ln(r))}\mathbf{1}_{]0,1[}(r)\Pi_{j=1}^n\mathbf{1}_{]0,1[}(x_j),$$

then, g_0 belongs to $L^p(d\gamma_a)$ and we have

$$||g_0||^p_{p,a} = \int_0^1 \frac{dr}{r(1-\ln(r))^p} = \int_{-\infty}^0 \frac{ds}{(1-s)^p} = \frac{1}{p-1}.$$

However, for every $(r,x) \in]0,1[\times]0,1[^n$,

$$\mathscr{R}_\mu(g_0)(r,x) = \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\mu)} r^{1-2\mu} \int_0^r (r^2-t^2)^{\mu-1} \frac{dt}{t(1-\ln(t))} = +\infty,$$

in particular $\mathscr{R}_\mu(g_0)$ does not belong to $L^p(d\gamma_a)$.
Case 2. Suppose that $2a+1 > p$ and let $\eta \in \mathbb{R}; -\frac{2a+1}{p} < \eta < -1$ and let

$$h_0(r,x) = r^\eta \mathbf{1}_{]0,1[}(r)\Pi_{j=1}^n \mathbf{1}_{]0,1[}(x_j).$$

Then the function h_0 lies in $L^p(d\gamma_a)$ and

$$||h_0||^p_{p,a} = \frac{1}{p\eta+2a+1}.$$

But, for every $(r,x) \in]0,1[\times]0,1[^n$,

$$\mathscr{R}_\mu(h_0)(r,x) = \frac{2\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\mu)} r^\eta \int_0^1 (1-t^2)^{\mu-1} t^\eta dt = +\infty.$$

Hence, for $2a+1 > p$, $\mathscr{R}_\mu$ does not map $L^p(d\gamma_a)$ into itself and this completes the proof of theorem. □

Combining Proposition (3.1), Theorem (3.2) and Theorem (3.3) , we claim the following interesting result.

Theorem 3.4. *For every $p \in [1,+\infty]$, the fractional operator $\mathscr{R}_\mu$ is bounded on $L^p(d\gamma_a)$ if and only if $2a+1 < p$ and in this case*

$$||\mathscr{R}_\mu||_{p,\gamma_a} = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{p-(2a+1)}{2p})}{\sqrt{\pi}\ \Gamma(\mu+\frac{p-(2a+1)}{2p})}.$$

Remark 3.5. The case $a = \mu$ in Theorem (3.4) is important because the measure $d\nu_\mu$ defined by the relation (3) is connected with the operators $D_j,\ 1 \leqslant j \leqslant n$ and Ξ and the Fourier-Hankel transform $\widetilde{\mathscr{F}}_{\mu-\frac{1}{2}}$ given by relation (2) and in this occurrence, $\mathscr{R}_\mu$ is bounded from $L^p(d\nu_\mu)$ into itself if and only if $2\mu+1 < p$ and we have

$$||\mathscr{R}_\mu||_{p,\nu_\mu} = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{p-(2\mu+1)}{2p})}{\sqrt{\pi}\ \Gamma(\mu+\frac{p-(2\mu+1)}{2p})}.$$

3.2. L^p-boundedness of the fractional transform $\mathscr{H}_\mu$. We denote by $r^{-2\mu}L^p(d\gamma_a)$ the space defined by $r^{-2\mu}L^p(d\gamma_a) = \{f :]0,+\infty[\times\mathbb{R}^n \longrightarrow \mathbb{C},\ f$ is measurable and the function $(r,x) \longmapsto r^{2\mu}f(r,x)$ belongs to $L^p(d\gamma_a)\}$
$r^{-2\mu}L^p(d\gamma_a)$ is equipped with the norm

$$N_{p,a}(f) = ||r^{2\mu}f||_{p,a}.$$

Theorem 3.6. *The operator $\mathscr{H}_\mu,\ \mu > 0$ is bounded from $r^{-2\mu}L^1(d\gamma_a)$ into $L^1(d\gamma_a)$ if and only if $2a+1>0$ and in this case*

$$N_{1,\gamma_a}(\mathscr{H}_\mu) = \sup_{||r^{2\mu}f||_{1,a}\leqslant 1} ||\mathscr{H}_\mu(f)||_{1,a} = \frac{\Gamma(\frac{2a+1}{2})}{2^\mu\ \Gamma(\mu+\frac{2a+1}{2})}.$$

Proof. Suppose that $a > -\frac{1}{2}$ and let $f \in r^{-2\mu}L^1(d\gamma_a)$. We have

$$\big|\mathscr{H}_\mu(f)(r,x)\big| \leq \frac{r^{2\mu}}{2^\mu\ \Gamma(\mu)}\int_1^\infty (t^2-1)^{\mu-1}\big|f(rt,x)\big|2tdt.$$

Applying Fubini-Tonnelli Theorem's, we get

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^n}\big|\mathscr{H}_\mu(f)(r,x)\big|d\gamma_a(r,x)\\
\leqslant\ & \frac{1}{2^\mu\ \Gamma(\mu)}\int_1^\infty (t^2-1)^{\mu-1}\Big(\int_0^\infty\int_{\mathbb{R}^n} r^{2\mu+2a}|f(tr,x)|drdx\Big)2tdt\\
=\ & ||r^{2\mu}f||_{1,a}\frac{1}{2^\mu\ \Gamma(\mu)}\int_1^\infty (t^2-1)^{\mu-1}t^{-(2\mu+2a+1)}2tdt.
\end{aligned}$$

By the change of variable $s=\frac{1}{t^2}$, we have

$$\frac{1}{2^\mu\ \Gamma(\mu)}\int_1^\infty (t^2-1)^{\mu-1}t^{-(2\mu+2a+1)}2tdt = \frac{\Gamma(\frac{2a+1}{2})}{2^\mu\ \Gamma(\mu+\frac{2a+1}{2})}.$$

This shows that for every $f \in r^{-2\mu}L^1(d\gamma_a)$, the function $\mathscr{H}_\mu(f)$ belongs to $L^1(d\gamma_a)$ and

$$||\mathscr{H}_\mu(f)||_{1,a} \leqslant \frac{\Gamma(\frac{2a+1}{2})}{2^\mu\ \Gamma(\mu+\frac{2a+1}{2})}||r^{2\mu}f||_{1,a}$$

On the other hand, for every nonnegative function $f \in r^{-2\mu}L^1(d\gamma_a)$, we have

$$||\mathscr{H}_\mu(f)||_{1,a} = \frac{\Gamma(\frac{2a+1}{2})}{2^\mu\ \Gamma(\mu+\frac{2a+1}{2})}||r^{2\mu}f||_{1,a}. \tag{25}$$

Hence, for $a > -\frac{1}{2}$, the fractional transform $\mathscr{H}_\mu$ is continuous from $r^{-2\mu}L^1(d\gamma_a)$ into $L^1(d\gamma_a)$ and

$$N_{1,\gamma_a}(\mathscr{H}_\mu) = \frac{\Gamma(\frac{2a+1}{2})}{2^\mu\ \Gamma(\mu+\frac{2a+1}{2})}.$$

For Converse, let $a \leqslant -\frac{1}{2}$ and let $f \in r^{-2\mu}L^1(d\gamma_a)$, f nonnegative function such that
$||r^{2\mu}f||_{1,a} = 1$. From relation (25)

$$||\mathscr{H}_\mu(f)||_{1,a} = +\infty,$$

which proves that for $a \leq -\frac{1}{2}$, the operator $\mathscr{H}_\mu$ does not map the space $r^{-2\mu}L^1(d\gamma_a)$ into $L^1(d\gamma_a)$. □

Theorem 3.7. *For every $p \in]1, +\infty[$, the fractional transform $\mathscr{H}_\mu$ is bounded from $r^{-2\mu}L^p(d\gamma_a)$ into $L^p(d\gamma_a)$ if and only if $2a + 1 > 0$ and in this case*

$$N_{p,\gamma_a}(\mathscr{H}_\mu) = \sup_{||r^{2\mu}f||_{p,a} \leqslant 1} ||\mathscr{H}_\mu(f)||_{p,a} = \frac{\Gamma(\frac{2a+1}{2p})}{2^\mu\ \Gamma(\mu + \frac{2a+1}{2p})}.$$

Proof. Let $a > -\frac{1}{2}$ and $f \in r^{-2\mu}L^p(d\gamma_a)$. By Minkouski's inequality, we have

$$\begin{aligned}
&||\mathscr{H}_\mu(f)||_{p,a} \\
\leqslant\ & \frac{1}{2^\mu\ \Gamma(\mu)} \int_1^\infty (t^2-1)^{\mu-1} \Big(\int_0^\infty \int_{\mathbb{R}^n} (r^{2\mu}|f(tr,x)|)^p r^{2a} dr dx \Big)^{\frac{1}{p}} 2t dt \\
=\ & ||r^{2\mu}f||_{p,a} \frac{1}{2^\mu\ \Gamma(\mu)} \int_1^\infty (t^2-1)^{\mu-1} t^{-\frac{2\mu p + 2a+1}{p}} 2t dt \\
=\ & \frac{\Gamma(\frac{2a+1}{2p})}{2^\mu\ \Gamma(\mu + \frac{2a+1}{2p})} ||r^{2\mu}f||_{p,a}.
\end{aligned}$$

Consequently, for $a > -\frac{1}{2}$, $\mathscr{H}_\mu$ is a bounded operator from $r^{-2\mu}L^p(d\gamma_a)$ into $L^p(d\gamma_a)$ and

$$N_{p,\gamma_a}(\mathscr{H}_\mu) \ \leqslant\ \frac{\Gamma(\frac{2a+1}{2p})}{2^\mu\ \Gamma(\mu + \frac{2a+1}{2p})}. \tag{26}$$

Let $\eta \in \mathbb{R}$, $\eta > 0$, and let

$$f_0(r,x) = r^{-2\mu - \frac{2a+\eta+1}{p}} \mathbf{1}_{[1,+\infty[}(r) \Pi_{j=1}^n \mathbf{1}_{]0,1[}(x_j).$$

The function f_0 belongs to $r^{-2\mu}L^p(d\gamma_a)$ and

$$||r^{2\mu}f_0||_{p,a} = (\frac{1}{\eta})^{\frac{1}{p}}.$$

Moreover,

$$\begin{aligned}
&|\mathscr{H}_\mu(f_0)(r,x)| \\
=\ & \mathscr{H}_\mu(f_0)(r,x) \\
\geq\ & \frac{1}{2^\mu\ \Gamma(\mu)} \Big(\int_r^\infty (t^2 - r^2)^{\mu-1} t^{-2\mu - \frac{2a+1+\eta}{p}} 2t dt \Big) \mathbf{1}_{[1,+\infty[}(r) \Pi_{j=1}^n \mathbf{1}_{]0,1[}(x_j) \\
=\ & \frac{\Gamma(\frac{2a+1+\eta}{2p})}{2^\mu\ \Gamma(\mu + \frac{2a+1+\eta}{2p})} r^{2\mu} f_0(r,x).
\end{aligned}$$

Thus,

$$||\mathscr{H}_\mu(f_0)||_{p,a} \geqslant \frac{\Gamma(\frac{2a+1+\eta}{2p})}{2^\mu\ \Gamma(\mu+\frac{2a+1+\eta}{2p})}||r^{2\mu}f_0||_{p,a}$$

and then, for every $\eta > 0$,

$$N_{p,\gamma_a}(\mathscr{H}_\mu) \geqslant \frac{\Gamma(\frac{2a+1+\eta}{2p})}{2^\mu\ \Gamma(\mu+\frac{2a+1+\eta}{2p})}.$$

This implies that

$$N_{p,\gamma_a}(\mathscr{H}_\mu) \quad \geqslant \quad \frac{\Gamma(\frac{2a+1}{2p})}{2^\mu\ \Gamma(\mu+\frac{2a+1}{2p})}. \tag{27}$$

Combining the relations (26) and (27), we deduce that for $a > -\frac{1}{2}$, the fractional transform $\mathscr{H}_\mu$ is a bounded operator from $r^{-2\mu}L^p(d\gamma_a)$ into $L^p(d\gamma_a)$ and that

$$N_{p,\gamma_a}(\mathscr{H}_\mu) = \frac{\Gamma(\frac{2a+1}{2p})}{2^\mu\ \Gamma(\mu+\frac{2a+1}{2p})}.$$

Now we prove that, for $a \geq \frac{1}{2}$, the operator $\mathscr{H}_\mu$ does not map the space $r^{-2\mu}L^p(d\gamma_{-\frac{1}{2}})$ into $L^p(d\gamma_{-\frac{1}{2}})$. We have two cases:

Case 1. Suppose that $2a+1=0$ and let

$$g_0(r,x) = \frac{1}{r^{2\mu}(1+\ln(r))^p}\mathbf{1}_{[1,+\infty[}(r)\Pi_{j=1}^n\mathbf{1}_{]0,1[}(x_j).$$

The function g_0 belongs to $r^{-2\mu}L^p(d\gamma_{-\frac{1}{2}})$ and

$$\begin{aligned} ||r^{2\mu}g_0||_{p,-\frac{1}{2}} &= \Big(\int_1^\infty \frac{dr}{r(1+\ln(r))^p}\Big)^{\frac{1}{p}} \\ &= \Big(\int_0^\infty \frac{du}{(1+u)^p}\Big)^{\frac{1}{p}} \\ &= \Big(\frac{1}{p-1}\Big)^{\frac{1}{p}}. \end{aligned}$$

But for every $(r,x) \in]1,+\infty[\times]0,1[^n$,

$$\mathscr{H}_\mu(g_0)(r,x) = \int_r^\infty (t^2-r^2)^{\mu-1}\frac{2t}{t^{2\mu}(1+\ln(r))}dt = +\infty.$$

This shows that for $a = -\frac{1}{2}$, the operator $\mathscr{H}_\mu$ does not map the space $r^{-2\mu}L^p(d\gamma_{-\frac{1}{2}})$ into $L^p(d\gamma_{-\frac{1}{2}})$.

Case 2. Finally, suppose that $a < -\frac{1}{2}$ and let $\eta \in \mathbb{R}$; $\frac{1}{2} < \eta < -a$.
Let

$$h_0(r,x) = r^{-2\mu-\frac{2a+2\eta}{p}}\mathbf{1}_{[1,+\infty[}(x)\Pi_{j=1}^n\mathbf{1}_{]0,1[}(x_j).$$

The function h_0 belongs to $r^{-2\mu}L^p(d\gamma_a)$, and

$$||r^{2\mu}h_0||_{p,a} = \Big(\int_1^{\infty} r^{-2\eta}dr\Big)^{\frac{1}{p}} = (\frac{1}{2\eta-1})^{\frac{1}{p}}.$$

However, for every $(r,x) \in]1,+\infty[\times]0,1[^n$,

$$\mathscr{H}_\mu(h_0)(r,x) = \frac{1}{2^\mu\ \Gamma(\mu)}\int_r^{\infty}(t^2-r^2)^{\mu-1}t^{-2\mu-\frac{2a+2\eta}{p}}2tdt = +\infty, \text{ because } a+\eta<0$$

Hence, for $a < -\frac{1}{2}$, the operator $\mathscr{H}_\mu$ does not map the space $r^{-2\mu}L^p(d\gamma_a)$ into $L^p(d\gamma_a)$.
The proof of theorem is complete. □

Remark 3.8. For every $a \in \mathbb{R}$, the fractional transform $\mathscr{H}_\mu$ does not map the space $r^{-2\mu}L^\infty(d\gamma_a)$ into itself.
In fact, the function $f(r,x) = r^{2\mu}\mathbf{1}_{[1,+\infty[}(r)$ belongs to $r^{-2\mu}L^\infty(d\gamma_a)$, but for every $(r,x) \in]0,+\infty[\times\mathbb{R}^n$

$$\mathscr{H}_\mu(f)(r,x) = \frac{1}{2^\mu\ \Gamma(\mu)}\int_r^{\infty}(t^2-r^2)^{\mu-1}t^{2\mu}2tdt = +\infty.$$

We conclude that for every $p \in [1,+\infty[$, the transform $\mathscr{H}_\mu$, $\mu > 0$, is bounded from $r^{-2\mu}L^p(d\gamma_a)$ into $L^p(d\gamma_a)$ if and only if $2a+1>0$ and

$$N_{p,\gamma_a}(\mathscr{H}_\mu) = \sup_{||r^{2\mu}f||_{p,a}\leqslant 1}||\mathscr{H}_\mu(f)||_{p,a} = \frac{\Gamma(\frac{2a+1}{2p})}{2^\mu\ \Gamma(\mu+\frac{2a+1}{2p})}.$$

In particular, for $a = \mu > 0$, the fractional transform $\mathscr{H}_\mu$ is bounded from $r^{-2\mu}L^p(d\nu_\mu)$ into $L^p(d\nu_\mu)$ and for every $f \in r^{-2\mu}L^p(d\nu_\mu)$,

$$||\mathscr{H}_\mu(f)||_{p,\nu_\mu} \leqslant \frac{\Gamma(\frac{2\mu+1}{2p})}{2^\mu\ \Gamma(\mu+\frac{2\mu+1}{2p})}||r^{2\mu}f||_{p,\nu_\mu}.$$

Competing Interests

The authors declares that they have no competing interests.

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