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OLD SYMMETRY PROBLEM REVISITED

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ABSTRACT. It is proved that if the problem $\nabla^2 u = 1$ in D, $u|_S = 0$, $u_N = m := |D|/|S|$ then D is a ball. There were at least two different proofs published of this result. The proof given in this paper is novel and short.

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1. Introduction

Let D be bounded smooth connected domain in $\mathbb{R}^3$, S be its boundary, N is the outer unit normal to S, u_N is the normal derivative of u on S, $|D|$ is the volume of D and $|S|$ is the suraface area of S. Various symmetry problems were considered in [1, 2].
Consider the problem

$$\nabla^2 u = 1 \quad in \quad D, \quad u|_S = 0, \quad u_N|_S = m = |D|/|S|. \tag{1}$$

Our result is the following:

Theorem 1.1. *If problem* (1) *is solvable then* D *is a ball.*

This result was proved by different methods in [3] and in [4]. The proof, given in the next section, is novel, short and is based on a new idea. We assume that $D \subset \mathbb{R}^2$ so that S is a curve. Then the ball is a disc.

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2. Proof of Theorem 1.1

Proof. Let s be the curve length, $\mathbf{s}$ be the point on S corresponding to the parameter s, $\{x(s), y(s)\}$ be the parametric representation of S, $\mathbf{s} = x(s)e_1 + y(s)e_2$, where $\{e_j\}|_{j=1,2}$ is a Cartesian basis in $\mathbb{R}^2$. It is known that $\frac{d\mathbf{s}}{ds} = \mathbf{t}(s)$ is the tangent unit vector to S at the point $\mathbf{s}$ and

$$\frac{d\mathbf{t}}{ds} = k(s)\nu(\mathbf{s}), \tag{2}$$

where $k(s) \geq 0$ is the curvature of S and $\nu(\mathbf{s})$ is the normal to S. Since $u_N = \nabla u \cdot N = m > 0$ on S the convexity of S does not change sign, so ν does not change sign, $k(s) > 0$ $\forall s \in S$ and $N(s) = -\nu(\mathbf{s})$ $\forall s \in S$. Differentiate the identity $u(x(s), y(s)) = 0$ with respect to s and get $\nabla u \cdot \mathbf{t} = \mathbf{0}$. Differentiate this identity and use (1)-(2) to get

$$u_{xx}t_1^2(s) + 2u_{xy}t_1(s)t_2(s) + u_{yy}t_2^2(s) + \nabla u \cdot k(s)\nu(s) = 0, \tag{3}$$

where $\mathbf{t} = t_1e_1 + t_2e_2$. Rewrite (3) as

$$u_{xx}t_1^2(s) + 2u_{xy}t_1(s)t_2(s) + u_{yy}t_2^2(s) = mk(s). \tag{4}$$

Equation (4) holds in every coordinate system obtained from $\{x, y\}$ by rotations. Clearly $u_{xx}(s), u_{yy}(s), u_{xy}(s)$ cannot vanish simultaneously due to (4). Also $u_{xx}(s), u_{yy}(s)$ cannot vanish simultaneously due to the first equation in (1).
Equation (4) holds in any coordinate system obtained from a fixed Cartesian system by rotations. Equation (1) on the boundary yields:

$$u_{xx} + u_{yy} = 1. \tag{5}$$

We prove that (4) and (5) are not compatible (lead to a contradiction) except when S is a circle.
Let $u_{xx} := p$, $u_{xy} := q$. Denote by A the 2×2 matrix with the elements $A_{11} = p$,$A_{22} = 1 - p$, where (5) was used, $A_{12} = A_{21} = q$. Let I be the identity matrix. The equation $\det(A - \lambda I) = \lambda^2 - \lambda - p^2 - q^2 + p = 0$ has two solutions, so the eigenvalues of A are:

$$\lambda_\pm = \frac{1}{2} \pm (\frac{1}{4} + p^2 + q^2 - p)^{1/2} = \frac{1}{2} \pm [(\frac{1}{2} - p)^2 + q^2]^{1/2}. \tag{6}$$

The corresponding eigenvectors are

$$e_1 = \{1, \gamma\}, \quad e_2 = \{-\gamma, 1\}, \quad \gamma := \frac{q}{p + \lambda_+ - 1}. \tag{7}$$

Note that $\lambda_+ + \lambda_- = 1$, $\lambda_+\lambda_- = -p^2 - q^2 + p$. Thus, $\lambda_+ > 0$. The eigenvectors are orthogonal: $e_1 \cdot e_2 = 0$ but not normalized: $\|e_1\|^2 = \|e_2\|^2 = 1 + \gamma^2$. Since $\|e_1\|^2$ is invariant under rotations of a Cartesian coordinate system, so is γ^2. Let $w := \{t_1, t_2\}$. Then (4) implies

$$(Aw, w) = mk(s) > 0. \tag{8}$$

Since e_1 and e_2 form an orthogonal basis in $\mathbb{R}^2$ one can find unique constants c_1, c_2 such that

$$c_1e_1 + c_2e_2 = w. \tag{9}$$

Solving this linear algebraic system for c_1, c_2 one gets:

$$c_1 = \frac{t_1 + \gamma t_2}{\Delta}, \quad c_2 = \frac{t_2 - \gamma t_1}{\Delta}, \tag{10}$$

where $\Delta = 1 + \gamma^2$ is the determinant of the matrix of the system (9).
Substitute w from (9) into (8) and get:

$$[c_1^2\lambda_+ + c_2^2\lambda_-](1 + \gamma^2) = mk(s) > 0, \tag{11}$$

where we have used the relations: $Ae_j = \lambda_j e_j$, $\lambda_1 := \lambda_+$, $\lambda_2 := \lambda_-$, $(e_1, e_2) = 0$, $\|e_j\|^2 = 1 + \gamma^2$, $(Ae_j, e_j) = \lambda_j(1 + \gamma^2)$, $j = 1, 2$. Using (10) one gets from (11):

$$(t_1 + \gamma t_2)^2\lambda_+ + (t_2 - \gamma t_1)^2\lambda_- = mk(s)(1 + \gamma^2) > 0. \tag{12}$$

We prove that (12) leads to a contradiction unless S is a circle.
Assume first that $\lambda_- < 0$ and recall that $\lambda_+ > 0$. Choose a point $s \in S$ and the Cartesian coordinate system such that $t_1(s) + \gamma(s)t_2(s) = 0$. This is possible since γ^2 is invariant under rotations and the only restriction on the real-valued t_1, t_2 is the relation $t_1^2 + t_2^2 = 1$. Since $\lambda_- < 0$ and $t_2 - \gamma t_1 \neq 0$, we have a contradiction with inequality (12).
Assume now that $\lambda_- \geq 0$ and $\lambda_- \neq \lambda_+$. Then the left side of (12) is not a constant as a function of $\{t_1, t_2\}$, that is, not a constant with respect to rotations of the coordinate system, while its right side is a constant. Thus, we have a contradiction.
Suppose finally that $\lambda_- = \lambda_+$. Then $\lambda_- = \lambda_+ = \frac{1}{2}$ at any $s \in S$. This implies by formula (6) that $p = \frac{1}{2}$, $u_{yy} = \frac{1}{2}$ and $q = 0$ on S for all $s \in S$. By formula (7) one gets $\gamma = 0$, $\|e_j\| = 1$. Consequently, by formula (4), it follows that $\kappa(s) = \frac{1}{2m}$. Thus, the curvature of S is a constant, so S is a circle of a radius a. Thus, $m = \frac{\pi a^2}{2\pi a} = \frac{a}{2}$, $k(s) = \frac{1}{a}$ and the solution to problem (1) is $u = \frac{|x|^2 - a^2}{4}$. Obviously this u solves equation (1) and satisfies the first boundary condition in (1). The second boundary condition is also satisfied: $u_N|_S = a/2$.
Theorem 1.1 is proved in the two-dimensional case. We leave to the reader to consider the three-dimensional case, see [5]. Theorem 1.1 is proved. □

Competing Interests

The author declares that he has no competing interests.

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