ASYMPTOTIC STABILITY AND BLOW-UP OF SOLUTIONS FOR THE GENERALIZED BOUSSINESQ EQUATION WITH NONLINEAR BOUNDARY CONDITION

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Abstract. In this paper, we consider initial boundary value problem of the generalized Boussinesq equation with nonlinear interior source and boundary absorptive terms. We establish both the existence of the solution and a general decay of the energy functions under some restrictions on the initial data. We also prove a blow-up result for solutions with positive and negative initial energy respectively.

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1. Introduction

In this paper, we consider the following initial boundary value problem for the generalized Boussinesq equation with a nonlinear Neumann condition

\begin{align}
\begin{cases}
    u_t - \Delta u_t - \Delta u + |u|^{q-2}u_t = f(u), \\
    u = 0, x \in \Gamma_0, \\
    \frac{\partial u}{\partial \nu} + g(u) = 0, x \in \Gamma_1, \\
    u(x, 0) = u_0(x), x \in \Omega,
\end{cases}
\end{align}

where $u = u(t, x)$ ($t \geq 0, x \in \Omega$), $\Delta$ denotes the Laplacian operator with respect to the $x$ variable, $\Omega$ is a bounded open subset of $\mathbb{R}^n (n \geq 1)$ of class $C^1$, $\partial \Omega = \Gamma_0 \cup \Gamma_1$, $mes(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and $\frac{\partial}{\partial \nu}$ denotes the unit outer normal.
derivative, \( q > 2 \) is a positive constant, and the initial datum \( u_0 \) is a given function with the compatibility boundary condition \( u_0 = 0 \) on \( \Gamma_0 \) and \( f(s) \) and \( g(s) \) are continuous functions. For sake of simplicity, in this paper, we consider \( f(s) = a|u|^{p-1}u, g(s) = b|u|^{k-1}u \), where \( p > 1, k > 1 \) and \( a = b = 1 \).

Problem (1) was derived in [1]. This problem describes an electric breakdown in crystalline semiconductors with allowance for the linear dissipation of bound- and free-charge sources [1, 2, 3], where the nonlinear Neumann boundary condition on the boundary of the semiconductor was introduced. According to the authors’ knowledge, there are few works on the study of problem (1). Korpusov and Sveshnikov [4] and Makarov [5] proved a local theorem on the existence of solutions to the following problem

\[
\begin{aligned}
&u_t - \Delta u_t - \Delta u + (|u|^{q_3}u)_t = |u|^{q_2}u, \\
&\frac{\partial u}{\partial \nu} + |u|^{q_1}u = 0, x \in \partial \Omega = \Gamma, \\
u(x, 0) = u_0(x), x \in \Omega
\end{aligned}
\]

by using the Galerkin method combined with the compactness method. By using the method of energy inequalities [6, 7], they also obtained sufficient conditions for the blow-up of solutions in a finite time interval and established upper and lower bounds for the blow-up time, provided the initial data satisfies

\[
\int_{\Omega} \left[ \frac{1}{q_2 + 2} |u_0|^{q_2 + 2} - \frac{1}{2} |\nabla u_0|^2 \right] dx - \frac{1}{q_1 + 2} \int_{\Gamma} |u_0|^{q_1 + 2} dx \\
\geq c_1 \left\{ \int_{\Omega} \left[ \frac{1}{2} |\nabla u_0|^2 + \frac{q_3 + 1}{q_3 + 2} |u_0|^{q_3 + 2} \right] dx + \frac{q_1 + 1}{q_1 + 2} \int_{\Gamma} |u_0|^{q_1 + 2} dx, \right\}
\]

where \( c_1 \) is a positive constant depending on \( q_1, q_2, q_3 \). In this paper, we consider both the existence of the solution and a general decay of the energy functions under some restrictions on the initial data. We also study blow-up condition of the solutions with positive and negative initial energy respectively.

Before we state and prove our results, let us recall some works related to the problem we address.

In the absence of the nonlinear diffusion term \( |u|^{q_2-2}u_t \) and \( g(u) = 0 \), problem (1) can be reduced to the following classical problem

\[
\begin{aligned}
&u_t - \Delta u_t - \Delta u = f(u), \\
&\frac{\partial u}{\partial \nu} = 0, \text{ or } u = 0, x \in \partial \Omega, \\
u(x, 0) = u_0(x), x \in \Omega
\end{aligned}
\]

The first equation in problenm(3) can be called Sobolev type equation, Sobolev-Galpern type equation, pseudo-parabolic equation, or the Benjamin Bona Mahony Burgers’ (BBM-Burgers) equation (for example, see [1, 3, 8, 9]. It also appears as a nonclassical diffusion equation in fluid mechanics, solid mechanics and heat conduction theory, for instance, see [10] and references therein. It’s well known that problem (3) has been studied by many authors. A powerful technique to treat problem (3) is the so called ”potential well method”, which
was established by Payne and Sattinger [11] and Sattinger [12], and then improved by Liu and Zhao [13] by introducing a family of potential wells. Recently, there are some interesting results about the global existence and blow-up of solutions for problem (3) with \( f(u) = u^p \) in [14] where a family of potential wells is introduced to prove global existence, nonexistence and asymptotic behavior of solutions with low initial energy, while for high initial energy, finite time blow-up of solutions is acquired by comparison principle. For other related works, we refer the readers to [1, 2, 3, 6, 7, 8, 10, 16, 15, 17, 18, 20, 21, 22, 23, 24] and the references therein. The obtained results show that global existence and nonexistence depend roughly on \( p \), the degree of nonlinearity in \( f \), the dimension \( n \), and the size of the initial data.

The equation in problem (1) with Dirichlet boundary condition (i.e. \( g(u) = 0 \)) has also been studied by many authors [1, 2, 3, 16, 25, 26, 27, 28]. Korpusov and Sveshnikov et al [1, 2, 3, 16, 25, 26] gave the local strong solution and sufficient close-to-necessary conditions for the blow-up of solutions with negative initial energy using the energy approach developed by Levine [6]. Furthermore, they also considered two different abstract Cauchy problems for equations of Sobolev type. Zhang et al [27, 28] showed the exponential growth and blow-up of solutions with negative or positive initial energy by constructing differential inequality. We also refer to [29, 30, 31, 32, 33, 34, 35] for related results.

For the following parabolic equation with a nonlinear boundary condition or dynamic boundary condition

\[
\begin{cases}
  u_t - \Delta u = f(u), \\
  u = 0, x \in \Gamma_0, \\
  \frac{\partial u}{\partial \nu} = -Q(u_t) + g(u), x \in \Gamma_1, \\
  u(x, 0) = u_0(x), x \in \Omega,
\end{cases}
\]

local well-posedness, global existence and blow-up results for the solutions have also been widely studied. For example, Levine and Smith [36] and Vitillaro [37, 38] studied local and global existence and nonexistence of the solutions to problem (4) by potential well theory. Also, we would like to mention the classical global existence and nonexistence results in [39, 40, 41, 42]. For problem (4) with \( Q = 0 \), as that in [39], if we interpret \( u \) as a heat distribution in the body \( \Omega \), and assume that \( u \geq 0 \) for the moment, noting that for ranges in which \(-f\) is positive we have "absorption" of heat, while when \(-f\) is negative we have "sources" of heat. The same holds for \(-g\): when \(-g\) is positive we have a flow of heat through the boundary of \( \Omega \) that extracts heat from the body, while in the opposite case, heat is flowing inside \( \Omega \). Then, for problem (4) with \( f(s) = |u|^{p-1}u \) and \( g(s) = |u|^{k-1}u \), \( f \) can be called "sources" term and \( g \) can be called "boundary absorptive" term. When term \( |u|^{q-2}u_t \) does not present in problem (1), the same boundary condition arises in the literature in connection with the wave equation, i.e. when the operator \( u_t - \Delta u \) in (1) is replaced by the wave operator \( u_{tt} - \Delta u \). Some related problems concerning wave equations with nonlinear damping and source terms have been considered.
In particular, Cavalcanti et al [44] deals with the problem
\[
\begin{cases}
u_{tt} - \Delta u = f(u), \\
\frac{\partial u}{\partial n} + u = -h(u_t) + g(u), x \in \partial \Omega, \\
u(x, 0) = u_0(x), u_t(0) = u_1, x \in \Omega,
\end{cases}
\]
where under some assumptions imposed on the damping and source terms, they showed the well-posedness of the problem and effective optimal decay rates for the solutions. They also established a blow-up result in the case where the boundary source dominates the boundary damping and initial data are large enough. In general, methods employed to study hyperbolic problems cannot be employed to study parabolic problems, and conversely. Nevertheless, the arguments of [44] can be conveniently adapted to problem (5) without $|u|^{q-2}u_t$.

In this paper, we will investigate the existence and nonexistence of global solutions to problem (1). More precisely, under appropriate assumptions imposed on the source and boundary absorption terms, we shall establish global existence of solutions by using the potential well method combined with a standard continuous argument. We will give sufficient conditions for the blow-up of solutions in a finite time interval under suitable initial data using differential inequality. It is different with the results in [4, 5]. We also give a general decay of the energy by an integral inequality in [54].

This paper is organized as follows. Section 2 is concerned with some notations and statement of assumptions. In Section 3, we prove global existence of solutions and the blow-up result for the solutions with positive and negative initial energy respectively. In Section 4, a general decay of the energy is proved.

2. Preliminaries

In this section, we present some materials needed in the proof of our results. We use the standard Lebesgue space $L^p(\Omega)(1 < p < \infty)$ and Sobolev space $H^1(\Omega)$ with their usual scalar products and norms. Moreover, we denote $||u||_{L^p(\Omega)} = ||u||_p$ and $||u||_{L^p(\Gamma_1)} = ||u||_{p, \Gamma_1}$ for $1 \leq p \leq \infty$, and the Hilbert space $H^{1}_{\Gamma_0}(\Omega) := \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$, $||u||^2_{H^{1}_{\Gamma_0}} = ||\nabla u||^2_2 + ||u||^2_{\Gamma_0}$, where $u|_{\Gamma_0}$ stands for the restriction of the trace of $u$ on $\partial \Omega$ to $\Gamma_0$, and in particular, we denote $||u||_2 = ||u||$ and $||u||_{2, \Gamma_1} = ||u||_{\Gamma_1}$. Since $\text{meas}(\Gamma_0) > 0$, a Poincare-type inequality holds and consequently $||\nabla u||$ is an equivalent norm in $H^{1}_{\Gamma_0}(\Omega)$. The constants $C$ used
throughout this paper are positive generic constants, which may be various in different occurrences.

We assume that

\[ 1 < p \leq \frac{n+2}{n-2}, \quad 1 < q \leq \frac{n}{n-2} \text{ if } n \geq 3; \]

\[ p > 1, q > 1 \text{ if } n = 1, 2; \quad p > \max\{q-1, k\} > 1. \quad (6) \]

Then, we have the Sobolev embedding \( H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{p+1}(\Omega) \) and the trace-Sobolev embedding \( H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{k+1}(\Gamma_1) \). In these cases, the embedding constants denote \( c_*, B_* \) respectively, i.e.

\[ ||u||_{p+1} \leq c_*||u||_{H^1_{\Gamma_0}(\Omega)}, \quad ||u||_{k+1, \Gamma_1} \leq B_*||u||_{H^1_{\Gamma_0}(\Omega)}. \quad (7) \]

A function \( u(x, t) \) of class \( H^1(0, T; H^1_{\Gamma_0}(\Omega)) \) is called a weak generalized solution of problem (1) if it satisfies the equation

\[
(u_t, \phi) + (\nabla u_t, \nabla \phi) + (\nabla u, \nabla \phi) + \int_\Omega |u|^{q-2}u_t \phi dx - \int_\Omega |u|^{p-1}u \phi dx \\
+ \int_{\Gamma_1} |u|^{k-1}u_t \phi dx + k \int_{\Gamma_1} |u|^{k-1}u_t \phi dx = 0
\]

for any \( \phi \in H^1_{\Gamma_0}(\Omega) \), and almost all \( t \in [0, T] \) and the initial condition \( u(x, 0) = u_0(x) \) (see [4, 5]).

**Theorem 2.1.** Let \( u_0 \in H^1(0, T; H^1_{\Gamma_0}(\Omega)) \) and \( p, q, k \) satisfy (9), then problem (1) has a unique weak generalized solution on \([0, T_0]\) for some \( T_0 > 0 \), and we have either \( T_0 = +\infty \) or \( T_0 < +\infty \) and

\[ \lim_{t \to T_0^-} \sup_{t \in [0, T_0]} ||u||^2_{H^1_{\Gamma_0}(\Omega)} = +\infty. \]

Theorem 2.1 can be easily established by combining the argument of [55], Theorem 1 and Theorem 2 in [4, 5], thus we omit it.

We define the functional that plays as the "potential energy"

\[
E(t) = E(u) = \frac{1}{2}||\nabla u||^2 - \frac{1}{p+1}||u||^{p+1} + \frac{1}{k+1}||u||^{k+1}_{k+1, \Gamma_1} \\
= \frac{1}{2}||u||^2_{H^1_{\Gamma_0}(\Omega)} - \frac{1}{p+1}||u||^{p+1}_{p+1} + \frac{1}{k+1}||u||^{k+1}_{k+1, \Gamma_1}, \quad (8)
\]

and the Nehari functional

\[ I(u) = ||u||^2_{H^1_{\Gamma_0}(\Omega)} - ||u||^{p+1}_{p+1} + ||u||^{k+1}_{k+1, \Gamma_1}. \]

We also have the following identity

\[
E'(t) = -\frac{1}{2}||u||^2_{H^1_{\Gamma_0}(\Omega)} - \int_\Omega |u|^{q-2}u^2 dx - k \int_{\Gamma_1} |u|^{k-1}u^2 dx \leq 0. \quad (9)
\]

In the sequel, a crucial role is played by the Nehari manifold to \( I \), which is

\[ N = \{ u \in H^1_{\Gamma_0}(\Omega) | I(u) = 0, ||u||_{H^1_{\Gamma_0}(\Omega)} \neq 0 \}, \]
and we can readily give the mountain-pass level \( d \) by \( d = \inf_{u \in N} E(u) \).

Next, we show some properties related to functions \( E(u) \) and \( I(u) \) in the following lemmas.

**Lemma 2.2.** Let \( u \in H^1_0(\Omega) \), \( \|u\|_{H^1_0(\Omega)} \neq 0 \) and (ii) hold, then

(i) \( \lim_{\lambda \to 0} E(\lambda u) = 0 \), \( \lim_{\lambda \to +\infty} E(\lambda u) = -\infty \);

(ii) In the interval \( 0 < \lambda < \infty \), there exists a unique \( \lambda_0 = \lambda_0(u) > 0 \) such that \( \frac{d}{d\lambda} E(\lambda u)|_{\lambda=\lambda_0} = 0 \);

(iii) \( E(\lambda u) \) is increasing on \( 0 < \lambda \leq \lambda_0 \), decreasing on \( \lambda_0 \leq \lambda < +\infty \) and takes the maximum at \( \lambda = \lambda_0 \);

(iv) \( I(\lambda u) > 0 \), for \( 0 < \lambda < \lambda_0 \); \( I(\lambda u) < 0 \), for \( \lambda > \lambda_0 \) and \( I(\lambda_0 u) = 0 \).

**Proof.** (i) The conclusion follows from

\[
E(\lambda u) = \frac{\lambda^2}{2} \|u\|^2_{H^1_0(\Omega)} - \frac{\lambda^{p+1}}{p+1} \|u\|^{p+1} + \frac{\lambda^{k+1}}{k+1} \|u\|^{k+1}_{k+1,\Gamma_1}.
\]

(ii) First, note that

\[
\frac{d}{d\lambda} E(\lambda u) = \lambda \|u\|^2_{H^1_0(\Omega)} - p \|u\|^{p+1}_{p+1} + \lambda^k \|u\|^{k+1}_{k+1,\Gamma_1} = 0, \lambda > 0
\]

is equivalent to

\[
\lambda^{p-1} \|u\|^{p+1}_{p+1} - \lambda^{k-1} \|u\|^{k+1}_{k+1,\Gamma_1} = 0, \lambda > 0
\]

Let

\[
h(\lambda) = \lambda^{p-1} \|u\|^{p+1}_{p+1} - \lambda^{k-1} \|u\|^{k+1}_{k+1,\Gamma_1}
\]

where \( h_1(\lambda) = \lambda^{p-k} \|u\|^{p+1}_{p+1} - \|u\|^{k+1}_{k+1,\Gamma_1} \). Note that \( h_1(\lambda) \) is increasing on \( 0 < \lambda < \infty \), \( \lim_{\lambda \to 0^+} h_1(\lambda) \leq 0 \), and \( \lim_{\lambda \to +\infty} h_1(\lambda) = +\infty \), and hence there exists a unique \( \lambda^* > 0 \) such that \( h_1(\lambda^*) = 0 \), thereby \( h(\lambda^*) = 0, h(\lambda) < 0 < \lambda < \lambda^*, h(\lambda) > 0 \) for \( \lambda^* < \lambda < \infty \). Hence, for any \( \|u\|_{H^1_0(\Omega)} > 0 \), there exists a unique \( \lambda_0 > \lambda^* \) such that (10) holds, and then (ii) holds.

(iii) Note that \( \frac{d}{d\lambda} E(\lambda u) = \lambda \|u\|^2_{H^1_0(\Omega)} - h(\lambda) \). From the proof of (ii), it follows that if \( 0 < \lambda < \lambda^* \), then \( h(\lambda) < 0 \); if \( \lambda^* < \lambda < \lambda_0 \), then \( 0 < h(\lambda) < \|u\|^2_{H^1_0(\Omega)} \); and if \( \lambda_0 < \lambda < \infty \), then \( h(\lambda) > \|u\|^2_{H^1_0(\Omega)} \). From this, the conclusion of (iii) holds.

(iv) The conclusion follows from the proof of (iii) and

\[
I(\lambda u) = \lambda^2 \|u\|^2_{H^1_0(\Omega)} - \lambda^{p+1} \|u\|^{p+1}_{p+1} + \lambda^{k+1} \|u\|^{k+1}_{k+1,\Gamma_1} = \lambda \frac{d}{d\lambda} E(\lambda u).
\]

This completes the proof of Lemma 2.2. \( \square \)
Now, we define

$$F(x) = \frac{1}{2} x^p - \frac{c_p^{p+1}}{p+1} x^{p+1} - \frac{B^{k+1}}{k+1} x^{k+1},$$

and let $r_0$ be the unique real root of equation $F'(x) = 0$. We easily verify that $r_0$ is the unique real root of equation $\phi(x) = 1$, where $\phi(x) = c_p^{p+1} x^{p+1} + B^{k+1} x^{k-1}$, then $\phi(r_0) = c_p^{p+1} r_0^{p+1} + B^{k+1} r_0^{k-1} = 1$. It can be checked that $r_0$ is a point of local maximum for $F(x)$ (see [44] for more details). Accordingly, let us define $E_1$ as

$$E_1 = F(r_0) = \frac{1}{2} r_0^2 - \frac{c_p^{p+1}}{p+1} r_0^{p+1} - \frac{B^{k+1}}{k+1} r_0^{k+1}.$$

**Lemma 2.3.** Let (6) hold, then (i) if $0 < \|u\|_{H^1_{\Gamma_0}(\Omega)} < r_0$, then $I(u) > 0$; (ii) if $I(u) < 0$, then $\|u\|_{H^1_{\Gamma_0}(\Omega)} > r_0$; (iii) if $I(u) = 0$ and $\|u\|_{H^1_{\Gamma_0}(\Omega)} \neq 0$, i.e. $u \in N$, then $\|u\|_{H^1_{\Gamma_0}(\Omega)} \geq r_0$.

**Proof.** (i) Since $\phi(x)$ is a strictly increasing function in $(0, r_0)$, from

$$0 < \|u\|_{H^1_{\Gamma_0}(\Omega)} < r_0,$$

we get $\phi(\|u\|_{H^1_{\Gamma_0}(\Omega)}) < \phi(r_0)$, and

$$I(u) = \|u\|_{H^1_{\Gamma_0}(\Omega)}^2 - \|u\|_{p+1}^{p+1} + \|u\|_{k+1, \Gamma_1}^{k+1}$$

$$\geq \|u\|_{H^1_{\Gamma_0}(\Omega)}^2 - \|u\|_{p+1}^{p+1} + \|u\|_{k+1, \Gamma_1}^{k+1},$$

$$= \|u\|_{H^1_{\Gamma_0}(\Omega)}^2 (1 - c_p^{p+1} \|u\|_{H^1_{\Gamma_0}(\Omega)}^{p+1} - B^{k+1} \|u\|_{H^1_{\Gamma_0}(\Omega)}^{k+1})$$

$$= \|u\|_{H^1_{\Gamma_0}(\Omega)}^2 (\phi(r_0) - \phi(\|u\|_{H^1_{\Gamma_0}(\Omega)})) > 0.$$

(ii) Condition $I(u) < 0$ gives

$$\phi(r_0) \|u\|_{H^1_{\Gamma_0}(\Omega)} = \|u\|_{H^1_{\Gamma_0}(\Omega)}^2$$

$$< \|u\|_{p+1}^{p+1} - \|u\|_{k+1, \Gamma_1}^{k+1} \|u\|_{p+1}^{p+1} + \|u\|_{k+1, \Gamma_1}^{k+1},$$

$$\leq (c_p^{p+1} \|u\|_{H^1_{\Gamma_0}(\Omega)}^{p+1} + B^{k+1} \|u\|_{H^1_{\Gamma_0}(\Omega)}^{k+1}) \|u\|_{H^1_{\Gamma_0}(\Omega)}^2 = \phi(\|u\|_{H^1_{\Gamma_0}(\Omega)}) \|u\|_{H^1_{\Gamma_0}(\Omega)}^2,$$

which implies $\|u\|_{H^1_{\Gamma_0}(\Omega)} \neq 0$ and $\|u\|_{H^1_{\Gamma_0}(\Omega)} > r_0$ by the monotonicity of $\phi$.

(iii) If $I(u) = 0$ and $\|u\|_{H^1_{\Gamma_0}(\Omega)} \neq 0$, then

$$\phi(r_0) \|u\|_{H^1_{\Gamma_0}(\Omega)} = \|u\|_{H^1_{\Gamma_0}(\Omega)}^2 = \|u\|_{p+1}^{p+1} - \|u\|_{k+1, \Gamma_1}^{k+1},$$

$$\leq \|u\|_{p+1}^{p+1} + \|u\|_{k+1, \Gamma_1}^{k+1} \leq \phi(\|u\|_{H^1_{\Gamma_0}(\Omega)}) \|u\|_{H^1_{\Gamma_0}(\Omega)}^2,$$

and from the monotonicity of $\phi$, we get $\|u\|_{H^1_{\Gamma_0}(\Omega)} > r_0$. □

**Lemma 2.4.** $d \geq d_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right)r_0^2 = \frac{p-1}{2(p+1)}r_0^2$. 

Proof. For $u \in N$ (or $I(u) = 0$ and $||u||_{H^1_{t_0}(\Omega)} \neq 0$), by Lemma 2.3, we have $||u||_{H^1_{t_0}(\Omega)} > r_0$. Hence

$$E(u) \geq \frac{1}{2} ||u||^2_{H^1_{t_0}(\Omega)} + \frac{1}{p+1} (||u||^{p+1}_{p+1} + ||u||^{k+1}_{k+1, \Gamma_1})$$

$$= \left( \frac{1}{2} - \frac{1}{p+1} \right) ||u||^2_{H^1_{t_0}(\Omega)} + \frac{1}{p+1} I(u)$$

$$= \left( \frac{1}{2} - \frac{1}{p+1} \right) ||u||^2_{H^1_{t_0}(\Omega)} \geq \left( \frac{1}{2} - \frac{1}{p+1} \right) \lambda^2_0,$$

which gives $d \geq d_0$. 

Remark 2.5. Noting the definition of $d$ and the fact that

$$E(u) = \frac{1}{2} ||u||^2_{H^1_{t_0}(\Omega)} - \frac{1}{p+1} ||u||^{p+1}_{p+1} + \frac{1}{k+1} ||u||^{k+1}_{k+1, \Gamma_1}$$

$$\geq \frac{1}{2} ||u||^2_{H^1_{t_0}(\Omega)} - \frac{c^{p+1}}{p+1} ||u||^{p+1}_{H^1_{t_0}(\Omega)} - \frac{B^{k+1}}{k+1} ||u||^{k+1}_{H^1_{t_0}(\Omega)} = F(||u||_{H^1_{t_0}(\Omega)}),$$

we know $d \geq E_1$.

Now we define the subsets of $H^1_{t_0}(\Omega)$ related to problem (1)-(3). Set

$$W = \{ u \in H^1_{t_0}(\Omega) \mid E(u) < d, I(u) > 0 \}, V = \{ u \in H^1_{t_0}(\Omega) \mid E(u) < d, I(u) > 0 \}.$$

(12)

Lemma 2.6. If $u_0 \in H^1_{t_0}(\Omega)$, $0 < E(0) < d$, and $u$ is a weak solution of problem (1)-(3), then (i) $u \in W$ if $I(u_0) > 0$ or $||u||_{H^1_{t_0}(\Omega)} = 0$; (ii) $u \in V$ if $I(u_0) < 0$.

Proof. We only prove (i), and the proof for (ii) is similar. We are going to prove that $u \in W$ for $0 < t < T_0$. From (3), we have

$$E(u(t)) + \int_0^t \int_{\Omega} |u|^q - 2 u_s^2 dx + k \int_{\Gamma_1} |u|^{k-1} u_s^2 ds = E(0) < d, \text{ for any } t \in [0, T_0],$$

which implies $E(u(t)) < d$. To prove that $u \in W$ for $0 < t < T_0$, we argue by contradiction. Indeed, if it is not the case, there would exist $t_0 \in (0, T_0)$ such that $u(t_0) \in N$, and by the definition of $d = \inf_{u \in N} E(u)$, one has $d < E(t_0) \leq d$, then we reach to a contradiction. 

3. Global existence and blow-up of solutions

In this section, we prove the global existence and blow-up of solutions to problem (1).

Theorem 3.1. Let $u_0 \in H^1_{t_0}(\Omega)$, $0 < E(0) < d$, $I(u_0) > 0$ or $||u||_{H^1_{t_0}(\Omega)} = 0$, and $p, q, k$ satisfies (3), then the weak solution $u$ to problem (1) in Theorem 2.1 can be extended to $(0, \infty)$. 

Proof. By Lemma 2.5, we have \( u \in W \), then \( I(u) > 0 \) and \( E(u) < d \) for all \( t \in (0, T_0) \). Therefore,

\[
d > E(u) = \frac{1}{2} ||u||^2_{H^1_{\Gamma_0}(\Omega)} - \frac{1}{p+1} ||u||^{p+1}_{p+1} + \frac{1}{k+1} ||u||^{k+1}_{k+1, \Gamma_1}
\]

\[
> \left( \frac{1}{2} - \frac{1}{p+1} \right) ||u||^2_{H^1_{\Gamma_0}(\Omega)} + \frac{1}{p+1} I(u)
\]

\[
> \left( \frac{1}{2} - \frac{1}{p+1} \right) ||u||^2_{H^1_{\Gamma_0}(\Omega)}
\]

for all \( t \in (0, T_0) \). Then, (13) and (7) imply

\[
||u||^2_{H^1_{\Gamma_0}(\Omega)} < \frac{2(p+1)d}{p-1} ||u||^{p+1}_{p+1}
\]

\[
< e^{p+1} \left( \frac{2(p+1)d}{p-1} \right) ||u||^{k+1}_{k+1, \Gamma_1}
\]

\[
< B^{k+1} \left( \frac{2(p+1)d}{p-1} \right)^{\frac{k+1}{k+1}}
\]

(14) for all \( t \in (0, T) \). By (8) and the definition of \( E(u) \), we have

\[
\frac{1}{2} ||u_t||^2 + \frac{1}{2} ||u||^2_{H^1_{\Gamma_0}(\Omega)} \leq E(0) + \frac{1}{p+1} ||u||^{p+1}_{p+1} - \frac{1}{k+1} ||u||^{k+1}_{k+1, \Gamma_1} < C < +\infty
\]

(15) for all \( t \in (0, T) \). It follows from (15) and from a standard continuous argument that local weak solution \( u \) furnished by Theorem 2.1 can be extended to the whole internal \([0, \infty)\), that is to say, \( u \) is a global solution.

\[\Box\]

Theorem 3.2. Suppose that assumption (6) holds, \( u(0) = u_0 \in H^1_{\Gamma_0}(\Omega) \) and \( u \) is a local solution of problem (1). If \( E(0) < 0 \), then the solution of the system (1) blows up in finite time.

Proof. We set

\[
H(t) = -E(t).
\]

(16)

By the definition of \( H(t) \) and (9),

\[
H'(t) = -E'(t) \geq 0.
\]

(17)

Consequently, by \( E(0) < 0 \), we have

\[
H(0) = -E(0) > 0.
\]

(18)

It is clear that by (17) and (18)

\[
0 < H(0) \leq H(t).
\]

(19)

By (16) and the expression of \( E(t) \),

\[
H(t) - \frac{1}{p+1} ||u||^{p+1}_{p+1} + \frac{1}{k+1} ||u||^{k+1}_{k+1, \Gamma_1} = -\frac{1}{2} ||\nabla u||^2 < 0.
\]

(20)
One implies

\[ 0 < H(0) \leq H(t) \leq \frac{1}{p + 1}||u||_{p+1}^{p+1} - \frac{1}{k + 1}||u||_{k+1, \Gamma_1}^{k+1}, \]

\[ \leq \frac{1}{p + 1}||u||_{p+1}^{p+1} \leq \frac{1}{p + 1}||u||_{p+1}^{p+1} + \frac{1}{k + 1}||u||_{k+1, \Gamma_1}^{k+1}. \]  

(21)

Let us define the functional

\[ L(t) = H^{1-\sigma}(t) + \frac{\epsilon}{2} || \nabla u||^2 + \frac{\epsilon}{2} ||u||^2, \]

(22)

where \( \epsilon > 0 \) will be fixed in later and \( 0 < \sigma \leq \frac{p+1-q}{p+1} \) (this can be done since \( q - 1 < p \)). By taking the time derivative of (21), using problem (11), and performing several integration by parts, we get

\[ L'(t) = (1 - \sigma)H^{-\sigma}(t)H'(t) + \epsilon \int_{\Omega} \nabla u \nabla u_t dx + \epsilon \int_{\Gamma_1} uu_t dx \]

\[ = (1 - \sigma)H^{-\sigma}(t)H'(t) + \epsilon \int_{\Omega} [uu_t - u \Delta u_t + \epsilon \int_{\Gamma_1} u_t \frac{\partial u}{\partial n} dx \]

\[ = (1 - \sigma)H^{-\sigma}(t)H'(t) + 2\epsilon H(t) + 2\epsilon E(t) - \epsilon || \nabla u||^2 \]

\[ + \epsilon ||u||_{p+1}^{p+1} + \epsilon ||u||_{k+1, \Gamma_1}^{k+1} - \epsilon \int_{\Omega} u|^{q-2} uu_t dx + \epsilon \int_{\Gamma_1} u_t||u||_{k+1}^{k+1} u dx \]

\[ = (1 - \sigma)H^{-\sigma}(t)H'(t) + 2\epsilon H(t) + \epsilon(1 - \frac{2}{p+1})||u||_{p+1}^{p+1} \]

\[ + \epsilon(1 + \frac{2}{k+1})||u||_{k+1, \Gamma_1}^{k+1} - \epsilon \int_{\Omega} u|^{q-2} uu_t dx + \epsilon \int_{\Gamma_1} u_t||u||_{k+1}^{k+1} u dx. \]  

(23)

To estimate the last two terms in the right-hand side of (23), by the following Young’s inequality

\[ ab \leq \delta^{-1} a^2 + \delta b^2, \]

we deduce that, for any \( \delta_1 > 0 \) and \( \delta_2 > 0 \),

\[ \int_{\Omega} |u|^{q-2} uu_t dx = \int_{\Omega} (|u|^q - |u|^q u_t) dx \leq \delta_1^{-1} \int_{\Omega} |u|^{q-2} u_t^2 dx + \delta_1 \int_{\Omega} |u|^q dx, \]

\[ \int_{\Gamma_1} |u|^{k-1} uu_t dx \]

\[ = \int_{\Gamma_1} (|u|^{k-1} u_t)(|u|^{k-1} u) dx \leq \delta_2^{-1} \int_{\Gamma_1} |u|^{k-1} u_t^2 dx + \delta_2 \int_{\Gamma_1} |u|^{k+1} dx. \]

Therefore, we have

\[ L'(t) \geq (1 - \sigma)H^{-\sigma}(t)H'(t) + 2\epsilon H(t) + \epsilon(1 - \frac{2}{p+1})||u||_{p+1}^{p+1} \]

\[ + \epsilon(1 + \frac{2}{k+1})||u||_{k+1, \Gamma_1}^{k+1} \]
By (25), (26) and (28), we have

\[- \int_{\Omega} |u|^q u_t^2 dx \geq -H'(t), \quad - \int_{\Gamma_1} |u|^{k-1} u_t^2 dx \geq -H'(t)\]

by (9) and (17), we have

\[L'(t) \geq [(1 - \sigma - \epsilon M_1) H^{-\sigma}(t) - \epsilon \delta_2^{-1}] H'(t) + 2 \epsilon H(t) + \epsilon (1 - \frac{2}{p + 1}) ||u||_{p+1}^{p+1}\]

\[+ \epsilon (1 + \frac{2}{k + 1} - \delta_2)||u||_{k+1, \Gamma_1}^{k+1} - \epsilon M_1^{-1} H^{-\sigma}(t)||u||_q^q,\]

Taking into account (21) and the embedding \(L^{p+1}(\Omega) \hookrightarrow L^q(\Omega)\), we get

\[H^{s}(t)||u||_q \leq C_1 ||u||_{p+1}^{(p+1)s} ||u||_q^q \leq C_2 ||u||_{p+1}^{(p+1)s+q},\]

for some positive constants \(C_1\) and \(C_2\). Now apply the inequality

\[x^l \leq (x + 1) \leq (1 + \frac{1}{z})(x + z), \quad x \geq 0, 0 \leq l \leq 1, z > 0,\]

in particular, taking \(x = ||u||_{p+1}^{p+1}, l = \frac{(p+1)s+q}{p+1}, z = H(0)\), we obtain

\[||u||_{p+1}^{p+1} = (||u||_{p+1}^{p+1})^l \leq (1 + \frac{1}{H(0)}) (||u||_{p+1}^{p+1} + H(0)) \leq C_3 ||u||_{p+1}^{p+1},\]

where we have used the fact that \(0 < \frac{q}{p+1} < 1, 0 < \sigma \leq \frac{p+1-q}{p+1}\), and (21).

By (25), (20) and (28), we have

\[L'(t) \geq [(1 - \sigma - \epsilon M_1) H^{-\sigma}(t) - \epsilon \delta_2^{-1}] H'(t) + 2 \epsilon H(t)\]

\[+ \epsilon (1 - \frac{2}{p + 1} - C_3 M_1^{-1}) ||u||_{p+1}^{p+1} + \epsilon (1 + \frac{2}{k + 1} - \delta_2)||u||_{k+1, \Gamma_1}^{k+1}.\]

Now, we take \(\delta_2\) such that \(1 + \frac{2}{k+1} - \delta_2 > 0\), and we take \(M_1\) large enough such that \(1 - \frac{2}{p+1} - C_3 M_1^{-1} = C_4 > 0\). Once \(M_1\) and \(\delta_2\) are fixed, we can pick \(\epsilon\) small enough such that

\[1 - \sigma - \epsilon M_1 > 0,\]

\[(1 - \sigma - \epsilon M_1) H^{-\sigma}(t) - \epsilon \delta_2^{-1} > (1 - \sigma - \epsilon M_1) H^{-\sigma}(0) - \epsilon \delta_2^{-1} > 0,\]

where we have used the fact that \(H^{-\sigma}(t) > H^{-\sigma}(0)\). Then there exist \(C_5 > 0\) such that (29) becomes

\[L'(t) \geq C_5 (H(t) + ||u||_{p+1}^{p+1} + ||u||_{k+1, \Gamma_1}^{k+1}).\]

Then, we have

\[L(t) \geq L(0) \geq 0.\]
On the other hand, by the definition of $L(t)$ and (20), we have
\[
L(t) = H^{1-\sigma}(t) - \epsilon(H(t) - \frac{1}{p+1}||u||_{p+1}^{p+1} + \frac{1}{k+1}||u||_{k+1,\Gamma_1}^{k+1}) + \frac{\epsilon}{2}||u||^2
\]
\[
\leq (1 - \epsilon)H^{1-\sigma}(t) + \frac{\epsilon}{p+1}||u||_{p+1}^{p+1} - \frac{\epsilon}{k+1}||u||_{k+1,\Gamma_1}^{k+1} + \frac{\epsilon}{2}||u||^2
\]
\[
\leq (1 - \epsilon)H^{1-\sigma}(t) + \frac{\epsilon}{p+1}||u||_{p+1}^{p+1} + \frac{\epsilon}{2}||u||^2,
\]
where we have used the fact $H(t) \geq H^{1-\sigma}(t)$ (this can be ensured by (18), (19), $0 < \sigma < 1$ and that $E(0)$ is sufficient negative).

By the inequality (27) with $x = ||u||_{p+1}^{\frac{\eta+1}{p+1}}, l = 1 - \sigma < 1, z = H^{\frac{1}{1-\sigma}}(0)$, we have
\[
||u||_{p+1}^{p+1} = (||u||_{p+1}^{\frac{\eta+1}{p+1}})^{1-\sigma} \leq (1 + \frac{1}{H^{\frac{1}{1-\sigma}}(0)})(||u||_{p+1}^{\frac{\eta+1}{p+1}} + H^{\frac{1}{1-\sigma}}(0)) \leq C_6||u||_{p+1}^{\frac{\eta+1}{p+1}}.
\]

Therefore, we get
\[
L(t) \leq (1 - \epsilon)H^{1-\sigma}(t) + C_6||u||_{p+1}^{\frac{\eta+1}{p+1}} + \frac{\epsilon}{2}||u||^2.
\]

Then, by the embedding $L^{p+1}(\Omega) \hookrightarrow L^2(\Omega)$, we have, for fixed $\epsilon$ sufficient small,
\[
L^{\frac{1}{1-\sigma}}(t) \leq C_7[H(t) + ||u||_{p+1}^{\frac{\eta+1}{p+1}} + ||u||_{p+1}^{\frac{2}{1-\sigma}}].
\]

Using again the inequality (27) with $x = ||u||_{p+1}^{\frac{\eta+1}{p+1}}, l = \frac{2}{(p+1)(1-\sigma)} < 1$ (since $\sigma < \frac{p+1-\eta}{p+1} < \frac{p-1}{p+1}$), $z = H(0)$, we have
\[
||u||_{p+1}^{\frac{2}{1-\sigma}} = (||u||_{p+1}^{\frac{\eta+1}{p+1}})^{(p+1)(1-\sigma)} \leq (1 + \frac{1}{H(0)})(||u||_{p+1}^{\frac{\eta+1}{p+1}} + H(0)) \leq C_8||u||_{p+1}^{\frac{\eta+1}{p+1}}.
\]

From (32) and (33), we obtain
\[
L^{\frac{1}{1-\sigma}}(t) \leq C_9(H(t) + ||u||_{p+1}^{\frac{\eta+1}{p+1}}) \leq C_9(H(t) + ||u||_{p+1}^{\frac{\eta+1}{p+1}} + ||u||_{k+1,\Gamma_1}^{k+1}).
\]

Combining with (30) and (34), we arrive that
\[
L'(t) \geq C_{10}L^{\frac{1}{1-\sigma}}(t).
\]

Integration of (35) between 0 and $t$ gives the desired results. The theorem is proved.

In the following, we will prove that the solution will blow up provided that the initial energy $E(0) > 0$. The next lemma will play an essential role in our proving and it is similar to a lemma used firstly by Vitillaro [56]. Now the main idea of the proof is from Lemma 9.1 in [14].
Lemma 3.3. Let \( u \) be a solution of problem (11). Suppose that the assumption of \( k, p \) hold. Further assume that \( E(0) < E_1 \) and \( \|u(0)\|_{H^{k}_{1,0}(\Omega)} > r_0 \). Then there exists a constant \( r_1 > r_0 \) such that \( \|u(t)\|_{H^{k}_{1,0}(\Omega)} \geq r_1 \), and

\[
\frac{1}{p+1} \|u\|_{p+1}^{p+1} + \frac{1}{k+1} \|u\|_{k+1,r_1}^{k+1} \geq \frac{1}{2} r_1^2 - F(r_1) = \frac{c_r^{p+1}}{p+1} r_1^{p+1} + \frac{B_{r_1}^{k+1}}{k+1} r_1^{k+1}.
\]

Proof. We observe from (11) that

\[
E(u(t)) \geq F(\|u\|_{H^{k}_{1,0}(\Omega)}).
\]

We have that \( F(r) \) is increasing for \( 0 < r < r_0 \), decreasing for \( r > r_0 \), \( F(r_0) = E_1 \), and \( \lim_{r \to +\infty} F(r) = -\infty \). Then, since \( d \geq E_1 > E(u(0)) = F(\|u(0)\|_{H^{k}_{1,0}(\Omega)}) \geq F(0) \), there exist \( r'_1 < r_0 < r_1 \), which verify

\[
F(r_1) = F(r'_1) = E(u(0)).
\]

Considering that \( E(t) \) is non-increasing, we have

\[
E(u(t)) \leq E(u(0)).
\]

From (37) and (38) we have

\[
F(\|u(0)\|_{H^{k}_{1,0}(\Omega)}) \leq E(u(0)) = F(r_1).
\]

Since \( \|u(0)\|_{H^{k}_{1,0}(\Omega)}, r_1 \in (r_0, +\infty) \) and \( F(r) \) is deceasing in this interval, from (39) one has

\[
\|u(0)\|_{H^{k}_{1,0}(\Omega)} \geq r_1.
\]

In the sequel, we will prove that

\[
\|u(t)\|_{H^{k}_{1,0}(\Omega)} \geq r_1.
\]

In fact, we will argue by contradiction. Supposing that (11) does not hold, then, there exists \( t^* \in (0, T_0) \) such that

\[
\|u(t^*)\|_{H^{k}_{1,0}(\Omega)} < r_1.
\]

If \( \|u(t^*)\|_{H^{k}_{1,0}(\Omega)} > r_0 \), then from (39), (37) and (42), we have

\[
E(u(t^*)) \geq F(\|u(t^*)\|_{H^{k}_{1,0}(\Omega)}) > F(r_1) = E(u(0)),
\]

which contradicts (38) and proves (11). Now, if \( \|u(t^*)\|_{H^{k}_{1,0}(\Omega)} \leq r_0 \), we have, taking (40) into account, that there exists \( r_2 \) which verifies

\[
\|u(t^*)\|_{H^{k}_{1,0}(\Omega)} \leq r_0 < r_2 < r_1 \leq \|u(0)\|_{H^{k}_{1,0}(\Omega)}.
\]

Consequently, from the continuity of \( \|u(.)\|_{H^{k}_{1,0}(\Omega)} \), there exists \( t' \in (0, t^*) \) verifying

\[
\|u(t')\|_{H^{k}_{1,0}(\Omega)} = r_2.
\]
From the last identity and from (36), (37) and (43), we obtain
\[ E(u(t')) \geq F(||u(t')||_{H^1_{\Gamma_0}(\Omega)}) > F(r_2) > F(r_1) = E(u(0)), \]
which also contradicts (38) and proves (41).

On the other hand, from the identity of the energy, it holds that
\[
\frac{1}{2}||u||^2_{H^1_{\Gamma_0}(\Omega)} \leq E(u(0)) + \frac{1}{p+1}||u||_{p+1}^{p+1} + \frac{1}{k+1}||u||_{k+1,\Gamma_1}^{k+1},
\]
which implies, from (37), (41) and by the definition of \( F \), that
\[
\frac{1}{p+1}||u||_{p+1}^{p+1} + \frac{1}{k+1}||u||_{k+1,\Gamma_1}^{k+1} \geq \frac{1}{2}||u||^2_{H^1_{\Gamma_0}(\Omega)} - E(u(0))
\geq \frac{1}{2}r_1^2 - F(r_1) = \frac{c_{p+1}}{p+1}r_1^{p+1} + B_{k+1}^{k+1}.
\]

\[ \square \]

**Theorem 3.4.** Suppose that the assumption (6) holds, \( u(0) = u_0 \in H^1_{\Gamma_0}(\Omega) \) and \( u \) is a local solution of the system (1), \( ||u_0||_{H^1_{\Gamma_0}(\Omega)} > r_0 \) and \( E(0) < E_1 \). Then the solution of problem (1) blows up.

**Proof.** We set
\[ H(t) = E_2 - E(t), \]
where \( E_2 \) is a constant and \( E(0) < E_2 < E_1 < d \). By the definition of \( H(t) \) and (9)
\[ H'(t) = -E'(t) \geq 0, \]
which implies that \( H(t) \) is non-decreasing, and, consequently,
\[ H(t) \geq H(0) = E_2 - E(0) > 0. \]
Considering Lemma 3.3, we have that \( ||u(t)||_{H^1_{\Gamma_0}(\Omega)} \geq r_1 \), for some \( r_1 > r_0 \). From this inequality, the definition of the energy and taking (45) into account, we deduce
\[
H(t) = E_2 - \frac{1}{2}||u||^2_{H^1_{\Gamma_0}(\Omega)} - \frac{1}{p+1}||u||_{p+1}^{p+1} - \frac{1}{k+1}||u||_{k+1,\Gamma_1}^{k+1}
\leq E_1 - \frac{1}{2}||u||^2_{H^1_{\Gamma_0}(\Omega)} + \frac{1}{p+1}||u||_{p+1}^{p+1} - \frac{1}{k+1}||u||_{k+1,\Gamma_1}^{k+1}
\leq E_1 - \frac{1}{2}r_1^2 + \frac{1}{p+1}||u||_{p+1}^{p+1} - \frac{1}{k+1}||u||_{k+1,\Gamma_1}^{k+1},
\]
which implies, having in mind that \( E_1 = F(r_0) = \frac{1}{2}r_0^2 - \frac{c_{p+1}}{p+1}r_0^{p+1} - \frac{k_{p+1}}{k+1}r_0^{k+1} \),
that

\[
H(t) \leq \frac{1}{2}r_0^2 - \frac{c_{p+1}}{p+1}r_0^{p+1} - \frac{B_k^{k+1}}{k+1}r_0^{k+1} - \frac{1}{2}r_0^2 + \frac{1}{p+1}||u||^{p+1}_{p+1} - \frac{1}{k+1}||u||^{k+1}_{k+1, \Gamma_1}
\]

\[
\leq - \frac{c_{p+1}}{p+1}r_0^{p+1} - \frac{B_k^{k+1}}{k+1}r_0^{k+1} + \frac{1}{p+1}||u||^{p+1}_{p+1} - \frac{1}{k+1}||u||^{k+1}_{k+1, \Gamma_1}
\]

\[
\leq \frac{1}{p+1}||u||^{p+1}_{p+1} - \frac{1}{k+1}||u||^{k+1}_{k+1, \Gamma_1}
\]

\[
\leq \frac{1}{p+1}||u||^{p+1}_{p+1} \leq \frac{1}{p+1}||u||^{p+1}_{p+1} + \frac{1}{k+1}||u||^{k+1}_{k+1, \Gamma_1}.
\]

(48) \]

Then we can prove the theorem similar to the proof of Theorem 3.2.

4. Asymptotic stability

In this section, we will state and prove the exponential decay of the solutions to problem (1). In this context, we have the following lemma.

**Lemma 4.1.** Let \( u \) be a solution to problem (1). Assume that assumption (3) holds and \( u_0 \in W \), then we have

\[
||u||^2_{H^1_{\Gamma_0}(\Omega)} \leq \frac{2(p+1)}{p-1}E(t) \leq \frac{2(p+1)}{p-1}E(0),
\]

(49)

\[
||u||^{k+1}_{k+1, \Gamma_1} \leq B_k^{k+1}\left(\frac{2(p+1)}{p-1}E(0)\right)^{k-2}||u||^2_{H^1_{\Gamma_0}(\Omega)},
\]

(50)

\[
||u||^{p+1}_{p+1} \leq c_{p+1}^*\left(\frac{2(p+1)}{p-1}E(0)\right)^{p-2}||u||^{2}_{H^1_{\Gamma_0}(\Omega)}.
\]

(51)

**Proof.** By Lemma 2.5, we have \( u \in W \) and \( I(u) > 0 \). We know from (3) and the definition of \( E(t) \) that

\[
E(0) \geq E(u) = \frac{1}{2}||u||^2_{H^1_{\Gamma_0}(\Omega)} - \frac{1}{p+1}||u||^{p+1}_{p+1} + \frac{1}{k+1}||u||^{k+1}_{k+1, \Gamma_1}
\]

\[
\geq \frac{1}{2}||u||^2_{H^1_{\Gamma_0}(\Omega)} - \frac{1}{p+1}||u||^{p+1}_{p+1} + \frac{1}{p+1}||u||^{k+1}_{k+1, \Gamma_1} + \frac{1}{k+1} - \frac{1}{p+1}||u||^{k+1}_{k+1, \Gamma_1}
\]

\[
\geq \frac{1}{2}||u||^2_{H^1_{\Gamma_0}(\Omega)} + \frac{1}{p+1}I(u) + \frac{1}{p+1} - \frac{1}{k+1}||u||^{k+1}_{k+1, \Gamma_1}
\]

\[
\geq \frac{1}{2}||u||^2_{H^1_{\Gamma_0}(\Omega)} + \frac{p-k}{(p+1)(k+1)}||u||^{k+1}_{k+1, \Gamma_1}.
\]

Thus we obtain (49). By the embedding \( H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{k+1}(\Gamma_1) \) and (49), we have

\[
||u||^{k+1}_{k+1, \Gamma_1} \leq B_k^{k+1}||u||^{k+1}_{H^1_{\Gamma_0}(\Omega)} \leq B_k^{k+1}\left(\frac{2(p+1)}{p-1}E(0)\right)^{k-2}||u||^{2}_{H^1_{\Gamma_0}(\Omega)}.
\]
Proof. Multiplying the first equation in problem (1) by \( \alpha \), we have

\[ ||u||^{p+1}_{H_0^1(\Omega)} \leq c_1 \alpha ||u||^{p+1}_{H_0^1(\Omega)} \leq c_1 \alpha \left( \frac{2(p+1)}{p-1} E(0) \right)^{p-2} ||u||^2_{H_0^1(\Omega)}, \]

Then, (50) holds. By the embedding \( H_1^0(\Omega) \hookrightarrow L^{p+1}(\Omega) \) and (49), we have

Then, we conclude (51). Hence, we complete the proof.

Now, we state an important lemma by Martinez[54].

Lemma 4.2. Let \( E : R^+ \to R^+ \) be a nonincreasing function. Assume that there exists \( \sigma > 0 \) for which \( \int_0^{\infty} E(t)dt \leq \sigma E(t) \) for any \( S \geq 0 \), then there exist two positive constants \( C \) and \( \xi \) independent of \( t \) such that:

\[ 0 < E(t) \leq C e^{-\xi t}. \]

Theorem 4.3. Assume that assumption (1) holds and \( u_0 \in W \). Moreover, assume that \( E(0) \leq d \) and \( B_1^{k+1} \left( \frac{2(p+1)}{p-1} E(0) \right)^{k-2(p+1)(k-1)} + c_1 \left( \frac{2(p+1)}{p-1} E(0) \right)^{p-2} \alpha < 1 \), then there exist two positive constants \( C \) and \( \xi \) independent of \( t \) such that:

\[ 0 < E(t) \leq C e^{-\xi t}. \]

Proof. Multiplying the first equation in problem (1) by \( u \), then integrating it over \( \Omega \times (S, T) \), and performing several integration by parts, we get:

\[ \int_S^T \int_\Omega \left[ u_t + \nabla u \nabla u_t + |u|^{q-2} uu_t \right] dx dt + \int_S^T ||u||^2_{H_0^1(\Omega)} dt \\
+ \int_S^T ||u||^{k+1}_{k+1, 1} dt + k \int_S^T \int_{\Gamma_1} |u|^{k-1} uu dx dt = \int_S^T ||u||^{p+1}_{p+1} dt. \]  \hspace{1cm} (52)

From the definition of \( E(t) \) and equation (52), we obtain

\[ 2 \int_S^T E(t) dt = \int_S^T \left[ ||u||^2_{H_0^1(\Omega)} + \frac{2}{k+1} ||u||^{k+1}_{k+1, 1} + \frac{2}{p+1} ||u||^{p+1}_{p+1} \right] dt \\
- \int_S^T \int_\Omega \left[ u_t + \nabla u \nabla u_t \right] dx dt - \int_S^T \int_\Omega |u|^{q-2} uu_t dx dt \\
- k \int_S^T \int_{\Gamma_1} |u|^{k-1} uu dx dt - k \int_S^T \int_{\Gamma_1} ||u||^{k+1}_{k+1, 1} dt + \frac{p-1}{p+1} \int_S^T \int_\Omega |u|^{p+1} dx dt. \]  \hspace{1cm} (53)

Now, we estimate every term on the right-hand side of (53).

Employing Hölder’s inequality, Young’s inequality, (49) and (4), the first and second terms on the right-hand side of (53) can be estimated as follows, for \( \delta_1 > 0 \),

\[ - \int_S^T \int_\Omega \left[ u_t + \nabla u \nabla u_t \right] dx dt \leq \delta_1 \int_S^T ||u||^2_{H_0^1(\Omega)} dt + C(\delta_1) \int_S^T ||u||^2_{H_0^1(\Omega)} dt \]

\[ \leq \delta_1 \frac{2(p+1)}{p-1} \int_S^T E(t) dt - C(\delta_1) \int_S^T E'(t) dt. \]  \hspace{1cm} (54)
By Hölder’s inequality, Young’s inequality, (49) and (9), the third terms on the right-hand side of (53) can be estimated as follows, for \( \delta_2 > 0 \),

\[
- \int_{S}^{T} \int_{\Omega} |u|^{q-2} uu_t dxdt = - \int_{S}^{T} \int_{\Omega} (|u|^q u_t)|u|^q dxdt \\
\leq \delta_2 \int_{S}^{T} ||u||_q^q dt + C(\delta_2) \int_{S}^{T} \int_{\Omega} |u|^{q-2}u_t^2 dxdt \\
\leq \delta_2 c^q_2 \frac{2(p+1)}{p-1} \frac{2(p+1)}{p-1} E(0)^{q-2} \int_{S}^{T} E(t) dt - C(\delta_2) \int_{S}^{T} E'(t) dt, \quad (55)
\]

where we used the embedding \( H^{k}_{1, \Gamma_0}(\Omega) \hookrightarrow L^q(\Omega) \) and (49).

Similar to the process of the proof of (55) and by (50), we have

\[
- \int_{S}^{T} \int_{\Gamma_1} |u|^{k-1} uu_t dxdt \leq \int_{S}^{T} \int_{\Gamma_1} |u|^{\frac{k+1}{k}}(\frac{k-1}{k} u_t)|u|^k dxdt \\
\leq \delta_3 \int_{S}^{T} ||u||_{k+1, \Gamma_1}^{k+1} dt + C(\delta_3) \int_{S}^{T} \int_{\Gamma_1} |u|^{k-1}u_t^2 dxdt \\
\leq \delta_3 c^{k+1}_s \frac{2(p+1)}{p-1} \frac{2(p+1)}{p-1} E(0)^{k-2} \frac{2(p+1)}{p-1} \int_{S}^{T} E(t) dt - C(\delta_3) \int_{S}^{T} E'(t) dt, \quad (56)
\]

As for the fifth term on the right-hand side of (53), by (50) and (9), we arrive at

\[
- \int_{S}^{T} \int_{\Gamma_1} |u|^{\frac{k}{k+1}} uu_t dxdt \\
\leq 2c^{k+1}_s \frac{2(p+1)}{p-1} \frac{2(p+1)}{p-1} E(0)^{k-2} \frac{2(p+1)k-1}{(p-1)(k+1)} \int_{S}^{T} E(t) dt.
\quad (57)
\]

For the sixth term on the right-hand side of (53), by (50) and (9), we get

\[
\frac{p-1}{p+1} \int_{S}^{T} ||u||_{p+1}^{p+1} dt \leq 2c^{p+1}_s \frac{2(p+1)}{p-1} E(0)^{p-2} \int_{S}^{T} E(t) dt. \quad (58)
\]

Then, combining these estimates (54)-(58), (53) becomes

\[
2 \int_{S}^{T} E(t) dt \\
\leq [\delta_1 \frac{2(p+1)}{p-1} + \delta_2 c^2 E(0)^{q-2} \frac{2(p+1)}{p-1}] (\frac{2(p+1)}{p-1} E(0))^{q-2} + \delta_3 c^{k+1}_s \frac{2(p+1)}{p-1} \frac{2(p+1)}{p-1} E(0)^{k-2} \frac{2(p+1)}{p-1} \int_{S}^{T} E'(t) dt + 2\delta_3 c^{k+1}_s \frac{2(p+1)}{p-1} \frac{2(p+1)k-1}{(p-1)(k+1)} \int_{S}^{T} E(t) dt.
\]

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\[ +2c_{p+1}\left(\frac{2(p+1)}{p-1}E(0)^{p-2}\right)\int_{S}^{T} E(t)dt \]
\[ -(C(\delta_1) + C(\delta_2) + C(\delta_3))\int_{S}^{T} E'(t)dt. \]  

(59)

Note \( B_{k+1}^{k+1}(2(p+1)E(0))^{k-2} \frac{(p+1)(k-1)}{(p-1)(k+1)} + c_{p+1}\frac{2(p+1)}{p-1}E(0)^{p-2} = \alpha < 1, \) and choose \( \delta_1 > 0, \delta_2 > 0, \delta_3 > 0 \) sufficiently small such that
\[ 2 - \delta_1 \frac{2(p+1)}{p-1} - \delta_2 c_{p+1}\frac{2(p+1)}{p-1} \left(\frac{2(p+1)}{p-1}E(0)^{p-2}\right) \]
\[ -\delta_3 B_{k+1}^{k+1}(2(p+1)E(0))^{k-2} \frac{2(p+1)}{p-1} = -2\alpha > 0. \]  

(60)

Hence, by (60), there exists a positive constant \( \sigma > 0 \) such that
\[ \int_{S}^{T} E(t)dt \leq \sigma E(S), \text{ for any } S \geq 0. \]

By letting \( T \) go to \(+\infty\) on the left hand in the aforementioned inequality, one can easily deduce that Lemma 4.2 is satisfied. Hence the conclusion of Theorem 4.3 is established. \( \square \)

By Lemma 4.1, we have the following result

**Corollary 4.4.** Under the assumption of Theorem 4.3, there exist two positive constants \( C \) and \( \xi \) independent of \( t \) such that:
\[ ||u||_{H^{1}_0(\Omega)} \leq Ce^{-\xi t}. \]

**Remark 4.5.** If \( g(u) \) is boundary source term and \( f(u) \) is absorptive term, we can also get the similar results.

5. Conclusions

This paper consider the initial boundary value problem of the generalized Boussinesq equation with nonlinear interior source and boundary absorptive terms. Under appropriate assumptions imposed on the source and boundary absorption terms, we establish global existence of solutions by using the potential well method combined with a standard continuous argument and we give sufficient conditions for the blow-up of solutions with positive and negative initial energy respectively in a finite time. It is different with the results in [4,5]. We also give a general decay of the energy by an integral inequality in [54].

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**Competing Interests**

The authors declare that they have no competing interests.
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