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Strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in Hilbert spaces

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Abstract: In this paper, we present a new non-convex hybrid iteration algorithm for common fixed points of a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the domains of Hilbert spaces.

Keywords: S iteration, nonexpansive mapping, hybrid algorithm, closed quasi-nonexpansive.

MSC: 47H05, 47H09, 47H10

1. Introduction

Fixed point theory of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings is an active area of interest and finds applications in many related fields like image recovery, signal processing and geometry of objects [1]. From time to time, some versions of theorems relating to fixed points of functions of special nature keep on appearing in almost in all branches of mathematics. Consequently, we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. For example, a fixed-point iteration scheme has been applied in IMRT optimization to pre-compute dose-deposition coefficient (DDC) matrix, see [2]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is currently.

The construction of fixed point theorems (e.g. Banach fixed point theorem) which not only claim the existence of a fixed point but yield an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some map f that is contracting with respect to some (complete) metric on X will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and is invariably used in most of the occasions, see [3]. But it only ensures weak convergence, see [4] but more often then not, we require strong convergence in many real world problems relating to Hilbert spaces, see [5]. So mathematician are in search for the modifications of the Mann's process to control and ensure the strong convergence, (see [4,6–12] and references therein).

Most probably the first noticeable modification of Mann's iteration process was proposed by Nakajo *et al.* in [8] in 2003. They introduced this modification for only one nonexpansive mapping, where as Kim and Xu introduced a modification for asymptotically nonexpansive mapping in 2006, see [9]. In the same year Martinez *et al.* in [10] introduced a modification of the Ishikawa iteration process for a nonexpansive mapping. They also gave modification of Halpern iteration method. Su *et al.* in [11] gave a monotone hybrid iteration process for nonexpansive mapping. Liu *et al.* in [12] gave a novel iteration method for finite family of quasi-asymptotically pseudo-contractive mapping in a Hilbert space. Let H be the fixed notation for Hilbert space and C be nonempty, closed and convex subset of it. First we recall some basic definitions that will accompany us throughout this paper.

Let $P_c(.)$ be the metric projection onto C. A mapping $T : C \to C$ is said to be non-expensive if $\|Tx - Ty\| \leq \|x - y\| \; \forall \; x, y \in C$. And $T : C \to C$ is said to be quasi-Lipschitz if

1. $FixT \neq \phi$,

2. For all $p \in FixT$, $\|Tx - p\| \leq L\|x - p\|$ where L is a constant $1 \leq L < \infty$.

If $L = 1$ then T is known as quasi-nonexpansive. It is well-known that T is said to be closed if for $n \to \infty$, $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$. T is said to be weak closed if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$. as $n \to \infty$. It is admitted fact that a mapping which is weak closed should be closed but converse is no longer true.

Let $\{T_n\}$ be a sequence of mappings having non-empty fixed points sets. Then $\{T_n\}$ is defined to be uniformly closed if for all convergent sequences $\{Z_n\} \subset C$ with conditions $\|Zx_n - Z_n\| \to 0, n \to \infty$ implies the limit of $\{Z_n\}$ belongs to $FixT_i$.

In 1953 [3], Mann proposed an iterative scheme given as:

$$x_{n+1} = (1 - \alpha_n)x_n n + \alpha_n T(x_n); n = 0, 1, 2, \ldots.$$

Guan *et al.* in [7] established the following non-convex hybrid iteration algorithm corresponding to Mann iterative scheme:

$$\begin{cases} x_0 \in C = Q_0, & \text{choosen arbitrarily,} \\ y_n = (1 - \alpha_n)x_n + \alpha_n T_n x_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq (1 + (L_n - 1)\alpha_n)\|x_n - z\| \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

They proved strong convergence results relating to common fixed points for a uniformly closed asymptotic family of countable quasi-Lipschitz mappings in H. They applied their results for the finite case to obtain fixed points. In this article, we establish a non-convex hybrid algorithm and prove strong convergence theorems about common fixed points related to a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the realm of Hilbert spaces. An application of this algorithm is also given.

We fix $\overline{co}C_n$ for closed convex closure of C_n for all $n \geq 1$, $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$, T_n for countable quasi-L_n-Lipschitz mappings from C into itself, and T be closed quasi-nonexpansive mapping from C into itself to avoid redundancy. We also present an application of our algorithm.

2. Main results

In this part we formulate our main results. We start with some basic definitions.

Definition 1. $\{T_n\}$ is said to be asymptotic, if $\lim_{n\to\infty} L_n = 1$

Proposition 2. *For $x \in H$ and $z \in C$, $z = P_C x$ iff we have $\langle x - z, z - y \rangle \geq 0$ for all $y \in C$.*

Proposition 3. *The common fixed point set F of above said T_n is closed and convex.*

Proposition 4. *For any given $x_0 \in H$, we have $p = P_C x_0 \Longleftrightarrow \langle p - z, x_0 - p \rangle \geq 0, \forall z \in C$.*

Theorem 5. *Suppose that $\alpha_n \in (0, 1]$, and $\beta_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, & \textit{choosen arbitrarily,} \\ y_n = (1 - \alpha_n)T_n x_n + \alpha_n T_n z_n, & n \geq 0, \\ z_n = (1 - \beta_n) + \beta_n T_n x_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq L_n(1 + (L_n - 1)\alpha_n\beta_n)\|x_n - z\|\} \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$.

Proof. We give our proof in following steps.
Step 1. We know that $\overline{co}C_n$ and Q_n are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$\begin{aligned}
\|y_n - p\| &= \|(1-\alpha_n)T_n x_n + \alpha_n T_n z_n - p\| \\
&= \|(1-\alpha_n)T_n x_n + \alpha_n T_n((1-\beta_n) + \beta_n T_n x_n) - p\| \\
&= \|(1-\alpha_n\beta_n)(T_n x_n - p) + (\alpha_n\beta_n)(T_n^2 x_n)\| \\
&\leq (1-\alpha_n\beta_n)\|T_n x_n - p\| + (\alpha_n\beta_n)\|T_n^2 x_n\| \\
&= L_n(1 + (L_n - 1)\alpha_n\beta_n)\|x_n - p\|
\end{aligned}$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. therefore, $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$.
Step 2. We show that $F \cap A \subset \overline{co}C_n \cap Q_n$ for all $n \geq 0$. it suffices to show that $F \cap A \subset Q_n$, for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$ we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $\overline{co}C_n \cap Q_n$, from Proposition 3, we have
$\langle x_{n+1} - z, x_{n+1} - x_0\rangle \leq 0, \forall z \in \overline{co}C_n \cap Q_n$
as $F \cap A \subset \overline{co}C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of Q_{n+1} implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{co}C_n \cap Q_n$ holds for all $n \geq 0$.
Step 3. We prove $\{x_n\}$ is bounded. Since F is a nonempty, closed, and convex subset of C, there exists a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0$, we have
$\|x_{n+1} - x_0\| \leq \|z - x_0\|$
for every $z \in \overline{co}C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{co}C_n \cap Q_n$, we get
$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$
for each $n \geq 0$. This implies that $\{x_n\}$ is bounded.
Step 4. We show that $\{x_n\}$ converges strongly to a point of C (we show that $\{x_n\}$ is a cauchy sequence). As $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Proposition 4), we have
$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$
for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0\rangle \leq 0$ and hence

$$\begin{aligned}
\|x_{n+m} - x_n\|^2 &= \|(x_{n+m} - x_0) - (x_n - x_0)\|^2 \\
&\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0\rangle \\
&\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0 \text{ as } n \to \infty
\end{aligned}$$

for any $m \geq 1$. Therefore $\{x_n\}$ is a cauchy sequence in C, then there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$.
Step 5. We show that $y_n \to q$, as $n \to \infty$. Let
$D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + L_n^2(L_n - 1)(L_n + 1)\}$.
From the definition of D_n, we have

$$\begin{aligned}
D_n &= \{z \in C : \langle y_n - z, y_n - z\rangle \leq \langle x_n - z, x_n - z\rangle + L_n^2(L_n - 1)(L_n + 1)\} \\
&= \{z \in C : \|y_n\|^2 - 2\langle y_n, z\rangle + \|z\|^2 \leq \|x_n\|^2 - 2\langle x_n, z\rangle + \|z\|^2 + L_n^2(L_n - 1)(L_n + 1)\} \\
&= \{z \in C : 2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + L_n^2(L_n - 1)(L_n + 1)\}
\end{aligned}$$

This shows that D_n is convex and closed, $n \in \mathbb{Z}^+ \cup \{0\}$. Next, we want to prove that $C_n \subset D_n$, $n \geq 0$.
In fact, for any $z \in C_n$, we have

$$\begin{aligned}
\|y_n - z\|^2 &\leq [L_n(1 + (L_n - 1)\alpha_n\beta_n)]^2\|x_n - z\|^2 \\
&= \|x_n - z\|^2 L_n^2 + L_n^2[2(L_n - 1)\alpha_n\beta_n + (L_n - 1)^2\alpha_n^2\beta_n^2]\|x_n - z\|^2 \\
&\leq \|x_n - z\|^2 L_n^2 + L_n^2[2(L_n - 1) + (L_n - 1)^2]\|x_n - z\|^2 \\
&= \|x_n - z\|^2 L_n^2 + L_n^2(L_n - 1)(L_n + 1)\|x_n - z\|^2.
\end{aligned}$$

From $C_n = \{z \in C : \|y_n - z\| \leq [L_n(1 + (L_n - 1)\alpha_n\beta_n)]\|x_n - z\|\} \cap A,\ n \geq 0$, we have $C_n \subset A$, $n \geq 0$. Since A is convex, we also have $\overline{co}C_n \subset A$, $n \geq 0$. Consider $x_n \in \overline{co}C_{n-1}$, we know that

$$\begin{aligned}\|y_n - z\| &\leq \|x_n - z\|^2 L_n^2 + L_n^2(L_n - 1)(L_n + 1)\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 + L_n^2(l_n - 1)(L_n + 1).\end{aligned}$$

This implies that $z \in D_n$ and hence $C_n \subset D_n$, $n \geq 0$. Sinnce D_n is convex, we have $\overline{co}(C_n) \subset D_n$, $n \geq 0$. Therefore $\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + L_n^2(L_n - 1)(L_n - 1) \to 0$ as $n \to \infty$. That is, $y_n \to q$ as $n \to \infty$.
Step 6. We show that $q \in F$. From the definition of y_n, we have $(1 + \alpha_n\beta_n T_n)\|T_n x_n - x_n\| = \|y_n - x_n\| \to 0$ as $n \to \infty$. Since $\alpha_n \in (a, 1] \subset [0, 1]$,from the above limit we have $\lim_n \to \infty\|T_n x_n - x_n\| = 0$.
Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.
Step 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exist a positive integer N, if $n > N$ then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to
$\|z_0 - x_n\|^2 = \|z_0 - x_n + x_n - x_0\|^2 = \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle$.
It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$ which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F$, this is a contradiction. This completes the proof. □

Now, we present an example of C_n which does not involve a convex subset.

Example 1. Take $H = R^2$, and a sequence of mappings $T_n : R^2 \to R^2$ given by $T_n : (t_1, t_2) \mapsto (\frac{1}{8}t_1, t_2)$, $\forall (t_1, t_2) \in R^2, \forall n \geq 0$.
It is clear that $\{T_n\}$ satisfies the desired definition of with $F = \{(t_1, 0) : t_1 \in (-\infty, +\infty)\}$ common fixed point set. Take $x_0 = (4, 0)$, $a_0 = \frac{6}{7}$, we have
$y_0 = \frac{1}{7}x_0 + \frac{6}{7}T_0 x_0 = (4 \times \frac{1}{7} + \frac{4}{8} \times \frac{6}{7}, 0) = (1, 0)$.
Take $1 + (L_0 - 1)a_0 = \sqrt{\frac{5}{2}}$, we have
$C_0 = \{z \in R^2 : \|y_0 - z\| \leq \sqrt{\frac{5}{2}}\|x_0 - z\|\}$.
It is easy to show that $z_1 = (1, 3)$, $z_2 = (-1, 3) \in C_0$. But
$z' = \frac{1}{2}z_1 + \frac{1}{2}z_2 = (0, 3) \overline{\in} C_0$,
since $\|y_0 - z\| = 2$, $\|x_0 - z\| = 1$. Therefore C_0 is not convex.

Corollary 6. *Assume that $\alpha_n \in (0, 1]$, and $\beta_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, & \textit{choosen arbitrarily}, \\ y_n = (1 - \alpha_n)Tx_n + \alpha_n Tz_n, & n \geq 0, \\ z_n = (1 - \beta_n) + \beta_n Tx_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

Proof. Take $T_n \equiv T$, $L_n \equiv 1$ in Theorem 5, in this case, C_n is convex and closed and , for all $n \geq 0$, by using Theorem1.9, we obtain our desired result. □

Corollary 7. *Assume that $\alpha_n \in (0, 1]$, and $\beta_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, & \textit{choosen arbitrarily}, \\ y_n = (1 - \alpha_n)Tx_n + \alpha_n Tz_n, & n \geq 0, \\ z_n = (1 - \beta_n) + \beta_n Tx_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

3. Application

Here, we give an application of our result for the following case of finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let
$\|T_i^j x - p\| \leq k_{i,j}\|x-p\|, \forall x \in C, p \in F,$
where F is common fixed point set of $\{T_n\}_{n=0}^{N-1}$,$\lim_j \to \infty k_{i,j} = 1$ for all $0 \leq i \leq N-1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is uniformly $L - Lipschitz$, if
$\|T_i^j x - T_i^j y\| \leq L_{i,j}\|x-y\|, \forall x, y \in C$
for all $i \in \{0, 1, 2, ..., N-1\}$, $j \geq 1$, where $L \geq 1$.

Theorem 8. *Let $\{T_n\}_{n=0}^{N-1} : C \to C$ be a finite uniformly L-Lipschitz family of asymptotically quasi-nonexpansive mappings with nonempty common fixed point set F. Assume that $\alpha_n \in (0,1]$, and $\beta_n \in [0,1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, & \text{choosen arbitrarily,} \\ y_n = (1-\alpha_n)T_{i(n)}^{j(n)}x_n + \alpha_n T_{i(n)}^{j(n)}z_n, & n \geq 0, \\ z_n = (1-\beta_n) + \beta_n T_{i(n)}^{j(n)}x_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq k_{i(n),j(n)} & \\ \qquad (1 + (k_{i(n),j(n)} - 1)\alpha_n\beta)\|x_n - z\|\} \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n}x_0, & \end{cases}$$

converges strongly to $P_F x_0$.

We can drive the prove from the following two conclusions.

Conclusion 9. *$\{T_{n=0}^{N-1}\}_{n=0}^{\infty}$ is a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself.*

Conclusion 10. *$F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, where $F(T)$ denotes the fixed point set of the mappings T.*

Corollary 11. *Let $T : C \to C$ be a L-Lipschitz asymptotically quasi-nonexpansive mappings with nonempty common fixed point set F. Assume that $\alpha_n \in (0,1]$, and $\beta_n \in [0,1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, & \text{choosen arbitrarily,} \\ y_n = (1-\alpha_n)T^n x_n + \alpha_n T^n z_n, & n \geq 0, \\ z_n = (1-\beta_n) + \beta_n T^n x_n, & n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq k_n(1 + (k_n - 1)\alpha_n\beta)\|x_n - z\|\} \cap A, & n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, & n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n}x_0, & \end{cases}$$

converges strongly to $P_F x_0$.

Proof. Take $T_n \equiv T$ in Theorem 8, we get the desired result. □

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