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Article

Existence and uniqueness results for Navier problems with degenerated operators

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Abstract: In this article, we prove the existence and uniqueness of solutions for the Navier problem $\Delta\left[\omega_1(x)|\Delta u|^{p-2}\Delta u + \nu_1(x)|\Delta u|^{q-2}\Delta u\right] - div\left[\omega_2(x)|\nabla u|^{p-2}\nabla u + \nu_2(x)|\nabla u|^{s-2}\nabla u\right] = f(x) - div(G(x))$, in Ω, with $u(x) = \Delta u = 0$, in $\partial\Omega$, where Ω is a bounded open set of $\mathbb{R}^N$ for $N \geq 2$, $\frac{f}{\omega_2} \in L^{p'}(\Omega,\omega_2)$ and $\frac{G}{\nu_2} \in \left[L^{s'}(\Omega,\nu_2)\right]^N$.

Keywords: Degenerate nonlinear elliptic equations, weighted Sobolev space.

MSC: 35J60, 35J70.

1. Introduction

The main purpose of this paper (see Theorem 7) is to establish the existence and uniqueness of solutions for the Navier problem

$$(P)\begin{cases} Lu(x) = f(x) - \operatorname{div}(G(x)), & \text{in } \Omega, \\ u(x) = \Delta u(x) = 0, & \text{in } \partial\Omega, \end{cases}$$

where

$$Lu(x) = \Delta\left[\omega_1(x)\,|\Delta u|^{p-2}\Delta u + \nu_1(x)\,|\Delta u|^{q-2}\Delta u\right] - \operatorname{div}\left[\omega_2(x)|\nabla u|^{p-2}\nabla u + \nu_2(x)\,|\nabla u|^{s-2}\nabla u)\right],$$

$\Omega \subset \mathbb{R}^N$ is a bounded open set, $\dfrac{f}{\omega_2} \in L^{p'}(\Omega,\omega_2)$, $\dfrac{G}{\nu_2} \in [L^{s'}(\Omega,\nu_2)]^N$, ω_1, ω_2, ν_1 and ν_2 are four weight functions (i.e., ω_i and ν_i, $i = 1,2$ are locally integrable functions on $\mathbb{R}^N$ such that $0 < \omega_i(x), \nu_i(x) < \infty$ a.e. $x \in \mathbb{R}^N$), Δ is the Laplacian operator, $1 < q, s < p < \infty$, $1/p + 1/p' = 1$ and $1/s + 1/s' = 1$.

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1–8]). The type of a weight depends on the equation type.

A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by B.Muckenhoupt in the early 1970's (see [7]). These classes have found many useful applications in harmonic analysis (see [9] and [10]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^N$ often belong to A_p (see [8] and [11]). There are, in fact, many interesting examples of weights (see [6] for p-admissible weights).

In the non-degenerate case (i.e. with $\omega(x) \equiv 1$), for all $f \in L^p(\Omega)$ the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x), \text{ in } \Omega \\ u(x) = 0, \text{ in } \partial\Omega \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [12]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x), \text{ in } \Omega \\ u(x) = 0, \text{ in } \partial\Omega \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [13]), where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the degenerated p-Laplacian has been studied in [11].

The paper is organized as follow. In Section 2 we present the definitions and basic results. In Section 3 we prove our main result about existence and uniqueness of solutions for problem (P).

2. Definitions and basic results

Let Ω be an open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable, a.e. in Ω positive and finite functions $\omega = \omega(x)$, $x \in \Omega$. Elements of $\mathcal{W}(\Omega)$ will be called weight functions. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will be denoted by μ_ω. Thus, $\mu_\omega(E) = \int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^N$.

Definition 1. Let $1 \le p < \infty$. A weight ω is said to be an A_p-weight, if there is a positive constant $C = C(p,\omega)$ such that, for every ball $B \subset \mathbb{R}^N$

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,dx\right)^{p-1} \le C, \ \text{ if } \ p > 1,$$
$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\operatorname*{ess\,sup}_{x\in B}\frac{1}{\omega(x)}\right) \le C, \ \text{ if } \ p = 1,$$

where $|.|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$.

If $1 < q \le p$, then $A_q \subset A_p$ (see [5,6,8] for more information about A_p-weights). As an example of an A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if $-N < \alpha < N(p-1)$ (see [8], Chapter IX, Corollary 4.4). If $\varphi \in BMO(\mathbb{R}^N)$, then $\omega(x) = e^{\alpha\,\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [9]).

Remark 1. If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|}\right)^p \le C\,\frac{\mu_\omega(E)}{\mu_\omega(B)}$$

for all measurable subsets E of B (see 15.5 strong doubling property in [6]). Therefore, $\mu_\omega(E) = 0$ if and only if $|E| = 0$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Definition 2. Let ω be a weight. We shall denote by $L^p(\Omega,\omega)$ $(1 \le p < \infty)$ the Banach space of all measurable functions f defined in Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p\omega(x)\,dx\right)^{1/p} < \infty.$$

We denote $[L^p(\Omega,\omega)]^N = L^p(\Omega,\omega) \times ... \times L^p(\Omega,\omega)$.

Remark 2. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega,\omega) \subset L^1_{\text{loc}}(\Omega)$ (see [8], Remark 1.2.4). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 3. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p < \infty$, k be a nonnegative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega,\omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \le |\alpha| \le k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p\omega(x)\,dx + \sum_{1\le|\alpha|\le k}\int_\Omega |D^\alpha u(x)|^p\omega(x)\,dx\right)^{1/p}. \tag{1}$$

We also define the space $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (1). We have that the spaces $W^{k,p}(\Omega,\omega)$ and $W_0^{k,p}(\Omega,\omega)$ are Banach spaces (see Proposition 2.1.2 in [8]).

The dual space of $W^{1,p}_0(\Omega,\omega)$ is the space $[W^{1,p}_0(\Omega,\omega)]^* = W^{-1,p'}(\Omega,\omega)$,

$$W^{-1,p'}(\Omega,\omega) = \{T = f - \operatorname{div}(G) : G = (g_1, ..., g_N), \frac{f}{\omega}, \frac{g_j}{\omega} \in L^{p'}(\Omega,\omega)\}.$$

It is evident that a weight function ω which satisfies $0 < C_1 \leq \omega(x) \leq C_2$, for a.e. $x \in \Omega$, gives nothing new (the space $W^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $W^{k,p}(\Omega)$). Consequently, we shall be interested in all above such weight functions ω which either vanish somewhere in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

We need the following basics results.

Theorem 4. *(The weighted Sobolev inequality) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let ω be an A_p-weight, $1 < p < \infty$. Then there exists positive constants C_Ω and δ such that for all $u \in W^{1,p}_0(\Omega,\omega)$ and $1 \leq \eta \leq N/(N-1) + \delta$*

$$\|u\|_{L^{\eta p}(\Omega,\omega)} \leq C_\Omega \| |\nabla u| \|_{L^p(\Omega,\omega)}. \tag{2}$$

Proof. Its suffices to prove the inequality for functions $u \in C^\infty_0(\Omega)$ (see Theorem 1.3 in [4]). To extend the estimates (2) to arbitrary $u \in W^{1,p}_0(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C^\infty_0(\Omega)$ functions tending to u in $W^{1,p}_0(\Omega,\omega)$. Applying the estimates (2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^p(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2). □

Lemma 5. *(a) Let $1 < p < \infty$, then exists a constant $C_p > 0$ such that for all $\xi, \eta \in \mathbb{R}^N$,*

$$\big| |\xi|^{p-2}\,\xi - |\eta|^{p-2}\eta \big| \leq C_p\, |\xi - \eta| (\, |\xi| + |\eta|)^{p-2}.$$

(b) Let $1 < p < \infty$. There exist two positive constants α_p and β_p such that for every $\xi, \eta \in \mathbb{R}^N$ $(N \geq 1)$

$$\alpha_p(\, |\xi| + |\eta|)^{p-2} |\xi - \eta|^2 \leq \langle\, |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta \rangle \leq \beta_p(\, |\xi| + |\eta|)^{p-2} |\xi - \eta|,$$

where $\langle .,. \rangle$ denotes here the Euclidian scalar product in $\mathbb{R}^N$.

Proof. See Proposition 17.2 and Proposition 17.3 in [13]. □

3. Weak Solutions

Let $\omega_1, \omega_2 \in A_p$ and $\nu_1, \nu_2 \in \mathcal{W}(\Omega)$, $1 < q, s < p < \infty$. We denote by X the space $X = W^{2,p}(\Omega,\omega_1) \cap W^{1,p}_0(\Omega,\omega_2)$ with the norm

$$\|u\|_X = \left(\int_\Omega |\nabla u|^p\, \omega_2\, dx + \int_\Omega |\Delta u|^p\, \omega_1\, dx \right)^{1/p}.$$

In this section we prove the existence and uniqueness of weak solutions $u \in X$ to the Navier problem

$$(P) \begin{cases} Lu(x) = f(x) - \operatorname{div}(G(x)), \text{ in } \Omega, \\ u(x) = \Delta u = 0, \text{ in } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $\dfrac{f}{\omega_2} \in L^{p'}(\Omega,\omega_2)$ and $\dfrac{G}{\nu_2} \in [L^{s'}(\Omega,\nu_2)]^N$, $G = (g_1, ..., g_N)$.

Definition 6. We say that $u \in X$ is a weak solution for problem (P) if

$$\begin{aligned} &\int_\Omega |\Delta u|^{p-2} \Delta u\, \Delta\varphi\, \omega_1\, dx + \int_\Omega |\Delta u|^{q-2} \Delta u\, \Delta\varphi\, \nu_1\, dx + \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla\varphi \rangle\, \omega_2\, dx \\ &+ \int_\Omega |\nabla u|^{s-2} \langle \nabla u, \nabla\varphi \rangle\, \nu_2\, dx = \int_\Omega f\, \varphi\, dx + \int_\Omega \langle G, \nabla\varphi \rangle\, dx, \end{aligned} \tag{3}$$

for all $\varphi \in X$, with $f/\omega_2 \in L^{p'}(\Omega, \omega_2)$ and $G/\nu_2 \in [L^{s'}(\Omega, \nu_2)]^N$, where $\langle ., . \rangle$ denotes here the Euclidean scalar product in $\mathbb{R}^N$.

Remark 3. (a) Since $1 < q, s < p < \infty$ and if $\dfrac{\nu_1}{\omega_1} \in L^{p/(p-q)}(\Omega, \omega_1)$ and $\dfrac{\nu_2}{\omega_2} \in L^{p/(p-s)}(\Omega, \omega_2)$, there exist two constants $M_1, M_2 > 0$ such that

$$\|u\|_{L^q(\Omega,\nu_1)} \le M_1 \|u\|_{L^p(\Omega,\omega_1)} \text{ and } \|u\|_{L^s(\Omega,\nu_2)} \le M_2 \|u\|_{L^p(\Omega,\omega_2)}$$

where $M_1 = \left[\displaystyle\int_\Omega \left(\frac{\nu_1}{\omega_1} \right) \omega_1 \, dx \right]^{(p-q)/pq}$ and $M_2 = \left[\displaystyle\int_\Omega \left(\frac{\nu_2}{\omega_2} \right) \omega_2 \, dx \right]^{(p-s)/ps}$. In fact, since $1 < q, s < p < \infty$, we have $r = p/q > 1$ and $r' = p/(p-q)$,

$$\begin{aligned} \|u\|^q_{L^q(\Omega,\nu_1)} &= \int_\Omega |u|^q \, \nu_1 \, dx = \int_\Omega |u|^q \frac{\nu_1}{\omega_1} \, \omega_1 \, dx \\ &\le \left(\int_\Omega |u|^{qr} \, \omega_1 \, dx \right)^{1/r} \left(\int_\Omega \left(\frac{\nu_1}{\omega_1} \right)^{r'} \omega_1 \, dx \right)^{1/r'} \\ &= \left(\int_\Omega |u|^p \, \omega_1 \, dx \right)^{q/p} \left(\int_\Omega \left(\frac{\nu_1}{\omega_1} \right)^{p/(p-q)} \omega_1 \, dx \right)^{(p-q)/p}. \end{aligned}$$

Hence, $\|u\|_{L^q(\Omega,\nu_1)} \le M_1 \, \|u\|_{L^p(\Omega,\omega_1)}$. Analogously, we obtain $\|u\|_{L^s(\Omega,\nu_2)} \le M_2 \, \|u\|_{L^p(\Omega,\omega_2)}$.
(b) Using the estimate in (a) we have

$$\begin{aligned} \left| \int_\Omega |\Delta u|^{q-2} \Delta u \, \Delta\varphi \, \nu_1 \, dx \right| &\le \int_\Omega |\Delta u|^{q-1} \, |\Delta\varphi| \, \nu_1 \, dx \\ &\le \left(\int_\Omega |\Delta u|^{(q-1)q'} \nu_1 \, dx \right)^{1/q'} \left(\int_\Omega |\Delta\varphi|^q \, \nu_1 \, dx \right)^{1/q} \\ &= \left(\int_\Omega |\Delta u|^q \, \nu_1 \, dx \right)^{(q-1)/q} \left(\int_\Omega |\Delta\varphi|^q \, \nu_1 \, dx \right)^{1/q} \\ &= \|\Delta u\|^{q-1}_{L^q(\Omega,\nu_1)} \|\Delta\varphi\|_{L^q(\Omega,\nu_1)} \\ &\le M_1^{q-1} \, \|\Delta u\|^{q-1}_{L^p(\Omega,\omega_1)} \, M_1 \, \|\Delta\varphi\|_{L^p(\Omega,\omega_1)} \\ &\le M_1^q \, \|u\|_X \, \|\varphi\|_X, \end{aligned}$$

and, analogously, we also have

$$\left| \int_\Omega |\nabla u|^{s-2} \langle \nabla u, \nabla\varphi \rangle \, \nu_2 \, dx \right| \le M_2^s \, \|u\|_X \, \|\varphi\|_X.$$

Theorem 7. *Let $\omega_i \in A_p$, $\nu_i \in \mathcal{W}(\Omega)$ $(i = 1, 2)$, $1 < q, s < p < \infty$. Suppose that*
(a) $\dfrac{\nu_1}{\omega_1} \in L^{p/(p-q)}(\Omega, \omega_1)$ and $\dfrac{\nu_2}{\omega_2} \in L^{p/(p-s)}(\Omega, \omega_2)$;
(b) $f/\omega_2 \in L^{p'}(\Omega, \omega_2)$ and $G/\nu_2 \in [L^{s'}(\Omega, \nu_2)]^N$.
Then the problem (P) has a unique solution $u \in X$ and

$$\|u\|_X \le \left[C_\Omega \left\| \frac{f}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} + M_2 \left\| \frac{|G|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \right]^{1/(p-1)},$$

where C_Ω is the constant in Theorem 4 and M_2 is the constant in Remark 3 (a).

Proof. (I) *Existence*. By Theorem 4 (with $\eta = 1$), we have that

$$\left| \int_\Omega f \, \varphi \, dx \right| \le \left(\int_\Omega \left| \frac{f}{\omega_2} \right|^{p'} \omega_2 \, dx \right)^{1/p'} \left(\int_\Omega |\, \varphi|^p \, \omega_2 \, dx \right)^{1/p}$$

$$\leq \quad C_\Omega \left\| \frac{f}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} \| \, |\nabla \varphi| \, \|_{L^p(\Omega,\omega_2)} \leq C_\Omega \left\| \frac{f}{\omega_2} \right\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_X, \tag{4}$$

and by Remark 3 (a)

$$\begin{aligned} \left| \int_\Omega \langle G, \nabla\varphi\rangle \, dx \right| dx &\leq \int_\Omega |\langle G, \nabla\varphi\rangle| \, dx \leq \int_\Omega |G||\nabla\varphi| \, dx = \int_\Omega \frac{|G|}{\nu_2} |\nabla\varphi| \, \nu_2 \, dx \\ &\leq \left\| \frac{|G|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \| \, |\nabla\varphi| \, \|_{L^s(\Omega,\nu_2)} \leq M_2 \left\| \frac{|G|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \| \, |\nabla\varphi| \, \|_{L^p(\Omega,\omega_2)} \\ &\leq M_2 \left\| \frac{|G|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \|\varphi\|_X. \end{aligned} \tag{5}$$

Define the functional $J : X \to \mathbb{R}$ by

$$\begin{aligned} J(\varphi) &= \frac{1}{p} \int_\Omega |\Delta\varphi|^p \, \omega_1 \, dx + \frac{1}{q} \int_\Omega |\Delta\varphi|^q \, \nu_1 \, dx \\ &\quad + \frac{1}{p} \int_\Omega |\nabla\varphi|^p \, \omega_2 \, dx + \frac{1}{s} \int_\Omega |\nabla\varphi|^s \, \nu_2 \, dx - \int_\Omega f \, \varphi \, dx - \int_\Omega \langle G, \nabla\varphi\rangle \, dx. \end{aligned}$$

Using (4), (5), Remark 3(a) and Young's inequality ($a\,b \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$), we have that

$$\begin{aligned} J(\varphi) &\geq \frac{1}{p} \int_\Omega |\Delta\varphi|^p \, \omega_1 \, dx + \frac{1}{q} \int_\Omega |\Delta\varphi|^q \, \nu_1 \, dx + \frac{1}{p} \int_\Omega |\nabla\varphi|^p \, \omega_2 \, dx + \frac{1}{s} \int_\Omega |\nabla\varphi|^s \, \nu_2 \, dx \\ &\quad - \left\| \frac{f}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} \|\varphi\|_{L^p(\Omega,\omega_2)} - \left\| \frac{|G|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \| \, |\nabla\varphi| \, \|_{L^s(\Omega,\nu_2)} \\ &\geq \frac{1}{p} \int_\Omega |\nabla\varphi|^p \, \omega_2 \, dx + \frac{1}{s} \int_\Omega |\nabla\varphi|^s \, \nu_2 \, dx - C_\Omega \left\| \frac{f}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} \| \, |\nabla\varphi| \, \|_{L^p(\Omega,\omega_2)} \\ &\quad - \left\| \frac{|G|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \| \, |\nabla\varphi| \, \|_{L^s(\Omega,\nu_2)} \\ &\geq \frac{1}{p} \int_\Omega |\nabla\varphi|^p \, \omega_2 \, dx + \frac{1}{s} \int_\Omega |\nabla\varphi|^s \, \nu_2 \, dx - \frac{C_\Omega^{p'}}{p'} \left\| \frac{f}{\omega_2} \right\|^{p'}_{L^{p'}(\Omega,\omega_2)} \\ &\quad - \frac{1}{p} \| \, |\nabla\varphi| \, \|^p_{L^p(\Omega,\omega_2)} - \frac{1}{s'} \left\| \frac{|G|}{\nu_2} \right\|^{s'}_{L^{s'}(\Omega,\nu_2)} - \frac{1}{s} \| \, |\nabla\varphi| \, \|^s_{L^s(\Omega,\nu_2)} \\ &= -\frac{C_\Omega^{p'}}{p'} \left\| \frac{f}{\omega_2} \right\|^{p'}_{L^{p'}(\Omega,\omega_2)} - \frac{1}{s'} \left\| \frac{|G|}{\nu_2} \right\|^{s'}_{L^{s'}(\Omega,\nu_2)} \end{aligned}$$

that is, J is bounded from below. Let $\{u_n\}$ be a minimizing sequence, that is, a sequence such that

$$J(u_n) \;\to\; \inf_{\varphi \in X} J(\varphi).$$

Then for n large enough, we obtain

$$\begin{aligned} 0 \geq J(u_n) &= \frac{1}{p} \int_\Omega |\Delta u_n|^p \, \omega_1 \, dx + \frac{1}{q} \int_\Omega |\Delta u_n|^q \, \nu_1 \, dx + \frac{1}{p} \int_\Omega |\nabla u_n|^p \, \omega_2 \, dx + \frac{1}{s} \int_\Omega |\nabla u_n|^s \, \nu_2 \, dx \\ &\quad - \int_\Omega f \, u_n \, dx - \int_\Omega \langle G, \nabla u_n\rangle \, dx, \end{aligned}$$

and we have

$$\frac{1}{p} \int_\Omega |\Delta u_n|^p \, \omega_1 \, dx + \frac{1}{p} \int_\Omega |\nabla u_n|^p \, \omega_2 \, dx$$

$$\leq \frac{1}{p}\int_\Omega |\Delta u_n|^p\,\omega_1\,dx + \frac{1}{q}\int_\Omega |\Delta u_n|^q\,\nu_1\,dx + \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega_2\,dx + \frac{1}{s}\int_\Omega |\nabla u_n|^s\,\nu_2\,dx$$
$$\leq \int_\Omega f\,u_n\,dx + \int_\Omega \langle G, u_n\rangle\,dx. \tag{6}$$

Hence, by Theorem 4 (with $\eta = 1$), Remark 3(a) and (6), we obtain

$$\begin{aligned}
\|u_n\|_X^p &= \int_\Omega |\Delta u_n|^p\,\omega_1\,dx + \int_\Omega |\nabla u_n|^p\,\omega_2\,dx \\
&\leq p\left(\int_\Omega f\,u_n\,dx + \int_\Omega \langle G, \nabla u_n\rangle\,dx\right) \\
&\leq p\left(\left\|\frac{f}{\omega_2}\right\|_{L^{p'}(\Omega,\omega_2)} \|u_n\|_{L^p(\Omega,\omega_2)} + \left\|\frac{|G|}{\nu_2}\right\|_{L^{s'}(\Omega,\nu_2)} \|\,|\nabla u_n|\,\|_{L^s(\Omega,\nu_2)}\right) \\
&\leq p\left(C_\Omega \left\|\frac{f}{\omega_2}\right\|_{L^{p'}(\Omega,\omega_2)} \|\,|\nabla u_n|\,\|_{L^p(\Omega,\omega_2)} + M_2 \left\|\frac{|G|}{\nu_2}\right\|_{L^{s'}(\Omega,\nu_2)} \|\,|\nabla u_n|\,\|_{L^p(\Omega,\omega_2)}\right) \\
&\leq p\left(C_\Omega \left\|\frac{f}{\omega_2}\right\|_{L^{p'}(\Omega,\omega_2)} + M_2 \left\|\frac{|G|}{\nu_2}\right\|_{L^{q'}(\Omega,\nu_2)}\right) \|u_n\|_X.
\end{aligned}$$

Hence,

$$\|u_n\|_X \leq \left[p\left(C_\Omega \left\|\frac{f}{\omega_2}\right\|_{L^{p'}(\Omega,\omega_2)} + M_2 \left\|\frac{|G|}{\nu_2}\right\|_{L^{s'}(\Omega,\nu_2)}\right)\right]^{1/(p-1)}.$$

Therefore $\{u_n\}$ is bounded in X. Since X is reflexive, there exists a subsequence, still denoted by $\{u_n\}$, and a function $u \in X$ such that $u_n \rightharpoonup u$ in X. Since,

$$X \ni \varphi \mapsto \int_\Omega f\,\varphi\,dx + \int_\Omega \langle G, \nabla\varphi\rangle\,dx,$$

and

$$X \ni \varphi \mapsto \|\Delta\varphi\|^p_{L^p(\Omega,\omega_1)} + \|\Delta\varphi\|^q_{L^q(\Omega,\nu_1)} + \|\,|\nabla\varphi|\,\|^p_{L^p(\Omega,\omega_2)} + \|\,|\nabla\varphi|\,\|^s_{L^s(\Omega,\nu_2)},$$

are continuous then J is continuous. Moreover since $1 < q, s < p < \infty$ we have that J is convex and thus lower semi-continuous for the weak convergence. It follows that

$$J(u) \leq \liminf_n J(u_n) = \inf_{\varphi \in X} J(\varphi),$$

and thus u is a minimizer of J on X (see Theorem 25.C and Corollary 25.15 in [14]). For any $\varphi \in X$ the function

$$\begin{aligned}
\lambda \mapsto\ & \frac{1}{p}\int_\Omega |\Delta(u+\lambda\varphi)|^p\,\omega_1\,dx + \frac{1}{q}\int_\Omega |\Delta(u+\lambda\varphi)|^q\,\nu_1\,dx + \frac{1}{p}\int_\Omega |\nabla(u+\lambda\,\varphi)|^p\,\omega_2\,dx \\
& + \frac{1}{s}\int_\Omega |\nabla(u+\lambda\varphi)|^s\,\nu_2\,dx - \int_\Omega (u+\lambda\,\varphi)\,f\,dx - \int_\Omega \langle G, \nabla(u+\lambda\,\varphi)\rangle\,dx
\end{aligned}$$

has a minimum at $\lambda = 0$. Hence,

$$\frac{d}{d\lambda}\Big(J(u+\lambda\,\varphi)\Big)\Big|_{\lambda=0} = 0,\ \forall\,\varphi \in X.$$

We have

$$\frac{d}{d\lambda}\left(|\,\nabla(u+\lambda\,\varphi)|^p\,\omega_2\right) = p\,\{|\nabla(u+\lambda\,\varphi)|^{p-2}(\langle\nabla u, \nabla\varphi\rangle + \lambda\,|\nabla\varphi|^2)\}\,\omega_2,$$

and

$$\frac{d}{d\lambda}\left(|\,\Delta(u+\lambda\,\varphi)|^p\,\omega_1\right) = p\,|\Delta u + \lambda\Delta\varphi|^{p-2}(\,\Delta u + \lambda\Delta\varphi)\,\Delta\varphi\,\omega_1,$$

and we obtain

$$0 = \frac{d}{d\lambda}\Big(J(u+\lambda\,\varphi)\Big)\Big|_{\lambda=0} = \left[\frac{1}{p}\left(p\int_\Omega |\nabla(u+\lambda\,\varphi)|^{p-2}(\langle\nabla u, \nabla\varphi\rangle + \lambda\,|\nabla\varphi|^2)\,\omega_2\,dx\right.\right.$$

$$
\begin{aligned}
&+p\int_\Omega |\Delta u+\lambda\Delta\varphi|^{p-2}(\Delta u+\lambda\Delta\varphi)\,\Delta\varphi\,\omega_1\,dx\Big)+\frac{1}{s}\Big(s\int_\Omega |\nabla(u+\lambda\,\varphi)|^{s-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda\,|\nabla\varphi|^2)\,\nu_2\,dx\Big)\\
&+\frac{1}{q}\Big(q\int_\Omega |\Delta u+\lambda\Delta\varphi|^{q-2}(\Delta u+\lambda\Delta\varphi)\,\Delta\varphi\,\nu_1\,dx\Big)-\int_\Omega \varphi\,f\,dx-\int_\Omega\langle G,\nabla\varphi\rangle\,dx\Big]\Big|_{\lambda=0}\\
=\;&\int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega_1\,dx+\int_\Omega |\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,\omega_2\,dx+\int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\nu_1\,dx\\
&+\int_\Omega |\nabla u|^{s-2}\langle\nabla u,\nabla\varphi\rangle\,\nu_2\,dx-\int_\Omega f\,\varphi\,dx-\int_\Omega\langle G,\nabla\varphi\rangle\,dx.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega_1\,dx+\int_\Omega |\nabla u|^{p-2}\langle\nabla u\,\nabla\varphi\rangle\,\omega_2\,dx+\int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\nu_1\,dx+\int_\Omega |\nabla u|^{s-2}\langle\nabla u,\nabla\varphi\rangle\,\nu_2\,dx\\
&=\int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx,
\end{aligned}
$$

for all $\varphi\in X$, that is, $u\in X$ is a solution of problem (P).
(II) *Uniqueness*. If $u_1,u_2\in X$ are two weak solutions of problem (P), we have

$$
\begin{aligned}
&\int_\Omega |\Delta u_1|^{p-2}\Delta u_1\,\Delta\varphi\,\omega_1\,dx+\int_\Omega |\Delta u_1|^{q-2}\Delta u_1\,\Delta\varphi\,\nu_1\,dx+\int_\Omega |\nabla u_1|^{p-2}\langle\nabla u_1,\nabla\varphi\rangle\,\omega_2\,dx\\
&+\int_\Omega |\nabla u_1|^{s-2}\langle\nabla u_1,\nabla\varphi\rangle\,\nu_2\,dx=\int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega |\Delta u_2|^{p-2}\Delta u_2\,\Delta\varphi\,\omega_1\,dx+\int_\Omega |\Delta u_2|^{q-2}\Delta u_2\,\Delta\varphi\,\nu_1\,dx+\int_\Omega |\nabla u_2|^{p-2}\langle\nabla u_2,\nabla\varphi\rangle\,\omega_2\,dx\\
&+\int_\Omega |\nabla u_2|^{s-2}\langle\nabla u_2,\nabla\varphi\rangle\,\nu_2\,dx=\int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx,
\end{aligned}
$$

for all $\varphi\in X$. Hence

$$
\begin{aligned}
&\int_\Omega\Big(|\Delta u_1|^{p-2}\Delta u_1-|\Delta u_2|^{p-2}\Delta u_2\Big)\Delta\varphi\,\omega_1\,dx+\int_\Omega\Big(|\Delta u_1|^{q-2}\Delta u_1-|\Delta u_2|^{q-2}\Delta u_2\Big)\Delta\varphi\,\nu_1\,dx\\
&+\int_\Omega\Big(|\nabla u_1|^{p-2}\langle\nabla u_1,\nabla\varphi\rangle-|\nabla u_2|^{p-2}\langle\nabla u_2,\nabla\varphi\rangle\Big)\,\omega_2\,dx+\int_\Omega\Big(|\nabla u_1|^{s-2}\langle\nabla u_1,\nabla\varphi\rangle\\
&-|\nabla u_2|^{s-2}\langle\nabla u_2,\nabla\varphi\rangle\Big)\nu_2\,dx=0.
\end{aligned}
$$

Taking $\varphi=u_1-u_2$, and using Lemma 5 (b) there exist positive constants $\alpha_p,\tilde{\alpha}_p,\alpha_q,\alpha_s$ such that

$$
\begin{aligned}
0\;=\;&\int_\Omega\Big(|\Delta u_1|^{p-2}\Delta u_1-|\Delta u_2|^{p-2}\Delta u_2\Big)(\Delta u_1-\Delta u_2)\,\omega_1\,dx\\
&+\int_\Omega\Big(|\Delta u_1|^{q-2}\Delta u_1-|\Delta u_2|^{q-2}\Delta u_2\Big)(\Delta u_1-\Delta u_2)\,\nu_1\,dx\\
&+\int_\Omega\Big(|\nabla u_1|^{p-2}\langle\nabla u_1,\nabla u_1-\nabla u_2\rangle-|\nabla u_2|^{p-2}\langle\nabla u_2,\nabla u_1-\nabla u_2\rangle\Big)\,\omega_2\,dx\\
&+\int_\Omega\Big(|\nabla u_1|^{s-2}\langle\nabla u_1,\nabla u_1-\nabla u_2\rangle-|\nabla u_2|^{s-2}\langle\nabla u_2,\nabla u_1-\nabla u_2\rangle\Big)\,\nu_2\,dx\\
=\;&\int_\Omega\Big(|\Delta u_1|^{p-2}\Delta u_1-|\Delta u_2|^{p-2}\Delta u_2\Big)(\Delta u_1-\Delta u_2)\,\omega_1\,dx\\
&+\int_\Omega\Big(|\Delta u_1|^{q-2}\Delta u_1-|\Delta u_2|^{q-2}\Delta u_2\Big)(\Delta u_1-\Delta u_2)\,\nu_1\,dx\\
&+\int_\Omega\langle\,|\nabla u_1|^{p-2}\nabla u_1-|\nabla u_2|^{p-2}\nabla u_2,\nabla u_1-\nabla u_2\rangle\,\omega_2\,dx
\end{aligned}
$$

$$
\begin{aligned}
& + \int_\Omega \langle\, |\nabla u_1|^{s-2}\nabla u_1 - |\nabla u_2|^{s-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle\, \nu_2\, dx \\
\geq\ & \alpha_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2} |\Delta u_1 - \Delta u_2|^2\, \omega_1\, dx + \tilde{\alpha}_p \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{p-2} |\nabla u_1 - \nabla u_2|^2\, \omega_2\, dx \\
& + \alpha_q \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{q-2} |\Delta u_1 - \Delta u_2|^2\, \nu_1\, dx + \alpha_s \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{s-2} |\nabla u_1 - \nabla u_2|^2\, \nu_2\, dx \\
\geq\ & \alpha_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2} |\Delta u_1 - \Delta u_2|^2\, \omega_1\, dx + \tilde{\alpha}_p \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{p-2} |\nabla u_1 - \nabla u_2|^2\, \omega_2\, dx.
\end{aligned}
$$

Therefore $\Delta u_1 = \Delta u_2$ and $\nabla u_1 = \nabla u_2$ a.e. and since $u_1, u_2 \in X$, then $u_1 = u_2$ a.e. (by Remark 1).
(III) *Estimate for* $\|u\|_X$.
In particular, for $\varphi = u \in X$ in Definition 6 we have

$$
\int_\Omega |\Delta u|^p\, \omega_1\, dx + \int_\Omega |\Delta u|^q\, \nu_1\, dx + \int_\Omega |\nabla u|^p\, \omega_2\, dx + \int_\Omega |\nabla u|^s\, \nu_2\, dx = \int_\Omega f\, u\, dx + \int_\Omega \langle G, \nabla u\rangle\, dx.
$$

Then, by Theorem 4 and Remark 3(a), we obtain

$$
\begin{aligned}
\|u\|_X^p &= \int_\Omega |\Delta u|^p\, \omega_1\, dx + \int_\Omega |\nabla u|^p\, \omega_2\, dx \\
&\leq \int_\Omega |\Delta u|^p\, \omega_1\, dx + \int_\Omega |\Delta u|^q\, \nu_1\, dx + \int_\Omega |\nabla u|^p\, \omega_2\, dx + \int_\Omega |\nabla u|^s\, \nu_2\, dx \\
&= \int_\Omega f\, u\, dx + \int_\Omega \langle G, \nabla u\rangle\, dx \\
&\leq \left\| \frac{f}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} \|u\|_{L^p(\Omega,\omega_2)} + \left\| \frac{|G|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \|\, |\nabla u|\, \|_{L^s(\Omega,\nu_2)} \\
&\leq C_\Omega \left\| \frac{f}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} \|\, |\nabla u|\, \|_{L^p(\Omega,\omega_2)} + M_2 \left\| \frac{|G|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \|\, |\nabla u|\, \|_{L^p(\Omega,\omega_2)} \\
&\leq \left(C_\Omega \left\| \frac{f}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} + M_2 \left\| \frac{|G|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \right) \|u\|_X.
\end{aligned}
$$

Therefore,

$$
\|u\|_X \leq \left(C_\Omega \left\| \frac{f}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} + M_2 \left\| \frac{|G|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \right)^{1/(p-1)}.
$$

□

Corollary 8. *Under the assumptions of Theorem 7 with* $2 \leq q, s < p < \infty$. *If* $u_1, u_2 \in X$ *are solutions of*

$$
(P_1) \begin{cases} Lu_1(x) = f(x) - div(G(x)), & in\ \Omega, \\ u_1(x) = \Delta u_1(x) = 0, & in\ \partial\Omega, \end{cases}
$$

and

$$
(P_2) \begin{cases} Lu_2(x) = \tilde{f}(x) - div(\tilde{G}(x)), & in\ \Omega, \\ u_2(x) = \Delta u_2(x) = 0, & in\ \partial\Omega, \end{cases}
$$

then

$$
\|u_1 - u_2\|_X \leq \frac{1}{\gamma^{1/(p-1)}} \left(C_\Omega \left\| \frac{f - \tilde{f}}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} + M_2 \left\| \frac{|G - \tilde{G}|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \right)^{1/(p-1)},
$$

where γ *is a positive constant,* C_Ω *and* M_2 *are the same constants of Theorem 7.*

Proof. If u_1 and u_2 are solutions of $(P1)$ and $(P2)$ then for all $\varphi \in X$ we have

$$
\int_\Omega |\Delta u_1|^{p-2}\Delta u_1\, \Delta\varphi\, \omega_1\, dx + \int_\Omega |\Delta u_1|^{q-2}\Delta u_1\, \Delta\varphi\, \nu_1\, dx + \int_\Omega |\nabla u_1|^{p-2}\langle \nabla u_1, \nabla\varphi\rangle\, \omega_2\, dx
$$

$$+\int_\Omega |\nabla u_1|^{s-2}\langle \nabla u_1, \nabla\varphi\rangle\, \nu_2\, dx - \left(\int_\Omega |\Delta u_2|^{p-2}\Delta u_2\, \Delta\varphi\, \omega_1\, dx + \int_\Omega |\Delta u_2|^{q-2}\Delta u_2\, \Delta\varphi\, \nu_1\, dx\right.$$
$$\left. + \int_\Omega |\nabla u_1|^{p-2}\langle \nabla u_2, \nabla\varphi\rangle\, \omega_2\, dx + \int_\Omega |\nabla u_2|^{s-2}\langle \nabla u_2, \nabla\varphi\rangle\, \nu_2\, dx\right) = \int_\Omega (f-\tilde f)\,\varphi\, dx + \int_\Omega \langle G-\tilde G, \nabla\varphi\rangle\, dx. \tag{7}$$

In particular, for $\varphi = u_1 - u_2$, we obtain in (7).
(i) By Lemma 5(b) and since $2 \le q, s < p < \infty$, there exist two positive constants α_p and α_q such that

$$\int_\Omega \left(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta_2|^{p-2}\Delta u_2\right)\Delta(u_1-u_2)\,\omega_1\, dx \ge \alpha_p \int_\Omega \left(|\Delta u_1| + |\Delta u_2|\right)^{p-2}|\Delta u_1 - \Delta u_2|^2\,\omega_1\, dx$$
$$\ge \alpha_p \int_\Omega |\Delta u_1 - \Delta u_2|^{p-2}|\Delta u_1 - \Delta u_2|^2\,\omega_1\, dx = \alpha_p\int_\Omega |\Delta(u_1-u_2)|^p\,\omega_1\, dx,$$

and analogously

$$\int_\Omega \left(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2\right)\Delta(u_1-u_2)\,\nu_1\, dx \ge \alpha_q\int_\Omega |\Delta(u_1-u_2)|^q\,\nu_1\, dx \ge 0.$$

(ii) Since $2 \le q, s < p < \infty$ and by Lemma 5(b), there exit two positive constants $\tilde\alpha_p$ and α_s such that

$$\int_\Omega \left(|\nabla u_1|^{p-2}\langle \nabla u_1, \nabla(u_1-u_2)\rangle - |\nabla u_2|^{p-2}\langle \nabla u_2, \nabla(u_1-u_2)\rangle\right)\omega_2\, dx$$
$$= \int_\Omega \langle |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla(u_1-u_2)\rangle\,\omega_2\, dx$$
$$\ge \tilde\alpha_p \int_\Omega (|\nabla u_1| + |\nabla u_2|)^{p-2}|\nabla u_1 - \nabla u_2|^2\,\omega_2\, dx$$
$$\ge \tilde\alpha_p\int_\Omega |\nabla u_1 - \nabla u_2|^{p-2}\,|\nabla u_1 - \nabla u_2|^2\,\omega_2\, dx = \tilde\alpha_p\int_\Omega |\nabla(u_1-u_2)|^p\,\omega_2\, dx,$$

and analogously,

$$\int_\Omega \left(|\nabla u_1|^{s-2}\langle \nabla u_1, \nabla(u_1-u_2)\rangle - |\nabla u_2|^{s-2}\langle \nabla u_2, \nabla(u_1-u_2)\rangle\right)\nu_2\, dx \ge \alpha_s\int_\Omega |\nabla(u_1-u_2)|^s\,\nu_2\, dx \ge 0.$$

(iii) By Remark 3 (a) we have

$$\left|\int_\Omega (f-\tilde f)\,(u_1-u_2)\, dx + \int_\Omega \langle G-\tilde G, \nabla(u_1-u_2)\rangle\, dx\right|$$
$$\le \left(C_\Omega\left\|\frac{f-\tilde f}{\omega_2}\right\|_{L^{p'}(\Omega,\omega_2)} + M_2\left\|\frac{|G-\tilde G|}{\nu_2}\right\|_{L^{s'}(\Omega,\nu_2)}\right)\|u_1-u_2\|_X.$$

Hence, with $\gamma = \min\{\alpha_p, \tilde\alpha_p\}$, we obtain

$$\gamma\,\|u_1-u_2\|_X^p \le \alpha_p\int_\Omega |\Delta(u_1-u_2)|^p\,\omega_1\, dx + \tilde\alpha_p\int_\Omega |\nabla(u_1-u_2)|^p\,\omega_2\, dx$$
$$\le \left(C_\Omega\left\|\frac{f-\tilde f}{\omega_2}\right\|_{L^{p'}(\Omega,\omega_2)} + M_2\left\|\frac{|G-\tilde G|}{\nu_2}\right\|_{L^{s'}(\Omega,\nu_2)}\right)\|u_1-u_2\|_X.$$

Therefore,

$$\|u_1-u_2\|_X \le \frac{1}{\gamma^{1/(p-1)}}\left(C_\Omega\left\|\frac{f-\tilde f}{\omega_2}\right\|_{L^{p'}(\Omega,\omega_2)} + M_2\left\|\frac{|G-\tilde G|}{\nu_2}\right\|_{L^{s'}(\Omega,\nu_2)}\right)^{1/(p-1)}.$$

□

Corollary 9. *Assume $2 \le q,s < p < \infty$. Let the assumptions of Theorem 7 be fulfilled, and let $\{f_m\}$ and $\{G_m\}$ be sequences of functions satisfying $\dfrac{f_m}{\omega_2} \to \dfrac{f}{\omega_2}$ in $L^{p'}(\Omega,\omega_2)$ and $\left\|\dfrac{|G_m - G|}{\nu_2}\right\|_{L^{s'}(\Omega,\nu_2)} \to 0$ as $m \to \infty$. If $u_m \in X$ is a solution of the problem*

$$(P_m)\begin{cases} Lu_m(x) = f_m(x) - div(G_m(x)), \text{ in } \Omega, \\ u_m(x) = \Delta u_m(x) = 0, \text{ in } \partial\Omega, \end{cases}$$

then $u_m \to u$ in X and u is a solution of problem (P).

Proof. By Corollary 8 we have

$$\|u_m - u_r\|_X \le \frac{1}{\gamma^{1/(p-1)}}\left(C_\Omega\left\|\frac{f_m - f_r}{\omega_2}\right\|_{L^{p'}(\Omega,\omega_2)} + M_2\left\|\frac{|G_m - G_r|}{\nu_2}\right\|_{L^{s'}(\Omega,\nu_2)}\right)^{1/(p-1)}.$$

Therefore $\{u_m\}$ is a Cauchy sequence in X. Hence, there is $u \in X$ such that $u_m \to u$ in X. We have that u is a solution of problem (P). In fact, since u_m is a solution of (P_m), for all $\varphi \in X$ we have

$$\begin{aligned}
&\int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega_1\,dx + \int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\nu_1\,dx + \int_\Omega |\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,\omega_2\,dx + \int_\Omega |\nabla u|^{s-2}\langle\nabla u,\nabla\varphi\rangle\,\nu_2\,dx \\
&= \int_\Omega \left(|\Delta u|^{p-2}\Delta u - |\Delta_m|^{p-2}\Delta u_m\right)\Delta\varphi\,\omega_1\,dx + \int_\Omega\left(|\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\right)\Delta\varphi\,\nu_1\,dx \\
&+ \int_\Omega\left(|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\right)\omega_2\,dx \\
&+ \int_\Omega\left(|\nabla u|^{s-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{s-2}\langle\nabla u_m,\nabla\varphi\rangle\right)\nu_2\,dx \\
&+ \int_\Omega |\Delta u_m|^{p-2}\Delta u_m\,\Delta\varphi\,\omega_1\,dx + \int_\Omega |\Delta u_m|^{q-2}\Delta u_m\,\Delta\varphi\,\nu_1\,dx + \int_\Omega |\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\,\omega_2\,dx \\
&+ \int_\Omega |\nabla u_m|^{s-2}\langle\nabla u_m,\nabla\varphi\rangle\,\nu_2\,dx \\
=\ & I_1 + I_2 + I_3 + I_4 + \int_\Omega f_m\varphi\,dx + \int_\Omega \langle G_m,\nabla\varphi\rangle\,dx \\
=\ & I_1 + I_2 + I_3 + I_4 + \int_\Omega f\,\varphi\,dx + \int_\Omega \langle G,\nabla\varphi\rangle\,dx + \int_\Omega (f_m - f)\,\varphi\,dx + \int_\Omega \langle G_m - G,\nabla\varphi\rangle\,dx, \qquad (8)
\end{aligned}$$

where

$I_1 = \int_\Omega \left(|\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\right)\Delta\varphi\,\omega_1\,dx,$

$I_2 = \int_\Omega \left(|\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\right)\Delta\varphi\,\nu_1\,dx,$

$I_3 = \int_\Omega \left(|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\right)\omega_2\,dx,$

$I_4 = \int_\Omega \left(|\nabla u|^{s-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{s-2}\langle\nabla u_m,\nabla\varphi\rangle\right)\nu_2\,dx.$

We have that:

(1) By Lemma 5 (a) there exists $C_p > 0$ such that

$$\begin{aligned}
|I_1| &\le \int_\Omega \left||\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\right|\,|\Delta\varphi|\,\omega_1\,dx \\
&\le C_p\int_\Omega |\Delta u - \Delta u_m|\,(|\Delta u| + |\Delta u_m|)^{p-2}\,|\Delta\varphi|\,\omega_1\,dx.
\end{aligned}$$

Let $r = p/(p-2)$. Since $\dfrac{1}{p} + \dfrac{1}{p} + \dfrac{1}{r} = 1$, by the Generalized Hölder inequality we obtain

$$|I_1| \le C_p\left(\int_\Omega |\Delta u - \Delta u_m|^p\,\omega_1\,dx\right)^{1/p}\left(\int_\Omega |\Delta\varphi|^p\,\omega_1\,dx\right)^{1/p}\left(\int_\Omega (|\Delta u| + |\Delta u_m|)^{(p-2)r}\,\omega_1\,dx\right)^{1/r}$$

$$\leq C_p \|u - u_m\|_X \, \|\varphi\|_X \| |\Delta u| + |\Delta u_m| \|_{L^p(\Omega,\omega_1)}^{(p-2)}.$$

Now, since $u_m \to u$ in X, then exists a constant $M > 0$ such that $\|u_m\|_X \leq M$. Hence,

$$\| |\Delta u| + |\Delta u_m| \|_{L^p(\Omega,\omega_1)} \leq \|u\|_X + \|u_m\|_X \leq 2M. \tag{9}$$

Therefore,

$$|I_1| \quad \leq \quad C_p \, (2M)^{p-2} \, \|u - u_m\|_X \, \|\varphi\|_X = C_1 \, \|u - u_m\|_X \, \|\varphi\|_X.$$

Analogously, there exists a constant C_3 such that

$$|I_3| \leq C_3 \|u - u_m\|_X \, \|\varphi\|_X.$$

(2) By Lemma 5 (a) there exists a positive constant C_q such that

$$\begin{aligned} |I_2| \quad &\leq \quad \int_\Omega \big| |\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m \big| \, |\Delta\varphi| \, \nu_1 \, dx \\ &\leq \quad C_q \int_\Omega |\Delta u - \Delta u_m| \, (|\Delta u| + |\Delta u_m|)^{q-2} \, |\Delta\varphi| \, \nu_1 \, dx. \end{aligned}$$

Let $\alpha = q/(q-2)$ (if $2 < q < p < \infty$). Since $\dfrac{1}{q} + \dfrac{1}{q} + \dfrac{1}{\alpha} = 1$, by the Generalized Hölder inequality we obtain

$$\begin{aligned} |I_2| \quad &\leq \quad C_q \left(\int_\Omega |\Delta u - \Delta u_m|^q \, \nu_1 \, dx \right)^{1/q} \left(\int_\Omega |\Delta\varphi|^q \, \nu_1 \, dx \right)^{1/q} \left(\int_\Omega (|\Delta u| + |\Delta u_m|)^{(q-2)\alpha} \, \nu_1 \, dx \right)^{1/\alpha} \\ &= \quad C_q \, \|\Delta u - \Delta u_m\|_{L^q(\Omega,\nu_1)} \, \|\Delta\varphi\|_{L^q(\Omega,\nu_1)} \| |\Delta u| + |\Delta u_m| \|_{L^q(\Omega,\nu_1)}^{q-2}. \end{aligned}$$

Now, by Remark 3(a) and (9) we have

$$\begin{aligned} |I_2| \quad &\leq \quad C_q \, M_1 \|\Delta u - \Delta u_m\|_{L^p(\Omega,\omega_1)} \, M_1 \, \|\Delta\varphi\|_{L^p(\Omega,\omega_1)} M_1^{q-2} \, \| |\Delta u| + |\Delta u_m| \|_{L^p(\Omega,\omega_1)}^{q-2} \\ &\leq \quad C_q \, M_1^q \|u - u_m\|_X \|\varphi\|_X \, (2M)^{q-2} \\ &= \quad C_2 \, \|u - u_m\|_X \, \|\varphi\|_X. \end{aligned}$$

Analogously, if $2 < s < p < \infty$, there exists a positive constant C_4 such that

$$|I_4| \leq C_4 \, \|u - u_m\|_X \, \|\varphi\|_X.$$

In case $q = 2$ and $s = 2$, we have $|I_2|, |I_4| \leq M_1^2 \, \|u - u_m\|_X \, \|\varphi\|_X$.
Therefore, we have $I_1, I_2, I_3, I_4 \to 0$ when $m \to \infty$.
(3) We also have

$$\left| \int_\Omega (f_m - f) \, \varphi \, dx + \int_\Omega \langle G_m - G, \nabla\varphi \rangle \, dx \right| \left(C_\Omega \left\| \frac{f_m - f}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} + M_2 \left\| \frac{|G_m - G|}{\nu_2} \right\|_{L^{s'}(\Omega,\nu_2)} \right) \|\varphi\|_X \to 0,$$

when $m \to \infty$.
Therefore, in (8), we obtain when $m \to \infty$ that

$$\begin{aligned} &\int_\Omega |\Delta u|^{p-2}\Delta u \, \Delta\varphi \, \omega_1 \, dx + \int_\Omega |\Delta u|^{q-2}\Delta u \, \Delta\varphi \, \nu_1 \, dx + \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla\varphi \rangle \, \omega_2 \, dx \\ &+ \int_\Omega |\nabla u|^{s-2} \langle \nabla u, \nabla\varphi \rangle \, \nu_2 \, dx = \int_\Omega f \, \varphi \, dx + \int_\Omega \langle G, \nabla\varphi \rangle \, dx, \end{aligned}$$

i.e., u is a solution of problem (P). $\square$

Example 1. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$, $\omega_1(x,y) = (x^2+y^2)^{-1/2}$, $\omega_2(x,y) = (x^2+y^2)^{-1/4}$ ($\omega_i \in A_4$, $p=4$ and $q=s=3$), $\nu_1(x,y) = (x^2+y^2)^{-1/3}$, $\nu_2(x,y) = (x^2+y^2)^{1/8}$, $f(x,y) = \dfrac{\cos(xy)}{(x^2+y^2)^{1/6}}$ and $G(x,y) = \left(\dfrac{\sin(x+y)}{(x^2+y^2)^{1/6}}, \dfrac{\sin(xy)}{(x^2+y^2)^{1/6}}\right)$. By Theorem 7 , the problem

$$\begin{cases} \Delta\Big[(x^2+y^2)^{-1/2}\,|\Delta u|^2\Delta u + (x^2+y^2)^{-1/3}|\Delta u|\Delta u\Big] \\ -\operatorname{div}\Big[(x^2+y^2)^{-1/4}|\nabla u|^2\nabla u + (x^2+y^2)^{-1/8}|\nabla u|\nabla u\Big] \\ = f(x) - \operatorname{div}(G(x)), \ \text{ in } \Omega \\ u(x) = \Delta u = 0, \ \text{ in } \partial\Omega \end{cases}$$

has a unique solution $u \in W^{2,4}(\Omega,\omega_1) \cap W_0^{1,4}(\Omega,\omega_2)$.

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